

Discrete-Set

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Source And Learning Material:

<https://www.youtube.com/playlist?list=PLHXZ90QGMqxersk8fUxiUMSIx0DBqsKZS>

https://www.amazon.com/Discrete-Mathematics-Applications-Susanna-Epp/dp/1337694193/ref=sr_1_2?keywords=discrete+math&qid=1645043639&sr=8-2

1 What is Set

A **set** is a collection of objects. Individual object of set is element \in . Individual object of element that is not in The set is called not in element denoted with \notin .

1.1 Examples

- A = Employees in a company
John \in A
- B = {1,3,4,7}
3 \in A
 $\pi \notin$ B
- Z = Integers
3 \in Z

1.2 Order And Repetition Don't Matter

$$\begin{aligned}\{1, 2, 3, 4, 5, 6, 7\} &= \{2, 5, 4, 3, 1, 7, 6\} \\ &= \{2, 2, 2, 2, 5, 4, 3, 1, 1, 7, 6, 6, 6\}\end{aligned}$$

2 Subsets

- Every Element of A is also in B = $A \subseteq B$

2.0.1 Example 1

$$A = \{1, 3\}$$

$$B = \{1, 3, 4, 6\}$$

$$A \subseteq B$$

2.0.2 Example 2

$$A = \{1, 8\}$$

$$B = \{1, 3, 4, 6\}$$

$$A \not\subseteq B$$

because 8 is not in B

3 Set-Roster Notion

Replace ... with continuous clear pattern

Example:

Positive even integers

$$\{0, 2, 4, 6, \dots\}$$

all even integers

$$\{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$$

4 Set-Builder Notation

General form: $\{ x \mid P(x) \}$

x variable

"|" such that

P(x) Property is true

Example even integers

$$= \{ x \mid x = \text{Twice an integer} \}$$

Example square root

$$\{ x \mid \sqrt{x} \in \mathbb{Z} \}$$

5 Empty Set

Notion: $\{\}$ or \emptyset

$\{\emptyset\}$

Is $\emptyset \subset \{1,2,3\}$?

Recall: $A \subset B$ means:

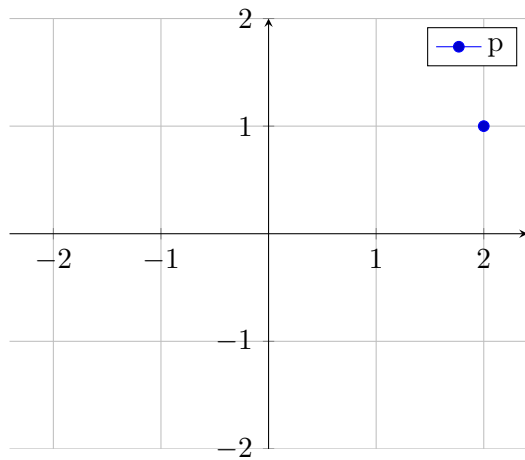
if $x \in A$, then $x \in B$

This is *vacuously* true!

6 Ordered Pairs (a, b)

- Order Matters
- $(a,b) = (c,d)$ if $a=c$ & $b=d$
- a & b could come from different sets

Defn: The cartesian product $A \times B$ is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$



$p = (2,1) \in \mathbb{R} \times \mathbb{R}$

first component is x axis and second component is y axis

Example:

$\{a,b\} \times \{0,1\}$

$A \times B = \{(a,1), (a,0), (b,0), (b,1)\}$

7 Relations

Ex: $a < b$

compares two integers

some pairs have this relationship $2 < 5$

some pairs don't $5 \not\leq 2$

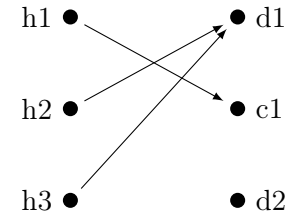
note:

