

# Paper Title

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**Abstract:** *The abstract must summarise the main content of the paper, and may be up to approximately 100 words in length. Unlike previous OSME publications, the abstract will be included in the proceedings. This abstract needs not be the same as that submitted for acceptance to the conference.*

## 1 Introduction

### Outline

## 2 Extruded Surfaces (Preliminaries)

## 3 Cross Sections

We introduce a new method of origami construction, using cross section diagrams. Instead of beginning our construction from a 2-dimensional sheet of paper, we consider a 1-dimensional cross section moving forwards in time. A simple example is demonstrated in Figure 2.

### Conservation of length

**Definition 1.** A cross section is defined as an ordered list of line segments  $\langle s_1, s_2, \dots, s_n \rangle$ , such that for every segment  $s_i$  (except the last), the right endpoint of  $s_i$  coincides with the left endpoint of  $s_{i+1}$ . Each line segment is associated with a velocity vector  $\mathbf{v}_i$  of unit magnitude. Each segment is also associated with an unit orientation vector  $\mathbf{o}_i$ , which is the unit vector from the left endpoint to the right endpoint.

**Lemma 2.** All points on a single segment move with the same velocity  $\mathbf{v}_i$ .

### 3.1 Joints

**Definition 3.** A cross section with  $n$  segments is also associated with a list of joints  $\langle J_1, \dots, J_{n-1} \rangle$ , where  $J_i$  corresponds to the right endpoint of  $s_i$ . The velocity of a joint  $J_i$  is  $\mathbf{J}_i = \mathbf{v}_i + \mathbf{v}_{i+1}$  (i.e. the sum of the velocities of the corresponding segments).

**Definition 4.** Note that the trajectory of the joints form a crease ... angle is angle between direction vectors. Creases are also created when a segment changes direction. ... angle is change in direction vector.

**Definition 5.** A joint plane is the plane that coincides with both segments  $l$  and  $r$  associated with a particular joint  $J$ .

**Definition 6.** Consider a joint  $J$  associated with segments  $l$  and  $r$ , and joint plane  $\mathcal{P}$ , where  $\mathbf{v}_l$  and  $\mathbf{v}_r$  are the velocities of segments  $l$  and  $r$ . We define  $\mathbf{v}_l^{\parallel}$  and  $\mathbf{v}_l^{\perp}$  as the components of  $\mathbf{v}_l$  coinciding with, and orthogonal to the joint plane respectively. Similarly, we define  $\mathbf{v}_r^{\parallel}$  and  $\mathbf{v}_r^{\perp}$ , as the components of  $\mathbf{v}_r$ . Further, note that  $\mathbf{v}_l^{\parallel}$  and  $\mathbf{v}_r^{\parallel}$  have to be orthogonal to  $\mathbf{o}_l$  and  $\mathbf{o}_r$  respectively.

**Claim 7.** For a joint  $J$  associated with segments  $l$  and  $r$ ,  $\mathbf{v}_l^{\perp} = \mathbf{v}_r^{\perp}$ .

**Corollary 8.** For a joint  $J$  associated with segments  $l$  and  $r$ ,  $\|\mathbf{v}_l^{\parallel}\| = \|\mathbf{v}_r^{\parallel}\|$ .

**Claim 9.** A joint  $J$  corresponding to segments  $s$  and  $t$  is valid if the resulting evolution of the joint preserves distances.

### 3.2 Time Travel

In the process of time travel, the lengths of segments may change. This can also be visualized as movement of the corresponding joint along one of the segments. The movement of a joint increases the length of one of its associated segments, and decreases the length of the other segment by the same amount (this ensures that the total length is preserved). Most velocity sequences are invalid. For example...

We will use some abuse of notation, by using  $\mathbf{v}_i, \mathbf{o}_i$  etc. to refer to the corresponding parameters of  $s_i$ .

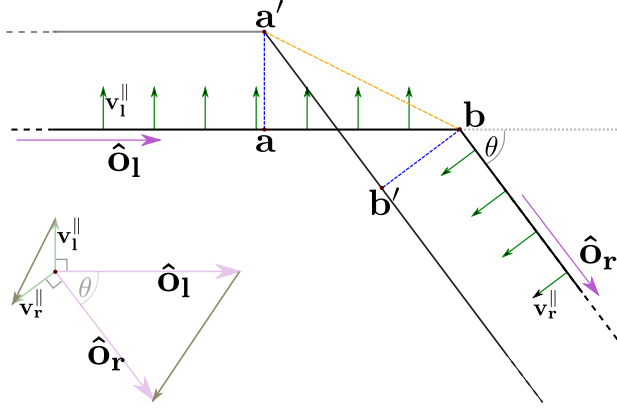
**Definition 10.** For ever segment  $s$ , we associate a left velocity  $L_s$ , which indicates the rate at which  $s$  grows from its left endpoint. Similarly, we define a right velocity  $R_s \dots$

**Claim 11.** The right velocity of  $s_i$  is the negation of the left velocity of  $s_{i+1}$  i.e.  $R_i = -L_{i+1}$ . Furthermore, the left velocity of the first segment, and the right velocity of the last segment should be zero, i.e.  $L_0 = R_n = 0$  This ensures that the total length of the cross section does not change.

