

# Discrete-Set

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Source And Learning Material:

<https://www.youtube.com/playlist?list=PLHXZ90QGMqxersk8fUxiUMSIx0DBqsKZS>

[https://www.amazon.com/Discrete-Mathematics-Applications-Susanna-Epp/dp/1337694193/ref=sr\\_1\\_2?keywords=discrete+math&qid=1645043639&sr=8-2](https://www.amazon.com/Discrete-Mathematics-Applications-Susanna-Epp/dp/1337694193/ref=sr_1_2?keywords=discrete+math&qid=1645043639&sr=8-2)

## 1 What is Set

A **set** is a collection of objects. Individual object of set is element  $\in$ . Individual object of element that is not in The set is called not in element denoted with  $\notin$ .

### 1.1 Examples

- A = Employees in a company  
John  $\in$  A
- B = {1,3,4,7}  
3  $\in$  A  
 $\pi \notin$  B
- Z = Integers  
3  $\in$  Z

### 1.2 Order And Repetition Don't Matter

$$\begin{aligned}\{1, 2, 3, 4, 5, 6, 7\} &= \{2, 5, 4, 3, 1, 7, 6\} \\ &= \{2, 2, 2, 2, 5, 4, 3, 1, 1, 7, 6, 6, 6\}\end{aligned}$$

## 2 Subsets

- Every Element of A is also in B =  $A \subseteq B$

### 2.0.1 Example 1

$$A = \{1, 3\}$$

$$B = \{1, 3, 4, 6\}$$

$$A \subseteq B$$

### 2.0.2 Example 2

$$A = \{1, 8\}$$

$$B = \{1, 3, 4, 6\}$$

$$A \not\subseteq B$$

*because*  $8$  is not in B

## 3 Set-Roster Notion

Replace ... with continuous clear pattern

Example:

Positive even integers

$$\{0, 2, 4, 6, \dots\}$$

all even integers

$$\{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$$

## 4 Set-Builder Notation

General form:  $\{ x \mid P(x) \}$

x variable

"|" such that

P(x) Property is true

Example even integers

$$= \{ x \mid x = \text{Twice an integer} \}$$

Example square root

$$\{ x \mid \sqrt{x} \in \mathbb{Z} \}$$

## 5 Empty Set

Notion:  $\{\}$  or  $\emptyset$

$\{\emptyset\}$

Is  $\emptyset \subset \{1,2,3\}$ ?

Recall:  $A \subset B$  means:

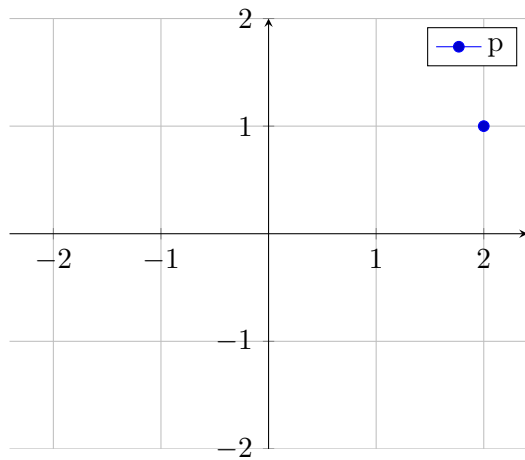
if  $x \in A$ , then  $x \in B$

This is *vacuously* true!

## 6 Ordered Pairs $(a, b)$

- Order Matters
- $(a,b) = (c,d)$  if  $a=c$  &  $b=d$
- $a$  &  $b$  could come from different sets

Defn: The cartesian product  $A \times B$  is the set of all ordered pairs  $(a,b)$  where  $a \in A$  and  $b \in B$



$p = (2,1) \in \mathbb{R} \times \mathbb{R}$

first component is x axis and second component is y axis

Example:

