Brief Communications

Nonlinear Bending of Beams With Concentrated Loads

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A simple numerical method is proposed for analyzing nonlinear bending of beams. The success of the application of this method to the nonlinear problems is demonstrated by two examples: (1) a cantilever beam with a concentrated load at the free end, and (2) a simply supported beam subjected to a nonsymmetrical load. The results are checked by other known solutions.

Introduction

The analysis of bending of beams is based on the Bernoulli–Euler theory which states that the bending moment at any point of a beam is proportional to the corresponding curvature. In most engineering problems the slope of the beam is small and its square can be neglected in comparison with unity, thus, the expression of the curvature is linearized. Results based on this approximation, however, cannot be applied in the case of large deflections, since they may lead to unconservative errors.

The problems of a cantilever carrying a vertical load at the free end have been solved by Barton (**1**), and Bisshopp and Drucker (**2**). Frisch–Fay (**3**) obtained the solution for a cantilever under two concentrated loads. The solutions for a simply supported beam subjected to a central concentrated load were obtained by Conway (**4**). Analyzing the same beam subjected to nonsymmetrical loads, Wang, Lee and Zienkiewicz (**5**) made calculations for both a concentrated load and a partial uniformly distributed load, and the results were checked by experiments.

In the works mentioned above, the beams are subjected to concentrated loads of not more than two. However, as the number of concentrated loads increases, the available known methods become tedious and complicated. This paper presents a simple numerical method for analyzing nonlinear bending of beams subjected to concentrated loads. We first analyze a cantilever beam with a concentrated load at the free end. The result is compared with a known exact solution (**2**). The proposed method is then applied to a simply supported beam subjected to a nonsymmetrical load and the result is checked by the only known approximate solution (**5**).

Example 1. Cantilever Beam with Concentrated Load at Free End

Consider the first example, as shown in Fig. 1. The cross-section of the beam is assumed to be constant. Taking the origin at the free end B, the Bernoulli–

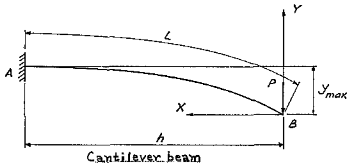

Fig. 1

Euler equation gives, for an initially straight beam,

$$d\theta/ds = -M/D. \tag{1}$$

Multiplying both sides of Eq. 1 by dx and writing $\cos\theta$ for dx/ds, we have $\cos\theta\, d\theta = -(M/D)\, dx$. Integrating the above expression between any two limits, m and n, yields

$$\sin\theta_n - \sin\theta_m = -\int_m^n (M/D)\, dx \tag{2}$$

where D is the flexural ridigity of the beam, x; the horizontal distance measured from the free end, θ; the slope, M; the bending moment; and

$$\int_m^n (M/D)\, dx,$$

the area of (M/D)—diagram along the X axis of the beam between m and n.

If one of the angles, θ_m, in the above expression is given, then the slope at all sections can be determined from Eq. 2 after the moment is computed at each section.

With θ's known at all points, the value of h and $y_{\max}$ (Fig. 1) can be calculated from the relationships

$$L = \int_0^h \sec\theta\, dx \tag{3}$$

and

$$y_{\max} = \int_0^h \tan\theta\, dx \tag{4}$$

where L is the given length of the beam.

In this example, the horizontal distance of the beam, h, is divided into eight equal intervals. The lower limit, m, is chosen at the fixed end A where the slope is zero. The complete operation of determining the slopes at all points can be carried out in a compact tabulation, shown in Table I. Using numerical integration by Simpson's rule, Eqs. 3 and 4 yield $h = 0.563L$ and $y_{\max} = 0.745L$. The value of α upon substituting $h = 0.563L$ and $Ph^2/D = 1.92$ is $\alpha = PL^2/D = 6.05$.

TABLE I.

Computation of θ's for Example 1.

																Common Factors
$x = 1$	⅞		¾		⅝		½		⅜		¼		⅛		0	h
$M/D = 1$	0.875		0.750		0.625		0.500		0.375		0.250		0.125		0	Ph/D
$\sin\theta_n - \sin\theta_m = 1.875$		1.625		1.375		1.125		0.875		0.625		0.375		0.125		$Ph^2/(16D)$
$\sin\theta_m = \sin\theta_A = 0$																
$\sin\theta_n = 0$	1.875		3.500		4.875		6.000		6.875		7.500		7.875		8.000	$Ph^2/(16D)$
$\sin\theta = 0$	0.225		0.420		0.585		0.720		0.825		0.900		0.945		0.960	$(Ph^2/D = 1.92)$
$\theta = 0°$	13°0′		24°50′		35°48′		46°3′		55°35′		64°9′		70°55′		73°44′	

TABLE II.

