Aggregating Distortions in Networks with Multi-Product Firms[1]

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[1] The views expressed are those of the authors and do not necessarily represent the views of the Central Bank of Chile or its board members.

Motivation

- Resource misallocation is a driver of aggregate TFP differences cross countries and over time
- Despite emphasis on micro data, the literature often assumes firms produce a single product
- However, most transactions involve multiproduct firms (e.g., 99% in B2B in Chile)
- Resource allocation across products within firms can affect aggregate allocative efficiency
- Challenge: Assign inputs to products when firms use joint-production technologies

How do multiproduct firms with joint-production technology affect allocative efficiency?

This Paper

Theory: Framework to measure allocative efficiency with multiproduct firms and joint production

- Result: technological constraints in adjusting product mix attenuate reallocation effects on TFP
- Sufficient statistics: firm-level PPF curvature using product prices

Meausurement:

- Data: Product-level prices and quantities for the universe of formal B2B transactions in Chile
- Joint production test
- Product-level markup estimation

Application: Aggregate TFP growth around Covid for Chile:

- Allocative efficiency explains the bulk of aggregate TFP growth
- Ignoring joint production overestimates allocative efficiency gains
- Parametric counterfactual on distance to Pareto-efficient frontier

Selected Related Literature

1. Multiproduct Firms

Klette and Kortum (2004); Bernard et al. (2010, 2011); Mayer et al. (2014); De Loecker et al. (2016); Hottman et al. (2016); Mayer et al. (2021); Wang and Yang (2023)

Study misallocation with multi-product firms that engage in joint production

2. Joint Production

Diewert (1971); Lau (1972); Hall (1973); Almunia et al. (2021); Dhyne et al. (2022); Ding (2023); Carrillo et al. (2023)

Provide empirical evidence consistent with joint production

3. Aggregation & Inefficiencies in Production Networks

Hulten (1978); Basu and Fernald (2002); Restuccia and Rogerson (2008); Petrin and Levinsohn (2012); Jones (2013); Liu (2019); Baqaee and Farhi (2020); Bigio and La'O (2020); Kikkawa (2022); Baqaee et al. (2023)

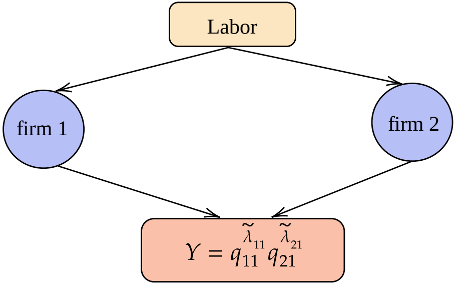
Generalize theory to allow joint production and quantify using IO product-level data

Table of Contents

Illustrative Example

- Demonstrate that aggregate TFP can change without technological progress
- Illustrate how the change in TFP varies with and without joint production
- Show how sufficient statistics work
- The intuition from the simple example survives in a general setup

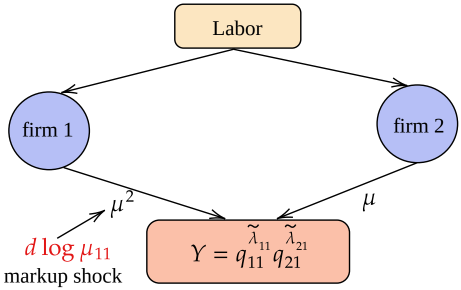
Case A: Single Product Firms



- First index: firm. Second index: firm's product
- Production functions: $q_{11} = L_{11}, \quad q_{21} = L_{21}$
- Each firm sets a marked up price: $p_{i1} = \mu_{i1} mc_{i1}$
- Household offer labor, owns firms and consumes:

  $Y = q_{11}^{\tilde{\lambda}_{11}} q_{21}^{\tilde{\lambda}_{21}}$
- $TFP = Y/L$

Case A: Single Product Firms Markup Shock TFP Change

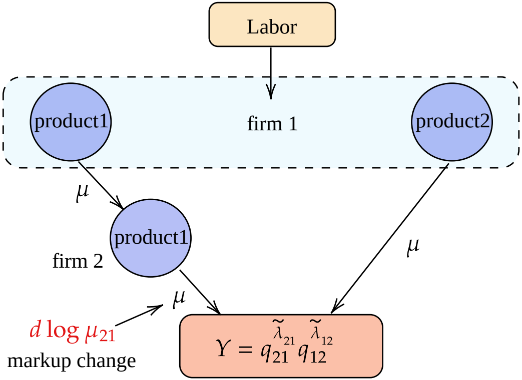


- Firm 1 underproduces due to a higher markup
- First order TFP response to markup change:

$$\Delta \log TFP = \tilde{\lambda}_{11} \underbrace{\left(\frac{\bar{\mu}}{\mu_{11}} - 1 \right)}_{<0} d \log \mu_{11}$$

- where $\bar{\mu}$ is the harmonic mean markup
- An increase in markups for products with mean or above mean markup decreases TFP

Case B: Multiproduct Firms Markup Shock TFP Change



- Cumulative markups: $\Gamma_{11} = \mu^2$ and $\Gamma_{12} = \mu$, $\bar{\Gamma}_1$ is firm 1 harmonic mean cumulative markup
- Define $\tilde{\lambda}_{11} = \tilde{\lambda}_{21}$ to capture the (indirect) cost effect of firm 1's product 1 on final demand
- First order TFP response to markup change:

$$\Delta \log TFP = \tilde{\lambda}_{21} \underbrace{\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1 \right)}_{<0} d \log \mu_{21}$$

- An increase in cumulative markups for products with high cumulative markup decreases TFP

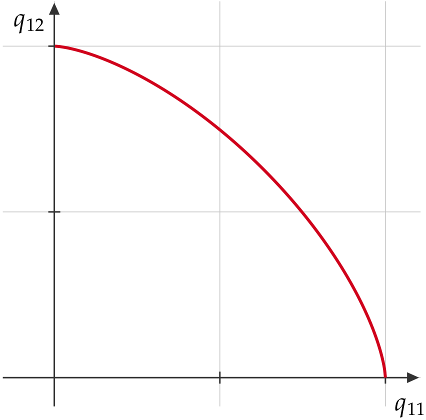
Production Possibility Frontier with Joint Production

Multiproduct firms use shared inputs to make multiple outputs (Diewert (1971); Lau (1972); Hall (1973)) details

Constant-Elasticity of Transformation (CET, σ) (CES DUAL)

How easy it is to transform one output into another
(dual of CES in inputs)

$$\underbrace{\left(q_{11}^{\frac{\sigma+1}{\sigma}}+q_{12}^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}}}_{\text{output bundle}}=L$$



