Chapter 1

Rigid body

1.1 Kinematics

Let p_i represent an arbitrary point on the rigid body 'i' that is shown in Figure 1.1 and c_i the origin of the frame f_i which is rigidly attached to the body (translates and rotates with it). The frame F is the inertial frame of reference.

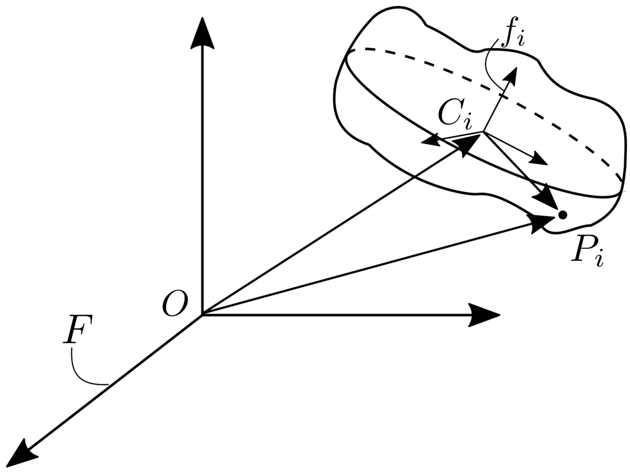

Figure 1.1: Rigid Body

1.1.1 Position

The position of the arbitrary point "p_i" with respect to the inertial frame is defined as

$$\underline{r}^F_{op_i/F} = \underline{r}^F_{oc_i/F} + R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i}, \tag{1.1}$$

where $R^F_{f_i}$ is the rotation matrix of body frame f_i with respect to the inertial frame F.

1.1.2 Velocity

The velocity of the the arbitrary point P_i with respect to the inertial frame is defined as

$$\dot{\underline{r}}^F_{op_i/F} = \dot{\underline{r}}^F_{oc_i/F} + \frac{d}{dt}(R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i}),$$

$$\dot{\underline{r}}^F_{op_i/F} = \dot{\underline{r}}^F_{oc_i/F} + \dot{R}^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i} + R^F_{f_i}\ \dot{\underline{r}}^{f_i}_{c_ip_i/f_i}. \tag{1.2}$$

We know that for a rigid body the distance between to points remains constant meaning that $\dot{\underline{r}}^{f_i}_{c_ip_i/f_i} = 0$. The time derivative of the rotation matrix can be defined as

$$\dot{R}^F_{f_i} = S(\underline{\omega}^F_{f_i/F})\ R^F_{f_i}, \tag{1.3}$$

where, $\underline{\omega}^F_{f_i/F}$ the rotational velocity of f_i frame with resepct to F frame and $S(\underline{a})$ skew-symetrix matrix defined as

$$S(\underline{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \quad \text{where} \quad \underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Given the above equation (1.2) becomes

$$\begin{aligned} \dot{\underline{r}}^F_{op_i/F} &= \dot{\underline{r}}^F_{oc_i/F} + S(\underline{\omega}^F_{f_i/F}) R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i} \\ &= \dot{\underline{r}}^F_{oc_i/F} - S(R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i})\ \underline{\omega}^F_{f_i/F} \\ &= \dot{\underline{r}}^F_{op_i/F} = \dot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\ \underline{r}^{f_i}_{c_ip_i/f_i})\ (R^F_{f_i})^T\ \underline{\omega}^F_{f_i/F}. \end{aligned}$$

or

$$\dot{\underline{r}}^F_{op_i/F} = \dot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\ \underline{r}^{f_i}_{c_ip_i/f_i})\ \underline{\omega}^{f_i}_{f_i/F}. \tag{1.4}$$

It can be proven that the rotational velocity of the body frame with respect to the inertial frame (expressed in the body frame) can be written as:

$$\underline{\omega}^{f_i}_{f_i/F} = G(\underline{\theta}_i)\ \dot{\underline{\theta}}_i = G_i\ \dot{\underline{\theta}}_i, \quad G_i = G(\underline{\theta}_i), \tag{1.5}$$

where $\underline{\theta}_i$ is the vector with the parameters that describe the orientation of the body frame with respect to the inertial frame (euler angles, euler parameters, rodrigues parameters, etc.).