h: human

d: dog

c: cat

m: monkey



h4 ● ● m1

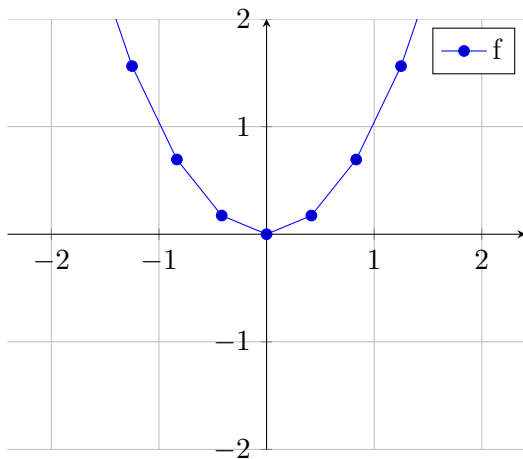
defn: A Relation R between A and B is a subset of $A \times B$

ie ordered pairs

$(a,b) \in A \times B$

8 Functions

ex: $f(x) = x^2$



function do something to every input in my domain and produce output for each input

domain: set of all possible input

range: set of all possible output

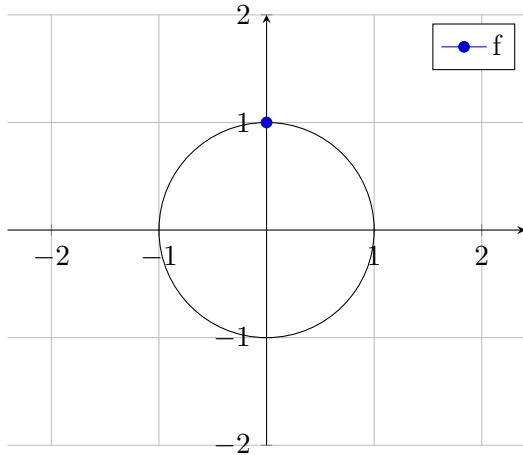
Defn: A function F between A and B
is a relation between A and B such that:
subset of $A \times B$

1. For every $x \in A$ there is an element $y \in B$ such that $(x,y) \in F$
Which means
For every input x , there is some output y , $F(x)=y$

2. If $(x,y) \in F$ and $(x,z) \in F$ then $y = z$

Example: Is this relation?

Consider the relation C where $(x,y) \in C$ if $x^2 + y^2 = 1$. Is this a function?



This is relation but not a function because there is more than one output associated with one input

9 Statement

A statement is a sentence that is either true or false

Examples:

p: " $5 > 2$ " = True

q: " $2 > 5$ " = False

r: " $x > 2$ " = Not a statement because we don't know what x is

9.1 New statement from old

$\neg p$ means not p

$p \wedge q$ means p and q

$p \vee q$ means p or q

Example:

"My shirt is gray but my shorts are not"

p = My shirt is gray

q = My shorts are gray

$p \wedge \neg q$

9.2 Truth Table for $(\neg p) \vee (\neg q)$

p	q	$\neg p$	$\neg q$	
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Def: Two statements are logically equivalent if they have the same truth table

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

Def: A tautology t is a statement that is always true

Example:

That dog is a mammal

t : tautology

p : some statement

t	p	$t \vee p$
T	T	T
T	F	T

Def: A contrandiction c is a statement that is always false

Example:

That dog is a reptile

c : contrandiction

p : some statement

c	p	$c \wedge p$
F	T	F
F	F	F

so $c \wedge p$ is a contrandiction

10 Demorgan's Law & Logical Equivalent

p: Trefor is a unicorn

q: Trefor is a goldfish

$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$?

It's NOT the case that Trefor is either a unicorn OR a goldfish.

is equivalent to:

Trefor is NOT a unicorn AND is NOT a goldfish.

