Discrete-Set

Amar Panji Senjaya

[2022-01-28 Fri 09:15]

Source And Learning Material:
`https://www.youtube.com/playlist?list=PLHXZ9OQGMqxersk8fUxiUMSIxODBqsKZS`
`https://www.amazon.com/Discrete-Mathematics-Applications-Susanna-Epp/dp/1337694193/ref=sr_1_2?keywords=discrete+math&qid=1645043639&sr=8-2`

1 What is Set

A **set** is a collection of objects. Individual object of set is element $\in$. Individual object of element that is not in The set is called not in element denoted with $\notin$.

1.1 Examples

- A = Employees in a company
  John $\in$ A

- B = {1,3,4,7}
  3 $\in$ A
  $\pi \notin$ B

- Z = Integers
  3 $\in$ Z

1.2 Order And Repetition Don't Matter

```
{1, 2, 3, 4, 5, 6, 7} = {2, 5, 4, 3, 1, 7, 6}
                      = {2, 2, 2, 2, 5, 4, 3, 1, 1, 7, 6, 6, 6}
```

2 Subsets

- Every Element of A is also in B = A $\subseteq B$

2.0.1 Example 1

A = {1, 3}
B = {1, 3, 4, 6}
A $\subseteq B$

2.0.2 Example 2

A = {1, 8}
B = {1, 3, 4, 6}
A $\not\subseteq B$
because $\in$ 8 is not in B

3 Set-Roster Notion

Replace ... with continuos clear pattern
Example:
Positive even integers
{0, 2, 4, 6, ...}
all even integers
{..., -6, -4, -2, 0, 2, 4, 6, ...}

4 Set-Builder Notation

General form: { x | P(x) }
x variable
"|" such that
P(x) Property is true
Example even integers
= { x | x = Twice an integer }
Example square root
{ x | $\sqrt{x} \in$ Z }

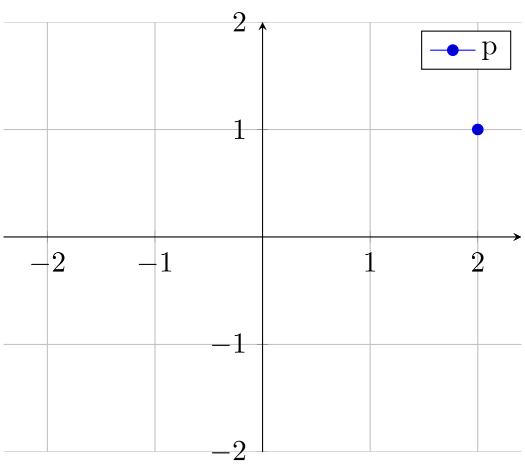
5 Empty Set

Notion: {} or $\emptyset$
{$\emptyset$}
Is $\emptyset \subset$ {1,2,3}?
Recall: A $\subset$ B means:
if x $\in$ A, then x $\in$ B
This is *vacuously* true!

6 Ordered Pairs (a, b)

- Order Matters
- (a,b) = (c,d) of a=c & b=d
- a&b could come from different sets

<u>Defn</u>: The cartesian product A $\times$ B is the set of all ordered pairs (a,b) where a $\in$ A and b $\in$ B



p = (2,1) $\in$ R x R
first component is x axis and second component is y axis

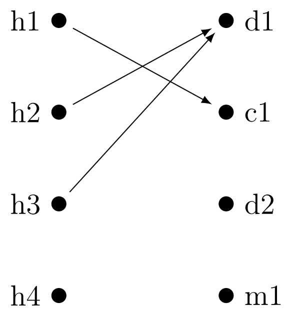
Example:
{a,b} $\times$ {0,1}
A $\times$ B = {(a,1), (a,0), (b,0), (b1)}