Given expression (1.5), equation (1.4) becomes

$$\dot{\underline{r}}^F_{op_i/F} = \dot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\ \underline{r}^{f_i}_{c_ip_i/f_i})\ G_i\ \dot{\underline{\theta}}_i.$$

Finally, if we define the generalized rigid body coordinates as

$$\underline{q}_{r_i} = \begin{bmatrix} \underline{r}^F_{oc_i/F} \\ \underline{\theta}_i \end{bmatrix}, \tag{1.6}$$

then equation (1.4) becomes

$$\dot{\underline{r}}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \dot{\underline{q}}_{r_i}, \tag{1.7}$$

where,

$$L_r(\underline{q}_{r_i}) = L_{r_i} = \begin{bmatrix} I_{3\times3} & -R^F_{f_i}\ S(\ \underline{r}^{f_i}_{c_ip_i/f_i})\ G_i \end{bmatrix}. \tag{1.8}$$

1.1.3 Acceleration

The acceleration of the the arbitrary point P_i with respect to the inertial frame is defined as

$$
\begin{aligned}
\ddot{\underline{r}}^F_{op_i/F} &= \frac{d}{dt}(\dot{\underline{r}}^F_{oc_i/F} + S(\underline{\omega}^F_{f_i/F}) R^F_{f_i}\ \underline{r}^{f_i}_{c_i p_i/f_i}) \\
&= \ddot{\underline{r}}^F_{oc_i/F} + \dot{\underline{\omega}}^F_{f_i/F} \times (R^F_{f_i}\ \underline{r}^{f_i}_{c_i p_i/f_i}) + \underline{\omega}^F_{f_i/F} \times \frac{d}{dt}(R^F_{f_i}\ \underline{r}^{f_i}_{c_i p_i/f_i}).
\end{aligned}
$$

Based on the above, the expression for the acceleration can take the following form

$$
\ddot{\underline{r}}^F_{op_i/F} = \ddot{\underline{r}}^F_{oc_i/F} + R^F_{f_i}\ S(\underline{\alpha}^{f_i}_{f_i/F})\ \underline{r}^{f_i}_{c_i p_i/f_i} + R^F_{f_i}\ (S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_i p_i/f_i} \tag{1.9}
$$

where

$$
\underline{\alpha}^{f_i}_{f_i/F} = \dot{\underline{\omega}}^{f_i}_{f_i/F} = \frac{d}{dt}(G(\underline{\theta}_i)\ \dot{\underline{\theta}}_i) = \dot{G}_i\ \dot{\underline{\theta}}_i + G_i\ \ddot{\underline{\theta}}_i, \tag{1.10}
$$

the expression for the angular acceleration of the body. Substituting (1.10) into (1.9) leads to

$$
\begin{aligned}
\ddot{\underline{r}}^F_{op_i/F} &= \ddot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_i p_i/f_i})\ \underline{\alpha}^{f_i}_{f_i/F} + R^F_{f_i}\ (S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_i p_i/f_i} \\
&= \ddot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_i p_i/f_i})(\dot{G}_i\ \dot{\underline{\theta}}_i + G_i\ \ddot{\underline{\theta}}_i) + R^F_{f_i}\ (S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_i p_i/f_i}
\end{aligned}
$$

or

$$
\ddot{\underline{r}}^F_{op_i/F} = \ddot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_i p_i/f_i})\ \dot{G}_i\ \dot{\underline{\theta}}_i - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_i p_i/f_i})\ G_i\ \ddot{\underline{\theta}}_i + R^F_{f_i}\ (S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_i p_i/f_i}
$$

If we define

$$
\underline{a}_{v r_i}(\underline{q}_{r_i}, \dot{\underline{q}}_{r_i}) = R^F_{f_i}\ [(S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_i p_i/f_i} - S(\underline{r}^{f_i}_{c_i p_i/f_i})\ \dot{G}_i\ \dot{\underline{\theta}}_i], \tag{1.11}
$$

then the above expression can be written as

$$
\ddot{\underline{r}}^F_{op_i/F} = \ddot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_i p_i/f_i})\ G_i\ \ddot{\underline{\theta}}_i + \underline{a}_{v r_i}(\underline{q}_{r_i}, \dot{\underline{q}}_{r_i}).
$$

