Large deflection of a simply supported beam

Article · August 2010

3 authors:

Ali Abolfathi
University College London
23 PUBLICATIONS   40 CITATIONS

SEE PROFILE

Michael J. Brennan
São Paulo State University
448 PUBLICATIONS   8,626 CITATIONS

SEE PROFILE

Tim P. Waters
University of Southampton
108 PUBLICATIONS   1,970 CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:

Fault Diagnosis of Rolling Element Bearings and Gears View project

AECOM Crossrail View project

University of Southampton

Institute of Sound and Vibration Research

Dynamics Group

Large deflection of simply supported beam

By

A. Abolfathi, M.J. Brennan, and T. Waters

Abstract

The large deflection of a simply-supported beam loaded in the middle is a classic problem in mechanics which has been studied by many people who have implemented different methods to determine the solution, such as analytical exact solutions and the finite element method. The problem is investigated again here but the Galerkin method is used to obtain an approximate force-deflection characteristic of the beam. It is shown that the beam can be modelled with a Duffing-type stiffness with hardening nonlinearity. The exact solution and that from the finite element method are used to validate the results. The accuracy of the results and the suitability of the Sine function to model the deflected shape of the beam in the Galerkin method are investigated.

The large deflection of a simply-supported beam due to a pure bending moment is also investigated. The exact solution is obtained and the results are used to describe the behaviour of the beam.

1

Contents

1   Introduction

The common approach to study deflection and vibration of beams is to implement linear theory and to consider small rotations. This approach is valid for many practical cases, since the materials become plastic by increasing the deformation before the small rotation assumption becomes invalid. However, there are different cases in which beams can violate the assumption of small deflections. For example, thin bars under excessive load or vibration which can result in large deflections without going outside the elastic range of their material. Different methods, both numerical and analytical, are suggested to obtain the large deformations of beams. In vibration, the nonlinearity of large deflection appears in the stiffness term. A nonlinear stiffness dependent on power series of the deflection is the most suitable form to study the nonlinear vibration of beams. Hence, the Galerkin method has been used in the study reported here.

The problem of large deformations of beams has attracted a lot of attention and some different methods have been suggested to solve the problem. Frisch-Fay [1] has cited most of the work in his book "Flexible bars" up to its publication date. Bishop and Drucker [2] presented an analytical solution for the large deflection of a cantilever beam loaded at its tip in terms of elliptical integrals. Conway [3] found a solution for the large deflection of a simply supported beam. He considered the distance between the supports to be fixed without considering axial stresses. The problem was solved for two cases of the vertical reaction at the supports and for a perpendicular reaction force with friction. Gospodnetic [4] considered a thin elastic beam deflected by three symmetrically arranged knife-edged supports. In the absence of friction, he considered that the supports exert forces normal to the deflected beam and are situated at fixed distances from each other while the beam could slide. He also found the solution in terms of elliptical integrals. He plotted the ratio of the maximum deflection obtained from nonlinear theory and linear theory as a function of applied force and claimed it to be different from the analogous diagram given in reference [3].

Sundara Raja Iyengar and Lakshmana Rao [5] studied the same problem as the one in reference [3] and considered a uniform distributed load in addition to the concentrated load. They used a power series expansion for the angle along the beam length to find the solution for the problem.

3

Wang et. al. [6] used a numerical method to find the large deflection of a cantilever loaded along its length and a simply-supported beam loaded partially along its length or by a concentrated load. He used the finite difference method to find a solution for the bending differential equation along the length of the beam and compared the results with experimental results. Wang [7] introduced a different numerical method based on integrating along the horizontal axis for a cantilever beam loaded at its end and for a simply supported beam subject to a non-symmetrical load. In reference [8] equations for a cantilever and a simply supported beam with a distributed load were integrated to obtain the relationship between the angle along the beam, the projection in the axial direction and the load. Numerical methods were then implemented to solve the problem and deflections were calculated using numerical integration.

Beléndez [9] restated the analytical solution using an elliptical integral for a cantilever beam loaded at its tip and compared the results of analytical and numerical solutions with experimental results.

Thomsen [10] studied vibration of a simply-supported beam under a harmonic concentrated force on the middle. He considered simply-supported beams with moveable and immovable supports. The Galerkin method was used. The effect of shortening of projection length in axial direction was not considered which resulted in the erroneous conclusion that a simply supported beam with movable supports acts as a softening system.

In this study, a modification and improvement to the solution based on the Galerkin method is introduced, and it is shown that a simply-supported beam under a concentrated load at the centre can be modelled by a Duffing-type stiffness with hardening nonlinear behaviour. The report starts with a brief description of the theory of the basic principles of bending. Then the special case of a simply-supported beam loaded by a pure bending moment is studied. This case is used to provide a physical explanation for the displacements of beams in general. The solution of the simply-supported beam is the main part of this report. The exact solution is presented at first and is followed with an approximate solution using the Galerkin method. A physical explanation concerning the behaviour of beam is provided and the report then concludes with a discussion about the accuracy of method.

2 Beam deformation due to a bending moment

According to Gere and Timoshenko [11], "a beam is a structural member that is subjected to loads acting transversely to the longitudinal axis". A force applied to a beam causes a bending moment to develop inside the beam as well as shear and, or axial forces depending on the direction of the applied force. The main cause of the deformation in a beam is the bending moment and the effect of shear deformation can often be neglected; this is the case in this study. Under the action of a moment, the cross-section of a beam remains plane and is normal to the longitudinal axis of the beam. This is true for beams with any homogeneous material properties (elastic or inelastic as well as linear or nonlinear), and is also true for the large deflection of a beam under pure bending [12].

To determine the relationships between strain, deformation and bending moment, the procedure in reference [11] is followed. Consider the section of a beam shown in Fig 1a in which the bending moment is equal to M. The sign convention is given in Fig. **1**b. The strain in the beam element is given by,

$$\varepsilon_x = -\frac{h}{\rho} = -\kappa y \tag{1}$$

where ρ is radius of curvature and h is the distance from the neutral axis at which the strain is measured. The reciprocal of the radius of curvature is called the curvature and is represented by κ. The bending moment related to the curvature of the beam can be found by

$$M = \int_A \sigma_x h \, dA = \int_A E \varepsilon_x h \, dA \tag{2}$$

Substituting for ε_x from Eq. (1) gives

$$M = -E\kappa \int_A h^2 \, dA \tag{3}$$

The integral in Eq. (3) is the second moment of area denoted by I, so the relationship between the curvature and the bending moment can be written as,

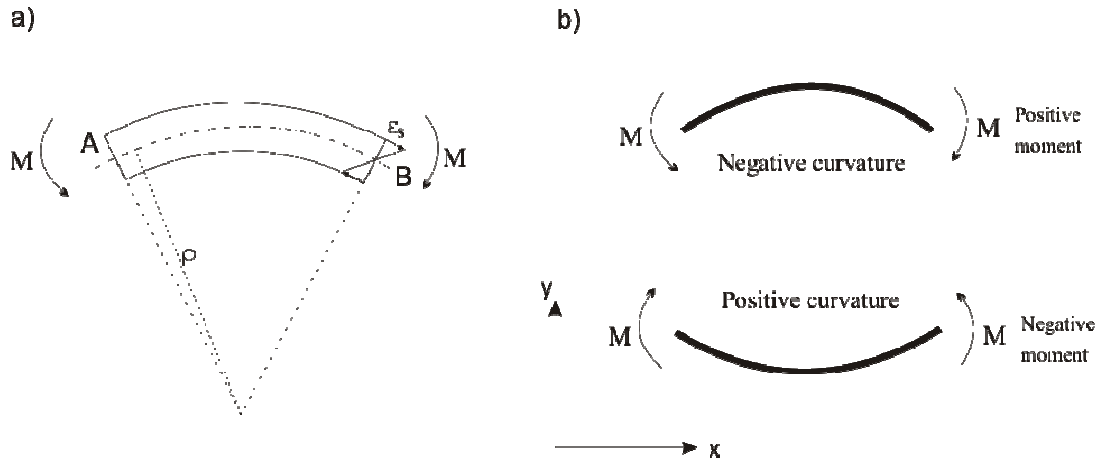$$\kappa = \frac{1}{\rho} = -\frac{M}{EI} \tag{4}$$

Fig. 1. a) Deflected beam due to pure bending moment, b) Sign convention for curvature

It should be noted that Eq. (4) is valid for large as well as small deflections. This equation is the key equation in the derivation of the expressions for beam deflection.