Production Possibility Frontier with Joint Production

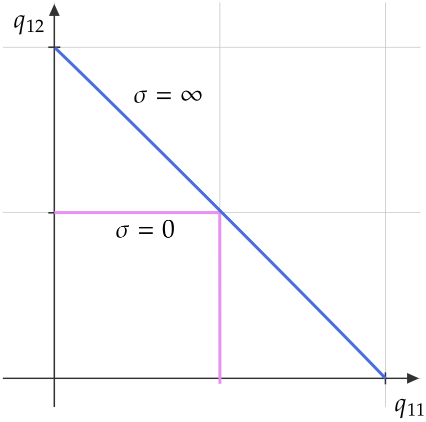
Multiproduct firms use shared inputs to make multiple outputs (Diewert (1971); Lau (1972); Hall (1973))

Constant-Elasticity of Transformation (CET, σ) (CES DUAL)

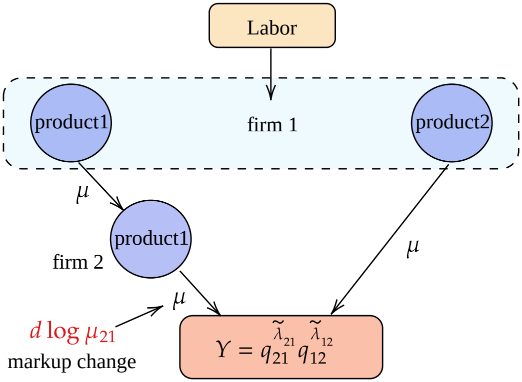
How easy it is to transform one output into another

$$\underbrace{\left(q_{11}^{\frac{\sigma+1}{\sigma}} + q_{12}^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}}}_{\text{output bundle}} = L$$

- $\sigma \to \infty$: perfect transformation (outputs are perfect substitutes)
- $\sigma \to 0$: fixed proportions (outputs are fixed and cannot be reallocated)



Case C: Joint Production Markup Shock TFP Change



$$\left(q_{11}^{\frac{\sigma+1}{\sigma}} + q_{12}^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}} = L$$

- First order TFP response to the same markup change:

$$\Delta \log TFP = \left(1 - \frac{1}{\sigma+1}\right) \tilde{\lambda}_{21} \left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right) d \log \mu_{21}$$

depends on ease of adjusting product mix, σ

$$\Delta \log TFP = \begin{cases} \tilde{\lambda}_{21} \left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right) d \log \mu_{21} & \text{if } \sigma \to \infty \\ 0 & \text{if } \sigma \to 0 \end{cases}$$

- Joint production attenuates TFP response

Using Prices Instead of PPF Curvature

- To derive sufficient statistics, rewrite the TFP response in a way that is not dependent on σ
- With joint production, relative price changes are associated with changes in production ratio:

$$d\log\left(p_{11}/p_{12}\right) = \frac{1}{\sigma} d\log\left(q_{11}/q_{12}\right)$$

Using Prices Instead of PPF Curvature

- With joint production, relative price changes are associated with changes in production ratio:

$$d\log(p_{11}/p_{12}) = \frac{1}{\sigma} d\log(q_{11}/q_{12})$$

- Solving for prices: $d\log(p_{11}/p_{12}) = -\frac{1}{\sigma+1} d\log\mu_{21}$, the TFP response is expressed by: deri

$$\Delta \log TFP = \tilde{\lambda}_{21}\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right) d\log\mu_{21}\left(1 - \frac{1}{\sigma+1}\right)$$

$$= \underbrace{\tilde{\lambda}_{21}\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right) d\log\mu_{21}}_{\text{Single-Product Term}} + \underbrace{d\log(p_{11}/p_{12})\,\tilde{\lambda}_{21}\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right)}_{\text{Multi-Product Term}}$$

Using Prices Instead of PPF Curvature

- With joint production, relative price changes are associated with changes in production ratio:

$$d\log\left(p_{11}/p_{12}\right)=\frac{1}{\sigma}d\log\left(q_{11}/q_{12}\right)$$

- Solving for prices: $d\log\left(p_{11}/p_{12}\right)=-\frac{1}{\sigma+1}d\log\mu_{21}$, the TFP response is expressed by: deri

$$\Delta\log TFP=\tilde{\lambda}_{21}\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}}-1\right)d\log\mu_{21}\left(1-\frac{1}{\sigma+1}\right)$$

$$=\underbrace{-\underbrace{d\log\Lambda}_{\text{Labor Share}}-\tilde{\lambda}_{21}d\log\mu_{21}}_{\text{Single-Product Term}}+\underbrace{Cov_{s_1}\left(d\log p_{(1,\cdot)},\frac{\bar{\Gamma}_1}{\Gamma_{(1,\cdot)}}\right)}_{\text{Multiproduct Term}}$$

- Single-product term: Different factor Marginal Revenue Products between firms
- Multiproduct term: Different factor MRP within firms with technological constraints

Sufficient Statistics for Simple Example

Proposition

In the simple example, TFP response to the markup shock can be expressed as

$$\Delta \log TFP = \underbrace{-\underbrace{d\log\Lambda}_{\text{Labor Share}} - \tilde{\lambda}_{21}\, d\log\mu_{21}}_{\text{Single-Product Term}} + \underbrace{Cov_{\lambda_1}\left(d\log p_{(1,\cdot)}, \frac{\bar{\Gamma}_1}{\Gamma_{(1,\cdot)}}\right)}_{\text{Multiproduct Term}}$$

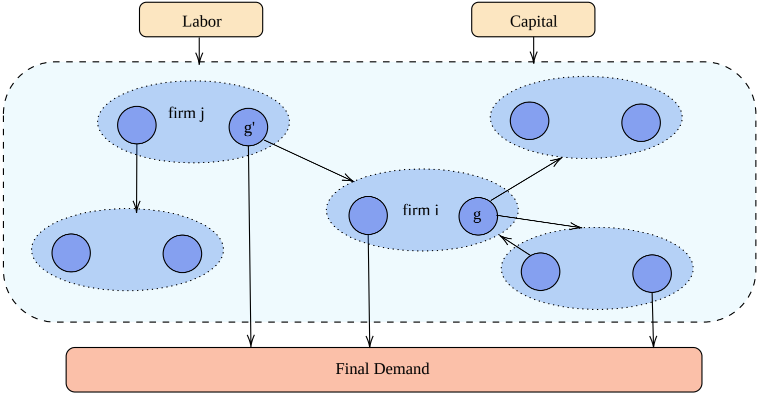
Single-Product term:
Change in factor shares discounting pure markup changes

Multiproduct term:
Firm-level product mix adjustments considering (potential) technological constraints

Table of Contents

General Theory for Measurement

- Introduce a general framework to derive sufficient statistics that can accommodate any number of products and arbitrary input-output linkages



General Setup

- Firm $i \in \mathcal{N}$ makes product $g \in \mathcal{G}$ using g' from firm j and factors as inputs
- CRS joint production with firm-level technology shock (technology shock can be product specific with additional assumption): details

$$F_i^Q\left(\underbrace{\{q_{ig}\}_{g\in\mathcal{G}}}_{\text{outputs}}\right) = A_i F_i^X\left(\underbrace{\{x_{i,jg'}\}_{j\in\mathcal{N},g'\in\mathcal{G}}}_{\text{product } g' \text{ from j}}, L_i, K_i\right)$$

- Firm i sets its products' price as $p_{ig} = \mu_{ig} mc_{ig}$, where μ_{ig} is the wedge (markup)
- Agnostic about microfoundations of markup (e.g., market power, sticky prices, etc.)