7 Relations

Ex: a $<$ b
compares two integers
some pairs have this relationship 2 $<$ 5

some pairs don't $5 \not< 2$

note:
h: human
d: dog
c: cat
m: monkey

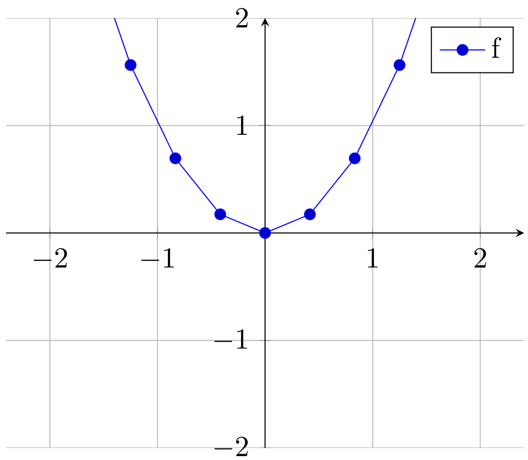


defn: A Relation R between A and B is a subset of A $\times$ B

ie ordered pairs
(a,b) $\in$ A $\times$ B

8 Functions

ex: $f(x) = x^2$



function do something to every input in my domain and produce output for each input

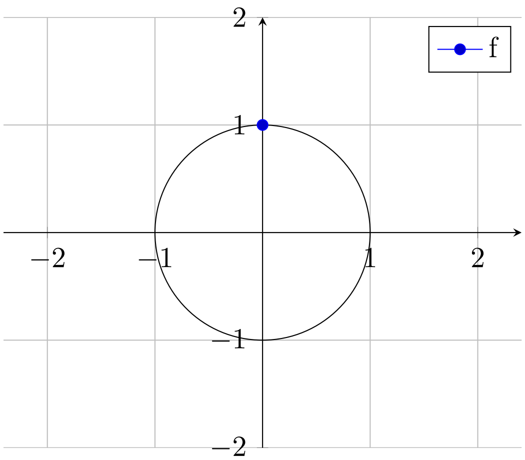
domain: set of all possible input
range: set of all possible output

Defn: A function F between A and B
is a relation between A and B such that:
subset of A $\times$ B

1. For every x $\in$ A there is an element y $\in$ B such that (x,y) $\in$ F
   Which means
   Fo every input x, there is some ouput y, F(x)=y

2. If (x,y) $\in$ F and (x,z) $\in$ F then y = z

Example: Is this relation?
Consider the relation C where (x,y) $\in$ C if $x^2 + y^2 = 1$. Is this a function?



This is relation but not a function because there is more than one output asociated with one input

9 Statement

A statement is a sentence that is either true or false

Examples:
p: "5 > 2" = True
q: "2 > 5" = False
r: "x > 2" = Not a statement because we don't know what x is

9.1 New statement from old

$\neg$ p means not p
p $\wedge$ q means p and q
p $\vee$ q means p or q

Example:
"My shirt is gray but my shorts are not"
p = My shirt is gray
q = My shorts are gray
p$\wedge$ $\neg$ q

9.2 Truth Table for $(\neg\ p) \vee (\neg\ q)$

p	q	$\neg$ p	$\neg$ q	
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Def: Two statements are logically equivalent if they have the same truth table

p	$\neg$ p	$\neg$ ($\neg$ p)
T	F	T
F	T	F

Def: A tautology t is a statement that is always true
Example:
That dog is a mammal

t: tautology
p: some statement

t	p	t $\vee$ p
T	T	T
T	F	T

Def: A contrandiction c is a statement that is always false
Example:
That dog is a reptile

c: contrandiction
p: some statement

c	p	c $\wedge$ p
F	T	F
F	F	F

so c $\wedge$ p is a contrandiction

10 Demorgan's Law & Logical Equivalent

p: Trefor is a unicorn
q: Trefor is a goldfish
$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$?
It's NOT the case that Trefor is either a unicorn OR a goldfish.
is equivalent to:
Trefor is NOT a unicorn AND is NOT a goldfish.