2.1. Deflection of a beam due to a pure bending moment

A beam loaded by a pure bending moment is a special case whose deflection can be easily found. Consider the beam shown in Fig. 2. The beam is of length L and is loaded by a bending moment M at each side. The distance s is measured along the deflected beam and x is the corresponding lateral distance; y_{max} is the transverse displacement of the beam at the centre and θ_0 denotes angle of the beam at each end.

For the element ds which is shown in Fig. 2a, Eq. (4) can be written as a function of $d\theta$ and ds as

$$\frac{d\theta}{ds} = -\frac{M}{EI} \tag{5}$$

The angle at the centre of the beam is zero because of symmetry. Integrating Eq. (5) from $x = 0$ to $x = \frac{L}{2}$ gives

$$\int_{\theta_0}^{0} d\theta = \int_{0}^{\frac{L}{2}} -\frac{M}{EI} ds \tag{6}$$

which can be solved to find the angle at each end of the beam,

6

$$\theta_0 = \frac{ML}{2EI} \tag{7}$$

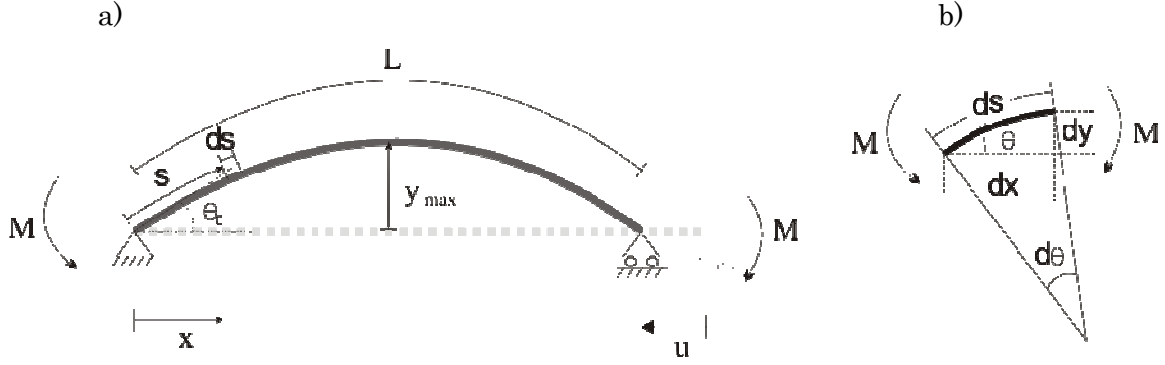a)   b)

Fig. 2. Simply supported beam loaded by bending moment, a) deflected beam, b) small section of beam of length ds

This shows that there is a linear relationship between the angle and the bending moment and rotational stiffness, $k_r = 2EI/L$. The transverse deflection of the beam can be determined from,

$$y(s) = \int dy = \int_0^s \sin\theta \, ds \tag{8}$$

The maximum transverse deflection occurs at L/2. Substituting for $s=L$/2 into Eq. (8), and noting the relationship between θ and s given in Eq. (5) results in

$$y_{max} = \int_0^{\frac{L}{2}} \sin\theta \, ds = -\frac{EI}{M}\int_{\theta_0}^0 \sin\theta \, d\theta \tag{9}$$

which evaluates to

$$y_{max} = \frac{EI}{M}(1 - \cos\theta_0) \tag{10}$$

Substituting for θ_0 from Eq. (7) and dividing through by L, gives the non-dimensional expression for y_{max} , which is,

$$\hat{y}_{max} = \frac{y_{max}}{L} = \frac{1}{2}\frac{1}{\widehat{M}}\left(1 - \cos\widehat{M}\right) \tag{11}$$

where $\widehat{M} = \frac{M}{(2EI/L)}$ is the non-dimensional bending moment. The lateral displacement of the tip of the beam can be determined in a similar way. The displacement of the tip is given by,

$$u = L - \int_0^L \cos\theta\, ds = L - \frac{EI}{M}\int_{\theta_0}^{-\theta_0} \cos\theta\ d\theta \tag{12}$$

The integral can be evaluated in a straightforward manner and the resulting equation divided through by L to give the non-dimensional displacement of the beam tip,

$$\hat{u} = \frac{u}{L} = 1 - \frac{1}{\widehat{M}}\sin\widehat{M} \tag{13}$$

The bending moment as a function of tip displacement is shown in Fig. **3** with the deformed shape of the beam illustrated for various loads. The radius of curvature is constant over the beam as a result of the constant bending moment along the beam. As a result, the beam forms an arc, then a circle as the bending moment increases. The stiffness of the beam $\widehat{M}/\hat{u}$, is softening up to when the angle is about 120° whereupon it becomes hardening.
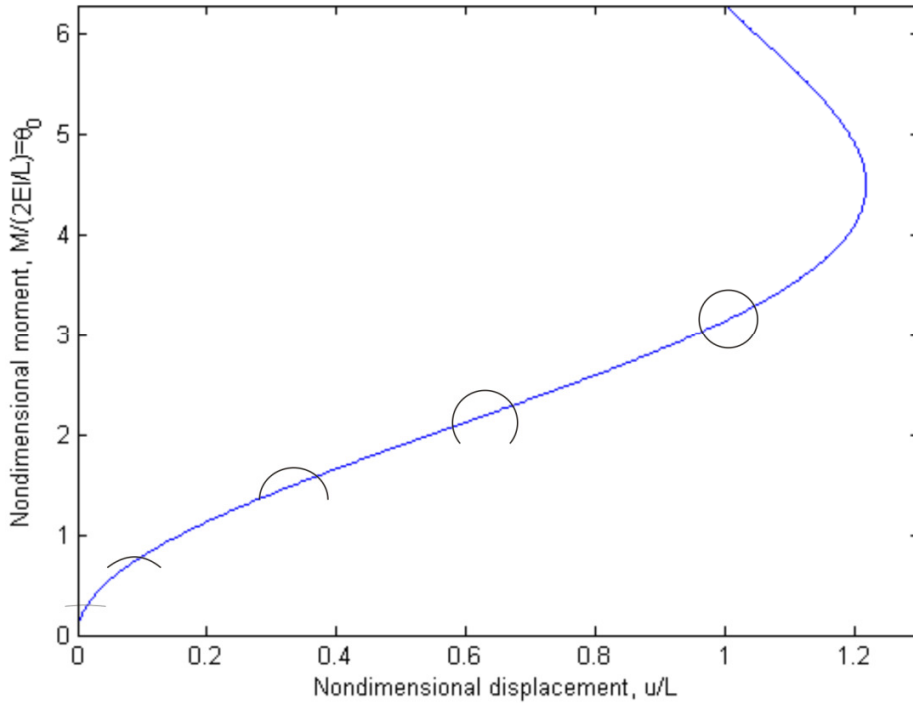

Fig. 3. Non-dimensional bending moment as a function of non-dimensional lateral displacement of the tip

