Chapter 1

Rigid body

1.1 Kinematics

Let P_i represent an arbitrary point on the rigid body 'i' that is shown in Figure 1.1 and C_i the center the body's center of mass. Furthermore, the frame f_i is rigidly attached to the body (translates and rotates with it), while the F frame is the inertial frame of reference.

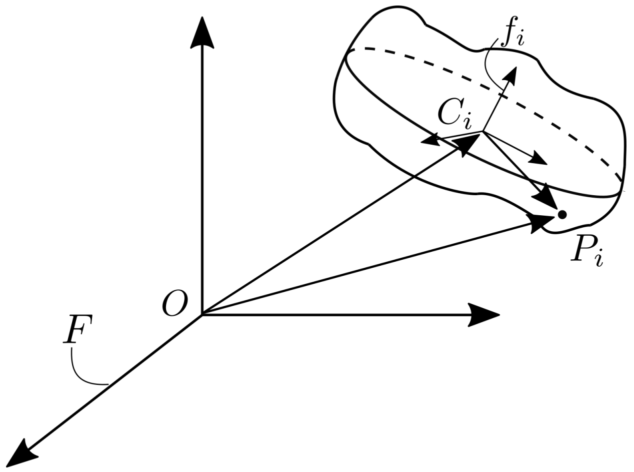

Figure 1.1: Rigid Body

The center of mass of the body can be defined as

$$\underline{r}^F_{oc_i/F} = \frac{1}{m_i}\int_{m_i} \underline{r}^F_{op_i/F}\; dm_i \quad \text{where} \quad m_i = \int_{m_i} dm_i.$$

1.1.1 Position

The position of the arbitrary point "p_i" with respect to the inertial frame is defined as

$$\underline{r}^F_{op_i/F} = \underline{r}^F_{oc_i/F} + R^F_{f_i}\; \underline{r}^{f_i}_{c_ip_i/f_i}, \tag{1.1}$$

where $R^F_{f_i}$ is the rotation matrix of body frame f_i with respect to the inertial frame F.

1.1.2 Velocity

The velocity of the the arbitrary point P_i with respect to the inertial frame is defined as

$$\underline{\dot{r}}^F_{op_i/F} = \underline{\dot{r}}^F_{oc_i/F} + \frac{d}{dt}(R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i}),$$

$$\underline{\dot{r}}^F_{op_i/F} = \underline{\dot{r}}^F_{oc_i/F} + \dot{R}^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i} + R^F_{f_i}\ \underline{\dot{r}}^{f_i}_{c_ip_i/f_i}. \tag{1.2}$$

We know that for a rigid body the distance between to points remains constant meaning that $\underline{\dot{r}}^{f_i}_{c_ip_i/f_i} = 0$. The time derivative of the rotation matrix can be defined as

$$\dot{R}^F_{f_i} = S(\underline{\omega}^F_{f_i/F})\ R^F_{f_i}, \tag{1.3}$$

where, $\underline{\omega}^F_{f_i/F}$ the rotational velocity of f_i frame with resepct to F frame and $S(\underline{a})$ skew-symetrix matrix defined as

$$S(\underline{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \quad \text{where} \quad \underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Given the above equation (1.2) becomes

$$\begin{aligned} \underline{\dot{r}}^F_{op_i/F} &= \underline{\dot{r}}^F_{oc_i/F} + S(\underline{\omega}^F_{f_i/F}) R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i} \\ &= \underline{\dot{r}}^F_{oc_i/F} - S(R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i})\ \underline{\omega}^F_{f_i/F} \\ &= \underline{\dot{r}}^F_{op_i/F} = \underline{\dot{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\ \underline{r}^{f_i}_{c_ip_i/f_i})\ (R^F_{f_i})^T\ \underline{\omega}^F_{f_i/F}. \end{aligned}$$

or

$$\underline{\dot{r}}^F_{op_i/F} = \underline{\dot{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\ \underline{r}^{f_i}_{c_ip_i/f_i})\ \underline{\omega}^{f_i}_{f_i/F}. \tag{1.4}$$

It can be proven that the rotational velocity of the body frame with respect to the inertial frame (expressed in the body frame) can be written as:

$$\underline{\omega}^{f_i}_{f_i/F} = G(\underline{\theta}_i)\ \underline{\dot{\theta}}_i = G_i\ \underline{\dot{\theta}}_i, \quad G_i = G(\underline{\theta}_i), \tag{1.5}$$

where $\underline{\theta}_i$ is the vector with the parameters that describe the orientation of the body frame with respect to the inertial frame (euler angles, euler parameters, rodrigues parameters, etc.).

Given expression (1.5), equation (1.4) becomes

$$\underline{\dot{r}}^F_{op_i/F} = \underline{\dot{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\ \underline{r}^{f_i}_{c_ip_i/f_i})\ G_i\ \underline{\dot{\theta}}_i.$$

Finally, if we define the generalized rigid body coordinates as

$$\underline{q}_{r_i} = \begin{bmatrix} \underline{\dot{r}}^F_{oc_i/F} \\ \underline{\theta}_i \end{bmatrix}, \tag{1.6}$$

then equation (1.4) becomes

$$\underline{\dot{r}}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \underline{\dot{q}}_{r_i}, \tag{1.7}$$

where,

$$L_r(\underline{q}_{r_i}) = L_{r_i} = \begin{bmatrix} I_{3\times3} & -R^F_{f_i}\ S(\ \underline{r}^{f_i}_{c_ip_i/f_i})\ G_i \end{bmatrix} \tag{1.8}$$

