Chapter 1

Rigid body

1.1 Kinematics

Let p_i represent an arbitrary point on the rigid body 'i' that is shown in Figure 1.1 and c_i the origin of the frame f_i which is rigidly attached to the body (translates and rotates with it). The frame F is the inertial frame of reference.

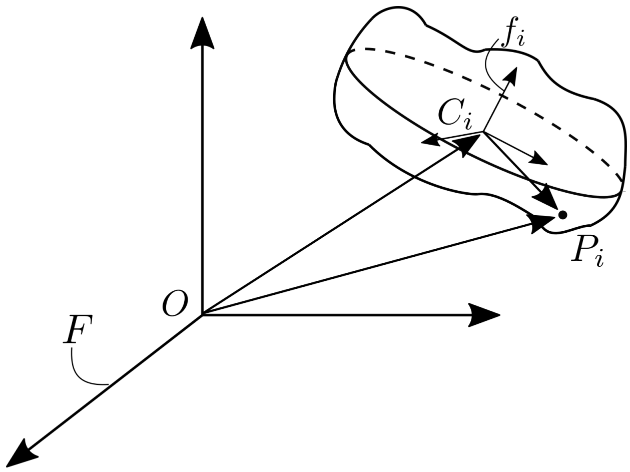

Figure 1.1: Rigid Body

1.1.1 Position

The position of the arbitrary point "p_i" with respect to the inertial frame is defined as

$$\underline{r}^F_{op_i/F} = \underline{r}^F_{oc_i/F} + R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i}, \tag{1.1}$$

where $R^F_{f_i}$ is the rotation matrix of body frame f_i with respect to the inertial frame F.

1.1.2 Velocity

The velocity of the the arbitrary point P_i with respect to the inertial frame is defined as

$$\dot{\underline{r}}^F_{op_i/F} = \dot{\underline{r}}^F_{oc_i/F} + \frac{d}{dt}(R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i}),$$

$$\dot{\underline{r}}^F_{op_i/F} = \dot{\underline{r}}^F_{oc_i/F} + \dot{R}^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i} + R^F_{f_i}\ \dot{\underline{r}}^{f_i}_{c_ip_i/f_i}. \tag{1.2}$$

We know that for a rigid body the distance between to points remains constant meaning that $\dot{\underline{r}}^{f_i}_{c_ip_i/f_i} = 0$. The time derivative of the rotation matrix can be defined as

$$\dot{R}^F_{f_i} = S(\underline{\omega}^F_{f_i/F})\ R^F_{f_i}, \tag{1.3}$$

where, $\underline{\omega}^F_{f_i/F}$ the rotational velocity of f_i frame with resepct to F frame and $S(\underline{a})$ skew-symetrix matrix defined as

$$S(\underline{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \quad \text{where} \quad \underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Given the above equation (1.2) becomes

$$\begin{aligned} \dot{\underline{r}}^F_{op_i/F} &= \dot{\underline{r}}^F_{oc_i/F} + S(\underline{\omega}^F_{f_i/F}) R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i} \\ &= \dot{\underline{r}}^F_{oc_i/F} - S(R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i})\ \underline{\omega}^F_{f_i/F} \\ &= \dot{\underline{r}}^F_{op_i/F} = \dot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\ \underline{r}^{f_i}_{c_ip_i/f_i})\ (R^F_{f_i})^T\ \underline{\omega}^F_{f_i/F}. \end{aligned}$$

or

$$\dot{\underline{r}}^F_{op_i/F} = \dot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\ \underline{r}^{f_i}_{c_ip_i/f_i})\ \underline{\omega}^{f_i}_{f_i/F}. \tag{1.4}$$

It can be proven that the rotational velocity of the body frame with respect to the inertial frame (expressed in the body frame) can be written as:

$$\underline{\omega}^{f_i}_{f_i/F} = G(\underline{\theta}_i)\ \dot{\underline{\theta}}_i = G_i\ \dot{\underline{\theta}}_i, \quad G_i = G(\underline{\theta}_i), \tag{1.5}$$

where $\underline{\theta}_i$ is the vector with the parameters that describe the orientation of the body frame with respect to the inertial frame (euler angles, euler parameters, rodrigues parameters, etc.).