#### Define Nodes

Consider a joint  $\mathbf{J}_i$  corresponding to segments  $l = s_i$  and  $r = s_{i+1}$ , at time  $t = 0$ . Henceforth, we will refer to  $L_l$  as  $L$ , and  $R_r$  as  $R$ . Without loss of generality, we assume that  $L < 0$ . At a later time  $t$ , let the new joint position be  $\mathbf{J}'_i$ . We define nodes  $a$  and  $b$  corresponding to  $\mathbf{J}_i$  and  $\mathbf{J}_{i+1}$  respectively. We also define the initial and final positions of  $a$  as  $\mathbf{a}$ , and  $\mathbf{a}'$ , and similarly for  $b$ , we define  $\mathbf{b}$  and  $\mathbf{b}'$ . Let  $d$  be the separation between nodes  $a$  and  $b$ . this setup is shown in Figure 1

First, note that  $\mathbf{a}, \mathbf{b}$  lie on the segment  $l$ , and  $\mathbf{a}', \mathbf{b}'$  lie on the segment  $r$ , which



**Figure 1:** A joint with segments  $l$  and  $r$ . The trajectory of the joint is shown in orange. The trajectories of  $a$  and  $b$  are shown in blue. The green arrows indicate  $\mathbf{v}^{\parallel}$

implies that  $\mathbf{b} - \mathbf{a} = d \cdot \hat{\mathbf{o}}_l$ , and  $\mathbf{b}' - \mathbf{a}' = d \cdot \hat{\mathbf{o}}_r$ .

$$\mathbf{b}' - \mathbf{a}' = (\mathbf{b} + t \cdot \mathbf{v}_r^{\parallel}) - (\mathbf{a} + t \cdot \mathbf{v}_l^{\parallel}) \quad (1)$$

$$\implies \mathbf{b}' - \mathbf{a}' = (\mathbf{b} - \mathbf{a}) + t \cdot (\mathbf{v}_r^{\parallel} - \mathbf{v}_l^{\parallel}) \quad (2)$$

$$\implies d \cdot \hat{\mathbf{o}}_r = d \cdot \hat{\mathbf{o}}_l + t \cdot (\mathbf{v}_r^{\parallel} - \mathbf{v}_l^{\parallel}) \quad (3)$$

$$\implies \mathbf{v}_r^{\parallel} - \mathbf{v}_l^{\parallel} = \frac{d}{t} \cdot (\hat{\mathbf{o}}_r - \hat{\mathbf{o}}_l) \quad (4)$$

$$= R \cdot (\hat{\mathbf{o}}_r - \hat{\mathbf{o}}_l) = -L \cdot (\hat{\mathbf{o}}_r - \hat{\mathbf{o}}_l) \quad (5)$$

This implies that  $\hat{\mathbf{o}}_l \times \hat{\mathbf{o}}_r$  is oriented opposite to  $\mathbf{v}_l^{\parallel} \times \mathbf{v}_r^{\parallel}$ . If the angle between the segments (between  $\hat{\mathbf{o}}_l$  and  $\hat{\mathbf{o}}_r$ ) is  $\theta$ , the magnitude of  $\hat{\mathbf{o}}_r - \hat{\mathbf{o}}_l$  is  $\sqrt{2 - 2\cos(\theta)}$ , and  $|\mathbf{v}_r^{\parallel} - \mathbf{v}_l^{\parallel}| = v \cdot \sqrt{2 - 2\cos(\pi - \theta)}$ . Here,  $v$  is magnitude of the plane velocity of  $J_i$ .

$$R = \frac{d}{t} = \frac{|\mathbf{v}_r^{\parallel} - \mathbf{v}_l^{\parallel}|}{|\hat{\mathbf{o}}_r - \hat{\mathbf{o}}_l|} \quad (6)$$

$$= v \cdot \frac{\sqrt{\sin^2(\pi/2 - \theta/2)}}{\sqrt{\sin^2 \theta/2}} \quad (7)$$

