Jacobian Matrix

Assume a vector $\underline{x} \in \mathbb{R}^m$ that describes the position and orientation of the end-effector with respect to the inertial frame. As we learned in the forward-kinematics, this is a function of the robot's indepedent joint coordinates $\underline{q} \in \mathbb{R}^n$.

This vector $\underline{x}$ can be defined using the "forward kinematics function" function $\underline{f} : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that

$$\underline{x} = \underline{f}(\underline{q}), \quad \text{where} \quad \underline{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad \text{and} \quad \underline{f}(\underline{q}) = \begin{bmatrix} f_1(\underline{q}) \\ \vdots \\ f_m(\underline{q}) \end{bmatrix}. \tag{1}$$

The time derivative of equation (1) is given as

$$\underline{\dot{x}} = \frac{d\underline{f}(\underline{q})}{dt} = \left(\frac{\partial \underline{f}(\underline{q})}{\partial \underline{q}}\right) \frac{\partial \underline{q}}{\partial t} = \left(\frac{\partial \underline{f}(\underline{q})}{\partial \underline{q}}\right) \underline{\dot{q}}. \tag{2}$$

We define the jacobian $\mathcal{J}$ of the system as

$$\mathcal{J}(\underline{q}) = \frac{\partial \underline{f}(\underline{q})}{\partial \underline{q}} = \begin{bmatrix} \frac{\partial \underline{f}(\underline{q})}{\partial q_1}, & \frac{\partial \underline{f}(\underline{q})}{\partial q_2} & \dots, \frac{\partial \underline{f}(\underline{q})}{\partial q_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(\underline{q})}{\partial q_1} & \dots & \frac{\partial f_1(\underline{q})}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(\underline{q})}{\partial q_1} & \dots & \frac{\partial f_m(\underline{q})}{\partial q_n} \end{bmatrix} \tag{3}$$

Combining equations (2) and (3) we get the mapping form joint velocities to end-effector velocities as

$$\underline{\dot{x}} = \mathcal{J}(\underline{q})\underline{\dot{q}}. \tag{4}$$

A systematic approach for deriving the jacobian (both geometric and analytical) for every robotic manipulator can be found in Mark W. Spong: Robot dynamics and control, Chapter 4. Follows an approach based on the calculus of rotation and skew-symmetric matrices.

An inverse mapping of the velocities can also be performed. For example given that we specify the translational and rotational velocity of the end-effector we can define the velocity of the joint coordinates as

$$\underline{\dot{q}} = \mathcal{J}^{-1}(\underline{q})\underline{\dot{x}} \tag{5}$$

However, the inverse of the jacobian $\mathcal{J}^{-1}$ does not always exist. Solution: Generalised Moore-Penrose inverse (MATLAB for $Ax = B$ use A\B or pinv(A)).

Singularities
Points of configuration space (all posible values of $\underline{q}$) for which the Jacobian is singular and thus has no inverse. Can be found by solving the algebraic equations (usually highly nonlinear):

$$\det \mathcal{J}(\underline{q}) = 0.$$

The solutions $\underline{q}_i$ of the above equation are called the singularity points. **Jacobian and dynamics**
Using the Jacobian we can also relate the *joint torques* and the *forces and moments at the end-effector* as:

$$\underline{\tau} = \mathcal{J}^T(\underline{q})\underline{F}.$$