General Setup

- Final demand as maximizer of homothetic aggregator, $Y = D(c_{ig}, \ldots, c_{NG})$: subject to:

$$\underbrace{\sum_{i\in\mathcal{N}}\sum_{g\in\mathcal{G}} p_{ig}c_{ig}}_{\text{Final goods expenditure}} = \underbrace{\sum_{f\in\{L,K\}} w_f L_f}_{\text{Factor income}} + \underbrace{\sum_{i\in\mathcal{N}}\sum_{g\in\mathcal{G}} (1-1/\mu_{ig})\, p_{ig}q_{ig}}_{\text{Profits}}$$

- Resource constraints: $q_{ig} = c_{ig} + \sum_{j\in\mathcal{N}} x_{jig}, \quad \sum_{i\in\mathcal{N}} L_i = L, \quad \sum_{i\in\mathcal{N}} K_i = K$

Definition (General Equilibrium)

Given productivities $\boldsymbol{A}$ *and product-level markups* $\boldsymbol{\mu}$*, an equilibrium consists of prices* p_{ig}, w_f*, intermediate input choices* $x_{ijg'}$*, factor inputs* (L_i, K_i)*, outputs* q_{ig}*, and consumption* c_{ig}*, such that (i) each firm minimizes its costs and sets prices as markups over marginal costs, (ii) final demand maximizes the aggregator subject to budget constraints, (iii) and all goods and factor markets clear.*

Growth Accounting with Multi-Product Firms

Proposition

To the first order, aggregate TFP is decomposed into allocative efficiency and technology :

$$\Delta TFP = \underbrace{\underbrace{\sum_i \tilde{\lambda}_i Cov_{\mathbf{s}_i}\left(d\log p_{(i,\cdot)}, \frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}}\right)}_{\text{Multiproduct Term}} \underbrace{-\sum_f \tilde{\Lambda}_f d\log \Lambda_f - \sum_i \tilde{\lambda}_i d\log \mu_i}_{\text{Single-Product Term}}}_{\Delta \text{Allocative Efficiency}} + \underbrace{\sum_i \tilde{\lambda}_i d\log A_i}_{\Delta \text{Technology(residual)}}$$

Single-Product term:
Change in factor shares discounting pure markup changes

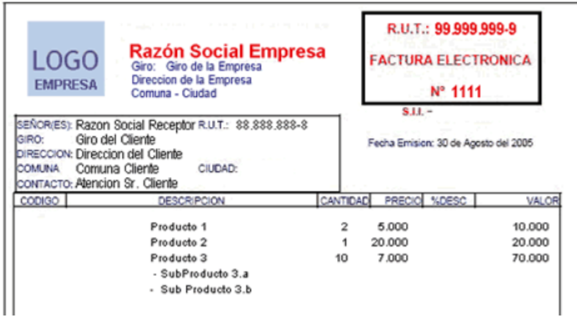
Multiproduct term:
Firm-level product mix adjustments considering (potential) technological constraints

Table of Contents

Data: Firm-to-Firm Transactions at the Product Level for Chile

Invoice Example

- Electronic invoices between formal firms universe, including services sectors
- Information on buyers, sellers, product, quantity, and price distribution
- Combined with balance sheets information


LOGO EMPRESA

Razón Social Empresa
Giro: Giro de la Empresa
Direccion de la Empresa
Comuna - Ciudad

R.U.T.: 99.999.999-9
FACTURA ELECTRONICA
Nº 1111
S.I.I. -

SEÑOR(ES): Razon Social Receptor R.U.T.: 88.888.888-8
GIRO: Giro del Cliente
DIRECCION: Direccion del Cliente
COMUNA: Comuna Cliente CIUDAD:
CONTACTO: Atencion Sr. Cliente

Fecha Emision: 30 de Agosto del 2005

CODIGO	DESCRIPCION	CANTIDAD	PRECIO	%DESC	VALOR
	Producto 1	2	5.000		10.000
	Producto 2	1	20.000		20.000
	Producto 3	10	7.000		70.000
	- SubProducto 3.a				
	- Sub Producto 3.b				

Construction of Sufficient Statistics

- Domar weights: from firm-to-firm transactions with balance sheets details
- Firm-product specific cumulative wedge Variance decomposition

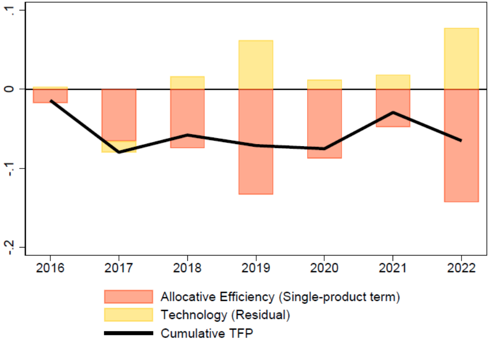
$$\Gamma_{ig} \equiv \underbrace{\frac{\tilde{\lambda}_{ig}}{\text{sales share}_{ig}}}_{\text{downstream wedge}} \times \underbrace{\mu_{ig}}_{\text{own markup}}$$

- Own markup: μ_{ig} estimation combining the production function approach (starting from Hall, 1988) with joint production (starting from Diewert (1973) details
- Factor shares & aggregate TFP are computed from aggregated microdata

Cumulative TFP Growth Decomposition Ignoring Multi-Product Term

$$\Delta \text{TFP} = \underbrace{\Delta \text{Technology}}_{\text{Residual}} + \underbrace{\Delta \text{Single-Product Term}}_{\text{Allocative Efficiency}} + \Delta \text{Multi-Product Term}$$

- **TFP shrinks (-7%)** in the period consistent with aggregate measures
- Negative allocative efficiency (-14%): firms with high-cumulative wedges shrink: low-cumulative wedges ones expand in terms of inputs usage
- Since TFP does not fall by as much, technology (residual) must be large and positive (7%)



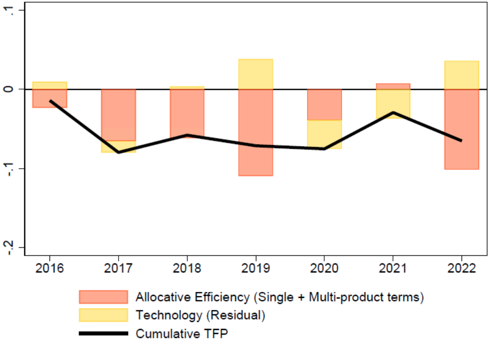
Cumulative TFP Growth Decomposition with Multi-Product Term

$$\Delta\text{TFP} = \underbrace{\Delta\text{Technology}}_{\text{Residual}} + \underbrace{\Delta\text{Single-Product Term} + \Delta\text{Multi-Product Term}}_{\text{Allocative Efficiency}}$$