Using the expression (1.8) we have

$$
\ddot{\underline{r}}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \ddot{\underline{q}}_{r_i} + \underline{a}_{v r_i}(\underline{q}_{r_i}, \dot{\underline{q}}_{r_i}). \tag{1.12}
$$

1.1.4 Virtual Displacement

We define the virtual displacement of an arbitrary point "p_i" of the rigid body "i" as

$$
\delta \underline{r}^F_{op_i/F} = \frac{\partial \underline{r}^F_{op_i/F}}{\partial \underline{q}_{r_i}}\ \delta \underline{q}_{r_i}. \tag{1.13}
$$

The velocity of this point has previously defined (equation (1.7)) as

$$\begin{aligned}
&\dot{\underline{r}}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \dot{\underline{q}}_{r_i} \\
\Rightarrow\ &\frac{\partial \underline{r}^F_{op_i/F}}{\partial t} = L_r(\underline{q}_{r_i})\ \frac{\partial \underline{q}_{r_i}}{\partial t} \\
\Rightarrow\ &\frac{\partial \underline{r}^F_{op_i/F}}{\partial \underline{q}_{r_i}} \frac{\partial \underline{q}_{r_i}}{\partial t} = L_r(\underline{q}_{r_i})\ \frac{\partial \underline{q}_{r_i}}{\partial t}
\end{aligned}$$

For indepedent $\frac{\partial \underline{q}_{r_i}}{\partial t}$ it is obvious that

$$\frac{\partial \underline{r}^F_{op_i/F}}{\partial \underline{q}_{r_i}} = L_r(\underline{q}_{r_i}). \tag{1.14}$$

Combining equations (1.13) and (1.14) we define the virtual displacement of point p_i as

$$\delta \underline{r}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \delta \underline{q}_{r_i} \tag{1.15}$$

1.2 Dynamics

There are several methods for developing the dynamic equations of motion of rigid bodies. In this section, the *principle of virtual work in dynamics* will be used to obtain the differential equations that govern the spatial motion of rigid bodies. Based on D'Alembert's principle, a body "i" (rigid or flexible) in a dynamic equilibrium obeys the following

$$\delta W_i^e = \delta W_i^{in}, \tag{1.16}$$

where, δW_i^e is the virtual work of the externaly applied forces to the body and δW_i^{in} is the virtual work of the the inertial forces.

1.2.1 Virtual work of inertial forces

For a continuum the virtual work of inertial forces is defined as

$$\delta W_i^{in} = \int_{m_i} \ddot{\underline{r}}^F_{op_i/F}\ \delta \underline{r}^F_{op_i/F}\ dm_i, \tag{1.17}$$

where p_i is an arbitrary point of the body "i".

For a rigid body it has proven that the following equations hold

$$\ddot{\underline{r}}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \ddot{\underline{q}}_{r_i} + \underline{a}_{v_{r_i}}(\underline{q}_{r_i}, \dot{\underline{q}}_{r_i}) \quad \text{and} \quad \delta \underline{r}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \delta \underline{q}_{r_i}.$$

Combining the above with equation (1.17) the virtual work of inertial forces can be written as

$$\delta W_i^{in} = \ddot{\underline{q}}^T_{r_i} \int_{m_i} (L_{r_i})^T\ L_{r_i}\ dm_i\ \delta \underline{q}_{r_i} + \int_{m_i} \underline{a}_{v_{r_i}}\ L_{r_i}\ dm_i\ \delta \underline{q}_{r_i}$$

If we define the mass matrix ,

$$M_{r_i} = \int_{m_i} (L_{r_i})^T \; L_{r_i} \; dm_i, \tag{1.18}$$

and the centrifugal-coriolis forces vector

$$\underline{f}_{v_{r_i}} = \underline{f}_{v_{r_i}}(\underline{q}_{r_i}, \underline{\dot{q}}_{r_i}) = -\int_{m_i} (L_{r_i})^T \; \underline{a}_{v_{r_i}} \; dm_i, \tag{1.19}$$

then the equation above becomes

$$\delta W_i^{in} = (\underline{\ddot{q}}_{r_i}^T \; M_{r_i} - \underline{f}_{v_{r_i}}^T)\delta \underline{q}_{r_i}. \tag{1.20}$$