The non-dimensional bending moment is plotted as a function of the transverse displacement and the lateral tip displacement in Fig. 4. The results obtained from finite element method using Ansys are also shown. It can be seen that these match well with the analytical results. The transverse stiffness $\widehat{M}/\widehat{y}$, is hardening from the beginning and does not soften for any value of bending moment, unlike the axial stiffness $\widehat{M}/\widehat{u}$, which is softening for relatively small bending moments and then has a hardening characteristic for very large bending moments. The reason of this behaviour is given in the next section by considering the displacement as a result of rotation of beam elements.
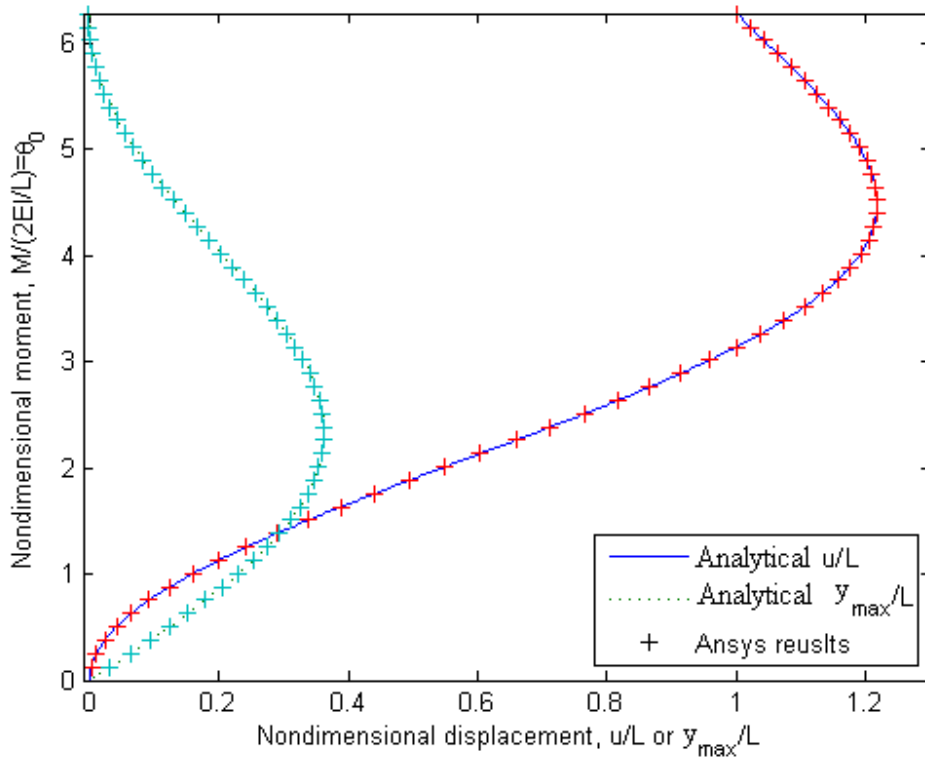

Fig. 4. Comparison of the analytical results with the results obtained from Ansys for the axial displacement of the tip of the beam and the transverse displacement at the centre of the beam.

9

2.2. Effect of rotation

Applying a bending moment to a beam causes it to deflect and take the shape of an arc. Although the rotational stiffness is constant, the cross stiffnesses in the axial and transverse directions are nonlinear, having softening and then hardening, and hardening behaviour respectively. Each segment of the beam rotates as a result of this deformation. Considering a single element, when it rotates its tip moves in both axial and transverse directions. Summing the displacements of all the elements gives the displacement of the beam in both directions. The effect of the angle of rotation on this displacement is the cause of the softening behaviour in the axial direction and the hardening behaviour in the transverse direction.

To illustrate this phenomenon, consider the line shown in Fig. **5**. The dotted line in Fig. **5**a has an angle 10° with the x axis. By rotating the line by a further 10°, the tip of the line moves to the right by a distance dx. Now, consider the same dotted line making an angle of 70° with the x axis. By rotating it by a further 10°, as before, the projection of displacement dx is larger than in the previous case. It can thus be seen that for an increasing angle the displacement in the x direction is proportionately larger and the displacement in the y direction is proportionately smaller.
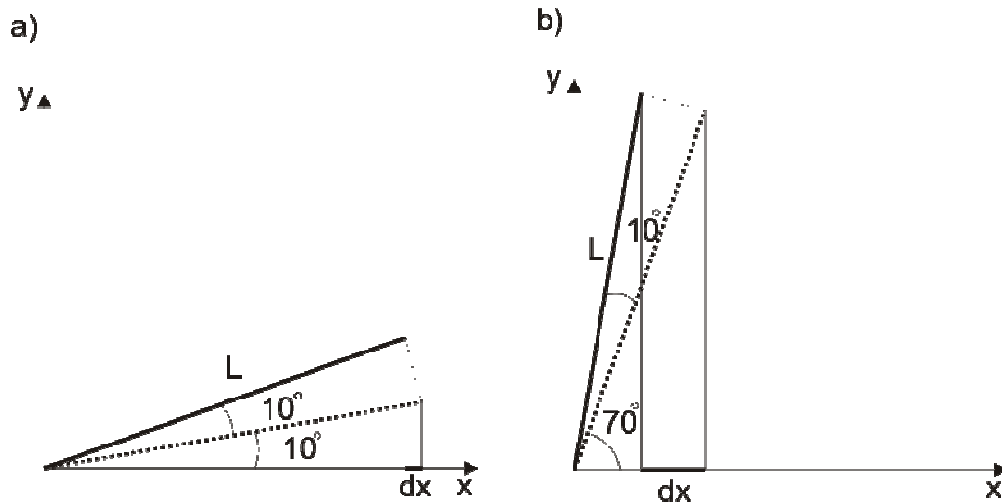

Fig. 5. Relationship between the angle and the displacement in the x direction.

The tip displacement of the line in the x direction is the difference between its projection onto the x axis for two different positions. This is related to the cosine of the angle and can be derived easily.

Consider now Eq. (12) which describes the axial displacement of the tip of the beam. If the beam is divided to n equal segments, the integral on the right hand side of Eq. (12) can be approximated by a summation over the n segments, so that

$$u = L - \sum_{m=0}^{n} \cos \theta_m \, \Delta s \tag{14}$$

Because of the constant radius of curvature, the angle, $\Delta\theta$, for each segment Δs is constant, so that

$$\Delta\theta = \frac{\Delta s}{\rho} \tag{15}$$

Considering that $\Delta s = L/n$, and $L = 2\rho\theta_0$ then

$$\Delta\theta = \frac{2\theta_0}{n} \tag{16}$$

So Eq. (14) can be rewritten as,

$$u = L - \sum_{m=0}^{n} \cos \left(\frac{2m\theta_0}{n}\right) \Delta s \tag{17}$$

Thus, it can be seen that the axial displacement of the tip of the beam is the difference between the length of the beam and the sum of all projections of beam segments onto the x axis. It can also be seen that the relationship between the axial displacement of the beam and the rotation of the beam behaves in a similar way to the line discussed at the beginning of this section. Note that the angle θ_0 is linearly related to the bending moment as shown by Eq. (7) so the axial displacement of the beam subject to a bending moment behaves as a softening spring as discussed previously. This explains the softening behaviour shown in Fig. **3** up to and angle θ_0 of about 120°. The hardening behaviour of the beam in the transverse displacement can be analysed in a similar way.

3. Simply supported beam loaded at the centre

An alternative method is needed to determine the large deformation of a simply supported beam loaded at its centre. The exact analytical solution is presented first. It is then used as a reference to validate the approximate solution derived using the Galerkin method in Section 3.2. The results are also compared with solutions determined using the finite element analysis software Ansys. The accuracy of the approximate method is also examined by comparing the shape function used with the actual deformed shape of the beam.

