Shepard Method and RBF

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Problem

Given data $(x_k, f_k), k = 1, \ldots, N$ with $x_k \in \mathbb{R}^d, f_k \in \mathbb{R}$, find a function $\tilde{f}(x)$ such that $\tilde{f}(x_k) \approx f_k$.

Scatter data

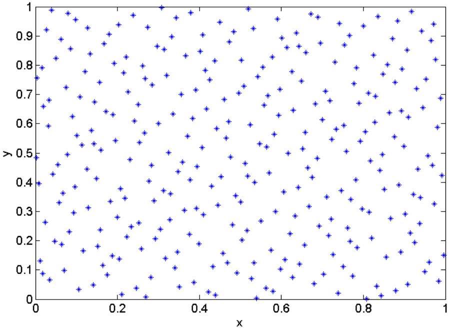

Figure: 300 scatter data points given by haltonseq.m in $[0,1]^2$

Scatter data (Contd.)

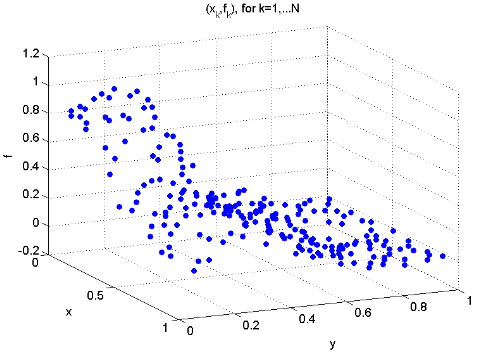

Figure: Given scatter data with locations and the value at locations

Methods

There are different methods to address given approximation problem. Here, we will discuss on the following methods:

- Radial Basis Function(RBF)
- Shepard Method

RBFs

Type	$\phi(r)$
linear	r
cubic	r^3
Gaussian	$\exp(-cr^2)$
polyharmonic	$r^{2n-1}, n \in \mathbb{N}$
polyharmonic	$r^{2n} \ln r, n \in \mathbb{N}$
multiquadrics(MQ)	$\sqrt{r^2 + c^2}$
thin-plate spline(TPS)	$r^2 \ln(r)$
inverse multiqudrics(IMQ)	$\frac{1}{\sqrt{r^2+c^2}}$

Table: Different types of Radial Basis Functions

CSRBFs

$\phi(r)$	Smoothness	d
$(1-r)_+^2$	C^0	3
$(1-r)_+^3$	C^0	5
$(1-r)_+^4(4r+1)$	C^2	3
$(1-r)_+^5(5r+1)$	C^2	5
$(1-r)_+^6(35r^2+18r+3)$	C^4	3
$(1-r)_+^7(16r^2+7r+1)$	C^4	5
$(1-r)_+^8(32r^3+25r^2+8r+1)$	C^6	3

Table: Wendland CSRBF with positive definiteness in the maximum dimension d of $\mathbb{R}^d$

RBF approximation

Find a function $\tilde{f}$ which is linear combination of translates $\phi(\|x - x_k\|_2)$ of a single **RBF**. i.e.

$$\tilde{f}(x) := \sum_{k=1}^{N} \alpha_k \phi(\|x - x_k\|_2) \tag{1}$$

Where, $x_1, \ldots x_N$ of $\mathbb{R}^d$ are centers or trial points.
such that

$$\sum_{k=1}^{N} \alpha_k \phi(\|x_i - x_k\|_2) \approx f(x_i), i = 1, \ldots N$$

Example

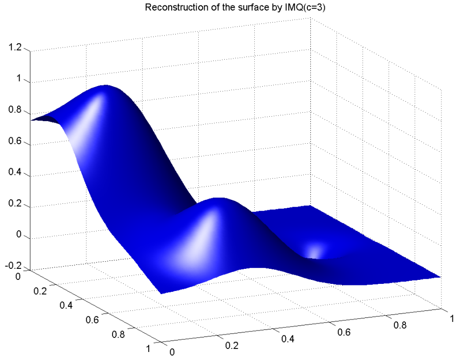

Figure: Reconstruction surface of the data given in Figure(1) by using IMQ(c=3)