- Adding the multi-product term attenuates the strong reallocation effects from 14% to 10%
- Technology (residual) shrinks to 3%

(1) Ignoring multiproduct overestimate AE by 40%

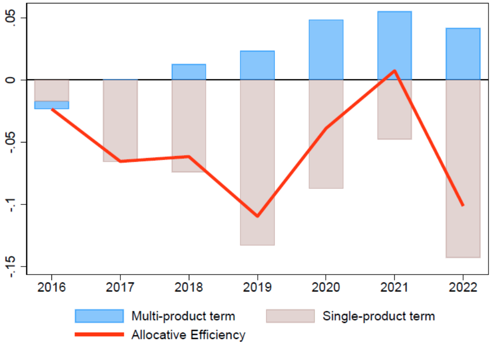
(2) Multiproduct term shirks the residual by 55%



Cumulative AE growth Decomposition with Multi-Product Term

$$\Delta\text{TFP} = \Delta\text{Technology} + \underbrace{\Delta\text{Single-Product Term} + \Delta\text{Multi-Product Term}}_{\text{Allocative Efficiency}}$$

- The **multiproduct term attenuates** the **single-product term** as the theory suggests
- Firms facing changes in product-specific demands cannot readjust their product mix as desired due to technological constraints



Conclusion

- We provide evidence consistent with multi-product firms engaging in joint production
- We develop sufficient statistics to measure resource misallocation with joint production
- We find changes in allocative efficiency explain the bulk of TFP growth in Chile for 2016-2022
- Ignoring joint production overestimates the importance of resource reallocation
- Policies aiming to correct resource misallocation should account for technological constraints

Table of Contents

Empirical Evidence of Joint Production and Parametric Counterfactual

- Sufficient statistics approach is useful for growth accounting given past observed data
- Develop a complementary parametric approach for counterfactuals, which requires the knowledge of the curvature of firm-level PPF
- Assume joint production function with constant elasticity of transformation, σ
- Present evidence consistent with joint production and estimate σ

Motivating Empirical Evidence of Joint Production

Assume:

- The cost function is $C(q_1,\ldots,q_G) = P_I\left(\sum_{i=1}^{G}\omega_g q_g^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}}$
- Demand for each product is : $q_g = D_g p_g^{-\theta_g}$
- Price of each product is set to maximize profits given $\{D_g\}$ and input prices variable markup

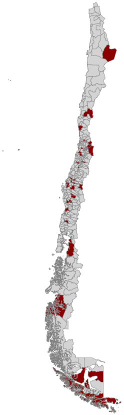
Proposition

A negative demand shock to product m ($d\log D_m < 0$) lowers quantity and raises the price of every other product $g \neq m$, if, and only if, $\sigma < \infty$.

- Intuitively, due to complementarities in production, a decrease in production of product m raises the cost of production of other products and decreases their production

Data Sources

- Provide evidence of joint production using product-level transaction data and product-specific demand shocks
- Data: Universe of electronic invoices between formal firms, aggregated at the firm-product level
- Demand shock: COVID lockdowns affecting buyers as product-specific demand shocks buyer regression



Locked down municipalities in March 2020

Simple Event Study Specification

- Define main product (m) to be the product with the highest revenue for firm i
- Estimate the impact of demand shocks to main product (m) on other products $g \neq m$
- Main product receives a demand shock if at least one buyer is in a locked down municipality in March 2020
- For all non-main products $g \neq m$,

$$\log X_{igt} = \sum_{j=-11}^{j=10} \beta_j \underbrace{Lockdown_{i,t-j}}_{\text{shock to main product}} + FE_{ig} + FE_t + \varepsilon_{igt}$$

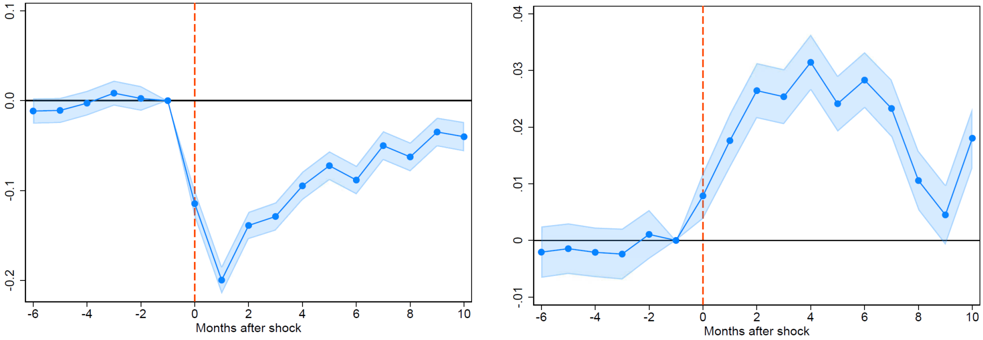
where $Lockdown_{i,t-j} = 1$ if firm i 's main product experienced the demand shock j months ago. X represents either quantity or price

Sample Restriction

- If your buyers are in a locked down area, you, buyers of your other products, or your suppliers may also be in locked down areas
- To deal with this, limit to a subsample of firms satisfying two conditions:
  (1) No direct supply shocks: Firms and their suppliers are not in locked down areas
  (2) No other demand shocks: Non-main product g has no buyers in locked down areas sample

Results

- Negative demand shock to main product reduces quantity and increases price of other products robustness



(a) log quantity (b) log price

Comparison with Alternative Within-Firm Spillover Mechanisms

- Venting Out (Almunia, Antràs, Lopez-Rodriguez, and Morales (2021))
  - Domestic demand $\downarrow \rightarrow$ Firm specific resources shift to exports $\rightarrow$ Exports $\uparrow$
  - Our results predict opposite sign
- Knowledge Linkages (Ding (2023))
  - US census: Product demand $\downarrow \rightarrow$ knowledge-linked products $\downarrow$
  - 5-year vs. monthly data and R&D intensity difference

Estimating Elasticity of Transformation σ

$$\Delta \log\left(\frac{p_{ig}}{p_{im}}\right) = \alpha + \beta \Delta \log\left(\frac{q_{ig}}{q_{im}}\right) + \gamma FE_{pt} + \xi_{igt}$$

	(1)	(2)	(3)
β	0.949***	1.207***	0.865***
	(0.0015)	(0.0586)	(0.0209)
Implied $\sigma (= 1/\beta)$	1.053	0.828	1.155
Time FE	Y	Y	N
Product $\times$ Time FE	N	N	Y
First Stage F stats	-	276.7	274.6

Notes: The standard errors are clustered at the firm level. Columns (1) report a result by ordinary least squares (OLS), while columns (2) and (3) report results by 2SLS. Product FE refer to harmonized product code. Three stars indicate statistical significance at the 1% level.

