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<u>Exercice 1 :</u>

A : array of n integers, sorted

$$A[i] = \begin{cases} A[i] & \text{for } i \in [0, n-1] \\ \infty & \text{for } i > n \end{cases}$$

So $A = [A[0], A[1], \ldots, A[n-1], \infty, \infty, \ldots, \infty]$

we can't apply binary search right away because we don't know how many ∞ are part of A.

So let's create subdivision of A.

1rst : $[A[0]]$

2nd : $[A[1], A[2]]$

3rd : $[A[3], \ldots, A[6]]$

$\vdots$

last : $[A[\frac{n-1}{2}], \ldots, A[n-1]]$

Then we look at every step if x ~~appea~~ is part of the sub array.

If yes, then we do a binary search on the sub array.

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Running time of this algorithm:

1^{rst} step : $O(\log n)$

2^{nd} step - binary search : $O(\log n)$

so total is $O(\log n)$

Exercice 2 :

A, B : 2 sorted array of size m and n.
Let's use recursion to solve this problem.

+ if one of the length of a or b is 0. Then the union of the two list is only the non zero list. So looking for k is equal at looking for k in the non zero list.
+ Then let's differenciate 2 cases :
  + $\frac{len(A) + len(B)}{2} < k$

    if $A\left[\frac{len(A)}{2}\right] > B\left[\frac{len(B)}{2}\right]$, it means than k cannot look into the first half of the array B.
    otherwise if $A\left[\frac{len(A)}{2}\right] < B\left[\frac{len(B)}{2}\right]$ then k cannot look in the first array of A.
    Then we repeat this method recursively.

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$$\frac{len(A) + len(B)}{2} > k$$

if $A\left[\frac{len A}{2}\right] > B\left[\frac{len B}{2}\right]$, it means we cannot have k looking inside the second half of A.
otherwise if $A\left[\frac{len A}{2}\right] < B\left[\frac{len B}{2}\right]$ then k cannot look in the second part of B.
Then we repeat this method recursively.

Exercice 3 :

2. let's write x and y :

$$x = \underbrace{x_{n-1} \; \dots \; x_{\frac{2n}{3}}}_{x_A} \; \underbrace{x_{\frac{2n}{3}-1} \; \dots \; x_{\frac{n}{3}}}_{x_B} \; \underbrace{x_{\frac{n}{3}-1} \; \dots \; x_0}_{x_C}$$

$$y = \underbrace{y_{n-1} \; \dots \; y_{\frac{2n}{3}}}_{y_A} \; \underbrace{y_{\frac{2n}{3}-1} \; \dots \; y_{\frac{n}{3}}}_{y_B} \; \underbrace{y_{\frac{n}{3}-1} \; \dots \; y_0}_{y_C}$$

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$$\begin{cases} x = x_A 10^{\frac{2n}{3}} + x_B 10^{\frac{n}{3}} + x_C \\ y = y_A 10^{\frac{2n}{3}} + y_B 10^{\frac{n}{3}} + y_C \end{cases}$$

$$\begin{aligned} xy = \; & x_A y_A \cdot 10^{4n/3} \\ & + (x_A y_B + x_B y_A) \cdot 10^{n} \\ & + (x_A y_C + x_B y_B + x_C y_A) \cdot 10^{2n/3} \\ & + (x_B y_C + x_C y_B) \cdot 10^{n/3} \\ & + x_C y_C \end{aligned}$$

with $(x_A y_B + x_B y_A) = \big((x_A + x_B)(y_A + y_B)\big) - x_A y_A - x_B y_B$

$(x_A y_C + x_C y_A) = \big((x_A + x_C)(y_A + y_C)\big) - x_A y_A - x_C y_C$

$(x_B y_C + x_C y_B) = \big((x_B + x_C)(y_B + y_C)\big) - x_B y_B - x_C y_C$

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so xy is the multiplication of 2 integers using only 6 multiplications of $\frac{n}{3}$ bit numbers.

$$\begin{cases} x_A y_A \\ x_B y_B \\ x_C y_C \\ (x_A + x_B)(y_A + y_B) \\ (x_A + x_C)(y_A + y_C) \\ (x_B + x_C)(y_B + y_C) \end{cases}$$

b) $T(n) = 3T\left(\frac{n}{3}\right) + 3T\left(\frac{n}{3}\right) + cn$

$T(n) = 6T\left(\frac{n}{3}\right) + cn$

per the master theorem.

$$\begin{cases} a = 6 \\ b = 3 \\ k = 1 \end{cases}$$

so $T(n) = O(n^{\log_3 6}) = O(n^{1,63})$.
which is longer than $O(n^{1,59})$.
so it is better to split into two parts.

c) if $T(n) = 5T\left(\frac{n}{3}\right) + cn$

then $T(n) = O(n^{\log_3 5}) = O(n^{1,46})$. In that case it is better to split into 3 parts.

