GEOMETRICALLY NONLINEAR AND DYNAMIC ANALYSIS OF EULER-BERNOULLI BEAMS USING ISOGEOMETRIC APPROACH

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ABSTRACT

This paper presents a numerical procedure for geometrically nonlinear and dynamic analysis of Euler-Bernoulli beams based on the framework of isogeometric approach. The method utilizes B-spline as the basis functions for both geometric representation and analysis. Only one deflection variable (without rotational degrees of freedom) is used for each control point. It allows us to use few degrees of freedom while retaining high accuracy of solution. Two numerical examples are provided to illustrate the effectiveness of present method.

Keywords: Isogeometric analysis; Euler-Bernoulli beam; rotation-free, geometric nonlinear, dynamic.

Introduction

The finite element method (FEM) is well known as the most powerful and reliable tool for computation and simulation of all engineering problems. In the standard FEM, a mesh generation needs to be used to transform the physics domain into an analysis-suitable geometry, i.e. a finite element mesh with the simple shapes (e.g. straight line; triangles, quadrilaterals; tetrahedral, hexahedral, etc, for 1D, 2D, 3D problems, respectively). Such finite element (FE) meshes are only an approximation to the original geometry and they lead to the geometrical error with curved domains. To overcome this disadvantage, Hughes *et al*. have recently proposed a novel computational method of so-called Isogeometric Analysis (Hughes, 2005).

The IGA uses basis functions which exactly describe the geometry to the approximate solutions. Being different from basis functions of the standard FEM based on Lagrange polynomial, isogeometric approach utilizes more general basis functions such as B-splines or Non-Uniform Rational B-splines (NURBS) that are common in Computer Aided Design (CAD) (Piegl,1997). The exact geometry is therefore maintained at the coarsest level of discretization and the re-meshing is performed on this coarsest level without any communication with CAD geometry. Isogeometric analysis has been widely applied to various practical problems such as linear and non-linear elasticity and plasticity behavior (Lguedj, 2008), structural vibrations (Cottrell, 2008), the plates and shells (Beirão da Veiga, 2012; Benson, 2010; Kiendl, 2010; Thai, 2012), structural shape optimization (Wall, 2008), and further improved NURBS approaches (Nguyen-Thanh, 2011), etc.

Beams – the most famous and simplest structures are widely used in civil and aerospace engineering. Among various beam theories, Euler - Bernoulli beam theory - EBT (also called as engineer's beam theory) was firstly established around in 1750 by Leonard Euler and Daniel Bernoulli with assumption that plane sections remain plane and perpendicular to the neutral axis during bending. In EBT, C^1-continuity of the approximation fields across element boundaries is needed and cubic Hermitian basis functions are therefore used. As a result, the conforming FE approximation has in general two degrees of freedom per each node: deflection and slope. In this paper, we study numerically Euler–Bernoulli beams using B-spline-based isogeometric approach combined with a rotation-free technique (Cottrell, 2006; Benson, 2012). The method uses only deflection degrees freedom (without rotational degrees of freedom). It is then applied for nonlinear and dynamic analysis of thin beams.

The paper is outlined as follows: in the next section the beam formulation based on B-spline basis function is presented. Section 3 devotes two benchmark numerical examples: geometric nonlinear analysis considering von Kármán strain for cantilever beam and dynamic analysis of simply supported beam undergoing harmonic excited force. Section 4 closes some remarking conclusions.

A novel beam formulation based B-spline basis function

Brief on the Euler – Bernoulli beam theory

In the Euler- Bernoulli theory of an isotropic beam, the von Kármán strains consist of an axial strain defined as (Reddy, 2004):

$$\varepsilon_{xx} = \frac{1}{2} w_{,x}^2 - z w_{,xx} = \varepsilon_{nl} + z\varepsilon_l \tag{1}$$

where w is the transverse deflection of the mid-plan of the beam; and all other strains are zeros.

Due to the assumption small strains, the Cauchy stress tensor will be used here. So the virtual strain energy can be expressed as:

$$\delta U_\varepsilon = \int_V \delta\varepsilon_{xx}\sigma_{xx} \mathrm{d}V = \int_L \left[(\frac{A}{2} w_{,x}^2)\delta w_{,x} w_{,x} + D\delta w_{,xx} w_{,xx} \right] \mathrm{d}l \tag{2}$$

where material components are defined as

$$\{A,\ D\} = \int_{-h/2}^{h/2} E\{1,\ z^2\}\mathrm{d}z \tag{3}$$

The virtual work done by the externally applied load

$$\delta W = -\int_L q\delta w \mathrm{d}l \tag{4}$$

where q is the distributed transverse load (measured per unit length).