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \wedge q)$	$(\neg p) \wedge (\neg q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Demorgan's Laws:

$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$

$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$

Double Negative:

$\neg(\neg p) \equiv p$

Identity Laws:

$p \vee c \equiv p$

$p \wedge t \equiv p$

Universal Bound Laws:

$p \vee t \equiv t$

$p \wedge c \equiv c$

Example:

$(\neg(p \vee \neg q)) \wedge t$

via DeMorgan's

$\equiv (\neg p \wedge \neg(\neg q)) \wedge t$

via Double Negative

$\equiv (\neg p \wedge q) \wedge t$

Via Identity

$\equiv \neg p \wedge q$

So $(\neg(p \vee \neg q)) \wedge t \equiv \neg p \wedge q$

11 Conditional Statement

Def: $p \Rightarrow q$ means:

"if p is TRUE then q is TRUE"

Whenever the hipotesis p is true then the conclution is also true.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$\neg p$	$\neg p \vee q$
F	T
F	F
T	T
T	T

$$p \Rightarrow q \equiv \neg p \vee q$$

Example:

If i study hard, then i will pass

$p =$ i study hard

$q =$ i will pass

$p \Rightarrow q$

Either I don't study hard, or i pass

$\neg p =$ I don't study hard

$q =$ i pass

11.1 When the hypothesis is false, the statement is vacuously true.

vacuously true meant the statement is true but true in a sort of unimportant or uninteresting or vacuous set.

Example:

If Trefor is a unicorn, then everyone get's an A

$p =$ Trefor is a unicorn

$q =$ everyone get's an A

$p \Rightarrow q$

Example:

Either Trefor is not a unicorn, or everyone get's an A

$\neg p =$ Trefor is not a unicorn

$q =$ everyone get's an A

$\neg p \vee q$

11.2 Negating a conditional

$$\begin{aligned}\neg (p \Rightarrow q) &\equiv \neg (\neg p \vee q) \\ &\equiv (\neg \neg p \wedge \neg q) \text{ Demorgan's law} \\ &\equiv p \wedge \neg q\end{aligned}$$

11.3 Contrapositive of a conditional:

$$\begin{aligned}p \Rightarrow q &\equiv \neg q \Rightarrow \neg p \\ p \Rightarrow q &\equiv \neg p \vee q \\ \neg q \Rightarrow \neg p &\equiv q \vee \neg p\end{aligned}$$

If i study hard, then i will pass $p \Rightarrow q$
Either I don't study hard, or i pass $\neg p \vee q$
If i don't pass, then i didn't study hard $\Rightarrow \neg p$
Either i pass, or I didn't study hard $q \vee \neg p$

11.4 The converse and the inverse of a statement

11.4.1 The converse statement

$p \Rightarrow q$ is the statement $q \Rightarrow p$
The converse statement is not logically equivalent

Example:

Not Logically equivalent

$p \Rightarrow q$ = If it's a dog, then it's a mammal = True

$q \Rightarrow p$ = If it's a mammal, then it's a dog = False

11.4.2 The inverse statement

$p \Rightarrow q$ is the statement $\neg p \Rightarrow \neg q$
The inverse statements is not logically equivalent

So inverse \equiv converse

Example

Not Logically equivalent

$p \Rightarrow q$ = If it's a dog, then it's a mammal = True

$\neg p \Rightarrow \neg q$ = If it's not a dog, then it's not a mammal = False

11.5 Biconditional statement

The Biconditional $p \iff q$
meansthatboth $p \Rightarrow q$ and $q \Rightarrow p$

Example

If i study hard, then I will pass = $p \Rightarrow q$

AND if I pass, then I studied hard = $q \Rightarrow p$

i will pass **if and only if** i study hard

if and only if = \Longleftrightarrow

11.6 Valid and Invalid Arguments

A valid argument is a list of premises from which the conclusion follows.

Example Argument:

If I do the dishes, then my wife will be happy with me.

I do the dishes.

Therefore, my wife is happy with me.

If **p**, then **q**.

p.

Therefore, **q**.

11.6.1 Modus Ponens

Modus Ponens is an argument of the form:

premise1 = If **p**, then **q**.

premise2 = **p**.

conclusion = Therefore, **q**.

Variables

p	q
T	T
T	F
F	F
F	T

Premises

$p \Rightarrow q$	p
T	T
F	T
T	F
T	F

Conclusion

$$\frac{q}{T}$$

X
X
X

11.6.2 Modus Tollens

Modus Tollens is an argument of the form:

if **p**, then **q**.

$\neg q$.

Therefore, $\neg p$.

example argument:

If i'm POTUS then i'm an American citizen.