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d). Using Toom-Cook multiplication algorithm:

let's write $x(t) = x_A t^2 + x_B t + x_C$

$y(t) = y_A t^2 + y_B t + y_C$

So $w(t) = x(t) \cdot y(t) = w_4 t^4 + w_3 t^3 + w_2 t^2 + w_1 t + w_0$.

let's pick 5 random points $\{-2, -1, 0, 1, \infty\}$

with $x(\infty) = \lim_{t \to \infty} \frac{p(t)}{t^{\deg(x)}}$

So

for $t = 0$ $\begin{cases} x(0) = x_C \\ y(0) = y_C \end{cases}$

for $t = 1$ $\begin{cases} x(1) = x_A + x_B + x_C \\ y(1) = y_A + y_B + y_C \end{cases}$

for $t = -1$ $\begin{cases} x(-1) = x_A - x_B + x_C \\ y(-1) = y_A - y_B + y_C \end{cases}$

for $t = -2$ $\begin{cases} x(-2) = 4x_A - 2x_B + x_C \\ y(-2) = 4y_A - 2y_B + y_C \end{cases}$

for $t = \infty$ $\begin{cases} x(\infty) = x_A \\ y(\infty) = y_A \end{cases}$

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(1) $w(0) = w_0 = x_c y_c$

(2) $w(1) = w_4 + w_3 + w_2 + w_1 + w_0 = (x_A + x_B + x_c)(y_A + y_B + y_c)$

(3) $w(-1) = w_4 - w_3 + w_2 - w_1 + w_0 = (x_A - x_B + x_c)(y_A - y_B + y_c)$

(4) $w(-2) = 16 w_4 - 8 w_3 + 4 w_2 - 2 w_1 + w_0 = (4x_A - 2x_B + x_c)(4y_A - 2y_B + y_c)$

(5) $w(\infty) = w_4 = x_A y_A$

So

$$\begin{pmatrix} w(0) \\ w(1) \\ w(-1) \\ w(-2) \\ w(\infty) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 16 & -8 & 4 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$

So

$$\begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1/3 & -1 & 1/6 & -2 \\ -1 & 1/2 & 1/2 & 0 & -1 \\ -1/2 & 1/6 & 1/2 & -1/6 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w(0) \\ w(1) \\ w(-1) \\ w(-2) \\ w(\infty) \end{pmatrix}$$

So $w_0 = x_c y_c$

$$w_1 = \frac{1}{2} x_c y_c + \frac{1}{3} \cdot (x_A + x_B + x_c)(y_A + y_B + y_c) - (x_A - x_B + x_c)(y_A - y_B + y_c)$$
$$+ \frac{1}{6} \cdot (4x_A - 2x_B + x_c)(4y_A - 2y_B + y_c) - 2 x_A y_A.$$

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$$w_2 = - x_C y_C + \frac{1}{2}(x_A + x_B + x_C)(y_A + y_B + y_C) + \frac{1}{2}(x_A - x_B + x_C)(y_A - y_B + y_C)$$
$$- x_A y_A$$

$$w_3 = -\frac{1}{2} x_C y_C + \frac{1}{6}(x_A + x_B + x_C)(y_A + y_B + y_C) + \frac{1}{2}(x_A - x_B + x_C)(y_A - y_B + y_C)$$
$$- \frac{1}{6}(4x_A - 2x_B + x_C)(4y_A - 2y_B + y_C) + 2 x_A y_A$$

$$w_4 = x_A y_A.$$

so far we used only 5 multiplications (Toom - Cook algorithm)

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Exercice 4 :

First phase : $\frac{2}{3}$ elements are sorted recursively, the $\frac{2}{3}$ First

Second phase : " , the $\frac{2}{3}$ last

Third phase : " , the $\frac{2}{3}$ first.

Recurrence for running time :

$$T(n) = 3T\left(\frac{2n}{3}\right)$$

so per the master theorem.

$$\begin{cases} a = 3 \\ b = \frac{3}{2} \\ k = 0 \end{cases} \quad \text{so } a > b^k \quad \text{so} \quad T(n) = O\left(n^{\log_{3/2} 3}\right) = O\left(n^{2.71}\right)$$

This algorithm is not efficient, compared to mergesort for exemple.

5. (30 points) In the table below, indicate the relationship between functions f and g for each pair (f, g) by writing **yes** or **no** in each box. For example, if $f = O(g)$ then write **yes** in the first box.

f	g	O	o	Ω	ω	Θ
$\log^5 n$	$10 \log^3 n$	no	no	yes	yes	no
$n^2 \log(2n)$	$n \log n$	no	no	yes	yes	no
$\sqrt{\log n}$	$\log \log n$	no	no	yes	yes	no
$n^2 + n^{1/3}$	$n^2 \log n + n^{5/2}$	yes	yes	no	no	no
$\sqrt{n} + 1500$	$n^{1/3} + \log n$	no	no	yes	yes	no
$\frac{3^n}{n^2}$	$2^n \log n$	no	no	yes	yes	no
$n^{\log n}$	2^n	yes	yes	no	no	no
2^n	$\frac{3^n}{n^{\log n}}$	yes	yes	no	no	no
n^n	$n!$	no	no	yes	yes	no
$\log n^n$	$\log n!$	yes	no	yes	no	yes