Meshless and other advanced numerical methods: Assignment 3

KANSA'S METHOD

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1 Assignment

Solve the following partial differential equation using Kansa's method:

$$\Delta u - x^2 y \frac{\partial u}{\partial x} = e^x + e^y - x^2 y e^x, (x, y) \in \Omega . \tag{1.1}$$

The problem must be solved using different boundary conditions.

1. BC_01
   Dirichlet conditions for all boundary points:

$$u = e^x + e^y . \tag{1.2}$$

2. BC_02
   Dirichlet conditions for boundary points with positive y-coordinate and Neumann conditions for boundary points with negative y-coordinate:

$$\begin{aligned} u &= e^x + e^y && \dots \quad y \geq 0, \\ \frac{\partial u}{\partial n} &= \left(\nabla\left(e^x + e^y\right)\right) \cdot \mathbf{n} && \dots \quad y < 0. \end{aligned} \tag{1.3}$$

The inverse multiquadrics (IMQ) must be used as a radial basis function:

$$\hat{f}(r) = \frac{1}{\sqrt{r^2 c^2 + 1}} . \tag{1.4}$$

The exact solution of the problem is given with the following equation:

$$u = e^x + e^y . \tag{1.5}$$

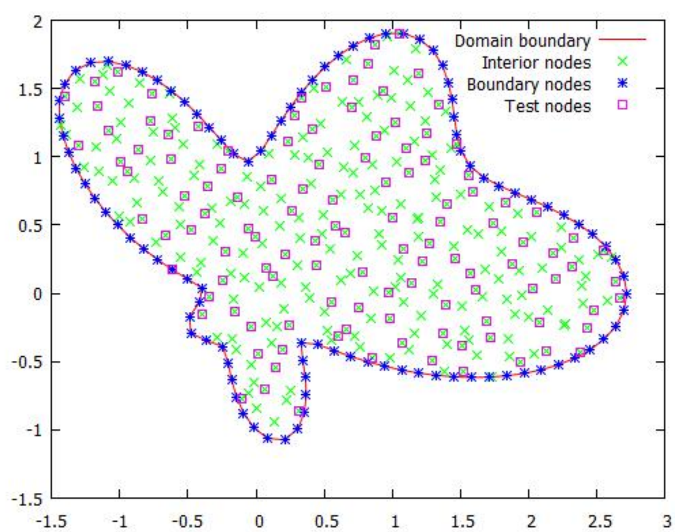

Figure 1: Nodes used as RBF centres and test points are shown.

The solution of the problem must performed using 300 uniformly distributed interior nodes and 100 boundary points (see Figure 1). For the evaluation of the method and calculation of the RMSE we used 120 test points shown on the right side of Figure 1.

2 Solution

2.1 RBFs and their derivatives

Because we are solving a second order partial differential equation we need to calculate the derivatives of the IMQ radial basis function:

1. RBF values:

$$\hat{f} = \sum_{i=1}^{n} \left(\frac{1}{\sqrt{1+r^2c^2}} \right) \tag{1.6}$$

2. RBF first order derivatives:

$$\frac{\partial \hat{f}}{\partial x} = \sum_{i=1}^{n} \left(\frac{-c^2\left(x-x_i\right)}{\left(\sqrt{1+r^2c^2}\right)^3} \right) \tag{1.7}$$

$$\frac{\partial \hat{f}}{\partial y} = \sum_{i=1}^{n} \left(\frac{-c^2\left(y-y_i\right)}{\left(\sqrt{1+r^2c^2}\right)^3} \right) \tag{1.8}$$

