Model Order Reduction of Finite Element Model

Deepika Srivastava
Ph.D. scholar, Department of Electrical Engineering
Motilal Nehru National Institute of Technology
Allahabad, India
deesrivastava@yahoo.co.in

Dr.Dinesh Chandra
Professor, Department of Electrical Engineering
Motilal Nehru National Institute of Technology
Allahabad, India
dinesh@mnnit.ac.in

Abstract—**Modal analysis is generally performed for computation of complex Finite Element models. In other disciplines, different techniques for Model Order Reduction have been developed in the previous two decades. This paper present the recent algorithmic advantages that lead to model order reduction methods that are applicable to finite element models. Efficiency of the proposed balance truncation approximation based model order reduction method with the help of a control system including as plant the FE model is demonstrated**

Index Terms—**Balanced Truncation Algorithm, Finite Element Model, Model Order Reduction.**

I. INTRODUCTION

Power of Solid Mechanics or "Continuum Mechanics" is very limited, only capable of solving some simple continuum mechanics problems, e.g. beam, shaft, plate/shell, with simple geometry and boundary. For complex structures these methods become extremely difficult and challenge. However, it becomes easier if the large continuum solid can be cut into many small and regular continuum elements. This is very similar to complex buildings constructed by many small regularly-shaped bricks.

There are many techniques are available for the mathematical modeling of dynamical systems, e. g. for simulating, optimizing or controlling purposes, we can employ linear systems of differential equations. For increasing accuracy of such models, with Finite Element method and the often occurring effect of slow convergence lead to a rapidly growing number of describing equations that eventually can no more be solved and used efficiently. Model Order Reduction (MOR) techniques are designed to answer this problem by replacing the large scale original model by a considerably smaller one, maintain characteristic properties of the former and approximate its transfer behavior as much accurate as possible [2].

II. FROM FEM TO STATE SPACE FORM

The computation of the compact state space form includes two major steps. First we apply a certain transformation to rewrite the second order differential algebraic equations (DAE) in first order form. After having derived the first order equations, model order reduction based on balanced truncation is applied to the transformed system to derive a much smaller representation of the system in state space form. This approach has several advantages. First of all, the resulting model is of first order regarding the differentiation order.

Second, it is in state space form and we circumvent the difficulties caused by the differential algebraic formulation of the original system representation in further application steps. Most importantly in contrast to the modal reduction approaches mentioned above, the balancing based model order reduction takes the input to output relation of the system into account. Exploiting this additional information, it is usually possible to and smaller reduced order models or get similarly sized reduced order models with a much higher accuracy.
For the moment let us consider a general second order system with state and velocity measurements.

$$M\ddot{z}(t) + E\dot{z}(t) + Kz(t) = Bu(t)$$

$$y(t) = C_1\, \dot{z}(t) + C_1\, z(t) + Du(t) \qquad (1)$$

Here M , E and R€ R nXn are the sparse FEM-matrices resulting from the modeling, B € R n x p is the input matrix describing the external access to the system and C1, C2€ R m X n represent the measurements. Correspondingly, u € Rp and y € Rm are the control input to the system and the measured output, D € R m represents the direct feed through from input to the output.`

III. FROM SECOND ORDER FEM TO FIRST ORDER COMPACT STATE SPACE FORM

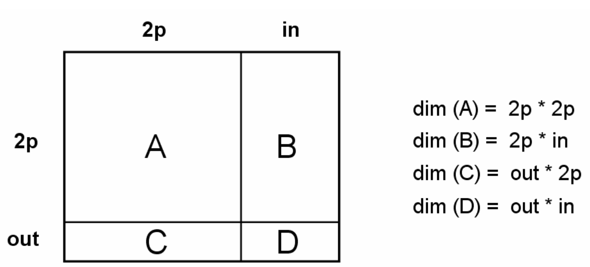

Fig 1 General matrix model dimensions

The state space model can be represented by equation (1), the state space model will have the following equations