Given expression (1.5), equation (1.4) becomes

$$\dot{\underline{r}}^F_{op_i/F} = \dot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\ \underline{r}^{f_i}_{c_ip_i/f_i})\ G_i\ \dot{\underline{\theta}}_i.$$

Finally, if we define the generalized rigid body coordinates as

$$\underline{q}_{r_i} = \begin{bmatrix} \dot{\underline{r}}^F_{oc_i/F} \\ \underline{\theta}_i \end{bmatrix}, \tag{1.6}$$

then equation (1.4) becomes

$$\dot{\underline{r}}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \dot{\underline{q}}_{r_i}, \tag{1.7}$$

where,

$$L_r(\underline{q}_{r_i}) = L_{r_i} = \begin{bmatrix} I_{3\times3} & -R^F_{f_i}\ S(\ \underline{r}^{f_i}_{c_ip_i/f_i})\ G_i \end{bmatrix}. \tag{1.8}$$

1.1.3 Acceleration

The acceleration of the the arbitrary point P_i with respect to the inertial frame is defined as

$$\begin{aligned}\ddot{\underline{r}}^F_{op_i/F} &= \frac{d}{dt}(\dot{\underline{r}}^F_{oc_i/F} + S(\underline{\omega}^F_{f_i/F})R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i}) \\ &= \ddot{\underline{r}}^F_{oc_i/F} + \dot{\underline{\omega}}^F_{f_i/F} \times (R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i}) + \underline{\omega}^F_{f_i/F} \times \frac{d}{dt}(R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i}).\end{aligned}$$

Based on the above, the expression for the acceleration can take the following form

$$\ddot{\underline{r}}^F_{op_i/F} = \ddot{\underline{r}}^F_{oc_i/F} + R^F_{f_i}\ S(\underline{\alpha}^{f_i}_{f_i/F})\ \underline{r}^{f_i}_{c_ip_i/f_i} + R^F_{f_i}\ (S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_ip_i/f_i} \tag{1.9}$$

where

$$\underline{\alpha}^{f_i}_{f_i/F} = \dot{\underline{\omega}}^{f_i}_{f_i/F} = \frac{d}{dt}(G(\underline{\theta}_i)\ \dot{\underline{\theta}}_i) = \dot{G}_i\ \dot{\underline{\theta}}_i + G_i\ \ddot{\underline{\theta}}_i, \tag{1.10}$$

the expression for the angular acceleration of the body. Substituting (1.10) into (1.9) leads to

$$\begin{aligned}\ddot{\underline{r}}^F_{op_i/F} &= \ddot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_ip_i/f_i})\ \underline{\alpha}^{f_i}_{f_i/F} + R^F_{f_i}\ (S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_ip_i/f_i} \\ &= \ddot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_ip_i/f_i})(\dot{G}_i\ \dot{\underline{\theta}}_i + G_i\ \ddot{\underline{\theta}}_i) + R^F_{f_i}\ (S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_ip_i/f_i}\end{aligned}$$

or

$$\ddot{\underline{r}}^F_{op_i/F} = \ddot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_ip_i/f_i})\ \dot{G}_i\ \dot{\underline{\theta}}_i - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_ip_i/f_i})\ G_i\ \ddot{\underline{\theta}}_i + R^F_{f_i}\ (S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_ip_i/f_i}$$

