Question 23. (a) $E(X) = 0P(0) + 1P(1) + 2P(2) = \frac{1}{5} + \frac{1}{10} + \frac{1}{30} + 2(\frac{3}{15}) = \frac{11}{15}$

(b) Z can take on these values: $-2, -1, 0, 1, 2$. Thus,

$$E(Z) = -2P(Z=-2) - 1P(Z=-1) + 0P(Z=0) + 1P(Z=1) + 2P(Z=2) = -2(\frac{1}{15}) - (\frac{1}{5}) + \frac{1}{30} + 2\frac{1}{15} = \frac{-1}{6}$$

Z^2 can take on these values: $0, 1, 4$. Thus,

$$E(Z^2) = 0P(Z^2=0) + 1P(Z^2=1) + 4P(Z^2=4) = (\frac{1}{5} + \frac{1}{30}) + 4(\frac{1}{15} + \frac{1}{15}) = \frac{23}{30}$$

Therefore,

$$\text{Var}(Z) = E(Z^2) - E(Z)^2 = \frac{23}{30} - (\frac{-1}{6})^2 = \frac{133}{180}$$

(c) Let $a = P(X=0, Y=0), b = P(X=0, Y=1)$.

$$\begin{aligned}
\text{Cov}(X,Y) &= E(XY) - E(X)E(Y) \\
&= \frac{-1}{6} - \frac{11}{15}E(Y) \\
&= \frac{-2}{45} \\
\Rightarrow E(Y) &= -1P(Y=-1) + 0P(Y=0) + 1P(Y=1) \\
&= -(\frac{2}{5} + \frac{1}{15}) + b \\
&= -\frac{-1}{6} \\
\Rightarrow b &= \frac{19}{30} \\
\Rightarrow a &= \frac{4}{15}
\end{aligned}$$

Since all the probabilities must add up to 1.

Question 24.

Question 25. (a) A student takes at least 10 trials to be correct for the first time iff they failed 9 times in a row from the beginning. Thus, we calculate the probability $P(\text{"failure"})^9 = 0.4^9 = \frac{512}{1953125} = 0.000262144$.

(b) Since the trials are independent, the first three trials have no impact on any subsequent trials. Therefore the student takes at least 13 trials to be correct given they fail the first three trials iff trials number 4-12 are all failures. Thus, we calculate the probability $P(\text{"failure"})^9 = 0.4^9 = \frac{512}{1953125} = 0.000262144$, same as part (a).

Question 26.

Question 27.

Question 28.

Question 29.

Question 30. First, let $X_j = \sum_{k=1}^{n_j} X_{j_k}$ for Bernoulli trials $X_{j_1}, \ldots, X_{j_{n_j}} \sim B(1,p)$. So, the characteristic function of $X_j \sim B(n_j, p)$ is

$$\begin{aligned}
\phi_{X_j}(t) &= E(e^{it\sum_{k=1}^{n_j} X_{j_k}}) \\
&= E(\prod_{k=1}^{n_j} e^{itX_{j_k}}) \\
&= \prod_{k=1}^{n_j} E(e^{itX_{j_k}}) \\
&= (E(e^{itX_{j_1}}))^{n_j} \\
&= (P(X_{j_1} = 1)e^{1it} + P(X_{j_1} = 0)e^{0it})^{n_j} \\
&= (pe^{it} + 1 - p)^{n_j}
\end{aligned}$$

Let $Y \sim B(\sum_{j=1}^{d} n_j, p)$. Then,

$$\begin{aligned}
\phi_{\sum_{j=1}^{d} X_j}(t) &= \prod_{j=1}^{d} \phi_{X_j}(t) \\
&= \prod_{j=1}^{d} (pe^{it} + 1 - p)^{n_j})^{n_j} \\
&= (pe^{it} + 1 - p)^{\sum_{j=1}^{d} n_j} \\
&= \phi_Y(t)
\end{aligned}$$

So by uniqueness, $\sum_{j=1}^{d} X_j \sim B(\sum_{j=1}^{d} n_j, p)$.

Question 31.