Table of Contents

Quantifying Distance to Frontier

- Use parametric model to measure TFP gains (distance to frontier) when removing all wedges:

$$\text{Distance to Frontier} \equiv \log Y(\mu)/Y(1)$$

- Parameterizing the general framework's production technologies using CET and CES specifications:

$$F_i^Q\left(\underbrace{\{q_{ig}\}_{g\in\mathcal{G}}}_{\text{outputs}}\right) = A_i F_i^X\left(\underbrace{\{x_{i,jg'}\}_{j\in\mathcal{N},g'\in\mathcal{G}}}_{\text{product } g' \text{ from j}}, L_i, K_i\right)$$

where

$$F_i^Q(\cdot) = \left(\sum \delta_{ig} q_{ig}^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}}, \quad F_i^X(\cdot) = \left(\sum \omega_{i,jg'} q_{i,jg'}^{\frac{\theta-1}{\theta}} + \omega_{i,L} L_i^{\frac{\theta-1}{\theta}} + \omega_{i,K} K_i^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}$$

distance to frontier

Calibration and Results

- Use Chilean invoice data to account for firm-to-firm linkages
- Comparison between estimated σ and $\sigma = \infty$ (independent product lines benchmark)
- For simplicity, we set $\theta = 2.5$ across inputs following Arkolakis et al. (2023)
- Joint production attenuates TFP losses

Specification	Distance to Frontier
Baseline Estimate ($\sigma = 1.2$)	12.3%
Independent Products ($\sigma \to \infty$)	18.7%

Decreasing Return to Scale in Firm Level

- Production function from Almunia et al. (2021)

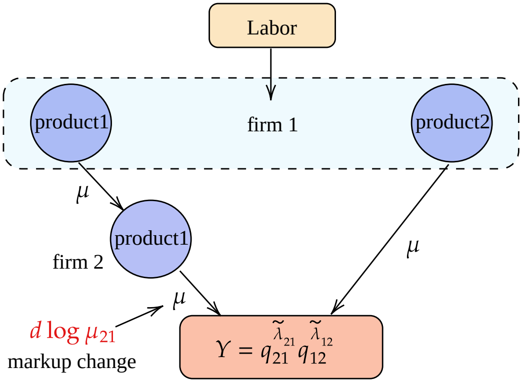
$$q_{11} + q_{12} = L_1^{\alpha}$$

- This is a special case of parametric version of our specification as $\sigma \to \infty$

$$\left(q_{11}^{\frac{\sigma+1}{\sigma}} + q_{12}^{\frac{\sigma+1}{\sigma}} \right)^{\frac{\sigma}{\sigma+1}} = L_1^{\alpha} K^{1-\alpha}$$

where K is firm specific factor

Derivation



$$\left(q_{11}^{\frac{\sigma+1}{\sigma}} + q_{12}^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}} = L$$

- The downstream markup change implies that sales of product 1 of firm 1 changes by $-d\log\mu_{21}$
  back
- Sales of product 2 do not change (due to Cobb-Douglas)
- Combining these with $d\log(p_{11}/p_{12}) = \frac{1}{\sigma} d\log(q_{11}/q_{12})$ implies

$$d\log(p_{11}/p_{12}) = -\frac{1}{\sigma+1} d\log\mu_{21}$$

Example with Taste Shock

- Instead of markup shock, consider taste shock: $d\tilde{\lambda}_2 = -d\tilde{\lambda}_1$ back
- Using $d\log(p_{11}/p_{12}) = \frac{1}{\sigma+1}\frac{1}{\tilde{\lambda}_2} d\log\tilde{\lambda}_1$, the TFP response can be expressed by:

$$\Delta \log TFP = \left(1 - \frac{1}{\sigma+1}\right)\frac{\tilde{\lambda}_1}{\tilde{\lambda}_2}\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right) d\log\tilde{\lambda}_1,$$

$$= \underbrace{\frac{\tilde{\lambda}_1}{\tilde{\lambda}_2}\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right) d\log\tilde{\lambda}_1}_{\text{Single-Product Term}} + \underbrace{\tilde{\lambda}_1\left(\frac{\bar{\Gamma}_1}{\Gamma_{11}} - 1\right) d\log p_{11}/p_{12}}_{\text{Multi-Product Term}}$$

- If $\sigma \to \infty$, multi-product term is zero
- If $\sigma \to 0$, $\Delta \log TFP$ is zero

Sufficient Statistics with Taste Shock

Proposition

In the simple example, TFP response to the taste shock is expressed by

$$\Delta \log TFP = \underbrace{Cov_{\boldsymbol{s}_1}\left(d\log p_{(1,\cdot)}, \frac{\bar{\Gamma}_1}{\Gamma_{(1,\cdot)}}\right)}_{\text{Multi-Product Term}} - \underbrace{\underbrace{d\log\Lambda}_{\text{Labor Share}}}_{\text{Single-Product Term}}$$

where $d\log p_{(1,\cdot)} = (d\log p_{11}, d\log p_{12})$, $\Gamma_{(1,\cdot)} = (\Gamma_{11}, \Gamma_{12})$, $\bar{\Gamma}_1 = \left(\tilde{\lambda}_{11}\Gamma_{11}^{-1} + \tilde{\lambda}_{12}\Gamma_{12}^{-1}\right)^{-1}$ *and* $\boldsymbol{s}_1 = \left(\tilde{\lambda}_{11}, \tilde{\lambda}_{12}\right)$

Joint Production

Let $J(\boldsymbol{q}, \boldsymbol{x})$ be the joint production function (Hall (1973))

$$J(\boldsymbol{q}, \boldsymbol{x}) = 0,$$

where $\boldsymbol{q}$; output vector and $\boldsymbol{x}$: input vector

Assumptions:

(1) CRS: $J(\boldsymbol{q}, \boldsymbol{x}) = 0$ implies $J(\lambda \boldsymbol{q}, \lambda \boldsymbol{x}) = 0$

(2) Separability between input and output bundles: $J(\boldsymbol{q}, \boldsymbol{x}) = -F^Q(\boldsymbol{q}) + F^X(\boldsymbol{x})$

Example Constant-Elasticity of Transformation and CES Input (CET-CES): cost function

$$\underbrace{\left(\sum_g q_g^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}}}_{\text{Output bundle}} = \underbrace{A\left(\omega_L L^{\frac{\theta-1}{\theta}} + \omega_K K^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}}_{\text{Input Bundle}}$$

back to simple example | back to network settings

Cost Function

Example

CET-CES Cost Function

$$C\left(\mathbf{q},w,r;A\right)=\frac{1}{A}\left(\sum_{g}\delta^{g}\left[q_{g}\right]^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}}\left(\alpha_{L}w^{1-\theta}+\alpha_{K}r^{1-\theta}\right)^{\frac{1}{1-\theta}},$$