Mass matrix

In equation (1.23), the mass matrix of the rigid body was defined as

$$M_{r_i} = \int_{m_i} (L_{r_i})^T \; L_{r_i} \; dm_i.$$

Given equation (1.8), this expression can be formulated as

$$\begin{aligned} M_{r_i} &= \int_{m_i} \begin{bmatrix} I_{3\times3} \\ -G_i^T \; (S(\underline{r}_{c_i p_i/f_i}^{f_i}))^T \; R_F^{f_i} \end{bmatrix} \begin{bmatrix} I_{3\times3} & -R_{f_i}^F \; S(\; \underline{r}_{c_i p_i/f_i}^{f_i}) \; G_i \end{bmatrix} \; dm_i \\ &= \int_{m_i} \begin{bmatrix} I_{3\times3} & -R_{f_i}^F \; S(\underline{r}_{c_i p_i/f_i}^{f_i}) \; G_i \\ -G_i^T \; (S(\underline{r}_{c_i p_i/f_i}^{f_i}))^T \; R_F^{f_i} & G_i^T \; (S(\underline{r}_{c_i p_i/f_i}^{f_i}))^T \; S(\underline{r}_{c_i p_i/f_i}^{f_i}) \; G_i \end{bmatrix} \; dm_i \\ &= \begin{bmatrix} m_{11}^{r_i} & m_{12}^{r_i} \\ m_{21}^{r_i} & m_{22}^{r_i} \end{bmatrix} \end{aligned}$$

where,

$$m_{11}^{r_i} = \int_{m_i} I_{3\times3} \; dm_i = m_i I_{3\times3} \tag{1.21a}$$

$$m_{12}^{r_i} = (m_{21}^{r_i})^T = -R_{f_i}^F \; \int_{m_i} S(\underline{r}_{c_i p_i/f_i}^{f_i}) \; dm_i \; G_i \tag{1.21b}$$

$$m_{22}^{r_i} = G_i^T \; \int_{m_i} (S(\underline{r}_{c_i p_i/f_i}^{f_i}))^T \; S(\underline{r}_{c_i p_i/f_i}^{f_i}) \; dm_i \; G_i \tag{1.21c}$$

We define the body's inertial tensor on the point c_i and with respect to the body frame f_i as

$$I_{c_i}^{f_i} = \int_{m_i} (S(\underline{r}_{c_i p_i/f_i}^{f_i}))^T \; S(\underline{r}_{c_i p_i/f_i}^{f_i}) \; dm_i. \tag{1.22}$$

Given that the frame f_i is rigidly attached to the body the above inertial tensor remains constant for the rigid body case. Then we have

$$M_{r_i} = \begin{bmatrix} m_i \; I_{3\times3} & m_{12} \\ m_{21} & G_i^T \; I_{c_i}^{f_i} \; G_i \end{bmatrix}. \tag{1.23}$$

Centrifugal-coriolis forces vector

The vector of centrifugal-coriolis forces was given by the equation (1.19) as

$$\underline{f}_{v_{r_i}}(\underline{q}_{r_i}, \underline{\dot{q}}_{r_i}) = -\int_{m_i} (L_{r_i})^T \; \underline{a}_{v_{r_i}} \quad dm_i$$

Substituting expressions (1.8) and (1.11)

$$\begin{aligned}
\underline{f}_{v_{r_i}} &= -\int_{m_i} \begin{bmatrix} I_{3\times 3} \\ -G_i^T \; (S(\underline{r}^{f_i}_{c_i p_i/f_i}))^T \; R^{f_i}_F \end{bmatrix} R^F_{f_i} \; [(S(\underline{\omega}^{f_i}_{f_i/F}))^2 \; \underline{r}^{f_i}_{c_i p_i/f_i} - S(\underline{r}^{f_i}_{c_i p_i/f_i}) \; \dot{G}_i \; \underline{\dot{\theta}}_i] \; dm_i && (1.24) \\
&= -\int_{m_i} \begin{bmatrix} R^F_{f_i} \; [(S(\underline{\omega}^{f_i}_{f_i/F}))^2 \; \underline{r}^{f_i}_{c_i p_i/f_i} - S(\underline{r}^{f_i}_{c_i p_i/f_i}) \; \dot{G}_i \; \underline{\dot{\theta}}_i] \\ -G_i^T \; (S(\underline{r}^{f_i}_{c_i p_i/f_i}))^T \; [(S(\underline{\omega}^{f_i}_{f_i/F}))^2 \; \underline{r}^{f_i}_{c_i p_i/f_i} - S(\underline{r}^{f_i}_{c_i p_i/f_i}) \; \dot{G}_i \; \underline{\dot{\theta}}_i] \end{bmatrix} \; dm_i. && (1.25)
\end{aligned}$$