3.1. Exact solution

A concentrated load p is applied to the middle of the simply supported beam shown in Fig. **6**. As before, the transverse displacement of the beam is denoted by y and y_{max} is the maximum deflection. The bending moment at each cross section of the beam at a distance x from the left-hand end is given by,

$$M = \frac{px}{2} \qquad \text{for } x < \frac{l}{2} \tag{18a}$$
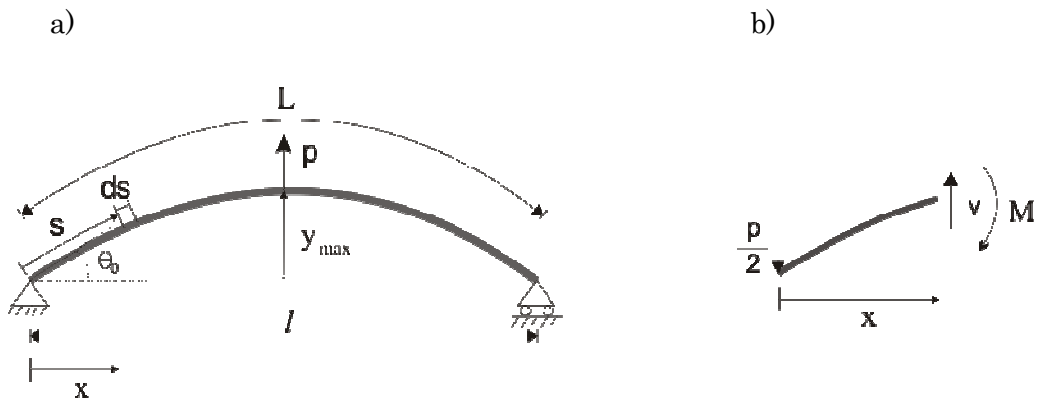
$$M = \frac{p(l-x)}{2} \qquad \text{for } x > \frac{l}{2} \tag{18b}$$

a)   b)

Fig. 6. Simply supported beam loaded at the centre, a) schematic of the beam, b) a small section of the beam

12

The solution presented here is similar to that given in references [2, 9]. By substituting the expression for the bending moment from Eq. (18a) into Eq. (5) gives,

$$\frac{d\theta}{ds} = -\frac{px}{2EI} \quad \text{for } x < \frac{l}{2} \tag{19}$$

Because of the symmetry only half of the beam is considered in the following analysis. The equation for $x > l/2$ would only be slightly different from the equations derived. Differentiating Eq. (19) with respect to s gives,

$$\frac{d^2\theta}{ds^2} = -\frac{p}{2EI}\frac{dx}{ds} \tag{20}$$

Now,

$$\frac{dx}{ds} = \cos\theta \tag{21}$$

Combining Eqs. (20) and (21) gives,

$$\frac{d^2\theta}{ds^2} + \frac{p}{2EI}\cos\theta = 0 \tag{22}$$

Multiplying Eq. (22) by $d\theta/ds$ gives, after some rearranging,

$$\frac{d}{ds}\left[\frac{1}{2}\left(\frac{d\theta}{ds}\right)^2 + \frac{p}{2EI}\sin\theta\right] = 0 \tag{23}$$

Integrating Eq. (23) gives,

$$\left(\frac{d\theta}{ds}\right)^2 + \frac{p}{EI}\sin\theta = c \tag{24}$$

where c is a constant. From Eq. (19) it can be seen that when $x = 0$ then $d\theta/ds = 0$. Also at this position $\theta = \theta_0$. Thus the constant is given by,

$$c = \frac{p}{EI}\sin\theta_0 \tag{25}$$

and Eq. (24) can be rewritten as,

$$\frac{d\theta}{ds} = -\sqrt{\frac{p}{EI}}\sqrt{\sin\theta_0 - \sin\theta} \tag{26}$$

Note that the minus sign is because of the negative curvature illustrated in Fig 1b. The angle at the left-hand support θ_0 can be found by integrating Eq. (26) $s=0$ to $L/2$,

$$L = -2\sqrt{\frac{EI}{p}}\int_{\theta_0}^{0}\frac{d\theta}{\sqrt{\sin\theta_0 - \sin\theta}} \tag{27}$$

The solution of the integral in Eq. (27) can be found in terms of an elliptical integral whose solution exists as power series. Noting that $\cos\theta = dx/ds$ and $\sin\theta = dy/ds$, the following equations can be derived from Eq. (26) for x and y at any point on beam as a function of θ,

$$x = 2\sqrt{\frac{EI}{p}}\sqrt{\sin\theta_0 - \sin\theta} \tag{28}$$

$$y = -\sqrt{\frac{EI}{p}}\int_{\theta_0}^{\theta}\frac{\sin\theta\, d\theta}{\sqrt{\sin\theta_0 - \sin\theta}} \tag{29}$$

The angle θ is equal to zero at the middle of the beam where the transverse displacement is maximum. The length between the two supports and the maximum transverse displacement can be determined by setting $\theta = 0$ in Eqs. (28) and (29) to give,

$$l = 4\sqrt{\frac{EI}{p}}\sqrt{\sin\theta_0} \tag{30}$$

$$y_{max} = \sqrt{\frac{EI}{p}}\int_{0}^{\theta_0}\frac{\sin\theta\, d\theta}{\sqrt{\sin\theta_0 - \sin\theta}} \tag{31}$$

As described in many publications, the displacement of the beam can be found by way of elliptical integrals using a suitable variable transformation. The other approach is to evaluate the integrals using numerical methods by Matlab. To do this, the applied force p can be found by solving the integral in Eq. (27) for different values of θ_0. Having a set of forces p and the correspondence set of angles θ_0, the maximum deflection can be found from Eq. (31) by calculating the integral in this equation numerically for each pair of p and θ_0. The distance between the supports can be easily calculated using Eq. (30). The shape of the

deflected beam can be found from Eqs. (28) and (29) by considering a set of values of θ varying from θ_0 to zero for each specific value of p.

3.2. An approximate solution

For large deflections, the relationship between the radius of curvature and the derivatives of the transverse displacement, y with respect to x (denoted by the superscript $'$) is [13],

$$\frac{1}{\rho} = \frac{y''}{(1 + (y')^2)^{\frac{3}{2}}} \tag{32}$$

This can be expanded by Maclaurin series up to the second term for y' to give,

$$\frac{1}{\rho} = y'' \left(1 - \frac{3}{2}(y')^2\right) \tag{33}$$

Combining Eq. (33) with Eq. (4) results in,

$$-\frac{M}{EI} = y'' \left(1 - \frac{3}{2} y'^2\right) \tag{34}$$

which is often called second order beam theory in the literature. Equation (34) can be solved by Galerkin's method by considering a shape function for the beam. Here a sinusoidal shape function is chosen as it represents the shape of a simply supported beam in its first mode of vibration. Thus the transverse displacement of the beam can be approximated by

$$y(x) = y_{max} \sin(kx) \tag{35}$$

The coefficient k can be found by applying the boundary condition $y(l) = 0$. Hence

$$y_{max} \sin(kl) = 0 \tag{36}$$

Hence, for the static deflection of the beam,

$$k = \frac{\pi}{l} \tag{37}$$

To find the deflection y_{max}, Galerkin's method is applied to Eq. (34) to give,

$$\int_0^l -\frac{M}{EI} sin\left(\frac{\pi}{l}x\right) dx = \int_0^l y''\left(1 - \frac{3}{2}y'^2\right) sin\left(\frac{\pi}{l}x\right) dx \qquad (38)$$

By substituting for $y(x)$ from Eq. (35) and M from Eq. (18a) gives,

$$\int_0^{\frac{l}{2}} \frac{px}{2EI} sin\left(\frac{\pi}{l}x\right) dx + \int_{\frac{l}{2}}^l \frac{p(l-x)}{2EI} sin\left(\frac{\pi}{l}x\right) dx$$

$$= \int_0^l y_{max}\left(\frac{\pi}{l}\right)^2 sin^2\left(\frac{\pi}{l}x\right)\left(1 - \frac{3}{2}\left(\frac{y_{max}\pi}{l}\right)^2 cos^2\left(\frac{\pi}{l}x\right)\right) dx \qquad (39)$$

Evaluating the integrals gives,

$$\frac{l^2 p}{2EI\pi^2} + \frac{l^2 p}{2EI\pi^2} = \frac{8y_{max}l^2\pi^2 - 3y_{max}^3\pi^4}{16l^3} \qquad (40)$$

which can be rearranged to give,

$$p = \frac{EI\pi^4}{2l^3}\left(y_{max} - \frac{3\pi^2}{8l^2}y_{max}^3\right) \qquad (41)$$

where right hand side of Eq. (41) is the same as the stiffness term of equation (3.158) in reference [10]. It may appear at first glance that the beam exhibits softening behaviour because of the negative sign in front of the cubic term. However, the length between the supports is not fixed and reduces in size because the beam is inextensible and the support on the right hand side slides to the left with increasing beam displacement. To determine whether the beam has a hardening or softening characteristic the expression given in Eq. (41) needs to be rewritten in terms of the length of the beam L. To do this the relationship between L and I needs to be determined.

