Solving PDEs with Denoising Diffusion Restoration Models
Application to the Poisson Equation

Marty Mukherjee [*1] **Melissa Stadt** [*1] **Jun Liu** [1] **Anita Layton** [1]

Abstract

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1. Introduction

2. Background

3. Theoretical Findings

3.1. Distribution of u

Assumption 3.1. f is Lipschitz in the domain Ω with Lipschitz constant L_f.

Consider the following blind inverse problem

$$f = \Delta u + z \tag{1}$$

where $z \sim N(0, \sigma_f^2 L_f^2)$ where σ_f is a known quantity and L_f is the Lipschitz constant of f.

Theorem 3.2. *If $f = \Delta u + z$ where $z \sim N(0, \sigma_f^2 L_f^2)$, then*

$$u(x, y) \sim N(u_0, \sigma_f^2 L_f^2 K(x, y)), \tag{2}$$

where

$$K(x, y) = \iint_\Omega (\psi((x', y') - (x, y)))^2 dx' dy', \tag{3}$$

where $\psi(\cdot)$ is the Green's function in two dimensions

$$\psi(x, y) = \frac{\ln(\|(x, y)\|)}{2\pi} \tag{4}$$

[*]Equal contribution [1]Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada. Correspondence to: Marty Mukherjee <a29mukhe@uwaterloo.ca>, Melissa Stadt <mstadt@uwaterloo.ca>.

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3.2. Eigenvalues and Eigenfunctions of the Laplacian operator

Theorem 3.3. *The eigenfunction and eigenvalue pairs of the Laplacian operator Δ in a domain $\Omega = [0, 1]^2$ subject to the boundary conditions $u = 0$ on $\partial\Omega$ are of the form*

$$u_{n,m}(x, y) = \sin(n\pi x)\sin(m\pi y) \tag{5}$$

$$\lambda_{n,m} = -(n\pi)^2 - (m\pi)^2 \tag{6}$$

4. DDRM for solving PDEs

Sampling of x_T.

We consider the fast Fourier transform (FFT) of x_t and f and perform the diffusion in its spectral space. Define $\bar{x}_t^{(n,m)}$ and $\bar{f}^{(n,m)}$ as follows:

$$\bar{x}_t^{(n,m)} = \langle x_t, \sin(n\pi x)\sin(m\pi y)\rangle, \tag{7}$$

$$\bar{f}^{(n,m)} = \langle f, \sin(n\pi x)\sin(m\pi y)\rangle / \lambda_{n,m}. \tag{8}$$

Since none of the eigenvalues $\lambda_{m,n}$ are zero, our DDRM method for sampling x_T is as follows:

$$p_\theta^{(T)}(\overline{\mathbf{x}}_t^{(n,m)} \mid \mathbf{f}) = \mathcal{N}(\overline{\mathbf{f}}^{(n,m)}, \sigma_T^2 - L_f^2\sigma_y^2/\lambda_{n,m}^2) \tag{9}$$

Choose $\sigma_T > \sigma_y/(\pi\sqrt{2})$.

Sampling of x_t.

$$p_\theta^{(t)}\left(\overline{\mathbf{x}}_t^{(n,m)} \mid \mathbf{x}_{t+1}, \mathbf{f}\right) = \tag{10}$$

$$\begin{cases} \mathcal{N}\left(\overline{\mathbf{x}}_{\theta,t}^{(n,m)} + \sqrt{1-\eta^2}\sigma_t \frac{\overline{\mathbf{f}}^{(n,m)} - \overline{\mathbf{x}}_{\theta,t}^{(n,m)}}{\sigma_{\mathbf{f}}/s_i}, \eta^2\sigma_t^2\right), & \sigma_t < \frac{\sigma_{\mathbf{f}}}{s_i} \\ \mathcal{N}\left((1-\eta_b)\,\overline{\mathbf{x}}_{\theta,t}^{(n,m)} + \eta_b\overline{\mathbf{f}}^{(n,m)}, \sigma_t^2 - \frac{\sigma_{\mathbf{f}}^2}{s_i^2}\eta_b^2\right), & \sigma_t \geq \frac{\sigma_{\mathbf{f}}}{s_i} \end{cases} \tag{11}$$

5. Numerical Results

6. Conclusion

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References

Gilbarg, D. and Trudinger, N. S. *Elliptic Partial Differential Equations of Second Order, Second Edition*. Springer-Verlag, 1983.

A. Proof of theorem 3.2

Proof. (Gilbarg & Trudinger, 1983) In a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, any function $u \in C^2(\bar{\Omega})$ satisfies

$$\begin{aligned} u(x,y) = &\iint_\Omega \psi((x',y')-(x,y))\Delta u(x',y')dx'dy' \\ &+ \int_{\partial\Omega} u(x',y')\frac{\partial \psi}{\partial n}((x',y')-(x,y))dS_{x',y'} \\ &- \int_{\partial\Omega} \psi((x',y')-(x,y))\frac{\partial u}{\partial n}(x',y')dS_{x',y'}. \end{aligned}$$

Given that $\Delta u = f + z$ with $z \sim N(0, \sigma_y^2 L_f^2)$, the first integrand can be written as

$$\begin{aligned} &\iint_\Omega \psi((x',y')-(x,y))\Delta u(x',y')dx'dy' \\ &= \iint_\Omega \psi((x',y')-(x,y))f dx'dy' \\ &+ \iint_\Omega \psi((x',y')-(x,y))z dx'dy'. \end{aligned}$$

And so we have: $E[u(x,y)] = u_0$ where $\Delta u_0 = f$. And $Var[u(x,y)] = \sigma_f^2 L_f^2 K(x,y)$ where

$$K(x,y) = \iint_\Omega (\psi((x',y')-(x,y)))^2 dx'dy',$$

□

B. Proof of theorem 3.3

Proof. To find the eigenvalues and eigenfunctions of the Laplacian operator, we are essentially solving the following PDE

$$\Delta u = \lambda u$$

We use separation of parts to split u as $u(x,y) = X(x)Y(y)$.

$$\Delta u = X''(x)Y(y) + X(x)Y''(y) = \lambda X(x)Y(y)$$

$$\frac{X''}{X} + \frac{Y''}{Y} = \lambda$$

Here, we solve two eigenvalue problems, $X''/X = \lambda_1$ and $Y''/Y = \lambda_2$, which means $\lambda = \lambda_1 + \lambda_2$. Consider the boundary value problem for X.

$$\frac{X''}{X} = \lambda_1, X(0) = X(1) = 0$$

The eigenvalue and eigenfunction pairs are trivially of the form

$$X_n(x) = \sin(n\pi x), \lambda_{1,n} = -(n\pi)^2$$

By symmetry, we have

$$Y_m(y) = \sin(m\pi y), \lambda_{2,m} = -(m\pi)^2$$

Thus, the eigenvalue and eigenfunction pairs of the Laplacian operator are of the form

$$\begin{aligned} &u_{n,m}(x,y) = \sin(n\pi x)\sin(m\pi y) \\ &\lambda_{n,m} = -(n\pi)^2 - (m\pi)^2 \end{aligned}$$

□