978-1-4673-9745-2 ©2016 IEEE

$$\dot{x} = Ax + Bu \quad (2)$$
$$y = Cx + Du \quad (3)$$

Where the state vector **x**, the input vector **u** and the output vector **y** can be defined as:

$$x = \begin{Bmatrix} q \\ \ddot{q} \end{Bmatrix} \quad (4)$$

$$x = \{F\} \quad (5)$$

$$y = \begin{Bmatrix} q_{out} \\ \dot{q}_{out} \\ \ddot{q}_{out} \\ F \end{Bmatrix} \quad (6)$$

It is assumed that the number of states is **2p**, the number of inputs is **in** and the one of output is **out**, the matrices A, B, C and D. In following figure 2 represents the characteristics.

The dimensions of the state-space matrices (A, B, C, & D). Using FEM a system model generated is given as

$$M\ddot{q} + C\dot{q} + Kq = F \quad (7)$$

Where q represents the co-ordinates and

M = Mass matrix
K = Stiffness matrix
C = Damping matrix
F = Applied nodal forces (external forces)

Using the equations of motion of each element of the FEM analysis mass, stiffness and damping matrices are obtained. On combining the above matrices above equation can be written in state space form as:

According to equation (1), the acceleration vector can be written as follows:

$$\ddot{q} = -M^{-1} B\dot{q} - M^{-1}Kq - M^{-1} F \quad (8)$$

So state space matrices are defined below(it is assumed that only external forces can be applied to the model)

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}B \end{bmatrix} \quad (9)$$

$$B = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \quad (10)$$

$$C = \begin{bmatrix} I & 0 \\ 0 & I \\ -M^{-1}K & -M^{-1}B \\ 0 & 0 \end{bmatrix} \quad (11)$$

$$D = \begin{bmatrix} 0 \\ 0 \\ M^{-1} \\ I \end{bmatrix} \quad (12)$$

So convert equation of motion into a state-space model, and Matlab program would be able to read the matrices of the FE model.

This kind of model integrated in Matlab will give the dynamic response of the modeled structure under one or multiple inputs. It is to be noted that this computation of dynamic response of structures can also be achieved directly by using finite element code. It is also possible to link the calculation of the response with the active control algorithm. Computing the response of the system with Matlab (with a state space model) is very much useful and also, once the FE model is done, it cannot be modified, and it is often required to adjust some parameters of algorithm and to find their impact on the response. It will become much faster and simpler in Matlab. Moreover it would also be helpful to add a model of the actuator(s), to get better results.

IV. Mechanical Model Example:

Example 1 Mass, stiffness, damping matrix of second order system are respectively given as**:**

$$M = \begin{bmatrix} 0.2913 & 0.0153 \\ 0.0153 & 0.3387 \end{bmatrix} \quad (13)$$

$$K = \begin{bmatrix} 630.44 & -425.83 \\ -425.83 & 439.01 \end{bmatrix} \quad (14)$$

$$B = \begin{bmatrix} 6.9208 & -3.6833 \\ -3.6833 & 5.6759 \end{bmatrix} \quad (15)$$

Corresponding State Space Model of the above given system is given by;

$$A=\begin{bmatrix}-0.6322 & 0.7215 & -0.6942 & 0.2884\\ 0.4600 & -5.2410 & 7.3420 & -3.8520\\ 0.1755 & 4.1490 & -7.1350 & 5.4340\\ -0.2570 & -3.3970 & 2.8820 & -6.995\end{bmatrix} \quad (16)$$

$$B=\begin{bmatrix}-8.6340 & -8.3270\\ -6.6150 & 0.9404\\ 5.4950 & -3.5500\\ -2.6080 & -0.7901\end{bmatrix} \quad (17)$$

$$C = [-12.000 \quad 6.6810 \quad -6.5420 \quad 2.7250] \quad (18)$$

$$D = 0 \quad (19)$$

V. MODEL REDUCTION PROBLEM

Original complex model:

$$\frac{dx(t)}{dt} = f(x(t)) + Bu(t)$$

$$y(t) = Cx(t) \quad (20)$$

The Reduced model:

$$\frac{(dx^r(t))}{dt} = f^r(x^r(t) + B^r u(t)$$

$$y^r(t) = C^r x^r(t) \quad (21)$$

VI. BALANCED TRUNCATION

The basic idea of balance truncation analysis, a system is considered in first order state space representation

$$\Sigma: \begin{array}{l} E\dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{array} \quad (22)$$

Where E=In. It is assumed that all eigen values of A are lying in the left half of the complex plane.

We can summarized the basic idea of balance truncation analysis as follows. Taking input as the impulse function δ , and the calculation of response can be done as h(t)=CeAt for all t>0 The corresponding system Gramians are given as

$$P = \sum_t x(t)x(t)^* = \int_0^\infty e^{At}BB^*e^{A^*t}dt,$$

$$Q = \sum_t \eta(t)^* \eta(t) = \int_0^\infty e^{A^*t}C^*Ce^{At}dt. \quad (23)$$

State space transformations

$$\hat{x}(t) = Tx(t)$$

will give the congruent Gramians

$$\hat{P} = TPT^T, \quad \hat{Q} = T^{-T}QT^{-1},$$

So, the eigen values of are the same as the eigen values of PQ.

$$\hat{P}\hat{Q} = TPT^TT^{-T}QT^{-1}$$

Gramians P and Q are converted as the two Lyapunov equations

$$AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC = 0. \quad (24)$$

There is requirement of U and V to find the order for computing the reduced system factors, i.e. $P=UU^T$ and $Q=LL^T$. The singular values of the product U^TL can give the Hankel singular values. Singular value decomposition of U^TL with $U^TL=ZSY^T$

where S is a positive diagonal matrix and Y, Z are orthogonal matrices, the state space transformation $\hat{x} = T_b x$ with

$$T_b = S^{\frac{1}{2}}Z^TU^{-1}. \quad (25)$$

With this transformation it is found that the two Gramians are equal and diagonal, i.e.,$P_b=Q_b=S$ and with

$$A_b = T_bAT_b^{-1}, \quad B_b = T_bB, \quad C_b = CT_b^{-1}$$

we have

$$A_bS + SA_b^T + B_bB_b^T = 0, \quad A_b^TS + SA_b + C_b^TC_b = 0.$$

Hankel singular values are assumed in decreasing order, so model reduction can be done by simple truncation.

Let

$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_\ell \geq \sigma_{\ell+1} \geq \ldots \geq \sigma_n$,

$S_1 = \mathrm{diag}(\sigma_1, \ldots, \sigma_\ell)$, $\quad S = \mathrm{diag}(S_1, S_2)$ and

A_b, B_b, C_b accordingly as

$$A_b = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B_b = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C_b = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

such that $\tilde{A} = A_{11} \in \mathbb{R}^{\ell\times\ell}$, $\tilde{B} = B_1 \in \mathbb{R}^{\ell\times p}$ and $\tilde{C} = C_1 \in \mathbb{R}^{m\times\ell}$ then for the reduced order model

$$\tilde{\Sigma}: \begin{array}{l} \tilde{E}\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \\ \tilde{y}(t) = \tilde{C}\tilde{x}(t), \end{array} \quad (26)$$

we have the error estimate

$$\sigma_\ell \leq \|\Sigma - \tilde{\Sigma}\|_{H_\infty} \leq 2(\sigma_{\ell+1} + \ldots + \sigma_n). \quad (27)$$

Here $\|\Sigma\|_{H_\infty}$ is the maximum frequency response of the system, using singular value decomposition we can perform the truncation

$$U^T L = \begin{bmatrix} Z_1 & Z_2 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} Y_1^T \\ Y_2^T \end{bmatrix}, \tag{28}$$