Kinetic energy is given as:

$$\delta K = \int_V \rho\delta\dot{w}\dot{w}\mathrm{d}V = \int_L \rho S\delta\dot{w}\dot{w}\mathrm{d}l \tag{5}$$

where ρ, S are the mass density and area of cross section, respectively.

The Hamilton's principle for an elastic body (Reddy, 2004) forms:

$$\int_{t_1}^{t_2}\left(\delta K-\delta U_{\varepsilon}-\delta W\right)\mathrm{d}t=0 \tag{6}$$

Integrating by parts of Eq. (6) yields

$$\int_{t_1}^{t_2}\left(\int_V \rho\ddot{w}\delta w\mathrm{d}V+\delta U_{\varepsilon}+\delta W\right)\mathrm{d}t=0 \tag{7}$$

B-spline basis function

To build a B-spline in one dimension, we need to define two things. The first one is two positive integers: a polynomial degree p and number of control point n. And the second one is a knot vector $\Xi=\{\xi_1,\xi_2,...,\xi_{n+p+1}\}$ which is a non-decreasing sequence of parameter values ξ_i with $i=1,...n+p$. Where $\xi_i\in\mathrm{R}$ called i^{th} knot lies in the parametric space. If the first and the last knots are repeated $p+1$ times, the knot vector is called open. A B-spline basis function is C^{∞} continuous inside a knot span and C^{p-1} continuous at a single knot.

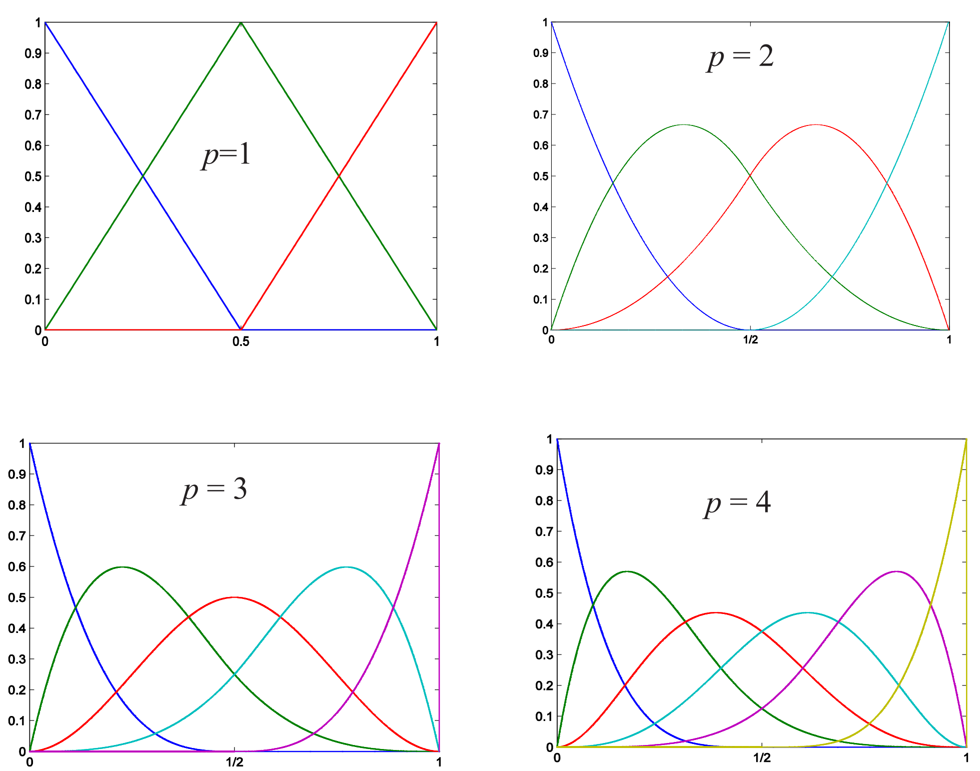
The B-spline basis functions $N_{i,p}(\xi)$ are defined recursively on the corresponding knot vector start with order $p=0$ as follows:

$$N_{i,0}(\xi)=\begin{cases}1 & if \quad \xi_i\le\xi<\xi_{i+1}\\ 0 & \text{otherwise}\end{cases}$$

and for $p\ge1$: $$N_{i,p}(\xi)=\frac{\xi-\xi_i}{\xi_{i+p}-\xi_i}N_{i,p-1}(\xi)+\frac{\xi_{i+p+1}-\xi}{\xi_{i+p+1}-\xi_{i+1}}N_{i+1,p-1}(\xi) \tag{8}$$