p	q	$\neg$ p	$\neg$ q	p $\vee$ q	$\neg$ (p $\wedge$ q)	($\neg$ p) $\wedge$ ($\neg$ q)
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Demorgan's Laws:
$\neg (p \vee q) \equiv (\neg p) \wedge (\neg q)$
$\neg (p \wedge q) \equiv (\neg p) \vee (\neg q)$

Double Negative:
$\neg(\neg p) \equiv p$

Identity Laws:
$p \vee c \equiv p$
$p \wedge t \equiv p$

Universal Bound Laws:
$p \vee t \equiv t$
$p \wedge c \equiv c$

Example:
$(\neg(p \vee \neg q)) \wedge t$
via DeMorgan's
$\equiv (\neg p \wedge \neg (\neg q)) \wedge t$
via Double Negative
$\equiv (\neg p \wedge q) \wedge t$
Via Identity
$\equiv \neg p \wedge q$

So $(\neg(p \vee \neg q)) \wedge t \equiv \neg p \wedge q$

11 Conditional Statement

Def: $p \Rightarrow q$ means:
"if p is TRUE then q is TRUE"
Whenever the hipotesis p is true then the conclution is also true.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$\neg$ p	$\neg p \vee q$
F	T
F	F
T	T
T	T

$p \Rightarrow q \equiv \neg p \vee q$

Example:
If i study hard, then i will pass
p = i study hard
q = i will pass
$p \Rightarrow q$

Either I don't study hard, or i pass
$\neg$ p = I don't study hard
q = i pass

11.1 When the hypothesis is false, the statement is vacuously true.

vacuously true meant the statement is true but true in a sort of unimportant or unintresting or vacuous set.

Example:
If Trefor is a unicorn, then everyone get's an A
p = Trefor is a unicorn
q = everyone get's an A
$p \Rightarrow q$

Example:
Either Trefor is not a unicorn, or everyone get's an A
$\neg$ p = Trefor is not a unicorn
q = everyone get's an A
$\neg p \vee q$

11.2 Negating a conditional

$\neg$ (p $\Rightarrow$ q) $\equiv$ $\neg$ ($\neg$ p $\vee$ q)
$\equiv$ ($\neg$ $\neg$ p $\wedge$ $\neg$ q) Demorgan's law
$\equiv$ p $\wedge$ $\neg$ q

11.3 Contrapositive of a conditional:

p $\Rightarrow$ q $\equiv$ $\neg$ q $\Rightarrow$ $\neg$ p
p $\Rightarrow$ q $\equiv$ $\neg$ p $\vee$ q
$\neg$ q $\Rightarrow$ $\neg$ p $\equiv$ q $\vee$ $\neg$ p

If i study hard, then i will pass p $\Rightarrow$ q
Either I don't study hard, or i pass $\neg$ p $\vee$
If i don't pass, then i didn't study hard $\Rightarrow$ $\neg$ p
Either i pass, or I didn't study hard q $\vee$ $\neg$ p

11.4 The converse and the inverse of a statement

11.4.1 The converse statement

p $\Rightarrow$ q is the statement q $\Rightarrow$ p
The converse statement is not logically equivalent

Example:
Not Logically equivalent
p $\Rightarrow$ q = If it's a dog, then it's a mammal = True
q $\Rightarrow$ p = If it's a mammal, then it's a dog = False

11.4.2 The inverse statement

p $\Rightarrow$ q is the statement $\neg$ p $\Rightarrow$ $\neg$ q
The inverse statements is not logically equivalent

So inverse $\equiv$ converse

Example
Not Logically equivalent
p $\Rightarrow$ q = If it's a dog, then it's a mammal = True
$\neg$ p $\Rightarrow$ $\neg$ q = If it's not a dog, then it's not a mammal = False

11.5 Biconditional statement

The Biconditional p $\iff q$
$meansthatbothp \Rightarrow$ q and q $\Rightarrow$ p

Example
If i study hard, then I will pass = $p \Rightarrow q$
AND if I pass, then I studied hard = $q \Rightarrow p$

i will pass **if and only if** i study hard
if and only if = $\iff$

11.6 Valid and Invalid Arguments

A valid argument is a list of premises from which the conclusion follows.