3. RBF second order derivatives:

$$\frac{\partial^2 \hat{f}}{\partial x^2} = \sum_{i=1}^{n} \left(\frac{3c^4\left[\left(x-x_i\right)^2+\left(y-y_i\right)^2\right]}{\left(\sqrt{1+r^2c^2}\right)^5} - \frac{2c^2}{\left(\sqrt{1+r^2c^2}\right)^3} \right) \tag{1.9}$$

$$\frac{\partial^2 \hat{f}}{\partial y^2} = \sum_{i=1}^{n} \left(\frac{3c^4\left[\left(x-x_i\right)^2+\left(y-y_i\right)^2\right]}{\left(\sqrt{1+r^2c^2}\right)^5} - \frac{2c^2}{\left(\sqrt{1+r^2c^2}\right)^3} \right) \tag{1.10}$$

For the Neumann boundary conditions we also need the derivatives of the RBF in the normal direction:

$$\frac{\partial \hat{f}}{\partial n} = \sum_{i=1}^{n} \left(\frac{\partial \hat{f}}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial \hat{f}}{\partial y}\frac{\partial y}{\partial n} = \frac{-c^2\left(x-x_i\right)}{\left(\sqrt{1+r^2c^2}\right)^3}\frac{\partial x}{\partial n} + \frac{-c^2\left(y-y_i\right)}{\left(\sqrt{1+r^2c^2}\right)^3}\frac{\partial y}{\partial n} \right), \tag{1.11}$$

where we define the derivatives in the normal direction with the following relations:

$$
\begin{aligned}
\frac{\partial x}{\partial n} &= n_x, \\
\frac{\partial y}{\partial n} &= n_y.
\end{aligned} \tag{1.12}
$$

We have choosen to use scaling in order to normalize the shape parameter. So the shape parameter in equations (1.6)-(1.11) is in fact a scaled value of initial shape parameter:

$$
c = \frac{c_{initial}}{\sigma}, \tag{1.13}
$$

where σ is defined as a scale and is calculated with the following equation:

$$
\sigma_i = \sqrt{\frac{\sum_{k=1}^{n}\left(\left(x_i - x_k\right)^2 + \left(y_i - y_k\right)^2\right)}{n-1}}, \tag{1.14}
$$

and n is the number of all RBF centres.

2.2 Kansa's method

The Kansa's method is formulated in the following way:

$$
\begin{bmatrix} \psi \\ \phi^D \\ \phi^N \end{bmatrix} \mathbf{w} = \begin{bmatrix} f \\ g^D \\ g^N \end{bmatrix}, \tag{1.15}
$$

The Kansa's matrix for the inner points has the following formulation:

$$
\left.
\begin{aligned}
\psi_{ij} &= \frac{6c^4 r^2}{\left(\sqrt{1+r^2c^2}\right)^5} - \frac{4c^2}{\left(\sqrt{1+r^2c^2}\right)^3} - x_j^{\,2} y_j \frac{-c^2\left(x_j - x_i\right)}{\left(\sqrt{1+r^2c^2}\right)^3} \\
f_j &= e^{x_j} + e^{y_j} - x_j^{\,2} y_j e^{x_j}
\end{aligned}
\right\} \; 1 \le j \le n_i, 1 \le i \le n . \tag{1.16}
$$

Where r is the Euclidian distance between points i and j. We will define the Kansa's matrices for boundary points in two different ways:

1. BC_01 - Dirichlet conditions for all boundary points

$$
\left.
\begin{aligned}
\phi_{ij}^{\;D} = \phi_{ij}^{\;N} &= \frac{1}{\sqrt{1+r^2c^2}} \\
g^D = g^N &= e^{x_j} + e^{y_j}
\end{aligned}
\right\} \; 1 \le j \le n_b^{\;D} \text{ and } 1 \le j \le n_b^{\;N}, 1 \le i \le n . \tag{1.17}
$$