Figure 1 illustrates a set of one-dimensional linear, quadratic, cubic and quartic B-spline basis functions according to open uniform knot vectors $\Xi=\{0,0,0.5,1,1\}$, $\Xi=\{0,0,0,0.5,1,1,1\}$, $\Xi=\{0,0,0,0,0.5,1,1,1,1\}$, $\Xi=\{0,0,0,0,0,0.5,1,1,1,1,1\}$, respectively. It is clear that the basis functions are C^{p-1} continuous. Thus, as $p\ge2$, the present approach always satisfies C^1-requirement in approximate formulations based on the EBT.

Figure 1. Some B-spline basis functions: linear, quadratic, cubic and quartic functions.



B-spline based approximate formulation

In the isogeometric analysis, the B-spline basis functions are used to describe the geometry and the deflection field:

$$w^h(\xi) = \sum_{A}^{n} N_A(\xi) w_A \tag{9}$$

where w_A is the deflection degree of freedom associated to control point A.

Substituting Eqs. (2), (4), (9) into Eq.(7) and noting that δw is arbitrary for any $t \in [t_1, t_2]$ thus, the Hamilton's principle can be rewritten as:

$$\mathbf{M}\ddot{\mathbf{w}} + (\mathbf{K}^l + \mathbf{K}^{nl})\mathbf{w} = \mathbf{F} \tag{10}$$

where $\mathbf{w}$ is the vector of nodal degrees of freedom and stiffness matrix, mass matrix and force vector are defined, respectively:

$$\begin{aligned} &K_{ij}^{l} = D\int_L N_{i,xx} N_{j,xx} \mathrm{d}l, \quad K_{ij}^{nl} = (\frac{A}{2} w_{,x}^2)\int_L N_i N_j \mathrm{d}l, \\ &M_{ij} = \rho S\int_L N_i N_j \mathrm{d}l, \quad F_i = q\int_L N_i \mathrm{d}l \end{aligned} \tag{11}$$

Numerical results

Elastostatic analysis

Let consider firstly a cantilever beam shown in Figure 2a, subjected to uniformly distributed load. In this example, the parameters are taken as EI=1, the length of beam L=1 and q=5. Because the beam involves no motion and the applied forces are independent of time, the inertia forces are negligible. Hence, Eq. (10) reduces to

$$(\mathbf{K}^l + \mathbf{K}^{nl})\mathbf{w} = \mathbf{F} \tag{12}$$

It can be seen that, the stiffness matrix in Eq. (12) associated with only deflection variables. As a result, the size of stiffness matrix is reduced significantly. This is more benefit for solving the nonlinear problem through the Newton-Raphson iteration procedure (Reddy, 2004).

Figure 2. Cantilever beam: (a) model; (b) meshing.

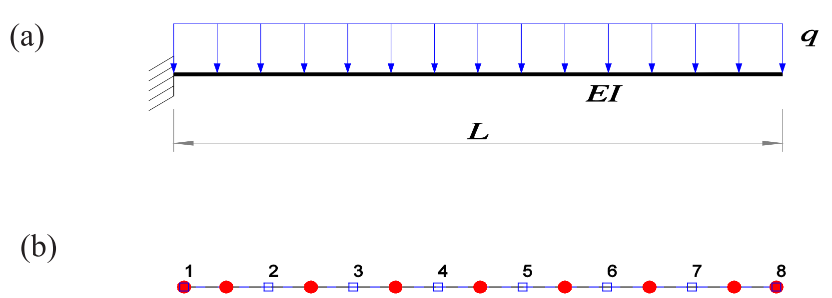


Figure 2b illustrates quadratic elements with each DOF per each control point (in circle). The displacement of the cantilever beam is plotted in Figure 3. As seen, the obtained result is in very excellent agreement with analytical one. As considering the nonlinear strain component in Eq. (1), the stiffness matrix is stiffer with the additional K^{nl} term. It makes therefore the deflection is reduced.