Using the transformation matrices $W := L^T Y_1^T S_{11}^{\frac{1}{2}}$, and $V := U^T Z_1 S_{11}^{\frac{1}{2}}$ we can find the reduced order system $\tilde{\Sigma}$ as

$$\tilde{A} = W^T A V, \quad \tilde{B} = W^T B, \quad \tilde{C} = CV.$$

In case

$$E \neq I_n$$

then the two Gramians give the solutions of the Lyapunov equations

$$APE^T + EPA^T + BB^T = 0, \quad A^T QE + E^T QA + C^T C = 0. \tag{29}$$

VII. Numerical Test and Result

(Redused Order Model Using TBR Technique)

The above taken model is reduced using balanced truncation technique and corresponding model is given as;

$$Ar = \begin{bmatrix} -0.6322 & 0.7215 & -0.6942 \\ -0.46 & -5.241 & 7.342 \\ 0.1755 & 4.149 & -7.135 \end{bmatrix} \tag{30}$$

$$Br = \begin{bmatrix} -8.634 & -8.327 \\ -6.615 & 0.9404 \\ 5.495 & -3.55 \end{bmatrix} \tag{31}$$

$$Cr = \begin{bmatrix} -12 & 6.681 & -6.542 \end{bmatrix} \tag{33}$$

$$Dr = 0 \tag{34}$$

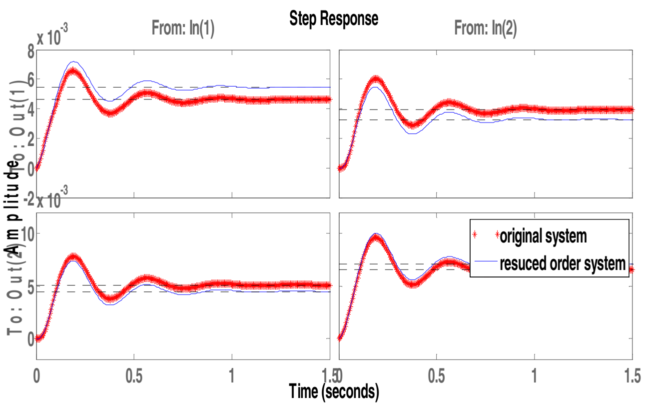

Fig 2.Stablity analysis of reduced order system

Example 2 Consider a 6th order stable discrete-time system.

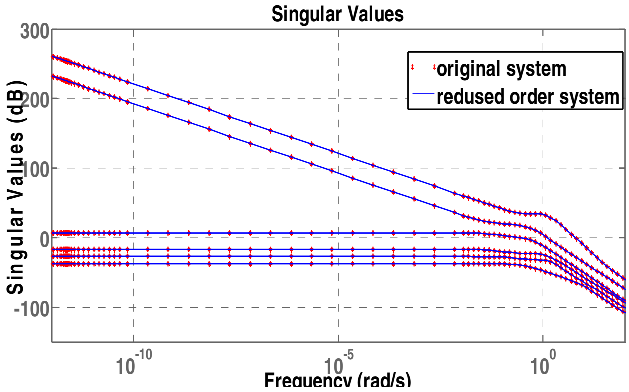

Fig 3. Singular values of original system and reduced order system

Stiffness matrix is given as;

$$K = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 9 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 \\ 9 & 8 & 7 & 5 & 4 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 5 & 6 & 7 \end{bmatrix} \tag{36}$$

Mass matrix is given as;

$$M = \begin{bmatrix} 1 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 1 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 \\ 9 & 8 & 7 & 1 & 4 & 3 \\ 1 & 2 & 3 & 1 & 5 & 6 \\ 1 & 2 & 3 & 5 & 1 & 7 \end{bmatrix} \tag{37}$$