Computation of θ's for Example 2.

																Common Factors
$M/D = 0$	0.75		1.50		1.25		1.00		0.75		0.50		0.25		0	$Ph/(8D)$
$\sin\theta_n - \sin\theta_m = -0.75$		−2.25		−2.75		−2.25		−1.75		−1.25		−0.75		−0.25		$Ph^2/(128D)$
$\sin\theta_n - \sin\theta_A = 0$	−0.75		−3.00		−5.75		−8.00		−9.75		−11.00		−11.75		−12.00	$Ph^2/(128D)$
$= 0$	−0.11719		−0.46875		−0.89844		−1.2500		−1.5234		−1.7188		−1.8359		−1.8750	$(Ph^2/D = 20)$
Assume $\theta_A' = +79°7'$																
$\theta' = 79°7'$	59°52′		30°53′		4°48′		−15°33′		−32°47′		−47°27′		−58°38′		−63°15′	
Eq. 5: $y = -0.0097$ h.																
Assume $\theta_A'' = +79°15'$																
$\theta'' = 79°15'$	59°55′		30°55′		4°49′		−15°31′		−32°45′		−47°25′		−58°36′		−63°12′	
Eq. 5: $y = -0.0054\,h$.																
Assume $\theta''' = +79°25'$																
$\theta''' = 79°25'$	59°58′		30°57′		4°51′		−15°29′		−32°43′		−47°22′		−58°32′		−63°8′	
Eq. 5: $y = +0.0002$ h.																

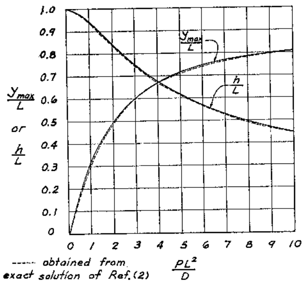

Fig. 2

The complete results of h and $y_{\max}$ for $\alpha = 1$ to 10 are shown in Fig. 2 and are in good agreement with those obtained by an exact analytical solution (2).

Example 2. Simply Supported Beam Subjected to Nonsymmetrical Load

The second example, shown in Fig. 3 can be solved by a procedure identical to that just presented. However, the angle at A in this case is unknown, hence, the slope at any section cannot be determined from Eq. 2.

To overcome this difficulty, a trial-and-error method is used. An initial value for θ_A is first assumed, the angle at all points can then be calculated from Eq. 2. From Fig. 3,

$$y = \int_0^h \tan\theta \, dx = 0. \tag{5}$$

By substituting the computed values of θ's into Eq. 5, the result, in general, will not yield to zero. Repetition of the trial will finally lead to the solution for θ_A.

In the present case, the horizontal distance b from the load P to the left support A is taken as $h/4$. Table II shows the computation for the determination of θ's for $Ph^2/D = 20$. Again, numerical integration of Eq. 3 yields $h = 0.608L$,

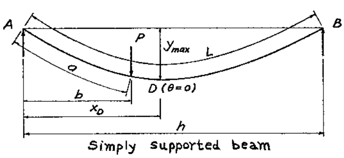

Fig. 3

α in this case is $\alpha = PL^2/D = 54$. The value of y_{max} can be determined from

$$y_{max} = \int_A^D \tan\theta \, dx \tag{6}$$

where D is the point at which $\theta = 0$. This results in $y_{max} = 0.347L$ at $x_D = 0.246L$. The load P is located along the beam at

$$a = \int_0^b \sec\theta \, dx = 0.370L$$

and the result is again checked with a known approximate solution (**5**) which gives $y_{max} = 0.345L$ and at $x_D = 0.246L$.

Conclusion

The proposed method can be applied successfully to the analysis of nonlinear bending of beams under the action of concentrated loads. Although only cases of single concentrated loads have been considered, these clearly do not present the limit of its applicability. From the examples given the proposed method offers the advantage of simplicity and ease in calculation, regardless of the number of concentrated loads. The method can also be easily extended to a beam of variable moment of inertia.

References

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