Figure 3. The deflection of cantilever beam

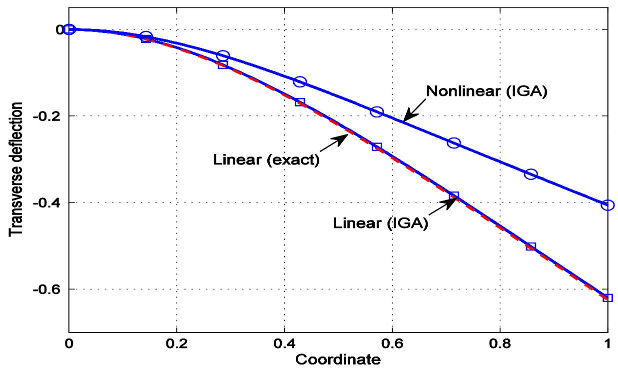
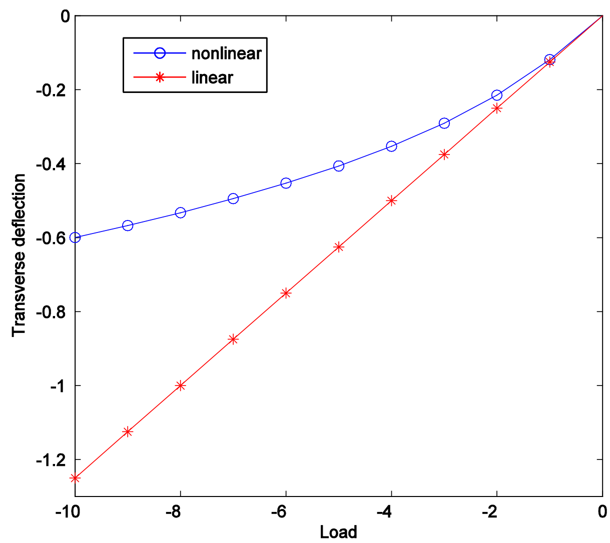


Figure 4 shows the variation of the deflection w at the free end of beam according to level of load. It can be recognized that the geometrically nonlinear results produce a stiffer reaction of the beam compared with the linear results. The stiffer behavior is explained by the additionally induced membrane stresses near the clamping point, which cannot be predicted by any linear theory.

Figure 4. The relationship between displacement at ***x=L*** **and applied load**

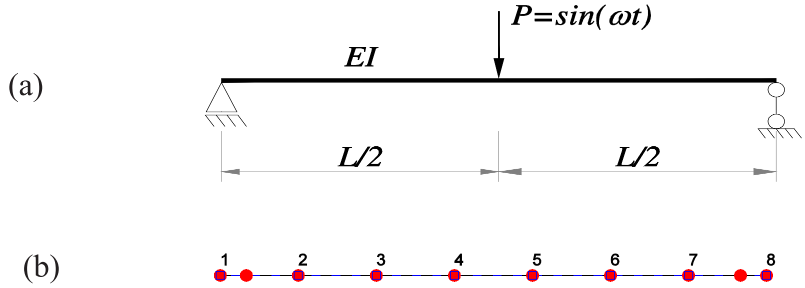


Force vibration analysis

In this section, we study a pined-pined thin beam (see Figure 5a) with the parameters given in the previous example. By removing the nonlinear part in Eq. (10), the dynamic equation of Euler-Bernoulli beam can be written as:

$$\mathbf{M}\ddot{\mathbf{w}} + \mathbf{K}^{l}\mathbf{w} = \mathbf{F} \quad (13)$$

Figure 5. Pinned-pinned beam: (a) model; (b) meshing



If the external force is zero, Eq. (13) becomes eigenvalue problem which finds the natural frequency $\omega \in \mathbf{R}$ satisfying the general free-vibration equation form

$$\left|\mathbf{K}^{l} - \omega^{2}\mathbf{M}\right| = 0 \qquad (14)$$

The first six frequencies of pined-pined thin beam are revealed in Figure 6. The general observations are: (1) with the same of degree of freedom (DOFs), the Hermitian element using cubic shape functions (Young, 2000) gains slightly more accurate solution than that of the quadratic element of IGA and worse than that of the cubic element with 7 subdivisions depicted as Figure 5b. (2) the present method can provide accurately high frequencies while using only coarse mesh with few DOFs. Figure 7 plots the first six mode shapes. It can be seen that the shapes of mode described through the control points in circle and are very smoothing.