Example Argument:
If I do the dishes, then my wife will be happy with me.
I do the dishes.
Therefore, my wife is happy with me.

If **p**, then **q**.
p.
Therefore, **q**.

11.6.1 Modus Ponens

Modus Ponens is an argument of the form:
premise1 = If **p**, then **q**.
premise2 = **p**.
conclusion = Therefore, **q**.

Variables

p	q
T	T
T	F
F	F
F	T

Premises

$p \Rightarrow q$	p
T	T
F	T
T	F
T	F

Conclusion

q
T
X
X
X

11.6.2 Modus Tollens

Modus Tollens is an argument of the form:
if **p**, then **q**.
$\neg$ q.
Therefore, $\neg$ p.

example argument:
If i'm POTUS then i'm an American citizen.
I'm not an American citizen.
Therefore, I'm not the POTUS.

p = i'm POTUS
q = i'm an American citizen.
$\neg$ q = I'm not an American citizen.
$\neg$ p = I'm not the POTUS.

11.6.3 Generalization

Generalization is an argument of the form:
p.
Therefore, p $\vee$ q.

Example:
i'm a canadian
Therefore, I'm a canadian or i'm a unicorn.

11.6.4 Specialization

Specialization is an argument of the form:
p $\wedge$ q.
Therefore, p.

Example:
I'm a canadian and I have a PhD
Therefore, I'm a Canadian.

11.6.5 Contradiction

Contradiction is an argument of the form:
$\neg$ p $\Rightarrow$ c
Therefore, p.

Example:
If i'm skilled at poker, then I will win.
I won money playing poker.
Therefore, I'm skilled at poker.

p = i'm skilled at poker
q = I will win.
q = I won money playing poker.
p = I'm skilled at poker.

p $\Rightarrow$ q
q
p

This argument is invalid argument because it's use converse statement.
We know converse statement is not logically equivalent.

12 Predicates and Quantified Statements

Recall: A statement is either TRUE or FALSE

12.1 Predicate

a **predicate** is a sentence depending on variables which becomes a sttemnt upon substituting values in the domain.

Example:
P(x): x is a factor of 12 with domain Z^+

P(6) True
P(5) False
P($\frac{1}{3}$) Nonsense! $\frac{1}{3} \notin Z^+$

12.2 The Truth set

The truth set of a predicate P(x):
$\{x \in D \mid P(x)\}$
i.e All values x in the domain where P(x) is true

Example:
P(x): x is a factor of 12 with domain Z^+
TS = {1, 2, 3, 4, 5, 6, 12}
$\subseteq \mathbb{Z}^+$

12.3 The Universal Quantifier

The Universal Quantifier $\forall$ means "for all"

Main Use "quantifying" predicates
$\forall$ x $\in$ D, P(x)
For all x in the domain, P(x) is true

Example:
Every dog is a mammal
$\forall$
D = set of dogs ; P(x): X is a mammal

12.4 The existensial quantifier

The existensial quantifier $\exists$ means "there exists"

Main use "quantifying" predicates
$\exists$ x $\in$ D, P(x)
There exists x in the domain, such that P(x)is true

Ex: Some person is the oldest in the world
$\exists \in$ {People in the world}; P(x): X is the oldest

statement P: "Roofus is a mammal"
predicate P(x): "x is a mammal"
statement Q: $\forall$ x $\in$ D, P(x): "every dog is a mammal"

12.5 Negating quantifier

Negate "$\forall$ x $\in$ Zˆ {+ }, x > 3"
$\exists$ x $\in$ Z^+ ,x $\ngtr$ 3
$\exists$ x $\in$ Z^+ ,x$\leq$ 3
$\neg$ P(x)

Negating a universal
$\neg$ ($\forall$ x D, P(x)) $\equiv$ x $\in$ D, $\neg$ P(x).

