Eigenvalues and Eigenvectors

If A is an $n \times n$ matrix, then a scalar λ is called an eigenvalue of A if there is a nonzero vector x such that $Ax = \lambda x$. If λ is an eigenvalue of A, then every nonzero vector x such that $Ax = \lambda x$ is called an eigenvector of A corresponding to λ.

$$Ax = \lambda x \quad \Rightarrow \quad (\lambda I - A)x = 0$$

$$\Rightarrow \det(\lambda I - A) = 0 \qquad \text{Characteristic equation.}$$

Example $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{bmatrix}$$

$$\det(\lambda I - A) = 0 \quad \Rightarrow \quad \begin{vmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{vmatrix} = 0$$

$$\lambda^2 - 3\lambda - 10 = 0 \quad \Rightarrow \quad (\lambda + 2)(\lambda - 5) = 0$$

$\lambda = -2, 5.$

For $\lambda = -2$, we have $\begin{bmatrix} -3 & -3 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

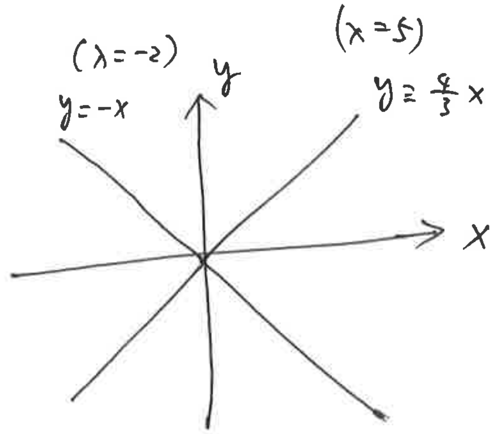
$\Rightarrow x = -t, \; y = t$

eigenvector: $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

For $\lambda = 5$, we have $\begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad x = \frac{3}{4} t, \quad y = t$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{4} t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$



Theorem: If A is a symmetric matrix, then eigenvectors from different eigenspace are orthogonal.

Spectral decomposition

If A is a symmetric matrix, then

$$A = [\vec{u}_1 \ \vec{u}_2 \ \cdots \ \vec{u}_n] \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}$$

$$= \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2 + \cdots + \lambda_n \vec{u}_n \vec{u}_n$$

which is called eigenvalue decomposition of A.

Example $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -3$, $\lambda_2 = 2$

with Corresponding eigenvectors $\vec{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Note that $\vec{x}_1 \cdot \vec{x}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1 \cdot 2 + (-2) \cdot 1 = 0$

$\Rightarrow \vec{x}_1 \perp \vec{x}_2$ (orthogonal).

Normalizing these basis vectors yields

$$\vec{u}_1 = \frac{\vec{x}_1}{\|\vec{x}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \quad \text{and} \quad \vec{u}_2 = \frac{\vec{x}_2}{\|\vec{x}_2\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

Eigenvalue decomposition of A is

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2 = (-3) \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix} + 2 \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$= (-3) \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix} + 2 \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$