- w and r are (factor) prices
- marginal cost

$$mc_{g}=\frac{1}{A}\frac{\partial C\left(\mathbf{q},w,r;A\right)}{\partial q_{g}}$$

back to joint production

Input-Output Objects: Revenue-Based

- To derive sufficient statistics, define Ω to be $(\mathcal{N}\mathcal{G}+\mathcal{F})\times(\mathcal{N}\mathcal{G}+\mathcal{F})$ input-output matrix: back

$$\Omega_{ig,jg'} = \frac{p_{jg'}x_{i,jg'}}{\underbrace{\mu_{ig}\left(\sum_{j,g'}p_{jg'}x_{i,jg'}+\sum_{f}w_f L_{if}\right)}_{\text{Revenue share of } jg'}},$$

- Leontief inverse matrix, Ψ:

$$\Psi \equiv (I-\Omega)^{-1},$$

- Domar weights, λ and:

$$\begin{aligned}\lambda' &\equiv b'\Psi,\\ &= b' + b'\Omega + b'\Omega^2 + \dots\end{aligned}$$

λ = sales share (over GDP)

Input-Output Objects: Revenue vs Cost-Based

- Ω and $\tilde{\Omega}$ are input output matrices: back

$$\Omega_{ig,jg'} = \underbrace{\frac{p_{jg'} x_{i,jg'}}{\mu_{ig}\left(\sum_{j,g'} p_{jg'} x_{i,jg'} + \sum_f w_f L_{if}\right)}}_{\text{Revenue share of } jg'}, \qquad \tilde{\Omega}_{ig,jg'} = \underbrace{\frac{p_{jg'} x_{i,jg'}}{\sum_{j,g'} p_{jg'} x_{i,jg'} + \sum_f w_f L_{if}}}_{\text{Cost share of } jg'}$$

- Leontief inverse matrices:

$$\Psi \equiv (I - \Omega)^{-1}, \qquad \tilde{\Psi} \equiv (I - \Omega)^{-1}$$

- Domar weights ($\tilde{\Lambda}$ for factors) :

$$\lambda' \equiv b'\Psi, \qquad \tilde{\lambda}' \equiv b'\tilde{\Psi}$$

If there is no markup, $\lambda = \tilde{\lambda}$ = sales share (over GDP)

Cumulative Wedge

- **Cumulative wedge** (for product g of firm i): back

$$\Gamma_{ig} \equiv \underbrace{\tilde{\lambda}_{ig}/\lambda_{ig}}_{\text{downstream wedge}} \times \underbrace{\mu_{ig}}_{\text{own wedge}},$$

- Captures indirect wedges through the downstream supply chain
- In the previous simple example: $\Gamma_{11} = \mu^2 > \Gamma_{12} = \mu$
- Firm-level (cost-based) Domar weights and their shares: $\tilde{\lambda}_i = \sum_g \tilde{\lambda}_{ig}, \quad s_{ig} = \frac{\tilde{\lambda}_{ig}}{\tilde{\lambda}_i}$

Decomposition of TFP into Allocative Efficiency and Technology

- $\mathcal{Y}(X, A)$: output Y given firm level productivities A and shares $X_{ijg} = x_{ijg}/q_{jg}$
- Change in response to shocks:

$$d\log Y = \underbrace{\frac{\partial \log \mathcal{Y}}{\partial \log X} d\log X}_{\Delta\text{Allocative Efficiency}} + \underbrace{\frac{\partial \log \mathcal{Y}}{\partial \log A} d\log A}_{\Delta\text{Technology}}$$

Decomposition of TFP into Allocative Efficiency and Technology

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- Efficient economy (set all markups equal 1) implies Hulten (1978):

$$\underbrace{d \log Y - \sum_f \tilde{\Lambda}_f d \log F_f}_{\Delta TFP} = \underbrace{0}_{\Delta \text{Allocative Efficiency}} + \underbrace{\sum \tilde{\lambda}_i d \log A_i}_{\Delta \text{Technology}}$$

back

Allocative Efficiency: Multi-Product Term

$$\text{Multi-Product Term} = \sum_i \tilde{\lambda}_i Cov_{\mathbf{s}_i}\left(\underbrace{d \log p_{(i,\cdot)}}_{\text{Price changes of firm } i\text{'s products}} , \underbrace{\frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}}}_{\text{Cumulative wedges of firm } i\text{'s products}} \right)$$

- Inside firm:
  - Joint production $\Rightarrow$ the opportunity cost to modify firm i's product mix is *not* constant
  - $\downarrow$ price of more distorted products, high Γ_{ig} $\Rightarrow$ beneficial reallocation
- Aggregation:
  - Sum up the covariance using firm's Domar weights, $\tilde{\lambda}_i$ back

Allocative Efficiency: Single-Product Term

$$\text{Single-Product Term} = -\sum_f \underbrace{\tilde{\Lambda}_f d \log \Lambda_f}_{\text{(1) Aggregate Factor share}} - \sum_i \underbrace{\tilde{\lambda}_i d \log \mu_i}_{\text{(2) Firm-level Markup}}$$

(1) A decline in factor shares $\Rightarrow$ resources shifting to high markup firms that are underproducing

(2) The direct change in factor shares due to markup changes: To isolate changes in allocative efficiency, need to subtract this back

Efficient Benchmark

- Efficient economy (set all markups equal 1) implies Hulten (1978):

$$\Delta \log TFP = \underbrace{0}_{\Delta \text{Allocative Efficiency}} + \underbrace{\sum \tilde{\lambda}_i d \log A_i}_{\Delta \text{Technology}}$$

back

Buyer Level Regression

$$\log \text{Total Intermediate Input Purchases}_{it} = \alpha Lockdown_{it} + FE_t + FE_i + \varepsilon_{it},$$

	(1)	(2)	(3)
Lockdown Dummy	-0.222***	-0.230***	-0.191***
	(0.0524)	(0.00521)	(0.0589)
Firm FE	Y	Y	Y
Time FE	N	N	Y
Sector $\times$ Time FE	N	Y	N
Restricted sample	N	N	Y
Observations	4,345,534	4,345,534	378,646

Notes: Standard errors are clustered at the country level. Columns (1) and (2) report results for the full sample, while column (3) reports results restricted to firms where none of the suppliers are located in the lockdown area. Three stars indicate statistical significance at the 1% level.

Purchases of intermediate inputs from lockdown counties decreased by about 20%. back

Summary Statistics

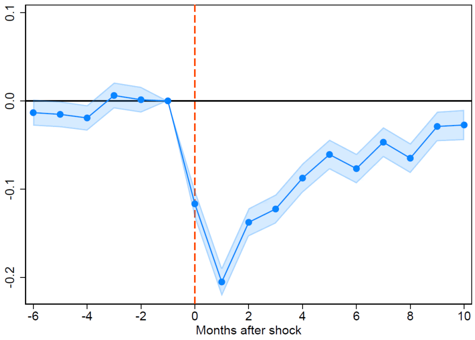
	Treatment Firms	Control Firms
Number of firms	26,411	96,321
Number of products sold	16	10
Number of producers	107	119
Number of buyers	59	26
Annual revenue (million pesos)	186	101

Notes: This Table presents the characteristics of treated firms (those whose major product buyers experienced lockdowns in March 2020) and control firms, showing values from February 2020, the month before the shock. The rows display the median of each statistic.