$\{a,b\} \times \{0,1\}$

$A \times B = \{(a,1), (a,0), (b,0), (b,1)\}$

## 7 Relations

Ex:  $a < b$

compares two integers

some pairs have this relationship  $2 < 5$

some pairs don't  $5 \not\prec 2$

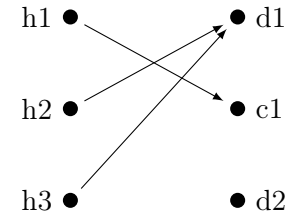
note:

h: human

d: dog

c: cat

m: monkey



h4 ●                      ● m1

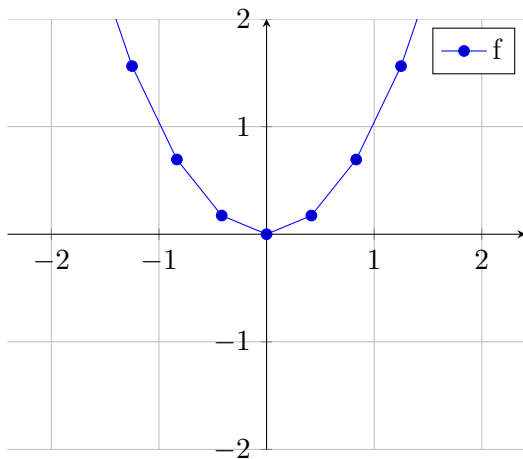
defn: A Relation R between A and B is a subset of  $A \times B$

ie ordered pairs

$(a,b) \in A \times B$

## 8 Functions

ex:  $f(x) = x^2$



function do something to every input in my domain and produce output for each input

domain: set of all possible input

range: set of all possible output

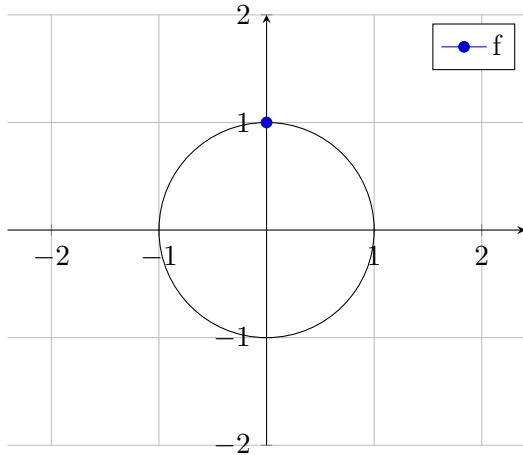
Defn: A function F between A and B  
is a relation between A and B such that:  
subset of  $A \times B$

1. For every  $x \in A$  there is an element  $y \in B$  such that  $(x,y) \in F$   
Which means  
For every input  $x$ , there is some output  $y$ ,  $F(x)=y$

2. If  $(x,y) \in F$  and  $(x,z) \in F$  then  $y = z$

Example: Is this relation?

Consider the relation  $C$  where  $(x,y) \in C$  if  $x^2 + y^2 = 1$ . Is this a function?



This is relation but not a function because there is more than one output associated with one input

## 9 Statement

A statement is a sentence that is either true or false

Examples:

p: " $5 > 2$ " = True

q: " $2 > 5$ " = False

r: " $x > 2$ " = Not a statement because we don't know what  $x$  is

### 9.1 New statement from old

$\neg p$  means not  $p$

$p \wedge q$  means  $p$  and  $q$

$p \vee q$  means  $p$  or  $q$

Example:

"My shirt is gray but my shorts are not"

$p$  = My shirt is gray

$q$  = My shorts are gray

$p \wedge \neg q$

## 9.2 Truth Table for $(\neg p) \vee (\neg q)$

p	q	$\neg p$	$\neg q$	
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Def: Two statements are logically equivalent if they have the same truth table

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

Def: A tautology  $t$  is a statement that is always true

Example:

That dog is a mammal

$t$ : tautology

$p$ : some statement

t	p	$t \vee p$
T	T	T
T	F	T

Def: A contrandiction  $c$  is a statement that is always false

Example:

That dog is a reptile

$c$ : contrandiction

$p$ : some statement

c	p	$c \wedge p$
F	T	F
F	F	F

so  $c \wedge p$  is a contrandiction

## 10 Demorgan's Law & Logical Equivalent

p: Trefor is a unicorn

q: Trefor is a goldfish

$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$ ?

