Singular Value Decomposition (SVD)

Theorem. If A is an $n \times n$ matrix of rank K, then A can be factored as

$$A = U \Sigma V^T$$

where U and V are $n \times n$ orthogonal matrices and Σ is an $n \times n$ diagonal matrix whose main diagonal has K positive entries and $n-K$ zeros.

Example.
$$\begin{bmatrix} \sqrt{3} & 2 \\ 0 & \sqrt{3} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}}_{V^T}.$$

In MATLAB,

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A = [1 2 3; 3 5 6; 2 4 6]
[U, S, V] = svd(A)
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$$U = \begin{bmatrix} -0.3162 & 0.3162 & -0.8944 \\ -0.7071 & -0.7071 & -0.0000 \\ -0.6325 & 0.6325 & 0.4472 \end{bmatrix}$$

$$S = \begin{bmatrix} 11.8033 & 0 & 0 \\ 0 & 0.8258 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

reduced singular value decomposition.

$$V = \begin{bmatrix} -0.3137 & -0.6542 & 0.6882 \\ -0.5675 & -0.4520 & -0.6882 \\ -0.7613 & 0.6064 & 0.2294 \end{bmatrix}$$

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A = U * S * V'
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Singular Value decomposition of nonsquare matrices

Let A be an $m \times n$ matrix.

main diagonal of A: a_{ii}, $i = 1, 2, \cdots, \min(m, n)$.

$$\begin{bmatrix} \otimes & \times & \times & \times & \times \\ \times & \otimes & \times & \times & \times \\ \times & \times & \otimes & \times & \times \end{bmatrix} \qquad \begin{bmatrix} \otimes & \times & \times & \times \\ \times & \otimes & \times & \times \\ \times & \times & \otimes & \times \\ \times & \times & \times & \otimes \\ \times & \times & \times & \times \end{bmatrix}$$

Theorem. If A is an $m \times n$ matrix of rank k, then A can be factored as

$$A = U \Sigma V^T$$

$$= \left[\vec{u}_1\ \vec{u}_2 \cdots \vec{u}_k \,\vdots\, u_{k+1} \cdots u_m\right] \left[\begin{array}{ccc|c} \sigma_1 & & O & \\ & \sigma_2 & & O_{k\times(n-k)} \\ O & & \ddots\ \sigma_k & \\ \hline & O_{(m-k)\times k} & & O_{(m-k)\times(n-k)} \end{array}\right] \begin{bmatrix} \vec{V}_1^T \\ \vec{V}_2^T \\ \vdots \\ V_k^T \\ \hline V_{k+1}^T \\ \vdots \\ V_n^T \end{bmatrix}$$

in which U, Σ and V have size $m \times m$, $m \times n$, and $n \times n$, respectively.

$\sigma_1, \sigma_2, \cdots \sigma_k$: singular values of A.

$\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_k$: left singular vectors of A.

$\vec{V}_1, \vec{V}_2, \cdots, \vec{V}_k$: right singular vectors of A

Example.

$$X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$[U, S, V] = \mathrm{svd}(X)$$

$$U = \begin{bmatrix} -0.1525 & -0.8226 & -0.3945 & -0.3800 \\ -0.3499 & -0.4214 & 0.2428 & 0.8007 \\ -0.5474 & -0.0201 & 0.6979 & -0.4614 \\ -0.7448 & 0.3812 & -0.5462 & 0.0407 \end{bmatrix}$$

$$S = \begin{bmatrix} 14.2691 & 0 \\ 0 & 0.6268 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} -0.6414 & 0.7672 \\ -0.7672 & -0.6414 \end{bmatrix}$$

The economy size decomposition

$$[U, S, V] = \mathrm{svd}(X, 0) \quad \text{or} \quad \mathrm{svd}(X, \text{'econ'})$$

$$U = \begin{bmatrix} -0.1525 & -0.8226 \\ -0.3499 & -0.4214 \\ -0.5474 & -0.0201 \\ -0.7448 & 0.3812 \end{bmatrix}$$

$$S = \begin{bmatrix} 14.2691 & 0 \\ 0 & 0.6268 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.6414 & 0.7672 \\ -0.7692 & -0.6414 \end{bmatrix}$$

Reduced singular value expansion

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_k \vec{u}_k \vec{v}_k^T$$

Example.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Since $K = 2$, the redued singular value decomposition of A is

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T$$

$$= \sqrt{3} \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} + (1) \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \sqrt{3} \begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \end{bmatrix} + (1) \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Data Compression and image processing

SVD Can be used to Compress visual information to reduce the required storage space and speedingup its electronic transmission.

The first step in Compression a visual image is to represent it as a numerical matrix from which the visual image can be recovered when needed.

A black and white picture Can be scanned as a rectangular array of pixels and then Stored as a matrix A by assigning each pixels a numerical value in accordance with its grey level. If 256 different gray level are used (0 = white , 255 = black), then the entries in the matrix would be integers between 0 and 255.

If the matrix A has Size $m \times n$, then one might store each of its mn entries individually. An alterative procedure is to compute the reduced SVD

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_k \vec{u}_k \vec{v}_k^T \qquad (*)$$

in which $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k$, and Store the σ's, the $\vec{u}$'s and $\vec{v}$'s. when needed, A Can be reconstructed.

Since each $\vec{u}_j$ has m entries and each $\vec{v}_j$ has n entries, the required storage space is $km + kn + k = k(m+n+1)$

Suppose that the singular values $\sigma_{r+1}, \sigma_{r+2}, \cdots, \sigma_K$ are sufficiently small, we can drop the corresponding term in (*).

$$A_r = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

A_r is called the rank r approximation of A.

The matrix requires storage space for only

$$rm + rn + r = r(m+n+1).$$

For example, given a picture with 1000×1000 pixels.

If we choose rank = 100 ; i.e. 100 singular values,

then the required storage space is $100(1000+1000+1) = 200{,}100$.

Comparing to the $1000 \times 1000 = 1{,}000{,}000$ storage space without compression, we can have a compression of almost 80%.