

Jacobian Matrix

Assume a vector $\underline{x} \in \mathbb{R}^m$ that describes the position and orientation of the end-effector with respect to the inertial frame. As we learned in the forward-kinematics, this is a function of the robot's independent joint coordinates $\underline{q} \in \mathbb{R}^n$.

This vector \underline{x} can be defined using the "forward kinematics function" function $\underline{f} : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that

$$\underline{x} = \underline{f}(\underline{q}), \quad \text{where } \underline{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad \text{and} \quad \underline{f}(\underline{q}) = \begin{bmatrix} f_1(\underline{q}) \\ \vdots \\ f_m(\underline{q}) \end{bmatrix}. \quad (1)$$

The time derivative of equation (??) is given as

$$\dot{\underline{x}} = \frac{d\underline{f}(\underline{q})}{dt} = \left(\frac{\partial \underline{f}(\underline{q})}{\partial \underline{q}} \right) \frac{\partial \underline{q}}{\partial t} = \left(\frac{\partial \underline{f}(\underline{q})}{\partial \underline{q}} \right) \dot{\underline{q}}. \quad (2)$$

We define the jacobian \mathcal{J} of the system as

$$\mathcal{J}(\underline{q}) = \frac{\partial \underline{f}(\underline{q})}{\partial \underline{q}} = \left[\frac{\partial f_1(\underline{q})}{\partial q_1}, \frac{\partial f_1(\underline{q})}{\partial q_2}, \dots, \frac{\partial f_1(\underline{q})}{\partial q_n}; \frac{\partial f_2(\underline{q})}{\partial q_1}, \frac{\partial f_2(\underline{q})}{\partial q_2}, \dots, \frac{\partial f_2(\underline{q})}{\partial q_n}; \dots; \frac{\partial f_m(\underline{q})}{\partial q_1}, \frac{\partial f_m(\underline{q})}{\partial q_2}, \dots, \frac{\partial f_m(\underline{q})}{\partial q_n} \right] = \begin{bmatrix} \frac{\partial f_1(\underline{q})}{\partial q_1} & \dots & \frac{\partial f_1(\underline{q})}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(\underline{q})}{\partial q_1} & \dots & \frac{\partial f_m(\underline{q})}{\partial q_n} \end{bmatrix} \quad (3)$$

Combining equations (??) and (??) we get the mapping form joint velocities to end-effector velocities as

$$\dot{\underline{x}} = \mathcal{J}(\underline{q})\dot{\underline{q}}. \quad (4)$$

A systematic approach for deriving the jacobian (both geometric and analytical) for every robotic manipulator can be found in Mark W. Spong: Robot dynamics and control, Chapter 4. Follows an approach based on the calculus of rotation and skew-symmetric matrices.

An inverse mapping of the velocities can also be performed. For example given that we specify the translational and rotational velocity of the end-effector we can define the velocity of the joint coordinates as

$$\dot{\underline{q}} = \mathcal{J}^{-1}(\underline{q})\dot{\underline{x}} \quad (5)$$

However, the inverse of the jacobian \mathcal{J}^{-1} does not always exist. Solution: Generalised Moore-Penrose inverse (MATLAB for $Ax = B$ use $A \setminus B$ or $\text{pinv}(A)$).

Singularities

Points of configuration space (all possible values of \underline{q}) for which the Jacobian is singular and thus has no inverse. Can be found by solving the algebraic equations (usually highly nonlinear):

$$\det \mathcal{J}(\underline{q}) = 0.$$

The solutions \underline{q}_i of the above equation are called the singularity points. **Jacobian and dynamics** Using the Jacobian we can also relate the *joint torques* and the *forces and moments at the end-effector* as:

$$\underline{\tau} = \mathcal{J}^T(\underline{q})\underline{F}.$$