It's NOT the case that Trefor is either a unicorn OR a goldfish.

is equivalent to:

Trefor is NOT a unicorn AND is NOT a goldfish.

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \wedge q)$	$(\neg p) \wedge (\neg q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Demorgan's Laws:

$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$

$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$

Double Negative:

$\neg(\neg p) \equiv p$

Identity Laws:

$p \vee c \equiv p$

$p \wedge t \equiv p$

Universal Bound Laws:

$p \vee t \equiv t$

$p \wedge c \equiv c$

Example:

$(\neg(p \vee \neg q)) \wedge t$

via DeMorgan's

$\equiv (\neg p \wedge \neg(\neg q)) \wedge t$

via Double Negative

$\equiv (\neg p \wedge q) \wedge t$

Via Identity

$\equiv \neg p \wedge q$

So  $(\neg(p \vee \neg q)) \wedge t \equiv \neg p \wedge q$



## 11 Conditional Statement

Def:  $p \Rightarrow q$  means:

"if p is TRUE then q is TRUE"

Whenever the hipotesis p is true then the conclution is also true.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$\neg p$	$\neg p \vee q$
F	T
F	F
T	T
T	T

$$p \Rightarrow q \equiv \neg p \vee q$$

Example:

If i study hard, then i will pass

$p$  = i study hard

$q$  = i will pass

$p \Rightarrow q$

Either I don't study hard, or i pass

$\neg p$  = I don't study hard

$q$  = i pass

### 11.1 When the hypothesis is false, the statement is vacuously true.

vacuously true meant the statement is true but true in a sort of unimportant or uninteresting or vacuous set.

Example:

If Trefor is a unicorn, then everyone get's an A

$p$  = Trefor is a unicorn

$q$  = everyone get's an A

$p \Rightarrow q$

Example:

Either Trefor is not a unicorn, or everyone get's an A

$\neg p$  = Trefor is not a unicorn

$q$  = everyone get's an A

$\neg p \vee q$

## 11.2 Negating a conditional

$$\begin{aligned}\neg (p \Rightarrow q) &\equiv \neg (\neg p \vee q) \\ &\equiv (\neg \neg p \wedge \neg q) \text{ Demorgan's law} \\ &\equiv p \wedge \neg q\end{aligned}$$

## 11.3 Contrapositive of a conditional:

$$\begin{aligned}p \Rightarrow q &\equiv \neg q \Rightarrow \neg p \\ p \Rightarrow q &\equiv \neg p \vee q \\ \neg q \Rightarrow \neg p &\equiv q \vee \neg p\end{aligned}$$

If i study hard, then i will pass  $p \Rightarrow q$   
Either I don't study hard, or i pass  $\neg p \vee q$   
If i don't pass, then i didn't study hard  $\neg q \Rightarrow \neg p$   
Either i pass, or I didn't study hard  $q \vee \neg p$

## 11.4 The converse and the inverse of a statement

### 11.4.1 The converse statement

$p \Rightarrow q$  is the statement  $q \Rightarrow p$   
The converse statement is not logically equivalent

Example:

Not Logically equivalent

$p \Rightarrow q$  = If it's a dog, then it's a mammal = True

$q \Rightarrow p$  = If it's a mammal, then it's a dog = False

### 11.4.2 The inverse statement

$p \Rightarrow q$  is the statement  $\neg p \Rightarrow \neg q$   
The inverse statements is not logically equivalent

So inverse  $\equiv$  converse

Example

Not Logically equivalent

$p \Rightarrow q$  = If it's a dog, then it's a mammal = True

$\neg p \Rightarrow \neg q$  = If it's not a dog, then it's not a mammal = False

## 11.5 Biconditional statement

The Biconditional  $p \iff q$   
*meansthatboth*  $p \Rightarrow q$  and  $q \Rightarrow p$

Example

If i study hard, then I will pass =  $p \Rightarrow q$

AND if I pass, then I studied hard =  $q \Rightarrow p$

i will pass **if and only if** i study hard

if and only if =  $\Longleftrightarrow$

## 11.6 Valid and Invalid Arguments

A valid argument is a list of premises from which the conclusion follows.