Example:
Negate "Someone in our class it taller than 7 feet"
$\exists$ x $\in$ D, P(x)

D = our class
P(x) = x is taller than 7 feet.

negation:
$\neg(\exists\ x \in D, P(x))$
everyone in our class is shorter than 7 feet
$\forall\ x \in D, \neg\ P(x)$

Negating an existensial
$\neg\ (\exists\ x \in D, P(x)) \equiv \forall\ x \in D, \neg\ P(x)$.

12.6 Negating logical statement

Example
Every interger has a larger integer
$\forall\ x \in Z, P(x)$

$\forall\ x \in Z, \exists\ y \in Z, y > x$
$P(x) = \exists\ y \in Z, y > x$

True: choose $y = x + 1 \in Z$

Negate: $\exists\ x \in Z, \neg\ P(x)$
$\exists\ x \in Z, \forall\ y \in Z$, y x

Example:
Some number in D is the largest
$\exists\ x \in D, P(x)$
$\exists\ x \in D, \forall\ y \in D$, X y

Negate: $\forall\ x \in D, \exists\ y \in D, x < y$

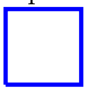
12.7 Universal Conditionals

Universal-Conditionals: $P(x) \Rightarrow Q(x)$
means $\forall\ x \in D, P(x) \Rightarrow Q(x)$

Example:
if x is the POTUS, then x is a US Citizen
P(x) = x is the POTUS
Q(x) = x is a US Citizen
D = people

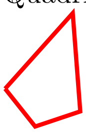
12.8 Necessary and Sufficient Conditions

Square



Rectangle

Quadrilateral



All Squares are Rectangles
if x is a Square, then x is a Rectangle
if A(x), then B(x)

A(x) is a sufficient condition for B(x)
i.e. "x being a square is sufficient to conclude x is a rectangle"

if x is a Rectangle, then x is a Quadrilateral
if B(x), then C(x)

if x is not a Quadrilateral then x is not a Rectangle
if $\neg$ C(x), then $\neg$ B(x)

sufficient to have a Square = A(x)
if we want a rectangle = B(x)
But necessary to have a Quadrilateral C(x)
A(x) $\Rightarrow$ B(x) $\Rightarrow$ C(x)
A(x) is a sufficient condition for B(x)
B(x) is a necessary condition for A(x)

Being a square is a sufficient condition fo being a rectangle
Being a rectangle is a necesary condition for being a square

13 Defining Even & Odd Integers

Informal Definition: n is an even integer if n can be written as twice an integer.
Formal Definition: n is an even integer if $\exists$ k $\in$ Z such that n = 2k

Informal Definition: n is an odd integer if n is an integer that is not even.
Formal Definition: n is an odd integer if $\exists$ k $\in$ Z such that n = 2k + 1

14 Mathematical Proofs

14.1 First Proof

Theorem: an even integer plus an odd integer is another odd integer
Proof:
Suppose m is even and n is odd.
$\exists$ $k_1 \in Z$ and $\exists$ $k_2 \in Z$ that $m = 2k_1$ and $n = 2k_2 + 1$

Then, $m + n = (2k_1)+(2k_2 + 1)$
$= 2(k_1 + k_2) + 1$
Let $k_3 = k_1 + k_2$, and note it is an integer.

Hence, $\exists$ $k_3 \in Z$ so that $m + n = 2k_3 + 1$
Thus m + n is odd.
$\square$

Direct Proofs of: $\forall$ x $\in$ D, P(x) $\Rightarrow$ Q(x)
1 State the Assumptions
2 Formally Define the assumptions
3 Manipulate
4 Arrive at Definition of conclusion
5 state the conclusion

means end of proof $\square$

14.2 Product of two even is even

Theorem: an **even integer** times an **even integer** is another **even integer**

Step 1: Define terms
Informal Definition: n is an **even integer** if n can be written as twice an integer.
Formal Definition: For n an integer:
n is even $\iff$ $\exists$ p $\in$ Z so that n = 2p

Step 2: State Theorem Formally
Theorem Formally: $\forall$ m, n $\in$ Z if m, n are even, then mn is even.

