Kansa's Method (RBF Collocation Method)

Consider the Poisson's equation

$$\Delta u = f(x,y), \qquad (x,y) \in \Omega, \tag{1}$$
$$u = g(x,y), \qquad (x,y) \in \partial\Omega, \tag{2}$$

where f and g are known functions. Assume the solution of above Poisson problem $u(x,y)$ can be approximated by the linear combination of radial basis function (MQ); i.e.,

$$u(x,y) \simeq \hat{u}(x,y) = \sum_{i=1}^{n} a_i \sqrt{r^2 + c^2} \tag{3}$$

where $r = \sqrt{(x-x_i)^2 + (y-y_i)^2}$ is the Euclidean distance between the center (x,y) and (x_i,y_i), and c is a free parameter. The determination of c is still an outstanding research topic. We choose n collocation points $\{(x_i,y_i)\}_1^n$ which contains n_i interior points $\{(x_i,y_i)\}_1^{n_i}$ and n_b boundary points $\{(x_i,y_i)\}_{n_i+1}^{n_i+n_b}$. Note that $n = n_i + n_b$. Observe that

$$\frac{d}{dx}\sqrt{(x-x_i)^2 + (y-y_i)^2 + c^2} = \frac{x-x_i}{\sqrt{(x-x_i)^2 + (y-y_i)^2 + c^2}} \tag{4}$$

$$\begin{aligned}\frac{d^2}{dx^2}\sqrt{(x-x_i)^2 + (y-y_i)^2 + c^2} &= \frac{d}{dx}\left(\frac{x-x_i}{\sqrt{(x-x_i)^2 + (y-y_i)^2 + c^2}}\right) \\ &= \frac{(y-y_i)^2 + c^2}{\left(\sqrt{(x-x_i)^2 + (y-y_i)^2 + c^2}\right)^3}\end{aligned} \tag{5}$$

Similarly,

$$\frac{d^2}{dy^2}\sqrt{(x-x_i)^2 + (y-y_i)^2 + c^2} = \frac{(x-x_i)^2 + c^2}{\left(\sqrt{(x-x_i)^2 + (y-y_i)^2 + c^2}\right)^3} \tag{6}$$

Substituting (5) and (6) into (1), we have

$$\sum_{i=1}^{n} a_i \frac{(y-y_i)^2 + c^2}{\left(\sqrt{(x-x_i)^2 + (y-y_i)^2 + c^2}\right)^3} + \sum_{i=1}^{n} a_i \frac{(x-x_i)^2 + c^2}{\left(\sqrt{(x-x_i)^2 + (y-y_i)^2 + c^2}\right)^3} = f(x,y)$$

which means

$$\sum_{i=1}^{n} a_i \frac{(x-x_i)^2 + (y-y_i)^2 + 2c^2}{\left(\sqrt{(x-x_i)^2 + (y-y_i)^2 + c^2}\right)^3} = f(x,y),$$

i.e.,

$$\sum_{i=1}^{n} a_i \frac{r^2 + 2c^2}{(r^2 + c^2)^{3/2}} = f(x,y), \quad (x,y) \in \Omega.$$

By Collocation method, we obtain n_i equations from the interior points:

$$\sum_{i=1}^{n} a_i \frac{(x_j-x_i)^2 + (y_j-y_i)^2 + 2c^2}{((x_j-x_i)^2 + (y_j-y_i)^2 + c^2)^{3/2}} = f(x_j,y_j), \qquad j = 1,2,...,n_i. \tag{7}$$

We choose n_b boundary points $\{(x_i,y_i)\}_{n_i+1}^{n_i+n_b}$. From (3), we obtain

$$\sum_{i=1}^{n} a_i \sqrt{(x_j-x_i)^2 + (y_j-y_i)^2 + c^2} = g(x_j,y_j), \qquad j = n_i+1,...,n. \tag{8}$$

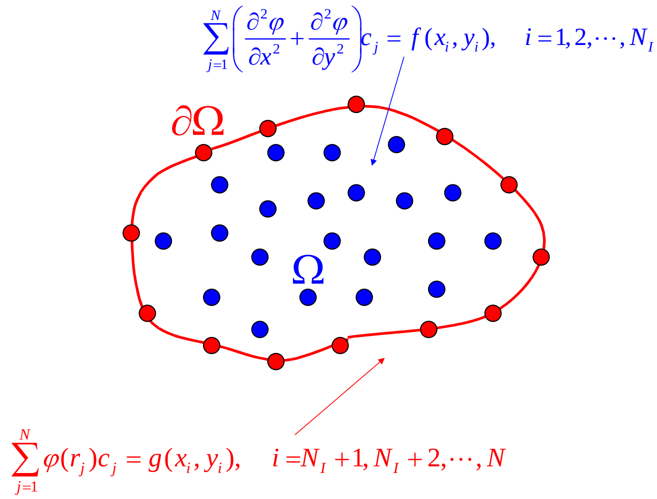

Figure 1: Kansa's method

(7)-(8) can be formulated in matrix form as follows:

$$\begin{bmatrix} \Psi \\ \phi \end{bmatrix} \mathbf{a} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$$

where Ψ and ϕ are two block matrices whose entries are

$$\begin{aligned} \Psi_{ij} &= \frac{(x_j - x_i)^2 + (y_j - y_i)^2 + 2c^2}{((x_j - x_i)^2 + (y_j - y_i)^2 + c^2)^{3/2}}, \qquad 1 \le j \le n_i, 1 \le i \le n, \\ \phi_{ij} &= \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + c^2}, \qquad 1 \le j \le n_b, 1 \le i \le n, \end{aligned}$$

and $\mathbf{f} = [f(x_1), f(x_2), ..., f(x_{n_i})]^T, \mathbf{g} = [g(x_{n_i+1}), g(x_{n_i+2}), ..., g(x_n)]^T$. Solving $n \times n$ system of equations (7)-(8), we obtain $\{a_i\}_1^n$. Then the approximate solution $\hat{u}(x, y)$ of the given Poisson equation can be approximated by (3).

For Neumann boundary condition, we have

$$\begin{aligned} \frac{\partial \hat{u}}{\partial n} &= \sum_{i=1}^{n} a_i \frac{\partial}{\partial n} \sqrt{r^2 + c^2} \\ &= \sum_{i=1}^{n} a_i \left(\frac{\partial}{\partial x} \sqrt{r^2 + c^2} \frac{\partial x}{\partial n} + \frac{\partial}{\partial y} \sqrt{r^2 + c^2} \frac{\partial y}{\partial n} \right) \\ &= \sum_{i=1}^{n} a_i \left(\frac{x - x_i}{\sqrt{r^2 + c^2}} \frac{\partial x}{\partial n} + \frac{y - y_i}{\sqrt{r^2 + c^2}} \frac{\partial y}{\partial n} \right), \quad (x, y) \in \partial\Omega \end{aligned}$$

The above mentioned method is simple but the challenge is how to find the optimal shape parameter c.