2. BC_02 - Dirichlet conditions for boundary points with positive y-coordinate and Neumann conditions for boundary points with negative y-coordinate

$$\left.\begin{aligned} \phi_{ij}{}^{D} &= \frac{1}{\sqrt{1+r^2c^2}} \\ g^{D} &= e^{x_j} + e^{y_j} \end{aligned}\right\} \quad 1 \le j \le n_b{}^{D}, 1 \le i \le n,$$

$$\tag{1.18}$$

$$\left.\begin{aligned} \phi_{ij}{}^{N} &= \frac{-c^2\left(x_j - x_i\right)}{\left(\sqrt{1+r^2c^2}\right)^3} n_{x_j} + \frac{-c^2\left(y_j - y_i\right)}{\left(\sqrt{1+r^2c^2}\right)^3} n_{y_j} \\ g^{N} &= \left(\nabla\left(e^{x_j} + e^{y_j}\right)\right) \cdot \mathbf{n}_j \end{aligned}\right\} \quad 1 \le j \le n_b{}^{N}, 1 \le i \le n.$$

We have used the Fortran code to perform the Kansa's method. We have used the LU decomposition to calculate the inverse of the interpolation matrix **A** in system (1.15).

3 Results

3.1 Determination of optimal shape parameter with RMSE

In the first step we used the 120 test points to evaluate the RMSE and the maximum error for different shape parameter. We tested the shape parameters in a range [0,5]. Results RMSE(c) are plotted in Figure 2 for both cases. We can see that the RMSE is minimal for the shape parameter is equal to about 0.85.

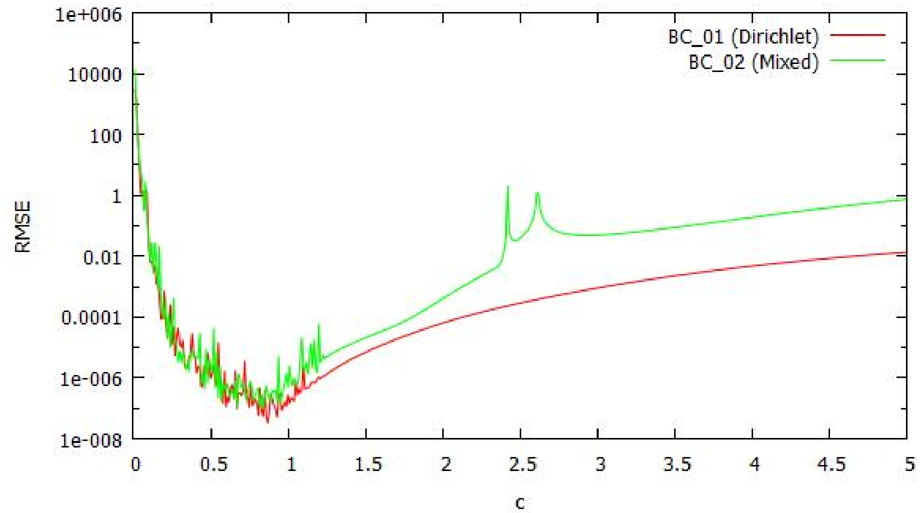

Figure 2: Plot of RMSE(c) for both cases (BC_01 and BC_02). We can see that error is at minimum for shape parameter 0.6 for both cases

Next we made a plot of error at all test points for the optimal shape parameter 0.85. Figure 3 shows the error for every test point for both cases (BC_01 and BC_02).