Now,

$$L = \int_0^L ds = \int_0^l \sqrt{1 + (y')^2} \, dx \qquad (42)$$

Substituting for y from Eq. (35) gives,

$$L = \int_0^l \sqrt{1 + \left(y_{max}\frac{\pi}{l}\right)^2 \cos^2\left(\frac{\pi}{l}x\right)}\, dx \qquad (43)$$

The integral in Eq. (43) can be evaluated as the sum of two elliptical integrals,

$$L = \frac{l}{\pi}\left(\int_0^{\frac{\pi}{2}} \sqrt{1 + \left(y_{max}\frac{\pi}{l}\right)^2 \sin^2\theta}\, d\theta + \sqrt{1 + \left(y_{max}\frac{\pi}{l}\right)^2}\int_0^{\frac{\pi}{2}} \sqrt{1 + \frac{y_{max}^2\pi^2}{l^2 + y_{max}^2\pi^2}\sin^2\theta}\, d\theta\right) \qquad (44)$$

Expanding the elliptic integrals as a series up to second term results in,

$$L = l + \frac{y_{max}^2\pi^2}{4l} \qquad (45)$$

Rearranging Eq. (45) the distance between the supports can be written in terms of the length of the beam as,

$$l = \frac{1}{2}\left(L \pm \sqrt{L^2 - y_{max}^2\pi^2}\right) \qquad (46)$$

Combining Eqs. (41) and (46) results in

$$p = \frac{2EI\pi^4\left(2y_{max}\left(L + \sqrt{L^2 - y_{max}^2\pi^2}\right)^2 - 3y_{max}^3\pi^2\right)}{(L + \sqrt{L^2 - y_{max}^2\pi^2})^5} \qquad (47)$$

Expanding Eq. (47) as a series up to third order gives

$$p = \frac{EI\pi^4}{2L^3}\left(y_{max} + \frac{3\pi^2}{8L^2}y_{max}^3\right) \qquad (48)$$

Comparing this with Eq. (41) it can be seem that the equations are identical in form with the exception of the sign change in front of the cubic term and the variable L replacing l. It can be seen that the beam has a hardening rather than a softening characteristic. Equations (47) and (48) can be written in non-dimensional form respectively as,

$$\hat{p} = 2\pi^4 \frac{\left(2\hat{y}_{max}\left(1 + \sqrt{1 - \hat{y}_{max}^2\pi^2}\right)^2 - 3\hat{y}_{max}^3\pi^2\right)}{\left(1 + \sqrt{1 - \hat{y}_{max}^2\pi^2}\right)^5} \qquad (49)$$

$$\hat{p} = \frac{\pi^4}{2}\left(1 + \frac{3\pi^2}{8}\hat{y}_{max}^{\;2}\right)\hat{y}_{max} \qquad (50)$$

where $\hat{p} = p/(EI/L^2)$ and $\hat{y}_{max} = y_{max}/L$.

Equations (49) and (50) are plotted in Fig. 7. together with results calculated from finite element analysis using Ansys. The force deflection plot obtained from Eq. (49) exhibits hardening then softening characteristics while the results from Ansys, Eq. (50) and the exact solution exhibit hardening behaviour. Both equations deviate from the exact solution at a non-dimensional transverse displacement of about 0.17. The behaviour of Eq. (49) occurs because the higher order terms of the expansion of the integral in Eq. (43) are neglected and an approximation for I is used in the derivation of Eq. (49).
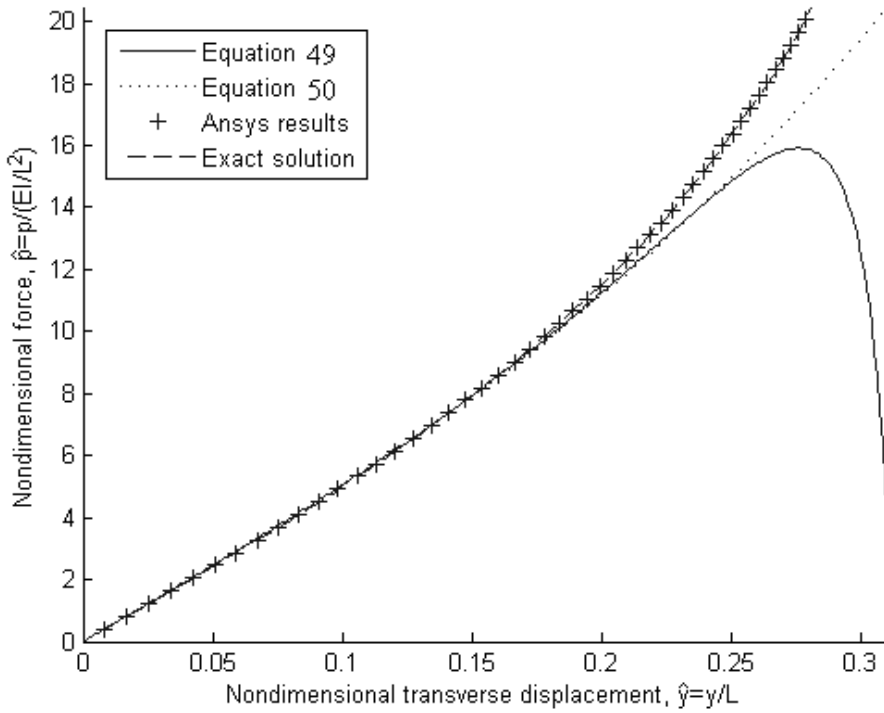

Fig. 7. Non-dimensional force as a function of non-dimensional transverse displacement

3.3. Discussion

The deflection of a beam is mainly due to the internal bending moment, and the effect of shear deformation can be neglected, which is the case in this study. To determine the

reason why the beam has a hardening characteristic, the behaviour of the bending moment, which is the main reason of the deflection, is investigated.

Suppose, a simply supported beam is loaded by a force equal to p_1 as shown in Fig. 8(a). The internal bending moment as a function of the distance from the fixed end x is shown in the same figure. The maximum bending moment is at the middle of the beam and is equal to $p_1 l_1/4$ where l_1 is the distance between the supports. By increasing the force to $p_2 = \alpha p_1$, the beam deflects more and the distance between the supports l_2 decreases compared to l_1. Although the force $p_2 = \alpha p_1$, the maximum bending moment $M_{\max 2} < \alpha M_{\max 1}$, because of the decreasing distance between the supports.
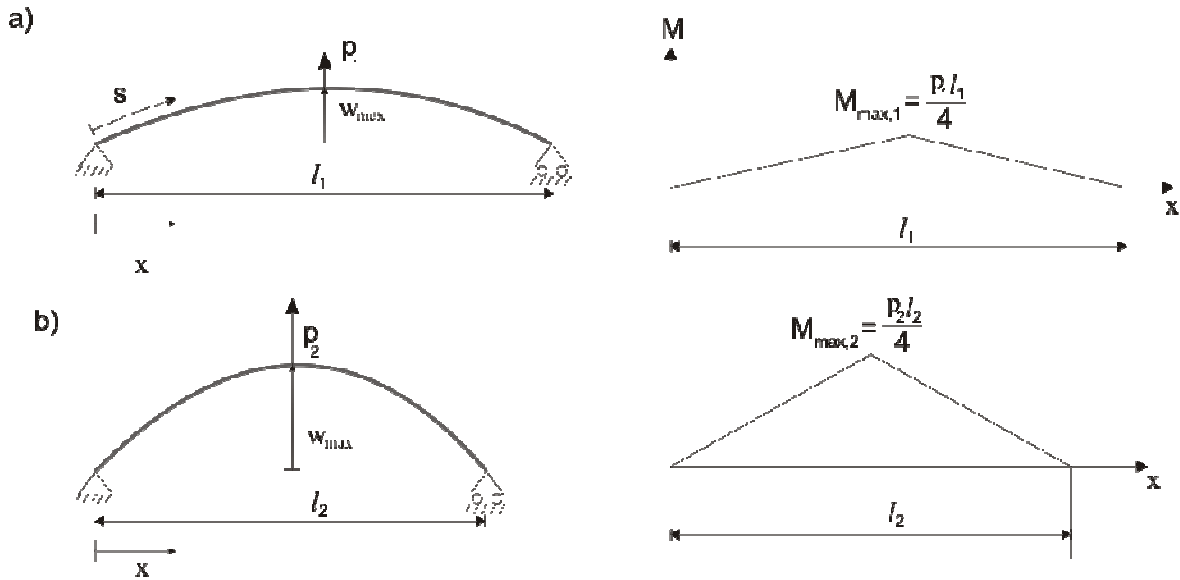