If we define

$$\underline{a}_{v_{r_i}}(\underline{q}_{r_i}, \dot{\underline{q}}_{r_i}) = R^F_{f_i}\ [(S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_ip_i/f_i} - S(\underline{r}^{f_i}_{c_ip_i/f_i})\ \dot{G}_i\ \dot{\underline{\theta}}_i], \tag{1.11}$$

then the above expression can be written as

$$\ddot{\underline{r}}^F_{op_i/F} = \ddot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_ip_i/f_i})\ G_i\ \ddot{\underline{\theta}}_i + \underline{a}_{v_{r_i}}(\underline{q}_{r_i}, \dot{\underline{q}}_{r_i}).$$

Using the expression (1.8) we have

$$\ddot{\underline{r}}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \ddot{\underline{q}}_{r_i} + \underline{a}_{v_{r_i}}(\underline{q}_{r_i}, \dot{\underline{q}}_{r_i}). \tag{1.12}$$

1.1.4 Virtual Displacement

We define the virtual displacement of an arbitrary point "p_i" of the rigid body "i" as

$$\delta\underline{r}^F_{op_i/F} = \frac{\partial\underline{r}^F_{op_i/F}}{\partial\underline{q}_{r_i}}\ \delta\underline{q}_{r_i}. \tag{1.13}$$

The velocity of this point has previously defined (equation (1.7)) as

$$\begin{aligned}
&\dot{\underline{r}}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \dot{\underline{q}}_{r_i} \\
&\Rightarrow \frac{\partial \underline{r}^F_{op_i/F}}{\partial t} = L_r(\underline{q}_{r_i})\ \frac{\partial \underline{q}_{r_i}}{\partial t} \\
&\Rightarrow \frac{\partial \underline{r}^F_{op_i/F}}{\partial \underline{q}_{r_i}} \frac{\partial \underline{q}_{r_i}}{\partial t} = L_r(\underline{q}_{r_i})\ \frac{\partial \underline{q}_{r_i}}{\partial t}
\end{aligned}$$

For indepedent $\frac{\partial \underline{q}_{r_i}}{\partial t}$ it is obvious that

$$\frac{\partial \underline{r}^F_{op_i/F}}{\partial \underline{q}_{r_i}} = L_r(\underline{q}_{r_i}). \tag{1.14}$$

Combining equations (1.13) and (1.14) we define the virtual displacement of point p_i as

$$\delta \underline{r}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \delta \underline{q}_{r_i} \tag{1.15}$$

1.2 Dynamics

There are several methods for developing the dynamic equations of motion of the rigid bodies. In this section, the *principle of virtual work in dynamics* will be used to obtain the differential equations that govern the spatial motion of the rigid bodies. Based on D'Alembert's principle, a body "i" (rigid or flexible) in a dynamic equilibrium obeys the following equation

$$\delta W^e_i = \delta W^{in}_i, \tag{1.16}$$

where, δW^e_i is the virtual work of the externaly applied forces to the body and δW^{in}_i is the virtual work the the inertial forces.

1.2.1 Virtual work of inertial forces

For a continuum the virtual work of inertial forces is defined as

$$\delta W^{in}_i = \int_{m_i} \ddot{\underline{r}}^F_{op_i/F}\ \delta \underline{r}^F_{op_i/F}\ dm_i, \tag{1.17}$$

where p_i is an arbitrary point of the body "i".

For a rigid body it has proven that

$$\ddot{\underline{r}}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \ddot{\underline{q}}_{r_i} + \underline{a}_{v_{r_i}}(\underline{q}_{r_i}, \dot{\underline{q}}_{r_i}) \quad \text{and} \quad \delta \underline{r}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \delta \underline{q}_{r_i}$$

Combining the above with equation (1.17) the virtual work of inertial forces can be written as

$$\delta W^{in}_i = \ddot{\underline{q}}^T_{r_i} \int_{m_i} (L_{r_i})^T\ L_{r_i}\ dm_i\ \delta \underline{q}_{r_i} + \int_{m_i} \underline{a}_{v_{r_i}}\ L_{r_i}\ dm_i\ \delta \underline{q}_{r_i}$$