If we define

$$\underline{f}_{v_{r_i}} = \begin{bmatrix} \underline{f}^r_{v_{r_i}} \\ \underline{f}^\theta_{v_{r_i}} \end{bmatrix},$$

then,

$$\underline{f}^r_{v_{r_i}} = -R^F_{f_i} \; (S(\underline{\omega}^{f_i}_{f_i/F}))^2 \left[\int_{m_i} \underline{r}^{f_i}_{c_i p_i/f_i} \; dm_i\right] + R^F_{f_i} \; \left[\int_{m_i} S(\underline{r}^{f_i}_{c_i p_i/f_i}) \; dm_i\right] \dot{G}_i \; \underline{\dot{\theta}}_i \qquad (1.26)$$

and

$$\begin{aligned}
\underline{f}^\theta_{v_{r_i}} = G_i^T \; &\left[\int_{m_i} (S(\underline{r}^{f_i}_{c_i p_i/f_i}))^T \; (S(\underline{\omega}^{f_i}_{f_i/F}))^2 \; \underline{r}^{f_i}_{c_i p_i/f_i} \; dm_i\right] \\
&- G_i^T \; \left[\int_{m_i} (S(\underline{r}^{f_i}_{c_i p_i/f_i}))^T \; S(\underline{r}^{f_i}_{c_i p_i/f_i}) \; dm_i\right] \; \dot{G}_i \; \underline{\dot{\theta}}_i
\end{aligned}$$

Given the expression for the inertial tensor as defined in equation (1.22), the above equation can be written as

$$\underline{f}^\theta_{v_{r_i}} = G_i^T \; \left[\int_{m_i} (S(\underline{r}^{f_i}_{c_i p_i/f_i}))^T \; (S(\underline{\omega}^{f_i}_{f_i/F}))^2 \; \underline{r}^{f_i}_{c_i p_i/f_i} \; dm_i\right] - G_i^T \; I^{f_i}_{c_i} \dot{G}_i \; \underline{\dot{\theta}}_i.$$

It can be easily proven that the integral in the above equation can be written as

$$\begin{aligned}
\int_{m_i} (S(\underline{r}^{f_i}_{c_i p_i/f_i}))^T \; (S(\underline{\omega}^{f_i}_{f_i/F}))^2 \; \underline{r}^{f_i}_{c_i p_i/f_i} \; dm_i &= -S(\underline{\omega}^{f_i}_{f_i/F}) \left[\int_{m_i} (S(\underline{r}^{f_i}_{c_i p_i/f_i}))^T \; S(\underline{r}^{f_i}_{c_i p_i/f_i}) \; dm_i\right] \underline{\omega}^{f_i}_{f_i/F} \\
&= -S(\underline{\omega}^{f_i}_{f_i/F}) \; I^{f_i}_{c_i} \; \underline{\omega}^{f_i}_{f_i/F}.
\end{aligned}$$

Combining the two equation above we get

$$\underline{f}^\theta_{v_{r_i}} = -G_i^T \; \left[S(\underline{\omega}^{f_i}_{f_i/F}) \; I^{f_i}_{c_i} \; \underline{\omega}^{f_i}_{f_i/F} + I^{f_i}_{c_i} \dot{G}_i \; \underline{\dot{\theta}}_i\right] \qquad (1.27)$$