Fig. 8. The bending moment in a simply supported beam loaded at the centre

Thus, as the force increases, the internal bending moment increases at a lower rate. Because of the direct relationship between the bending moment and the curvature, the beam exhibits hardening behaviour. In addition to this, the "rotation effect", which was described in section 2.2, is valid in this case as well, and this also causes the beam to have hardening characteristics.

3.4. Accuracy of the approximate solution

To find a solution for the large deflection of the simply supported beam, two approximations were made. The first one was to consider the deflected shape of the beam to be a Sine function. The second approximation was made in the determination of the length between the beam supports as a function of force and transverse displacement. This was considered to be a second order function of the length of the beam with the higher order terms being ignored. The effect of these approximations is investigated in this section.

The Sine function is considered to be a good candidate for the shape function for a simply supported beam in the Galerkin method. It satisfies the simply supported boundary condition and its first and second derivatives are easily determined. The shape of the beam modelled by the Sine function is compared with the exact shape of the deflected beam derived from the exact solution which is calculated numerically and is shown in Fig. 9. The dotted line represents the shape of the beam given by $\hat{y}_{max} \sin \pi \hat{x}$. The approximate shape is a reasonable representation of the actual shape of the beam even for large deflections. The Mean Square Error (MSE) between the actual and the approximate shape normalised by the area bounded by the curves of the deflected shape and the undeflected shape of the beam is shown in Fig. 10 as a function of the non-dimensional force applied to the beam. The error is small but increases rapidly for non-dimensional forces greater than about 12. As well as errors in the approximate displacement there are errors in the first and second derivatives of transverse displacement as well, and these are investigated next.
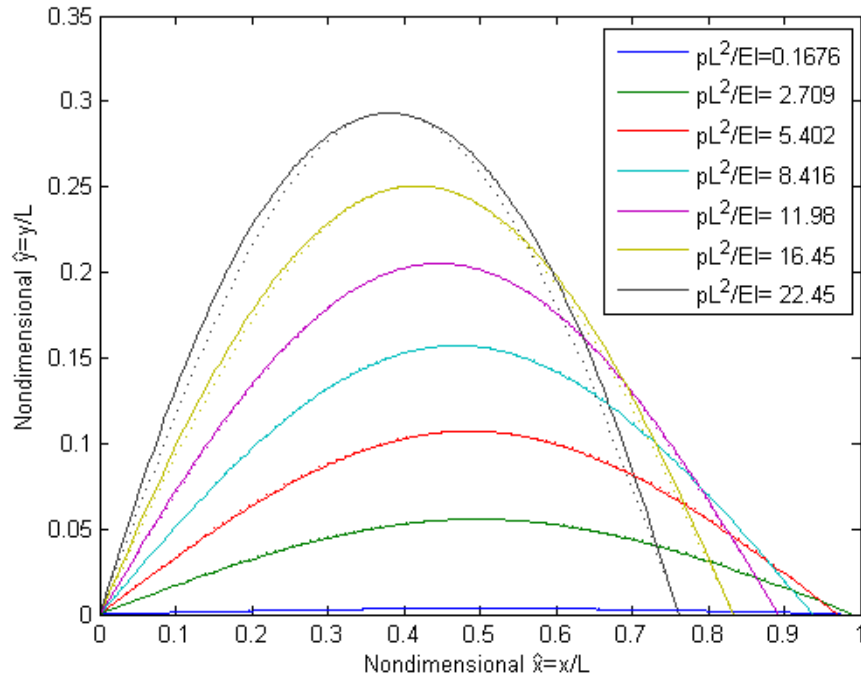
Fig. 9. Deflected shape of the beam for different values of load, solid lines (-): exact shape, dotted lines (...): shape given by a Sine function
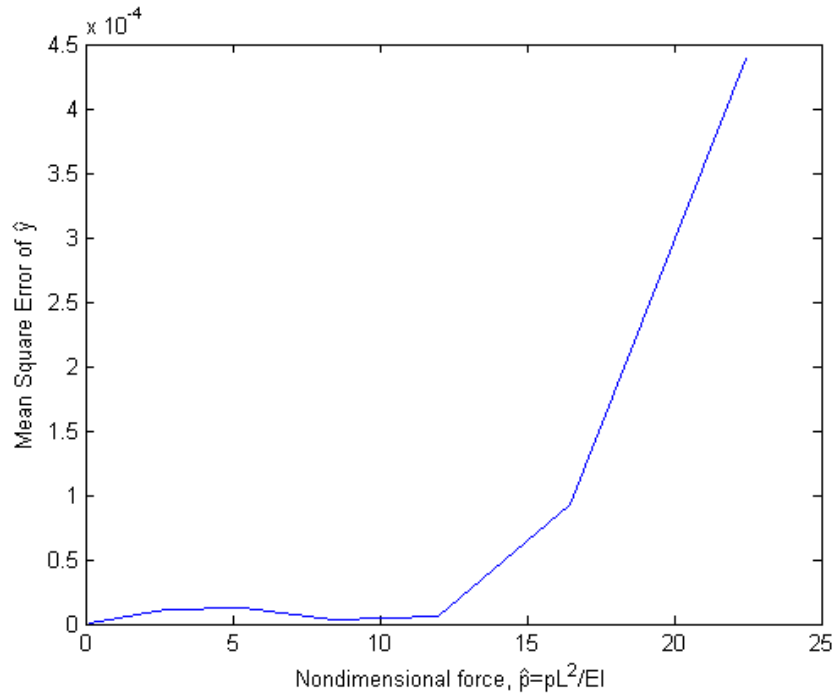

Fig. 10. Normalised mean square error between the actual and approximate shapes of the beam given in Fig. 10.

Examining Eq. (38), it can be seen that the first and second derivatives of the transverse displacement have been used to find the deflection of the beam. The first derivative of the transverse displacement with respect to x is shown in Fig. 11. The Sine function is a good approximation for small displacements, but deviates for higher loads especially close to the ends of the beam. The mean square error of the first derivative (slope) normalised by the area under the curve (as before) is plotted in Fig. 12. The MSE of the slope increases with the force similar to the MSE of the displacement.
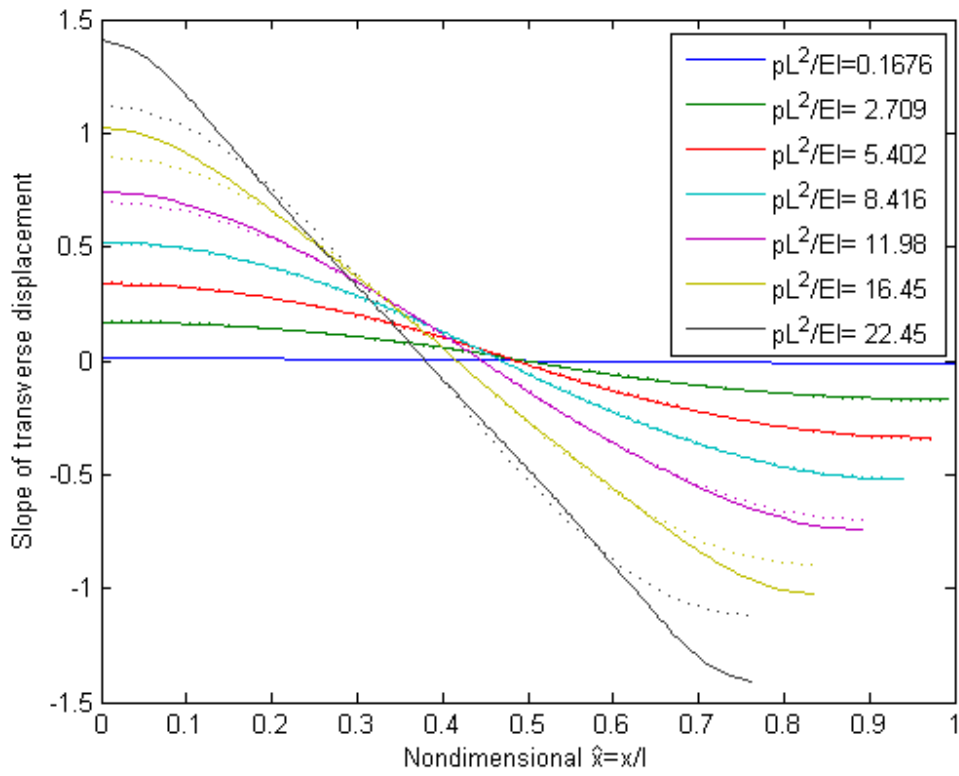