Example Argument:

If I do the dishes, then my wife will be happy with me.

I do the dishes.

Therefore, my wife is happy with me.

If **p**, then **q**.

**p**.

Therefore, **q**.

### 11.6.1 Modus Ponens

**Modus Ponens** is an argument of the form:

premise1 = If **p**, then **q**.

premise2 = **p**.

conclusion = Therefore, **q**.

Variables

p	q
T	T
T	F
F	F
F	T

Premises

$p \Rightarrow q$	p
T	T
F	T
T	F
T	F

Conclusion

$$\frac{q}{T}$$

X  
X  
X

### 11.6.2 Modus Tollens

**Modus Tollens** is an argument of the form:

if **p**, then **q**.

$\neg q$ .

Therefore,  $\neg p$ .

example argument:

If i'm POTUS then i'm an American citizen.

I'm not an American citizen.

Therefore, I'm not the POTUS.

$p$  = i'm POTUS

$q$  = i'm an American citizen.

$\neg q$  = I'm not an American citizen.

$\neg p$  = I'm not the POTUS.

### 11.6.3 Generalization

**Generalization** is an argument of the form:

$p$ .

Therefore,  $p \vee q$ .

Example:

i'm a canadian

Therefore, I'm a canadian or i'm a unicorn.

### 11.6.4 Specialization

**Specialization** is an argument of the form:

$p \wedge q$ .

Therefore,  $p$ .

Example:

I'm a canadian and I have a PhD

Therefore, I'm a Canadian.

### 11.6.5 Contradiction

**Contradiction** is an argument of the form:

$$\neg p \Rightarrow c$$

Therefore, p.

Example:

If i'm skilled at poker, then I will win.

I won money playing poker.

Therefore, I'm skilled at poker.

p = i'm skilled at poker

q = I will win.

q = I won money playing poker.

p = I'm skilled at poker.

$$p \Rightarrow q$$

q

p

This argument is invalid argument because it's use converse statement.

We know converse statement is not logically equivalent.

## 12 Predicates and Quantified Statements

Recall: A statement is either TRUE or FALSE

### 12.1 Predicate

a **predicate** is a sentence depending on variables which becomes a sttemnt upon substituting values in the domain.

Example:

P(x): x is a factor of 12 with domain  $Z^+$

P(6) True

P(5) False

P( $\frac{1}{3}$ ) Nonsense!  $\frac{1}{3} \notin Z^+$

### 12.2 The Truth set

**The truth set** of a predicate P(x):

$$\{x \in D \mid P(x)\}$$

i.e All values x in the domain where P(x) is true

Example:

$P(x)$ :  $x$  is a factor of 12 with domain  $\mathbb{Z}^+$

$TS = \{1, 2, 3, 4, 5, 6, 12\}$

$\subseteq \mathbb{Z}^+$

## 12.3 The Universal Quantifier

**The Universal Quantifier**  $\forall$  means "for all"

Main Use "quantifying" predicates

$\forall x \in D, P(x)$

For all  $x$  in the domain,  $P(x)$  is true

Example:

Every dog is a mammal

$\forall$

$D$  = set of dogs ;  $P(x)$ :  $X$  is a mammal

## 12.4 The existensial quantifier

**The existensial quantifier**  $\exists$  means "there exists"

Main use "quantifying" predicates

$\exists x \in D, P(x)$

There exists  $x$  in the domain, such that  $P(x)$  is true

Ex: Some person is the oldest in the world

$\exists \in \{\text{People in the world}\}; P(x)$ :  $X$  is the oldest

statement  $P$ : "Roofus is a mammal"

predicate  $P(x)$ : " $x$  is a mammal"

statement  $Q$ :  $\forall x \in D, P(x)$ : "every dog is a mammal"

## 12.5 Negating quantifier

Negate " $\forall x \in \mathbb{Z}^+, x > 3$ "

$\exists x \in \mathbb{Z}^+, x \not> 3$

$\exists x \in \mathbb{Z}^+, x \leq 3$

$\neg P(x)$

Negating a universal

$\neg (\forall x \in D, P(x)) \equiv \exists x \in D, \neg P(x)$ .