Figure 6. The first six frequencies ω of the pined-pined thin beam

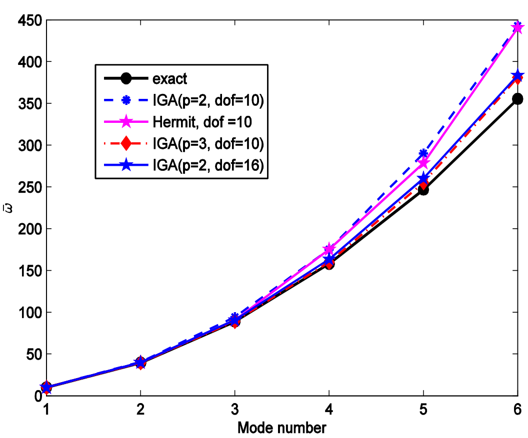
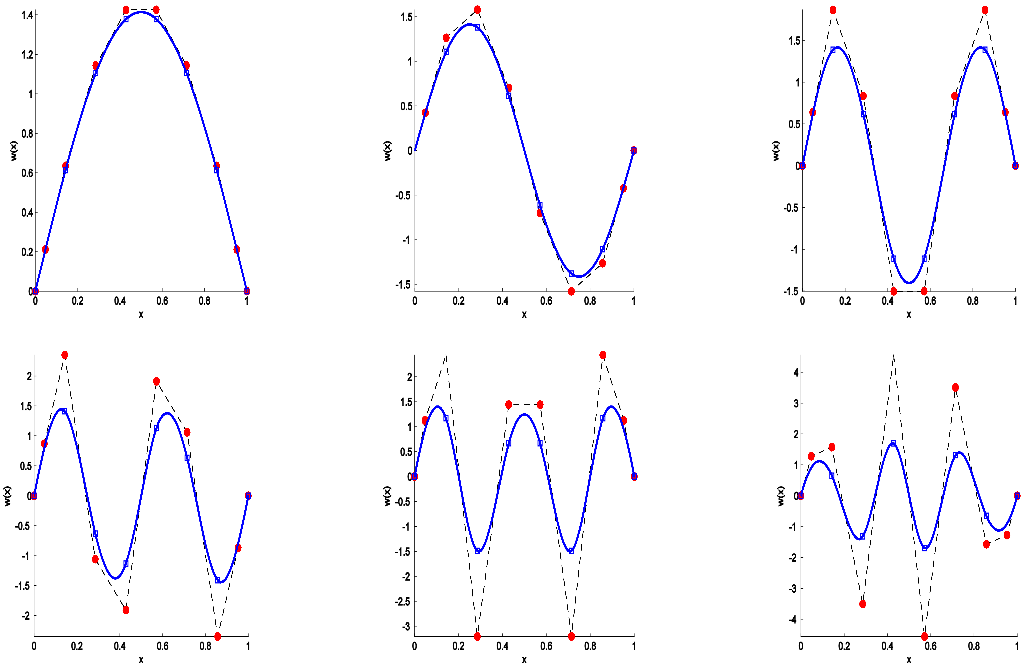
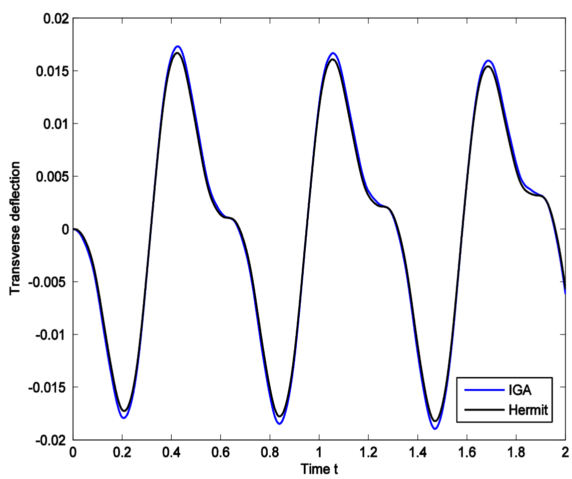


Figure 7. The first six mode shapes of the pined-pined beam



Now, let us consider a harmonic force excitation $P = \sin(20t)$ applied at the middle of the beam. This problem is solved using the Newmark method (Chopra, 2007) with time step $\Delta t = 10^{-3}$ s. Figure 8 reveals the deflection at $x=L/2$ of the pined-pined thin beam. It is seen that solution of the cubic element is a good competitor to the Hermitian finite element.