Fig. 11. Slope of transverse displacement, solid lines (-): exact solution, dotted lines (...): approximate solution
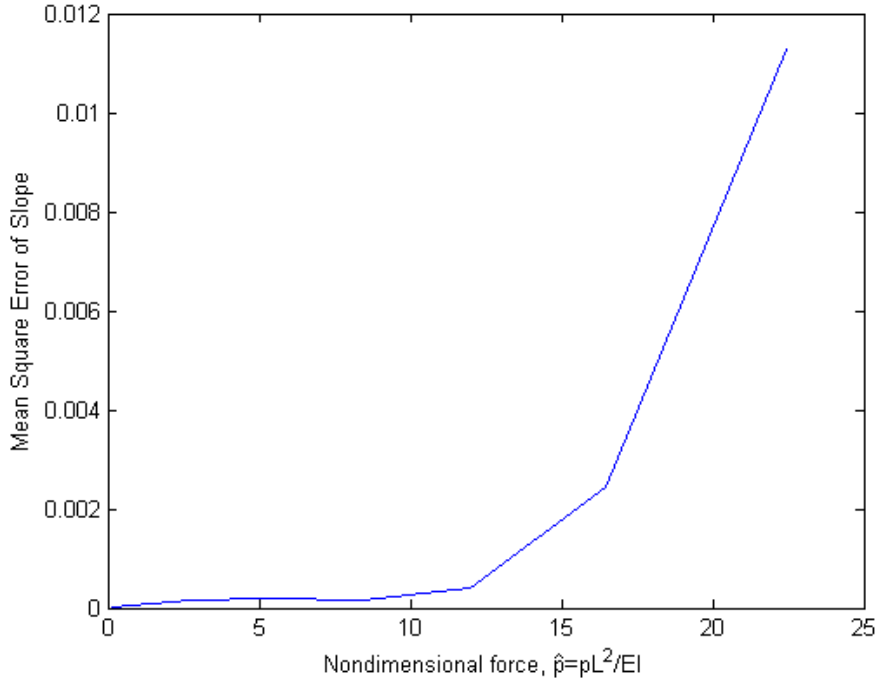
Fig. 12. Mean Square Error of slope normalised by the area under the curve

The same procedure is followed for the second derivative of the transverse displacement with respect to x. The second derivatives of the exact and approximate solutions are shown in Fig. 13. It can be seen that the Sine function is not a good approximate of the second derivative of the beam. The second derivative is not a smooth function any more, as there is a discontinuity at the centre of the beam. This is can be understood by examining Eq. (34), that the second derivative is a function of the bending moment as well as slope of the beam. The bending moment of a simply supported beam loaded by a force at the centre has a triangular shape with the maximum being in the middle. For small loads, where the slope is small, the second derivative is predominantly a function of the bending moment only. By increasing the force and as a result the deflection, the slope of the beam increases and can not be ignored in this case. This results in the second derivative having a pronounced discontinuity at the centre of the beam.

To obtain the approximate solution, the integral of the product of second derivative and first derivative of deformation is used. Although, the Sine function does not approximate well the second derivative of deformation, the area under the curve is very similar to the actual curve and thus the normalised MSE between the actual and the approximate solutions is small for small loads a shown in Fig. 14.
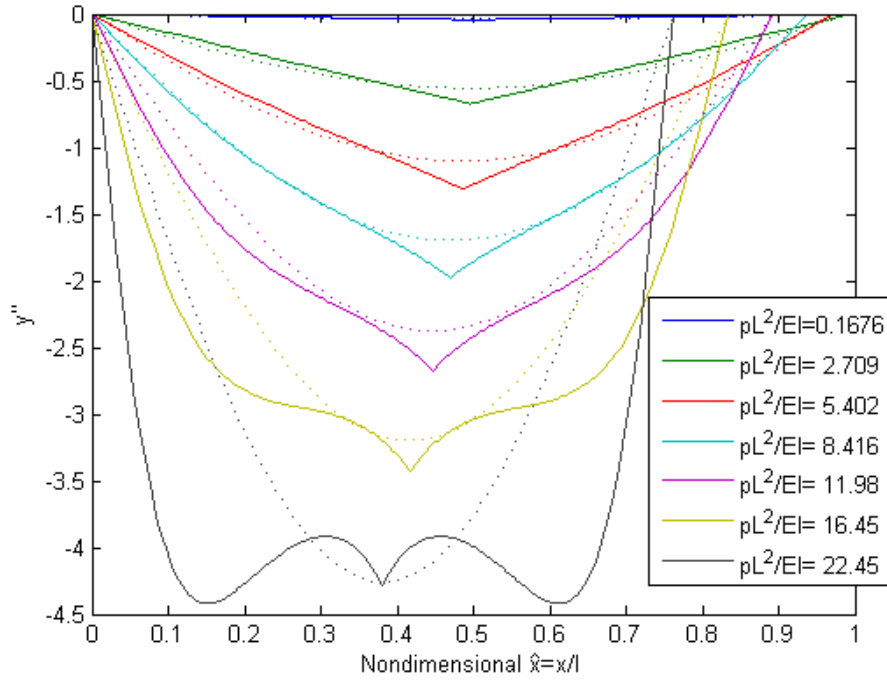
23

Fig. 13. Second derivative of the transverse displacement, solid lines (-): exact solution, dotted lines (...): sine function
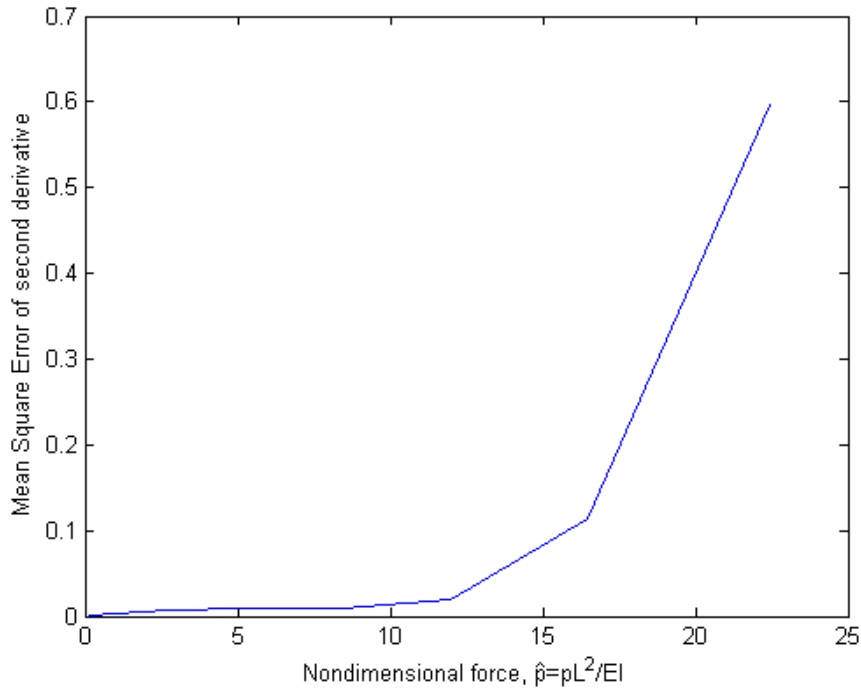