If we define the mass matrix ,

$$M_{r_i} = \int_{m_i} (L_{r_i})^T \; L_{r_i} \; dm_i, \tag{1.18}$$

and the centrifugal-coriolis forces vector

$$\underline{f}_{v_{r_i}} = \underline{f}_{v_{r_i}}(\underline{q}_{r_i}, \dot{\underline{q}}_{r_i}) = -\int_{m_i} (L_{r_i})^T \; \underline{a}_{v_{r_i}} \; \; dm_i, \tag{1.19}$$

then the equation above becomes

$$\delta W_i^{in} = (\ddot{\underline{q}}_{r_i}^T \; M_{r_i} - \underline{f}_{v_{r_i}}^T)\delta \underline{q}_{r_i}. \tag{1.20}$$

Mass matrix

In equation (1.23), the mass matrix of the rigid body was defined as

$$M_{r_i} = \int_{m_i} (L_{r_i})^T \; L_{r_i} \; dm_i.$$

Given equation (1.8), this expression can be formulated as

$$\begin{aligned}
M_{r_i} &= \int_{m_i} \begin{bmatrix} I_{3\times 3} \\ -G_i^T \; (S(\underline{r}_{c_i p_i/f_i}^{f_i}))^T \; R_F^{f_i} \end{bmatrix} \begin{bmatrix} I_{3\times 3} & -R_{f_i}^F \; S(\; \underline{r}_{c_i p_i/f_i}^{f_i}) \; G_i \end{bmatrix} \; dm_i \\
&= \int_{m_i} \begin{bmatrix} I_{3\times 3} & -R_{f_i}^F \; S(\underline{r}_{c_i p_i/f_i}^{f_i}) \; G_i \\ -G_i^T \; (S(\underline{r}_{c_i p_i/f_i}^{f_i}))^T \; R_F^{f_i} & G_i^T \; (S(\underline{r}_{c_i p_i/f_i}^{f_i}))^T \; S(\underline{r}_{c_i p_i/f_i}^{f_i}) \; G_i \end{bmatrix} \; dm_i \\
&= \begin{bmatrix} m_{11}^{r_i} & m_{12}^{r_i} \\ m_{21}^{r_i} & m_{22}^{r_i} \end{bmatrix}
\end{aligned}$$

where,

$$m_{11}^{r_i} = \int_{m_i} I_{3\times 3} \; dm_i = m_i I_{3\times 3} \tag{1.21a}$$

$$m_{12}^{r_i} = (m_{21}^{r_i})^T = -R_{f_i}^F \; \int_{m_i} S(\underline{r}_{c_i p_i/f_i}^{f_i}) \; dm_i \; G_i \tag{1.21b}$$

$$m_{22}^{r_i} = G_i^T \; \int_{m_i} (S(\underline{r}_{c_i p_i/f_i}^{f_i}))^T \; S(\underline{r}_{c_i p_i/f_i}^{f_i}) \; dm_i \; G_i \tag{1.21c}$$

We define the body's inertial tensor on the point c_i and with respect to the body frame f_i as

$$I_{c_i}^{f_i} = \int_{m_i} (S(\underline{r}_{c_i p_i/f_i}^{f_i}))^T \; S(\underline{r}_{c_i p_i/f_i}^{f_i}) \; dm_i. \tag{1.22}$$

Given that the frame f_i is rigidly attached to the body the above inertial tensor remains constant for the rigid body case. Then we have

$$M_{r_i} = \begin{bmatrix} m_i \; I_{3\times 3} & m_{12} \\ m_{21} & G_i^T \; I_{c_i}^{f_i} \; G_i \end{bmatrix}. \tag{1.23}$$

Centrifugal-coriolis forces vector

The vector of centrifugal-coriolis forces was given by the equation (1.19) as

$$\underline{f}_{v_{r_i}}(\underline{q}_{r_i}, \underline{\dot{q}}_{r_i}) = -\int_{m_i} (L_{r_i})^T \; \underline{a}_{v_{r_i}} \quad dm_i$$