Format: $\forall$ x $\in$ D, P(x) $\Rightarrow$ Q(x)

Step 3: Play around! to make some sense of why this is actually true
4 . 8 = 32
4 = 2 . 2
8 = 2 . 4
32 = 2(2) * 2(4)
= 2(2 . 2 . 4)

Step 4: actual Proof

Start with Assumptions

Apply Definitions

Use algebra, known theorems, logical inferences, etc

get to the conclusion

Proof:
Suppose m and n are even integers.
As m, n are even, $\exists$ r and $\exists$ s so that m = 2r and n = 2s.
Then, mn = (2r)(2s) by substitutiion
= 2(2rs) by algebra
Let t = 2rs, and note it is an integer.
Hence, $\exists$ t $\in$ Z so that mn=2t
Thus mn is even.
$\square$

14.3 Rational numbers definition

Informal Definition: n is a rational number it is a fraction. Ex: 3/7
Formal Definition: n is rational number if $\exists$ p $\in$ Z, $\exists$ q $\in$ Z/{0} such that n = $\frac{p}{q}$

Theorem: The sum of two **rational numbers** is another **rational number**
Proof: Suppose m and n are rational
$\exists$ p_1, p_2 $\in$ Z and $\exists$ q_1, q_2 $\in$ Z $\setminus$ {0} so that m = $\frac{p_1}{q_1}$ and n = $\frac{p_2}{q_2}$
Then, m + n = $\frac{p_1}{q_1}$ + $\frac{p_2}{q_2}$
= $\frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$
Let p_3 = p_{1q2} + p_{2q1} and q_3 = q_{1q2}
Hence, $\exists$ p_3 $\in$ Z, $\exists$ q3 $\in$ Z $\setminus$ {0} So
m + n = $\frac{p_3}{q_3}$
Thus m + n is rational
$\square$

14.4 Prove that divisibility is transitive

Divisibility:
12 is divisible by 3
12/3 = 4 $\in$ Z
12 = 3 . 4
4 $\in$ Z

12 is not divisible by 5
12/5 $\notin$ Z

$12 \neq 5$. P
p $\notin$ Z

Definition: For n and d integers, d $\neq 0$,
d|n $\iff if \exists$ k $\in$ Z such that n = dk

d|n means:
d divides n
n is divisible by d
n is a multiple of d
d is a factor of n

Theorem:
if a is divisble by b,
and b is divisble c,
then a is divisble by c

Theorem with notation:
b|a
c|b
c|a

Example illustration:
4 | 12
2 | 4
2 | 12

proof
Let b|a and c|b
$\exists$ s, t
a = sb, b = tc

want: a = cu u $\in$ Z
a = sb
sb = stc = c(st)

Then a = sb = s(tc)
= c(st)
st $\in$ Z
c | a $\square$

14.5 Disproving implications with counterexamples

Prove or disprove
For a, b $\in$ b $\in$ Z, $a^2 > b^2$ implies a > b.

$\pm\sqrt{a^2} > \sqrt{b^2} \neq$ a $>$ b

example:
$4^2 > 3^2$ = true
(-4)$^2 > 3^2$
16 $>$ 9
-4 $>$ 3 = not true
-4 $<$ **3** false by counter example

Method of Counterexample:
Aim: to prove P(x) $\Rightarrow$ Q(x) is false
$\forall$ x $\in$ D, P(x) $\Rightarrow$ Q(x)
$\exists$ x $\in$ D, $\neg$ (P(x) $\Rightarrow$ Q(x))

Method of counterexample:
Find one a $\in$ D where P(a) $\wedge$ $\neg$ Q(a)