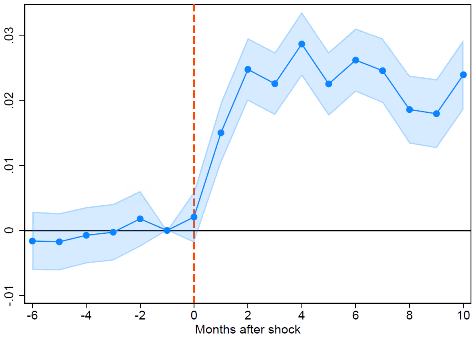
back

Results

- Use the samples with only above the 80th percentile of sales distribution robustness



(c) log quantity



(d) log price

DiD Specification

$$\log X_{igt} = \color{red}{\beta} \underbrace{Lockdown_{i,t}}_{\text{shock to main product}} + FE_{ig} + FE_t + \varepsilon_{igt}$$

where $Lockdown_{i,t} = 1$ if firm i 's main product experienced the demand shock and after March 2020. back

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
$\log q$	-0.117***		-0.117***		-0.102***		-0.106***		-0.125***		-0.159***	
	(0.0046)		(0.0046)		(0.0053)		(0.0188)		(0.0049)		(0.0062)	
$\log p$		0.0175***		0.0184***		0.0191***		0.0193***		0.0168***		0.0177***
		(0.0021)		(0.0022)		(0.0022)		(0.0096)		(0.0023)		(0.0024)
Input price control	N	N	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
Large firms	N	N	N	N	Y	Y	N	N	N	N	N	N
Only manufacturing firms	N	N	N	N	N	N	Y	Y	N	N	N	N
Time $\times$ product FE	N	N	N	N	N	N	N	N	Y	Y	N	N
Continuous treatment	N	N	N	N	N	N	N	N	N	N	Y	Y
Observations	7,693,066	7,693,066	7,669,848	7,669,848	4,394,166	4,394,166	1,399,617	1,399,617	7,594,918	7,594,918	7,693,066	7,693,066

Notes: Input price control indicates inclusion of the Tornqvist input price index. Large firms restricts to firms above the 80th percentile in total sales. Only manufacturing firms restricts to the manufacturing sector. Time $\times$ Product FE are time-varying fixed effects at the harmonized product level. Continuous treatment uses the share of transaction values to lockdown destinations in firm's main product instead of the binary lockdown variable. *** denote significance at 1%.

Variable Markup

- If Marshall's second law of demand hold (the higher q, the higher the markup, the smaller the pass-through), the same proposition hold
- Intuitively, markups offset some of the price changes back

Proposition

Under the marshall's second law of demand hold, A negative demand shock to product k ($d\log D_k < 0$) affects product $g \neq k$:

(i) If $\sigma < \infty$, quantity: $d\log q_g < 0$, price: $d\log p_g > 0$,

(ii) If $\sigma = \infty$, quantity: $d\log q_g = 0$, price: $d\log p_g = 0$

Data and Growth Accounting

(1) Sales, materials, investment: F29 (2014-2022)

(2) Wage bill, employment: DJ1887 (2014-2022)

(3) Capital: F22 (2014-2022)
- Capital stock using perpetual inventory methods combining capital stock with investment
- UCC: interest rate - inflation expectation + depreciation rate from LA-Klems database + external financing premium (5 percent)

(4) Product, I-O matrices and output and input prices: F2F electronic receipts (2014-2022)

(5) Official deflectors for aggregate real variables back

Data Cleaning

- The final sample does not include firms with a missing variable of sales, capital, wage bill, or materials
- Winzorized labor, capital and materials shares over sales at 1% of both tails of the distribution
- Firms with negative value added (sales minus materials), less than two workers, or capital less than 10.000 CLP (USD 15) are excluded

Multi-Product Firms Dominate Intermediate Transactions

- Multi-product firms account for 99 percent of intermediate transactions back

Percentile	Number of products by firm
1%	1
5%	2
10%	4
25%	36
50%	**475**
75%	2,459
90%	32,195
95%	37,422
99%	62,372

Table: Share of firm-to-firm transactions made by firms with no more than X product

Unweighted Distribution of the Number of Products

Percentile	Number of products by firm
1%	1
5%	1
10%	1
25%	2
50%	**7**
75%	26
90%	119
95%	290
99%	1,253

Table: Distribution of the Number of Products by Firm (Unweighted)

back

Markup Estimation

- Following Dhyne, Petrin, Smeets & Warzynski (2022) with CET output function
- Cobb-Douglas production function for input bundle using three inputs (K, L, M) (lower case variables denote logs)

$$Q_{it} = \beta_0 + \beta_K K_{it} + \beta_L L_{it} + \beta_M M_{it} + \omega_{it}$$

  where $Q_i = \left(\sum_{i=1}^{N} \left(q_{ig} \right)^{\frac{\sigma+1}{\sigma}} \right)^{\frac{\sigma}{\sigma+1}}$ using estimated σ
- GMM Estimation was performed separately by firm's sector
- Time invariant output elasticities recover product level markup back

$$\mu_{ig} = \frac{\beta_M^g}{\left(q_{ig} \right)^{\frac{\sigma+1}{\sigma}} / \sum_{g'=1}^{N} \left(q_{ig'} \right)^{\frac{\sigma+1}{\sigma}}} \frac{R_{ig}}{P_i^M M_i}$$

  where R_{ig} is the revenue from product g and $P_i^M M_i$ is the expenditure on materials.

Variance Decomposition

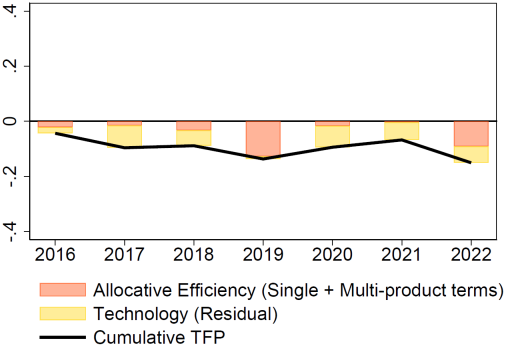
$$\Gamma_{ig} \equiv \frac{\tilde{\lambda}_{ig}}{salesshare_{ig}} \times \mu_{ig},$$

Table: Variance decomposition of $\log \Gamma$

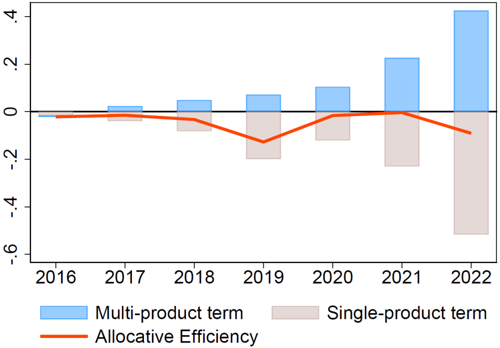
Year	Downstream distortions	Own markup	Covariance
2016	106.4%	1.8%	-8.2%
2017	107.4%	2.1%	-9.5%
2018	107.3%	2.2%	-9.5%
2019	107.6%	2.4%	-10.0%
2020	107.6%	2.6%	-10.2%
2021	107.3%	2.7%	-10.0%
2022	107.2%	2.8%	-10.0%