Example:

Negate "Someone in our class is taller than 7 feet"

$\exists x \in D, P(x)$

$D$  = our class  
 $P(x)$  =  $x$  is taller than 7 feet.

negation:  
 $\neg(\exists x \in D, P(x))$   
everyone in our class is shorter than 7 feet  
 $\forall x \in D, \neg P(x)$

Negating an existensial  
 $\neg(\exists x \in D, P(x)) \equiv \forall x \in D, \neg P(x)$ .

## 12.6 Negating logical statement

Example  
Every interger has a larger integer  
 $\forall x \in \mathbb{Z}, P(x)$

$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, y > x$   
 $P(x) = \exists y \in \mathbb{Z}, y > x$

True: choose  $y = x + 1 \in \mathbb{Z}$

Negate:  $\exists x \in \mathbb{Z}, \neg P(x)$   
 $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, y \leq x$

Example:  
Some number in  $D$  is the largest  
 $\exists x \in D, P(x)$   
 $\exists x \in D, \forall y \in D, x \geq y$

Negate:  $\forall x \in D, \exists y \in D, x < y$

## 12.7 Universal Conditionals

Universal-Conditionals:  $P(x) \Rightarrow Q(x)$   
means  $\forall x \in D, P(x) \Rightarrow Q(x)$

Example:  
if  $x$  is the POTUS, then  $x$  is a US Citizen  
 $P(x)$  =  $x$  is the POTUS  
 $Q(x)$  =  $x$  is a US Citizen  
 $D$  = people

## 12.8 Necessary and Sufficient Conditions

Square



Rectangle



Quadrilateral



All Squares are Rectangles

if  $x$  is a Square, then  $x$  is a Rectangle

if  $A(x)$ , then  $B(x)$

$A(x)$  is a sufficient condition for  $B(x)$

i.e. "x being a square is sufficient to conclude x is a rectangle"

if  $x$  is a Rectangle, then  $x$  is a Quadrilateral

if  $B(x)$ , then  $C(x)$

if  $x$  is not a Quadrilateral then  $x$  is not a Rectangle

if  $\neg C(x)$ , then  $\neg B(x)$

sufficient to have a Square =  $A(x)$

if we want a rectangle =  $B(x)$

But necessary to have a Quadrilateral  $C(x)$

$A(x) \Rightarrow B(x) \Rightarrow C(x)$

$A(x)$  is a sufficient condition for  $B(x)$

$B(x)$  is a necessary condition for  $A(x)$

Being a square is a sufficient condition for being a rectangle

Being a rectangle is a necessary condition for being a square

## 13 Defining Even & Odd Integers

Informal Definition:  $n$  is an even integer if  $n$  can be written as twice an integer.

Formal Definition:  $n$  is an even integer if  $\exists k \in \mathbb{Z}$  such that  $n = 2k$

Informal Definition:  $n$  is an odd integer if  $n$  is an integer that is not even.

Formal Definition:  $n$  is an odd integer if  $\exists k \in \mathbb{Z}$  such that  $n = 2k + 1$



## 14 First Proof

Theorem: an even integer plus an odd integer is another odd integer

Proof:

Suppose  $m$  is even and  $n$  is odd.

$\exists k_1 \in \mathbb{Z}$  and  $\exists k_2 \in \mathbb{Z}$  that  $m = 2k_1$  and  $n = 2k_2 + 1$

Then,  $m + n = (2k_1) + (2k_2 + 1)$

$= 2(k_1 + k_2) + 1$

Let  $k_3 = k_1 + k_2$ , and note it is an integer.

Hence,  $\exists k_3 \in \mathbb{Z}$  so that  $m + n = 2k_3 + 1$

Thus  $m + n$  is odd.

Direct Proofs of:  $\forall x \in D, P(x) \Rightarrow Q(x)$

1 State the Assumptions

2 Formally Define the assumptions

3 Manipulate

4 Arrive at Definition of conclusion

5 state the conclusion