I'm not an American citizen.

Therefore, I'm not the POTUS.

p = i'm POTUS

q = i'm an American citizen.

$\neg q$ = I'm not an American citizen.

$\neg p$ = I'm not the POTUS.

11.6.3 Generalization

Generalization is an argument of the form:

p .

Therefore, $p \vee q$.

Example:

i'm a canadian

Therefore, I'm a canadian or i'm a unicorn.

11.6.4 Specialization

Specialization is an argument of the form:

$p \wedge q$.

Therefore, p .

Example:

I'm a canadian and I have a PhD

Therefore, I'm a Canadian.

11.6.5 Contradiction

Contradiction is an argument of the form:

$$\neg p \Rightarrow c$$

Therefore, p.

Example:

If i'm skilled at poker, then I will win.

I won money playing poker.

Therefore, I'm skilled at poker.

p = i'm skilled at poker

q = I will win.

q = I won money playing poker.

p = I'm skilled at poker.

$$p \Rightarrow q$$

q

p

This argument is invalid argument because it's use converse statement.

We know converse statement is not logically equivalent.

12 Predicates and Quantified Statements

Recall: A statement is either TRUE or FALSE

12.1 Predicate

a **predicate** is a sentence depending on variables which becomes a sttemnt upon substituting values in the domain.

Example:

P(x): x is a factor of 12 with domain Z^+

P(6) True

P(5) False

P($\frac{1}{3}$) Nonsense! $\frac{1}{3} \notin Z^+$

12.2 The Truth set

The truth set of a predicate P(x):

$$\{x \in D \mid P(x)\}$$

i.e All values x in the domain where P(x) is true

Example:

$P(x)$: x is a factor of 12 with domain \mathbb{Z}^+

$TS = \{1, 2, 3, 4, 5, 6, 12\}$

$\subseteq \mathbb{Z}^+$

12.3 The Universal Quantifier

The Universal Quantifier \forall means "for all"

Main Use "quantifying" predicates

$\forall x \in D, P(x)$

For all x in the domain, $P(x)$ is true

Example:

Every dog is a mammal

\forall

D = set of dogs ; $P(x)$: X is a mammal

12.4 The existensial quantifier

The existensial quantifier \exists means "there exists"

Main use "quantifying" predicates

$\exists x \in D, P(x)$

There exists x in the domain, such that $P(x)$ is true

Ex: Some person is the oldest in the world

$\exists \in \{\text{People in the world}\}; P(x)$: X is the oldest

statement P : "Roofus is a mammal"

predicate $P(x)$: " x is a mammal"

statement Q : $\forall x \in D, P(x)$: "every dog is a mammal"

12.5 Negating quantifier

Negate " $\forall x \in \mathbb{Z}^+, x > 3$ "

$\exists x \in \mathbb{Z}^+, x \not> 3$

$\exists x \in \mathbb{Z}^+, x \leq 3$

$\neg P(x)$

Negating a universal

$\neg (\forall x \in D, P(x)) \equiv \exists x \in D, \neg P(x)$.

Example:

Negate "Someone in our class is taller than 7 feet"

$\exists x \in D, P(x)$

D = our class
 $P(x)$ = x is taller than 7 feet.

negation:
 $\neg(\exists x \in D, P(x))$
everyone in our class is shorter than 7 feet
 $\forall x \in D, \neg P(x)$

Negating an existensial
 $\neg(\exists x \in D, P(x)) \equiv \forall x \in D, \neg P(x)$.