Problem in RBF approximation

- Matrix size
- Singularity problem
- Ill condition problem

Original Shepard Method

Inverse weighted interpolation by Shepard[1968]

$$\tilde{f}(x) = \frac{\sum_{k=1}^{n} W_k(x) f_k}{\sum_{k=1}^{n} W_k(x)} \tag{2}$$

Where,
$W_k(x) = \frac{1}{\|x - x^k\|^p}$ is the weight function.
p is the positive real number the most frequent choice is $p = 2$.
The interpolant is arbitrarily often differentiable if $p > 1$.
If $p > 1$, the first derivative vanish at the data points, i.e., the interpolant has flat spots.

Centers-Test point Plot

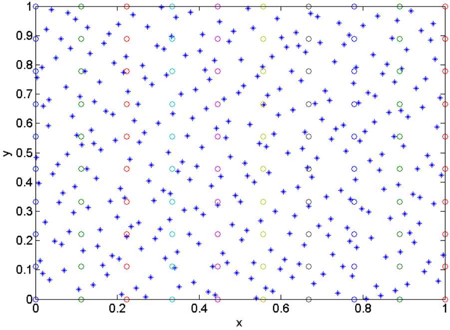

Figure: 300 halton points as centers are depicted by blue dots and 10×10 mesh grid as test points are depicted by different colors of circle

Shepard-Rbf

$$W_K(x) = \begin{cases} \frac{1}{\sqrt{r^2+c^2}} \\ \sqrt{r^2+c^2} \\ \exp(-cr^2) \end{cases}$$

where, c is a shape parameter and $r = \|x - x_k\|_2$

Modified Shepard Method

$$\tilde{f}(x) = \frac{\sum_{k=1}^{n} W_k(x) P_k}{\sum_{k=1}^{n} W_k(x)} \tag{3}$$

Where, $P_k(x)$ is a local approximant to the function $f(x)$ centered at $x^{(k)}$, with the property that $P_k(x^k) = f_k$. W_k are the weight functions

5000 halton points

Franke function reconstruction

weight function	c	Maximum Error	RMSE
$\frac{1}{r^c}$	3.8	4.2701931e-002	4.4337975e-003
$\exp(-cr^2)$	49.9	3.6801260e-002	3.7288021e-003
$\frac{1}{\sqrt{r^2+c^2}}$	0.1	6.2239628e-001	2.0765431e-001
$\sqrt{r^2+c^2}$	49.9	7.9426843e-001	2.8801391e-001

Table: Comparision of some shepard-rbf Error, parameter search within $[0.1, 50]$

5000 halton points

weight function	Elapsed time
$\frac{1}{r^c}$	20.027614
$\exp(-cr^2)$	10.126375
$\frac{1}{\sqrt{r^2+c^2}}$	8.4988990
$\sqrt{r^2+c^2}$	9.2344572

Table: Comparision of some shepard-rbf Elapsed time , parameter search within $[0.1, 50]$

9000 halton points

weight function	c	Maximum Error	RMSE
$\frac{1}{r^c}$	3.8	2.1755331e-002	3.1956403e-003
$\exp(-cr^2)$	49.9	3.0581758e-002	3.5173726e-003
$\frac{1}{\sqrt{r^2+c^2}}$	0.1	6.2250591e-001	2.0762788e-001
$\sqrt{r^2+c^2}$	49.9	7.9430642e-001	2.8801364e-001

Table: Comparision of some shepard-rbf Error with optimum shape parameter search interval $[0.1, 50]$

9000 halton points

weight function	Elapsed time
$\frac{1}{r^c}$	36.263805
$\exp(-cr^2)$	17.916682
$\frac{1}{\sqrt{r^2+c^2}}$	16.196735
$\sqrt{r^2+c^2}$	14.032800

