

# Chapter 1

## Rigid body: Kinematics & Dynamics

### 1.1 Kinematics

Let  $P_i$  represent an arbitrary point on the rigid body 'i' that is shown in Figure 1.1 and  $C_i$  the center of the body's center of mass. Furthermore, the frame  $f_i$  is rigidly attached to the body (translates and rotates with it), while the  $F$  frame is the inertial frame of reference.

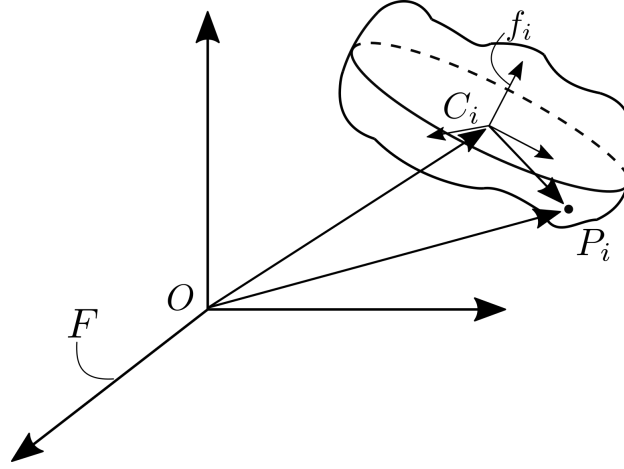


Figure 1.1: Rigid Body

The center of mass of the body can be defined as

$$\underline{r}_{OC_i/F}^F = \frac{1}{m_i} \int_{m_i} \underline{r}_{OP_i/F}^F dm_i \quad \text{where} \quad m_i = \int_{m_i} dm_i.$$

#### 1.1.1 Position

The position of the arbitrary point  $P_i$  with respect to the inertial frame is defined as

$$\underline{r}_{OP_i/F}^F = \underline{r}_{OC_i/F}^F + R_{f_i}^F \underline{r}_{C_i P_i/f_i}^{f_i}, \quad (1.1)$$

where  $R_{f_i}^F$  is the rotation matrix of body frame  $f_i$  with respect to the inertial frame  $F$ .

### 1.1.2 Velocity

The velocity of the arbitrary point  $P_i$  with respect to the inertial frame is defined as

$$\begin{aligned}\dot{\underline{r}}_{OP_i/F}^F &= \dot{\underline{r}}_{OC_i/F}^F + \frac{d}{dt}(R_{f_i}^F \underline{r}_{C_i P_i/f_i}^{f_i}), \\ \dot{\underline{r}}_{OP_i/F}^F &= \dot{\underline{r}}_{OC_i/F}^F + \dot{R}_{f_i}^F \underline{r}_{C_i P_i/f_i}^{f_i} + R_{f_i}^F \dot{\underline{r}}_{C_i P_i/f_i}^{f_i}.\end{aligned}\quad (1.2)$$

We know that for a rigid body the distance between points remains constant meaning that  $\dot{\underline{r}}_{C_i P_i/f_i}^{f_i} = 0$ . The time derivative of the rotation matrix can be defined as

$$\dot{R}_{f_i}^F = S(\underline{\omega}_{f_i/F}^F) R_{f_i}^F, \quad (1.3)$$

where,  $\underline{\omega}_{f_i/F}^F$  the rotational velocity of  $f_i$  frame with respect to  $F$  frame and  $S(\underline{a})$  skew-symmetric matrix defined as

$$S(\underline{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \quad \text{where } \underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Given the above equation (1.2) becomes

$$\begin{aligned}\dot{\underline{r}}_{OP_i/F}^F &= \dot{\underline{r}}_{OC_i/F}^F + S(\underline{\omega}_{f_i/F}^F) R_{f_i}^F \underline{r}_{C_i P_i/f_i}^{f_i} \\ &= \dot{\underline{r}}_{OC_i/F}^F - S(R_{f_i}^F \underline{r}_{C_i P_i/f_i}^{f_i}) \underline{\omega}_{f_i/F}^F \\ &= \dot{\underline{r}}_{OP_i/F}^F = \dot{\underline{r}}_{OC_i/F}^F - R_{f_i}^F S(\underline{r}_{C_i P_i/f_i}^{f_i}) (R_{f_i}^F)^T \underline{\omega}_{f_i/F}^F.\end{aligned}$$

or

$$\dot{\underline{r}}_{OP_i/F}^F = \dot{\underline{r}}_{OC_i/F}^F - R_{f_i}^F S(\underline{r}_{C_i P_i/f_i}^{f_i}) \underline{\omega}_{f_i/F}^{f_i}. \quad (1.4)$$

It can be proven that the rotational velocity of the body frame with respect to the inertial frame (expressed in the body frame) can be written as:

$$\underline{\omega}_{f_i/F}^{f_i} = G(\underline{\theta}_i) \dot{\underline{\theta}}_i = G_i \dot{\underline{\theta}}_i, \quad G_i = G(\underline{\theta}_i), \quad (1.5)$$

where  $\underline{\theta}_i$  is the vector with the parameters that describe the orientation of the body frame with respect to the inertial frame (euler angles, euler parameters, rodrigues parameters, etc.).

If we define the generalized rigid body parameters as

$$\underline{q}_{r_i} = \begin{bmatrix} \dot{\underline{r}}_{OC_i/F}^F \\ \underline{\theta}_i \end{bmatrix} \quad (1.6)$$