Figure 8. Deflection at $x=L/2$ of the pined-pined beam subjected harmonic force excitation



Conclusion

In this work, we used B-spline basis functions to analyze the Euler–Bernoulli beam problems. The control variable of the isogeometric element is only deflection degrees of freedom. With a half reduction of total of DOFs compared with the Hermitian element, the present method obtains very excellent agreement results without increase of computational cost. In addition, the described mode shapes are very smoothing even though a few control points are used. It is thus very promising to model more complicated structures in practice.

REFERENCES

[1] Hughes, T.J.R., Cottrell, J.A., Bazilevs, Y. (2005), Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement, C*omputer Methods in Applied Mechanics and Engineering*, Vol. 194 (39–41), pp. 4135–4195.

[2] Piegl, L.A.,and Tiller, W. (1997), *The NURBS book*, Springer Verlag second edition.

[3] Lguedj, T.E., Bazilevs, Y., Calo, V., Hughes, T. (2008), B and F projection methods for nearly incompressible linear and non-linear elasticity and plasticity using higher-order nurbs elements, *Computer Methods in Applied Mechanics and Engineering*, Vol. 197, pp. 2732-2762.

[4] Beirão da Veiga, L., Buffa, A., Lovadina, C., Martinelli, M., Sangalli, G. (2012), An isogeometric method for the Reissner–Mindlin plate bending problem, *Computer Methods in Applied Mechanics and Engineering*, Vol. 209-212, pp. 45-53.

[5] Benson, D.J., Bazilevs, Y., Hsu, M.C., Hughes, T.J.R. (2010), Isogeometric shell analysis: The Reissner–Mindlin shell, *Computer Methods in Applied Mechanics and Engineering*, Vol. 199(5-8), pp. 276–289.

[6] Kiendl, J., Bazilevs, Y., Hsu, M.C., Wchner, R., Bletzinger, K.U. (2010), The bending strip method for isogeometric analysis of Kirchhoff-Love shell structures comprised of multiple patches, *Computer Methods in Applied Mechanics and Engineering*, Vol. 199(37-40), pp. 2403–2416.

[7] Wall, W.A., Frenzel, M.A., Cyron C. (2008), Isogeometric structural shape optimization, *Computer Methods in Applied Mechanics and Engineering*, Vol. 197(33-40), pp. 2976–2988.

[8] Thai, Chien H., Nguyen-Xuan, H., Nguyen-Thanh, N., Le, T-H., Nguyen-Thoi, T., Rabczuk, T. (2012), *Static, free vibration, and buckling analysis of laminated composite Reissner-Mindlin plates using NURBS-based isogeometric approach*, *International Journal for Numerical Methods in Engineering*, Vol. 91, pp. 571-603.

[9] Nguyen-Thanh, N., Nguyen-Xuan, H. ,Bordas, S., Rabczuk, T. (2011), Isogeometric analysis using polynimial splines over hierarchical for two-dimensional elastic solids, *Computer Methods in Applied Mechanics and Engineering*, Vol. 200, pp. 1892–1908.

[10] Cottrell, J.A., Reali, A., Bazilevs, Y., Hughes, T.J.R. (2006), Isogeometric analysis of structural vibrations, *Computer Methods in Applied Mechanics and Engineering*, Vol. 195 (41–43), pp. 5257–5296.

[11] Benson, D.J., Bazilevs, Y., Hsu, M.C., Hughes, T.J.R. (2012), A large deformation, rotation-free, isogeometric shell. *Computer Methods in Applied Mechanics and Engineering*, Vol. 200, pp. 1367–1378.

[12] Reddy, J.N. (2004), *An Introduction to Nonlinear Finite Element Analysis*, Oxford university press.

[13] Young, W.K., and Hyochong B. (2000), *The finite element method using Matlab*, CRC Press; second edition.

[14] Chopra, A.K. (2007), *Dynamic of structures theory and applications to earthquake engineering*, Pearson Prentice Hall, third edition.