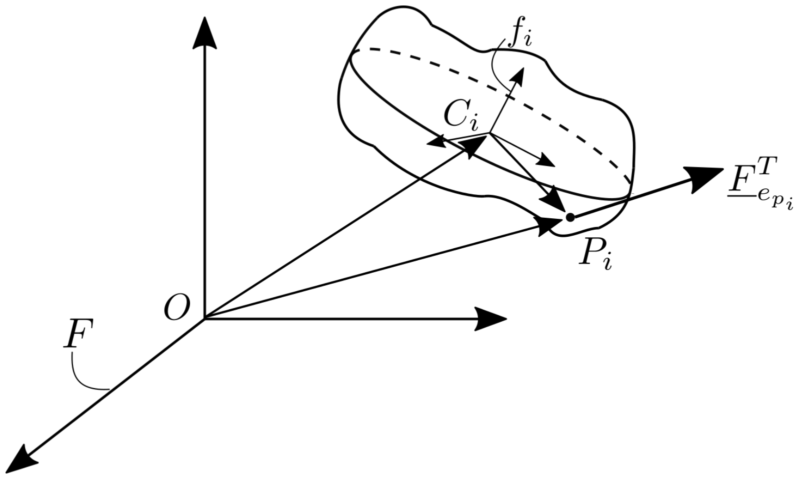

Figure 1.2: Rigid Body Under Force

1.3 Virtual work of external forces

Let $\underline{F}_{e_{p_i}}$ an external force that acts on the rigid body at point p_i as shown in Figure 1.2. Then the virtual work of the external forces this point can be defined as

$$\begin{aligned} \delta W_{p_i}^e &= (\underline{F}_{e_{p_i}}^F)^T \ \delta \underline{r}_{op_i/F}^F \\ &= (\underline{F}_{e_{p_i}}^F)^T \ L_r(\underline{q}_{r_i}) \ \delta \underline{q}_{r_i}, \end{aligned}$$

We define the generalized force on point p_i as

$$\underline{Q}_{r_{p_i}} = L_r^T(\underline{q}_{r_i}) \ \underline{F}_{e_{p_i}}^F \tag{1.28}$$

then the equation above becomes,

$$\delta W_{p_i}^e = \underline{Q}_{r_{p_i}}^T \ \delta \underline{q}_{r_i}. \tag{1.29}$$

Based on the definition (1.8) and the equation of the generalized force on point p_i ((1.28)), we get

$$\underline{Q}_{r_{p_i}} = \begin{bmatrix} I_{3\times 3} \\ -G_i^T(S(\underline{r}_{c_ip_i/f_i}^{f_i}))^T (R_{f_i}^F)^T \end{bmatrix} \underline{F}_{e_{p_i}}^F = \begin{bmatrix} \underline{F}_{e_{p_i}}^F \\ -G_i^T(S(\underline{r}_{c_ip_i/f_i}^{f_i}))^T \ \underline{F}_{e_{p_i}}^{f_i} \end{bmatrix}.$$

We define the moment with respect to point c_i, expressed in the body frame as

$$\underline{M}_{c_i}^{f_i} = S(\underline{r}_{c_ip_i/f_i}^{f_i}) \ \underline{F}_{e_{p_i}}^{f_i},$$

then the equation above becomes

$$\underline{Q}_{r_{p_i}} = \begin{bmatrix} \underline{F}_{e_{p_i}}^F \\ G_i^T \underline{M}_{c_i}^{f_i} \end{bmatrix}. \tag{1.30}$$

Let n_f be the number of forces acting on the body in different points p_i^j and n_m the number of moments with respect to point c_i. Then, the total virtual work of external forces and moments acting on the body "i" can be easily computed as

$$\delta W_i^e = \underline{Q}_{r_i}^T \ \delta \underline{q}_{r_i}, \quad \text{where} \quad \underline{Q}_{r_i} = \begin{bmatrix} \sum_{j=1}^{n_f} \left(\underline{F}_{e_i}^F\right)_j \\ G_i^T \sum_{j=1}^{n_m} \left(\underline{M}_{c_i}^{f_i}\right)_j \end{bmatrix}. \tag{1.31}$$

1.4 Equations of Motion

Based on equations (1.16), (1.20) and (1.31), the principle of virtual work for unconstrained motion can be defined as

$$\begin{aligned} &\delta W_i^e = \delta W_i^{in} \\ &\Rightarrow (\ddot{\underline{q}}_{r_i}^T \ M_{r_i} - \underline{f}_{v_{r_i}}^T)\delta \underline{q}_{r_i} = \underline{Q}_{r_i}^T \ \delta \underline{q}_{r_i} \\ &\Rightarrow (M_{r_i}\ddot{\underline{q}}_{r_i} - \underline{f}_{v_{r_i}})^T \ \delta \underline{q}_{r_i} = \underline{Q}_{r_i}^T \ \delta \underline{q}_{r_i} \\ &\Rightarrow (M_{r_i}\ddot{\underline{q}}_{r_i} - \underline{f}_{v_{r_i}} - \underline{Q}_{r_i})^T \ \delta \underline{q}_{r_i} = \end{aligned}$$