- Most of the variation stem from downstream distortions back

Growth Accounting with Production Function Approach



(e) Growth accounting with mutiproduct term



(f) Decomposition of allocative efficiency

back

Growth Accounting with Production Function Approach

$$Cov_{\mathbf{s}_i}\left(d\log p_{(i,\cdot)}, \frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}}\right) = Cov_{\mathbf{s}_i}\left(d\log \mu_{(i,\cdot)}, \frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}}\right) + Cov_{\mathbf{s}_i}\left(d\log mc_{(i,\cdot)}, \frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}}\right)$$

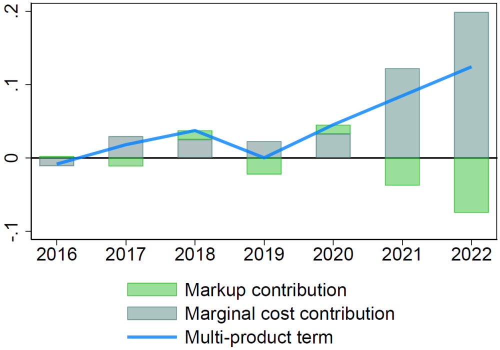
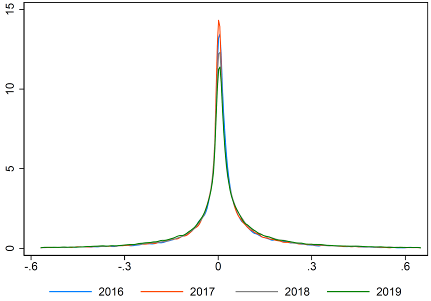


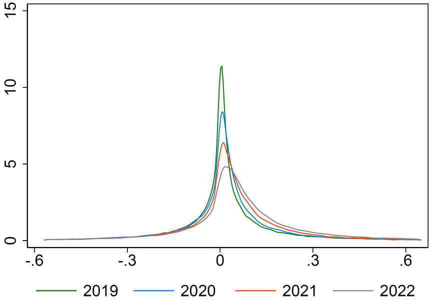
Figure: Decomposition of multiproduct term

Distributions of Multi-Product Term, $Cov_{\mathbf{s}_i}\left(d\log p_{(i,\cdot)}, \frac{\bar{\Gamma}_i}{\Gamma_{(i,\cdot)}}\right)$

- Stable distribution in normal times, but the distribution is skewed to the right after periods of turmoil
- Firms may have adjusted their product mixes significantly in response to the disturbance suggestive evidence



(a) 2016~2019



(b) 2019~2022

Large Product Mix Adjustments and Efficiency: Suggestive Evidence

- Firms may have adjusted their product mix significantly to respond to the recent supply chain disruption

back

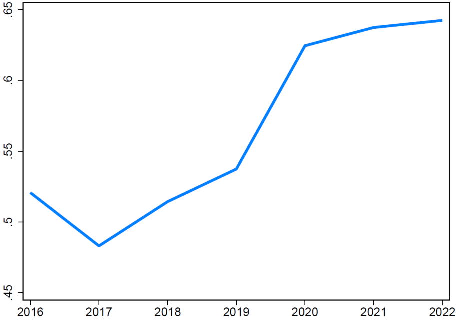


Figure: Variance of Product Quantity Changes: $Var_{\lambda_i}(d \log q_{ig})$

Distance to the Pareto-Efficient Frontier with Joint Production

Proposition

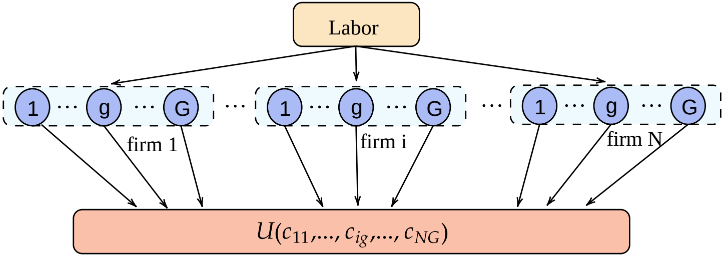
Starting at an efficient equilibrium, up to second order, and in response to the introduction of wedges, changes in TFP are given by Domar-weighted Harberger triangles:

$$\mathcal{L} = \frac{1}{2}\sum_{ig}\lambda_{ig}\underbrace{d\log q_{ig}}_{quantity}\underbrace{d\log \mu_{ig}}_{markup}$$

where λ_{ig} is the Domar weight of product g from firm i back

Horizontal Economy

- A representative consumer (CES with elasticity θ)
- N multi-product firms use labor to make G products using CET technology (elasticity σ)
- Markups μ_{ig} are heterogeneous across products and firms



Analytical Structural Result: Horizontal Economy

Proposition

Starting at an efficient equilibrium, up to second order, the distance to the frontier in the horizontal economy is given by:

$$\mathcal{L} = -\frac{1}{2}\theta\left(Var_{\boldsymbol{\lambda}}(d\log\mu_{ig}) - \frac{1}{\sigma+1}\mathbb{E}_{\bar{\boldsymbol{\lambda}}}\left\{Var_{\mathbf{s}_i}(d\log\mu_{ig})\right\}\right)$$

where $\boldsymbol{\lambda} = (\lambda_{11}, \lambda_{12}, \ldots, \lambda_{NG})$, $\bar{\boldsymbol{\lambda}} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ *with* $\lambda_i = \sum_g \lambda_{ig}$, *and* $\mathbf{s}_i = (\lambda_{i1}/\lambda_i, \lambda_{i2}/\lambda_i, \ldots, \lambda_{iG}/\lambda_i)$

- Joint production attenuates the distance to the frontier given wedges

Analytical Structural Result: Horizontal Economy

Proposition

Starting at an efficient equilibrium, up to second order, the distance to the frontier in the horizontal economy is given by:

$$\mathcal{L} = -\frac{1}{2}\theta\left(Var_{\boldsymbol{\lambda}}(d\log\mu_{ig}) - \frac{1}{\sigma+1}\mathbb{E}_{\bar{\boldsymbol{\lambda}}}\left\{Var_{\boldsymbol{s}_i}(d\log\mu_{ig})\right\}\right)$$

where $\boldsymbol{\lambda} = (\lambda_{11}, \lambda_{12}, \ldots, \lambda_{NG})$, $\bar{\boldsymbol{\lambda}} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ *with* $\lambda_i = \sum_g \lambda_{ig}$, *and* $\boldsymbol{s}_i = (\lambda_{i1}/\lambda_i, \lambda_{i2}/\lambda_i, \ldots, \lambda_{iG}/\lambda_i)$

- Joint production attenuates the distance to the frontier given wedges

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