Fig. 14: Mean Square Error of the second derivative normalised by the area under the curve

24

The second issue with the accuracy of the approximate solution is the accuracy of the solution for length between the supports. As mentioned in the previous section, the accuracy of Eq. (49) illustrated in Fig. 7 is linked to the accuracy of the expansion of the integral in Eq. (43) used to derive Eq. (49). Equation (45), which relates the length of the beam to the distance between the beam supports, can be written in a more complete form by including higher order terms to give,

$$L = l + \frac{y_{max}^2 \pi^2}{4l} - \frac{3y_{max}^4 \pi^4}{64l^3} + \frac{5y_{max}^6 \pi^6}{256l^5} - \frac{175y_{max}^8 \pi^8}{16384l^7} + \frac{441y_{max}^{10} \pi^{10}}{65536l^9} - \frac{4851y_{max}^{12} \pi^{12}}{1048576l^{11}} + \cdots \qquad (51)$$

Considering this equation, the distance between the beam supports l cannot be derived easily as a function of the beams length L. The integral in Eq. (43) can be solved numerically to compare the results with the series. A new variable a is introduced to help solve the integral numerically, so that

$$a = \frac{x}{l} \quad \rightarrow dx = l\, da \qquad (52)$$

and the integral in Eq. (43) can be rewritten as,

$$\frac{L}{l} = \int_0^1 \sqrt{1 + \left(\pi \frac{y_{max}}{l}\right)^2 \cos^2(\pi a)}\, da \qquad (53)$$

Solving this integral for values of y_{max}/l from zero to one gives the results presented in Fig. 15. Results of the expanded series using the first two, three and seven terms of Eq. (51) are plotted alongside the numerical solution as well. The expansion series results match with the numerical results well for small beam deflections, but deviate rapidly for large deflections. It can be seen that for the truncated series given in Eq. (45), which is used to give the analytical results, gives reasonable accuracy up to a value of $\frac{y_{max}}{l} < 0.2$.
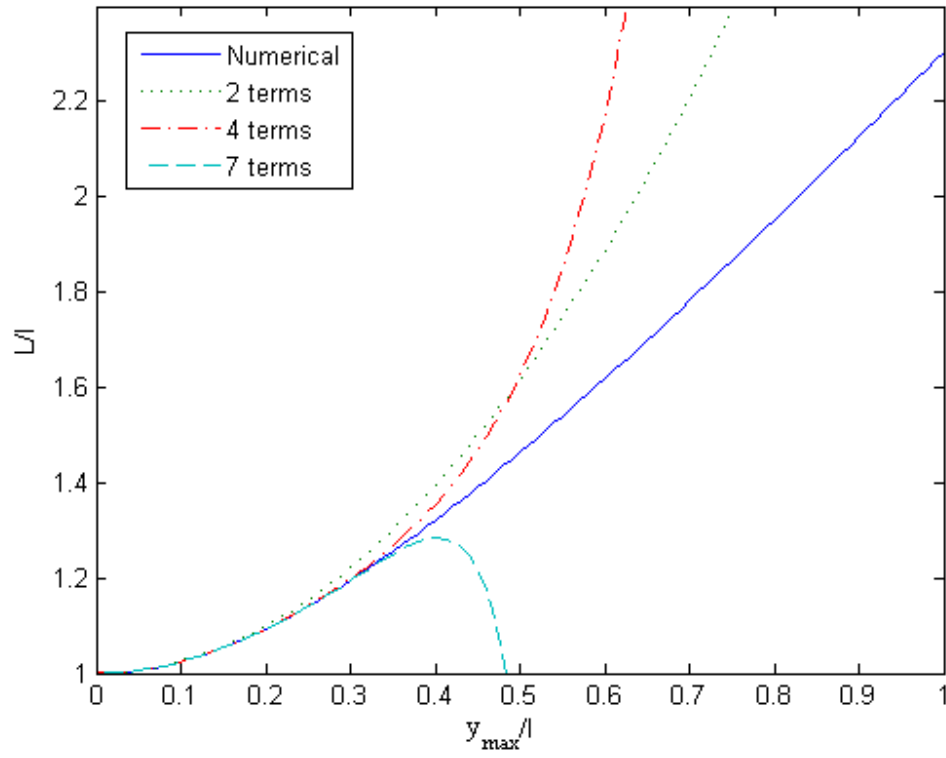
Fig. 15. Relationship between the distance between the beam supports, its length and the transverse displacement.

4. Conclusions

This report has described an investigation into the large deflections of beams. It has been shown that a simply-supported beam loaded in the middle can be modelled by a Duffing-like stiffness model with hardening nonlinearity. The Galerkin method has been used to obtain the approximate solution of the governing equation. The exact solution and the finite element method have been used to validate the results. The suitability of the sine function to be used as shape function in the Galerkin method for a simply-supported beam has also been investigated. It has been shown that this gives a good approximation for the force-deflection curves of the beam when it is loaded in the centre, provided that the maximum displacement is less than about 20% of the length of the beam.

The deformation of a simply supported beam loaded by pure bending moment has also investigated. The exact solution for large deflection in this case can easy to derive and the results are used to explain some aspects of beam behaviour. The rotational stiffness of the beam in this case is constant while the transverse stiffness is hardening.

5. References

1. Frisch-Fay, R., *Flexible bars*. 1962, London: Butterworths. 228p.
2. Bisshopp, K.E. and D.C. Drucker, *Large Deflection of Cantilever Beams.* Quarterly of Applied Mathematics, 1945. **3, No. 3**: p. 272-275.
3. Conway, H.D., *XCIV. The large deflection of simply supported beams.* Philosophical Magazine Series 7, 1947. **38**(287): p. 905 - 911.
4. Gospodnetic, D., *Deflection curve of a simply supported beam.* Transactions of the ASME, Journal of Applied Mechanics, 1959. **26**(4): p. 675-676.
5. Sundara Raja Iyengar, K.T. and S.K. Lakshmana Rao, *Large deflections of simply supported beams.* Journal of the Franklin Institute, 1955. **259**(6): p. 523-528.
6. Wang, T.M., S.L. Lee, and O.C. Zienkiewicz, *A numerical analysis of large deflections of beams.* International Journal of Mechanical Sciences, 1961. **3**(3): p. 219-228.
7. Wang, T.M., *Nonlinear bending of beams with concentrated loads.* Journal of the Franklin Institute, 1968. **285**(5): p. 386-390.
8. Wang, T.M., *Non-linear bending of beams with uniformly distributed loads.* International Journal of Non-Linear Mechanics, 1969. **4**(4): p. 389-395.
9. Belendez, T., C. Neipp, and A. Belendez, *Large and small deflections of a cantilever beam.* European Journal of Physics, 2002. **23**(3): p. 371-379.
10. Thomsen, J.J., *Vibrations and stability : advanced theory, analysis, and tools.* 2nd ed. 2003, Berlin ; New York: Springer. xxi, 404 p.
11. Gere, J.M. and S. Timoshenko, *Mechanics of materials.* 3rd ed. PWS-Kent series in engineering. 1990, Boston: PWS-Kent Pub. Co. xviii, 807 p.
12. Fazekas, G.A., *A note on the bending of euler beams.* Journal of engineering education, 1967. **57**(5): p. 393,394.
13. Bazant, Z.P. and L. Cedolin, *Stability of structures: elastic, inelastic, fracture, and damage theories.* The Oxford engineering science series 26. 1991, New York: Oxford University Press. xxiv, 984 p.

28