12.6 Negating logical statement

Example
Every interger has a larger integer
 $\forall x \in \mathbb{Z}, P(x)$

$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, y > x$
 $P(x) = \exists y \in \mathbb{Z}, y > x$

True: choose $y = x + 1 \in \mathbb{Z}$

Negate: $\exists x \in \mathbb{Z}, \neg P(x)$
 $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, y \leq x$

Example:
Some number in D is the largest
 $\exists x \in D, P(x)$
 $\exists x \in D, \forall y \in D, x \geq y$

Negate: $\forall x \in D, \exists y \in D, x < y$

12.7 Universal Conditionals

Universal-Conditionals: $P(x) \Rightarrow Q(x)$
means $\forall x \in D, P(x) \Rightarrow Q(x)$

Example:
if x is the POTUS, then x is a US Citizen
 $P(x)$ = x is the POTUS
 $Q(x)$ = x is a US Citizen
 D = people

12.8 Necessary and Sufficient Conditions

Square



Rectangle



Quadrilateral



All Squares are Rectangles

if x is a Square, then x is a Rectangle

if $A(x)$, then $B(x)$

$A(x)$ is a sufficient condition for $B(x)$

i.e. "x being a square is sufficient to conclude x is a rectangle"

if x is a Rectangle, then x is a Quadrilateral

if $B(x)$, then $C(x)$

if x is not a Quadrilateral then x is not a Rectangle

if $\neg C(x)$, then $\neg B(x)$

sufficient to have a Square = $A(x)$

if we want a rectangle = $B(x)$

But necessary to have a Quadrilateral $C(x)$

$A(x) \Rightarrow B(x) \Rightarrow C(x)$

$A(x)$ is a sufficient condition for $B(x)$

$B(x)$ is a necessary condition for $A(x)$

Being a square is a sufficient condition for being a rectangle

Being a rectangle is a necessary condition for being a square

13 Defining Even & Odd Integers

Informal Definition: n is an even integer if n can be written as twice an integer.

Formal Definition: n is an even integer if $\exists k \in \mathbb{Z}$ such that $n = 2k$

Informal Definition: n is an odd integer if n is an integer that is not even.

Formal Definition: n is an odd integer if $\exists k \in \mathbb{Z}$ such that $n = 2k + 1$

14 Mathematical Proof

14.1 First Proof

Theorem: an even integer plus an odd integer is another odd integer

Proof:

Suppose m is even and n is odd.

$\exists k_1 \in \mathbb{Z}$ and $\exists k_2 \in \mathbb{Z}$ that $m = 2k_1$ and $n = 2k_2 + 1$

Then, $m + n = (2k_1) + (2k_2 + 1)$

$= 2(k_1 + k_2) + 1$

Let $k_3 = k_1 + k_2$, and note it is an integer.

Hence, $\exists k_3 \in \mathbb{Z}$ so that $m + n = 2k_3 + 1$

Thus $m + n$ is odd.

Direct Proofs of: $\forall x \in D, P(x) \Rightarrow Q(x)$

1 State the Assumptions

2 Formally Define the assumptions

3 Manipulate

4 Arrive at Definition of conclusion

5 state the conclusion

14.2 Product of two even is even

Theorem: an **even integer** times an **even integer** is another **even integer**

Step 1: Define terms

Informal Definition: n is an **even integer** if n can be written as twice an integer.

Formal Definition: For n an integer:

n is even $\iff \exists p \in \mathbb{Z}$

So that $n = 2p$

Step 2: State Theorem Formally

Theorem Formally: $\forall m, n \in \mathbb{Z}$ if m, n are even,
then mn is even

Format: $\forall x \in D, P(x) \Rightarrow Q(x)$

Step 3: Play around!

$$4 \cdot 8 = 32$$

$$4 = 2 \cdot 2$$

$$8 = 2 \cdot 4$$

$$32 = 2(2) \cdot 2(4)$$

$$= 2(2 \cdot 2 \cdot 4)$$

Step 4: actual Proof

Start with Assumptions

Apply Definitions

Use algebra, known theorems, logical inferences, etc

get to the conclusion

Proof:

Suppose m and n are even integers.

As m, n are even, $\exists r$ and $\exists s$ so that $m = 2r$ and $n = 2s$.

Then, $mn = (2r)(2s)$ by substitutiion

$= 2(2rs)$ by algebra

Let $t = 2rs$, and note it is an integer.

Hence, $\exists t \in \mathbb{Z}$ so that $mn=2t$

Thus mn is even.

14.3 Rational numbers definition + first proof

14.4 Proving that divisibility is transitive

14.5 Disproving implications with counterexamples

14.6