Damping matrix is given by;

$$C = \begin{bmatrix} 1 & 3 & 4 & 5 & 3 & 7 \\ 1 & 2 & 3 & 4 & 3 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 \\ 9 & 8 & 7 & 1 & 4 & 3 \\ 1 & 2 & 8 & 1 & 5 & 6 \\ 1 & 2 & 3 & 5 & 1 & 7 \end{bmatrix} \tag{38}$$

This system is converted into state space and reduced to third order system using balance truncation reduction technique and checking the result is shown in following figure (3)

VIII. CONCLUSION

In this paper, a mechanical systems model equation of motion is converted to the equivalent high-order discrete-time system, which is farther transformed into continuous-time system model and its stable reduced order model is obtained using balanced truncation method.

REFERENCES

[1] Glaucio H. Paulino, “Introduction to FEM (History, Advantages and Disadvantages,” http://cee.ce.uiuc.edu/paulino

[2] Robert Cook et al., “Concepts and Applications of Finite Element Analysis,” John Wiley & Sons, 1989

[3] Robert Cook, “Finite Element Modeling For Stress Analysis,” John Wiley & Sons, 1995

[4] J. Tinsley Odenet al., “Finite Elements –An Introduction,” Prentice Hall, 1981

[5] A. Ebadi, M. Mirzaie and S. A. Gholamian, “A Comparison Between Electrical Circuit and Finite Element Modeling Methods for Performance Analysis of a Three-Phase Induction Motor under Voltage Unbalance ,” Iranian Journal of Electrical & Electronic Engineering, Vol. 8, No. 2, June 2012

[6] Ankireddypalli S. Reddy, Dr M. Vijaykumar, “Hottest Spot And Life Evaluation Of Power Transformer Design Using Finite Element Method”, Journal Of Theoretical And Applied Information Technology, ©2005 - 2008 Jatit.

[7] Andrew Lapthorn and Pat Bodger, ” A Comparison Between The Circuit Theory Model and Finite Element Model Reactive Components”, University of Canterbury. Electrical and Computer Engineering, Issue Date: 2009

[8] O. A. Mohammed1, Z. Liu1, S. Liu1, N. Y. Abed1, and L. J. Petersen2 “Finite Element Based Transformer Operational Model for Dynamic Simulations” Progress In Electromagnetics Research Symposium 2005, Hangzhou, China, August 22-26.

[9] F. Nilvetti, C. M. Pappalardo, “Mass, Stiffness and Damping Identification of a Two-Story Building Model” International Journal of Mechanical Engineering and Industrial Design - ISSN: 2280-6407 2012, 1(2): 19-35.

[10] William Gressick, John T. Wen, Jacob Fish, “Order Reduction for Large Scale Finite Element Models: a Systems Perspective”.

[11] Jagadeesh Chandra Prasad, 2dr. B.V. SANKER RAM, “Evaluation Of Synchronous Generator Reactance Using Finite Element Method (FEM)”, Journal Of Theoretical And Applied Information Technology Issn 1992 8645.

[12] Hilmi Lus,, Maurizio De Angelis, Raimondo Betti , Richard W. Longman, “Constructing Second–Order Models of Mechanical Systems From Identified State Space Realizations” Part II: Numerical Investigations.

[13] Y. M. Desai, P. Yu, N. Popplewell and A. H. Shah, “Perturbation-Based Finite Element Analyses of Transmission Line Galloping”,Journal of Sound and Vibration (1996) 191(4), 469–489

[14] Y. M. Desai, P. Yu, N. Popplewell and A. H. Shah ,“Finite Element Modelling Of Transmission Line Galloping”, Computrrs & Strucrum Vol. 57, No. 3, pp. 407420, 1995.

[15] Xiang Wang, Qing Wang, Zheng Zhang, Quan Chen and Ngai Wong “Balanced Truncation For Time-Delay Systems Via Approximate Gramians” 978-1-4244-7516-2/11/$26.00 ©2011 IEEE.