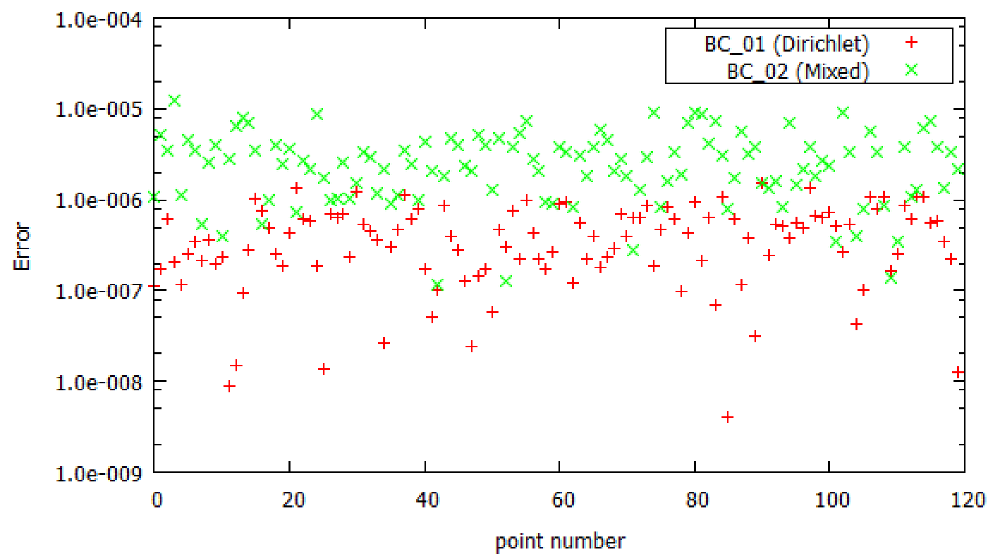

Figure 3: Plot of error for every test points.

3.2 Determination of optimal shape parameter with minimization and LOOCV

Next we performed the minimization to seek the optimal shape parameter. The function, which we choose to minimize, is the LOOCV, which was described in the report of Assignment 2. We used the golden section minimization procedure to find the global minimum. But the results are more unstable then in the previous assignment. We have used different search intervals ([0,1], [0,2], [0,3], [0,4], [0,5]) to find the optimal shape parameter. The resulting optimal shape parameter was dependent on the initial search interval, yet the RMSE was small for optimal shape parameters of each interval.

Table 1: Optimal shape parameter obtained with the minimization of the LOOCV for various initial search intervals.

Search interval	Optimal shape parameter and error			
	BC_01		BC_02	
	c	RMSE	c	RMSE
[0, 1]	0.4731	7.44E-07	0.3612	6.15E-07
[0, 2]	0.4627	6.05E-06	0.4629	4.26E-07
[0, 3]	0.5417	1.61E-07	0.5418	3.69E-07
[0, 4]	0.4467	7.66E-07	0.3806	6.89E-07
[0, 5]	0.5737	1.96E-07	0.5772	2.37E-07

On Figure 4 we have shown the L2 norm versus the shape parameter. The plot has many fluctuations in the area of the minimal error.

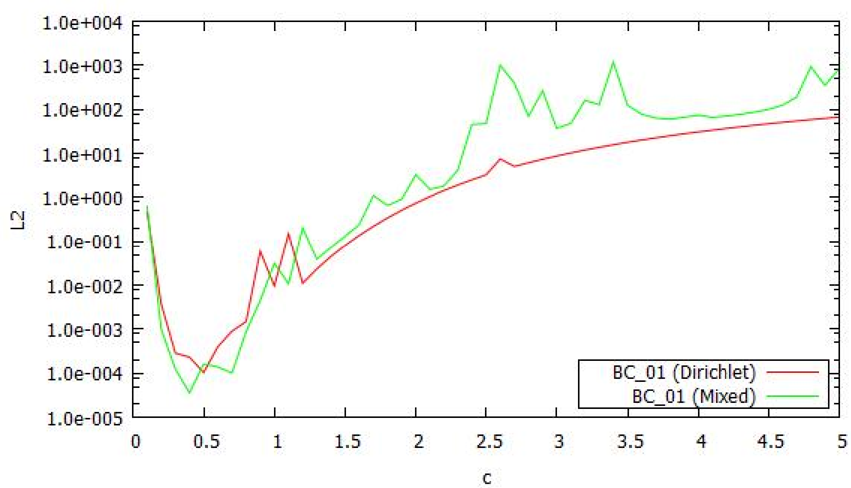

Figure 4: Plot of the L2 norm versus the shape parameter c.