Table: Comparision of some shepard-rbf Elapsed time

$W_k(x) = \exp(-cr^2)$

centers	c	Maximum Error	RMSE
5000	70.64	3.5939227e-002	3.2760768e-003
9000	97.52	1.8866434e-002	2.4207751e-003

centers	Elapsed time
5000	5.5022027
9000	9.0705696

Table: shepard-rbf error at optimum shape parameter search interval $[0.1, 2000]$

Error Plot

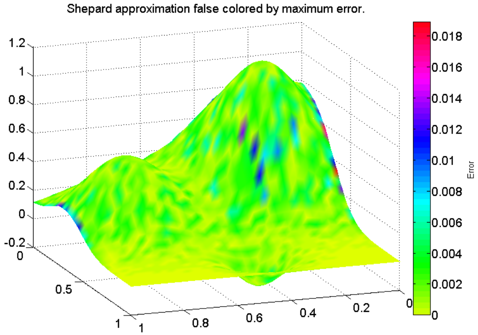

Figure: 9000 halton points as centers and 40×40 mesh grid plot

Error Plot

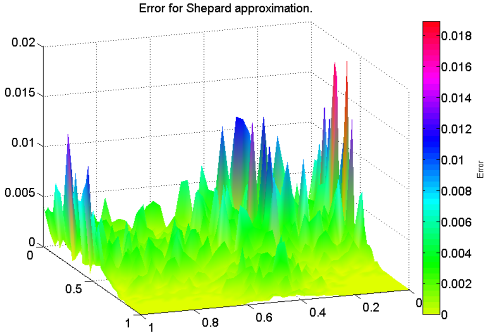

Figure: maximum Error plot with 9000 halton points as centers and 40×40 mesh grid

Shepard 3D

Franke function

centers	10,000 halton
testpoint	300 random
compact support	$\frac{1}{1.1675979e{+}001}$
Maximum Error	6.3931220e-002
RMSE	6.3358452e-003
Elapsed time	56.718323

Table: $W_k(x) = \phi(r) = (1-r)_+^6(35r^2+18r+3)$, compact support search interval $[0.001, 50]$

Shepard 3D

$$f(x,y,z) = 64x(1-x)y(1-y)z(1-z)$$

centers	10,000 halton
testpoint	300 random
compact support	$\frac{1}{1.3961674e+001}$
Maximum Error	5.0674751e-002
RMSE	8.4752704e-003
Elapsed time	61.992737

Table: $W_k(x) = \phi(r) = (1-r)_+^6(35r^2+18r+3)$, shape parameter search interval $[0.001, 50]$

Leave one out cross validation(LOOCV)

formula given by Rippa

$$e_k = \frac{c_k}{A^{-1}_{kk}};$$

where, e_k is the Error vector to be minimized.
c_k is the k^{th} coefficient in the expansion of the approximation.
A^{-1}_{kk} is the k^{th} diagonal element of the inverse of the corresponding interpolation matrix.

Gaussian rbf by cross validation

Franke function

N	shape parameter	Maximum Error	RMSE
343	6.0965519	3.4318758e-002	7.7272934e-004
1000	6.5627591	7.3487096e-005	1.0522727e-006
1200	6.2563583	9.2088373e-005	1.3946066e-006

Table: Gaussian rbf error with cross validation search for shape parameter in $[1, 50]$

Gaussian rbf by cross validation

N	time for cross validation	evaluation time
343	2.4576790	0.30508195
1000	39.385719	1.0710304
1200	96.475370	1.1619227

Table: Time for Gaussian rbf evaluation process with cross validation search for shape parameter in $[1, 50]$

Advantage of shepard

- We can work with huge data.
- It takes few amount of time than global rbf.
- Works for higher dimension
- No ill condition problem

Disadvantage of shepard

- relatively high error compared with global rbf using cross validation
- smothness problem

Conclusion

So, if we have small amount of data and we need high accuracy, Global rbf with cross validation is the best method for above mentioned examples. But, we should aware with the execution time for the cross validation.

References

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