In the case of unconstrained motion, the elements of the vector $\delta \underline{q}_{r_i}$ are independent. Thus, the above equation can be written as

$$M_{r_i}\ddot{\underline{q}}_{r_i} = \underline{f}_{v_{r_i}} + \underline{Q}_{r_i}. \tag{1.32}$$

or

$$\begin{bmatrix} m_i \ I_{3\times 3} & m_{12} \\ m_{21} & G_i^T \ I_{c_i}^{f_i} \ G_i \end{bmatrix} \begin{bmatrix} \ddot{\underline{r}}_{oc_i/F}^F \\ \ddot{\underline{\theta}}_i \end{bmatrix} = \begin{bmatrix} \underline{f}_{v_{r_i}}^r \\ \underline{f}_{v_{r_i}}^\theta \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{n_f} \left(\underline{F}_{e_i}^F\right)_j \\ G_i^T \sum_{j=1}^{n_m} \left(\underline{M}_{c_i}^{f_i}\right)_j \end{bmatrix}.$$

1.4.1 Equations of motion on the body's centroid

In the case that c_i is the center of mass of the body "i" then the equations of motion can be simplified significantly. The center of mass can be defined as

$$\underline{r}_{oc_i/F}^F = \frac{1}{m_i}\int_{m_i} \underline{r}_{op_i/F}^F \ dm_i \quad \text{where} \quad m_i = \int_{m_i} dm_i.$$

Given the above equation it is easily proven that

$$\begin{aligned} \int_{m_i} \underline{r}_{c_ip_i/f_i}^{f_i} \ dm_i &= R_F^{f_i} \int_{m_i} \underline{r}_{c_ip_i/f_i}^F \ dm_i \\ &= R_F^{f_i}(\int_{m_i} \underline{r}_{o_ip_i/F}^F \ dm_i - \int_{m_i} \underline{r}_{o_ic_i/F}^F \ dm_i) \\ &= R_F^{f_i}(m_i \ \underline{r}_{o_ic_i/F}^F - \underline{r}_{o_ic_i/F}^F \int_{m_i} dm_i) \\ &= \underline{0}. \end{aligned}$$

Furthermore, based on the above equation

$$\int_{m_i} S(\underline{r}^{f_i}_{c_i p_i/f_i})\ dm_i = \underline{0}.$$

Combining the above equations with equation (1.21) it is obvious

$$m^{r_i}_{12} = (m^{r_i}_{21})^T = -R^F_{f_i} \int_{m_i} S(\underline{r}^{f_i}_{c_i p_i/f_i})\ dm_i\ G_i = 0.$$

Similarly, from equation (1.26) we get

$$\underline{f}^r_{v_{r_i}} = -R^F_{f_i}\ (S(\underline{\omega}^{f_i}_{f_i/F}))^2 \left[\int_{m_i} \underline{r}^{f_i}_{c_i p_i/f_i}\ dm_i\right] + R^F_{f_i}\ \left[\int_{m_i} S(\underline{r}^{f_i}_{c_i p_i/f_i})\ dm_i\right] \dot{G}_i\ \dot{\underline{\theta}}_i = 0.$$

The equations of motion, in the case that the frame f_i is rigidly attached to the body's centroid, are defined as

$$\begin{bmatrix} m_i\ I_{3\times 3} & 0 \\ 0 & G_i^T\ I^{f_i}_{c_i}\ G_i \end{bmatrix} \begin{bmatrix} \ddot{\underline{r}}^F_{oc_i/F} \\ \ddot{\underline{\theta}}_i \end{bmatrix} = \begin{bmatrix} 0 \\ \underline{f}^{\theta}_{v_{r_i}} \end{bmatrix} + \begin{bmatrix} \sum\limits_{j=1}^{n_f} \left(\underline{F}^F_{e_i}\right)_j \\ G_i^T \sum\limits_{j=1}^{n_m} \left(\underline{M}^{f_i}_{c_i}\right)_j \end{bmatrix}. \tag{1.33}$$