1.1.3 Acceleration

The acceleration of the the arbitrary point P_i with respect to the inertial frame is defined as

$$
\begin{aligned}
\ddot{\underline{r}}^F_{op_i/F} &= \frac{d}{dt}(\dot{\underline{r}}^F_{oc_i/F} + S(\underline{\omega}^F_{f_i/F})R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i}) \\
&= \ddot{\underline{r}}^F_{oc_i/F} + \dot{\underline{\omega}}^F_{f_i/F} \times (R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i}) + \underline{\omega}^F_{f_i/F} \times \frac{d}{dt}(R^F_{f_i}\ \underline{r}^{f_i}_{c_ip_i/f_i}).
\end{aligned}
$$

Based on the above, the expression for the acceleration can take the following form

$$
\ddot{\underline{r}}^F_{op_i/F} = \ddot{\underline{r}}^F_{oc_i/F} + R^F_{f_i}\ S(\underline{\alpha}^{f_i}_{f_i/F})\ \underline{r}^{f_i}_{c_ip_i/f_i} + R^F_{f_i}\ (S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_ip_i/f_i} \tag{1.9}
$$

where

$$
\underline{\alpha}^{f_i}_{f_i/F} = \dot{\underline{\omega}}^{f_i}_{f_i/F} = \frac{d}{dt}(G(\underline{\theta}_i)\ \dot{\underline{\theta}}_i) = \dot{G}_i\ \dot{\underline{\theta}}_i + G_i\ \ddot{\underline{\theta}}_i, \tag{1.10}
$$

the expression for the angular acceleration of the body. Substituting (1.10) into (1.9) leads to

$$
\begin{aligned}
\ddot{\underline{r}}^F_{op_i/F} &= \ddot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_ip_i/f_i})\ \underline{\alpha}^{f_i}_{f_i/F} + R^F_{f_i}\ (S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_ip_i/f_i} \\
&= \ddot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_ip_i/f_i})(\dot{G}_i\ \dot{\underline{\theta}}_i + G_i\ \ddot{\underline{\theta}}_i) + R^F_{f_i}\ (S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_ip_i/f_i}
\end{aligned}
$$

or

$$
\ddot{\underline{r}}^F_{op_i/F} = \ddot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_ip_i/f_i})\ \dot{G}_i\ \dot{\underline{\theta}}_i - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_ip_i/f_i})\ G_i\ \ddot{\underline{\theta}}_i + R^F_{f_i}\ (S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_ip_i/f_i}
$$

If we define

$$
\underline{a}_{v_{r_i}}(\underline{q}_{r_i}, \dot{\underline{q}}_{r_i}) = R^F_{f_i}\ [(S(\underline{\omega}^{f_i}_{f_i/F}))^2\ \underline{r}^{f_i}_{c_ip_i/f_i} - S(\underline{r}^{f_i}_{c_ip_i/f_i})\ \dot{G}_i\ \dot{\underline{\theta}}_i], \tag{1.11}
$$

then the above expression can be written as

$$
\ddot{\underline{r}}^F_{op_i/F} = \ddot{\underline{r}}^F_{oc_i/F} - R^F_{f_i}\ S(\underline{r}^{f_i}_{c_ip_i/f_i})\ G_i\ \ddot{\underline{\theta}}_i + \underline{a}_{v_{r_i}}(\underline{q}_{r_i}, \dot{\underline{q}}_{r_i}).
$$

Using the expression (1.8) we have

$$
\ddot{\underline{r}}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \ddot{\underline{q}}_{r_i} + \underline{a}_{v_{r_i}}(\underline{q}_{r_i}, \dot{\underline{q}}_{r_i}). \tag{1.12}
$$

1.1.4 Virtual Displacement

We define the virtual displacement of an arbitrary point "p_i" of the rigid body "i" as

$$
\delta \underline{r}^F_{op_i/F} = \frac{\partial \underline{r}^F_{op_i/F}}{\partial \underline{q}_{r_i}}\ \delta \underline{q}_{r_i}. \tag{1.13}
$$

The velocity of this point has previously defined (equation (**??**)) as

$$
\begin{aligned}
&\dot{\underline{r}}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \dot{\underline{q}}_{r_i} \\
&\Rightarrow \frac{\partial \underline{r}^F_{op_i/F}}{\partial t} = L_r(\underline{q}_{r_i})\ \frac{\partial \underline{q}_{r_i}}{\partial t} \\
&\Rightarrow \frac{\partial \underline{r}^F_{op_i/F}}{\partial \underline{q}_{r_i}} \frac{\partial \underline{q}_{r_i}}{\partial t} = L_r(\underline{q}_{r_i})\ \frac{\partial \underline{q}_{r_i}}{\partial t}
\end{aligned}
$$

For indepedent $\frac{\partial \underline{q}_{r_i}}{\partial t}$ it is obvious that

$$
\frac{\partial \underline{r}^F_{op_i/F}}{\partial \underline{q}_{r_i}} = L_r(\underline{q}_{r_i}). \tag{1.14}
$$

Combining equations (1.13) and (1.14) we define the virtual displacement of point p_i as

$$
\delta \underline{r}^F_{op_i/F} = L_r(\underline{q}_{r_i})\ \delta \underline{q}_{r_i} \tag{1.15}
$$

1.2 Dynamics