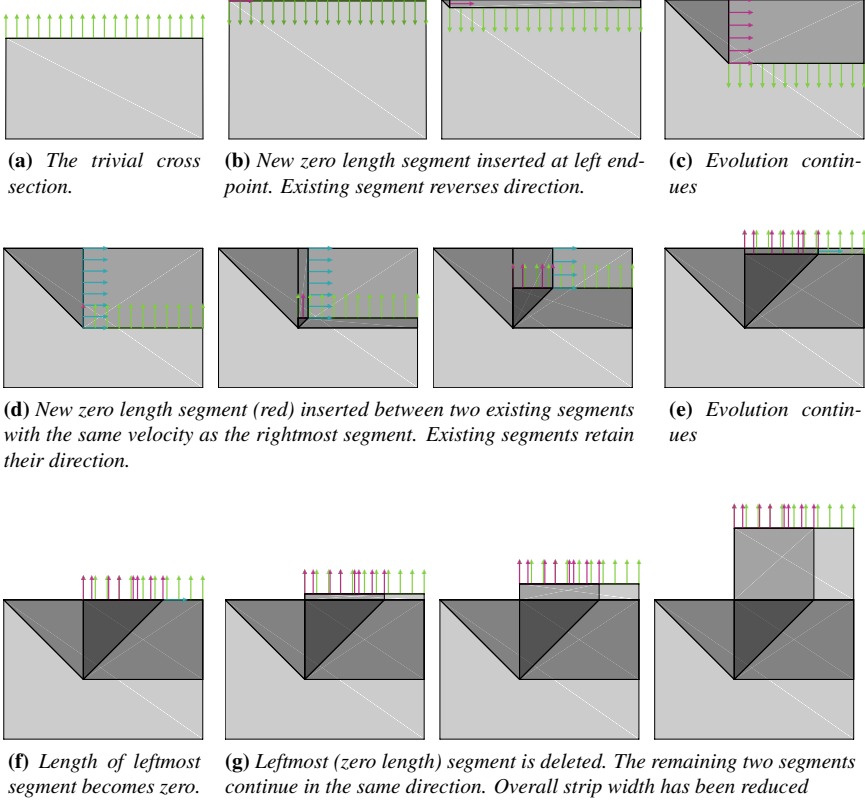
$$= v \cdot \cot\left(\frac{\theta}{2}\right) \quad (8)$$

### Surface traced out by cross section

This can result in a segment length becoming zero then delete. We may create a new segment at any point of zero length.

Note that time evolution is reversible.

### 3.3 Multiple Cross Sections



**Figure 2:** Cross Section evolution of a strip narrowing gadget.

*cite*

**Definition 12.** We consider a cross section  $C$  composed of segments  $\langle s_1, s_2, \dots, s_n \rangle$  with total length  $X$  (i.e.  $\sum |s_i| = X$ ). If we allow this cross section to evolve for time  $T$ , we obtain a cross section interval  $\mathcal{C}$  of length  $T$ . Let  $\mathcal{C}^I$  denote the initial cross section  $C = \langle s_1, s_2, \dots, s_m \rangle$ , and  $\mathcal{C}^F$  denote the final cross section  $\langle r_1, r_2, \dots, r_m \rangle$ .

**Definition 13.** Given two cross section intervals  $\mathcal{C}$  and  $\mathcal{D}$ , such that  $\mathcal{C}^F$  and  $\mathcal{D}^I$  are equivalent, we say that  $\mathcal{D}$  is next-compatible with  $\mathcal{C}$  and  $\mathcal{C}$  is previous-compatible with  $\mathcal{D}$ . Two cross sections  $C = \langle s_1, s_2, \dots, s_n \rangle$  and  $D = \langle r_1, r_2, \dots, r_m \rangle$  are equivalent if and only if  $C$  and  $D$  correspond to the same sequence of segments after the deletion of all zero length segments.

**Definition 14.** A cross section sequence is a sequence is an ordered list of cross section intervals  $\langle \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n \rangle$ , such that  $C_i$  is next-compatible with  $C_{i-1}$  for all

$i \in [n-1]$ . This is equivalent to stating that  $C_i$  is previous-compatible with  $C_{i+1}$  for all  $i \in [n-1]$ . Note that we do not care about the directions of the segments.

We will represent our full construction as a valid *cross section sequence*. Given a cross section sequence  $\langle \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n \rangle$ , the transition from  $\mathcal{C}_i$  to  $\mathcal{C}_{i+1}$  corresponds to the deletion of one or more length zero segments from  $\mathcal{C}_i$ , and the addition of one or more zero length segments to obtain  $\mathcal{C}_{i+1}$ . One simple example is shown in Figure 2.

### 3.4 Evolution Corresponds to Flat Paper

In this section we will demonstrate that the folding formed by cross section evolution is realizable from a sheet of flat paper. We note here that our construction may still result in self intersections.

We consider a cross section sequence  $\langle \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n \rangle$ , where each cross section interval  $\mathcal{C}_i$  has evolution time  $T_i$ . Say that the length of each cross section in the sequence is  $X$ . We will denote a strip of paper of size  $X \times L$  as an  $L$ -strip.

First, we will show that each  $\mathcal{C}_i$  corresponds to a  $T_i$ -strip. We will then use Theorem 14 to attach the sequence of  $T_i$ -strips, to form a complete  $X \times T$  sheet of paper, where  $T = \sum T_i$ .

#### 3.4.1 Cross Section Interval Forms a Strip from a Gluing of Trapezoids

In this section, we focus on a single cross section interval  $\mathcal{C}$  with segments  $\langle s_1, s_2, \dots, s_n \rangle$  evolving over time  $T$ . First consider the surface traced out by an individual segment  $s_i$ . Since the endpoints of  $s_i$  move in a straight line, each segment traces a trapezoid.

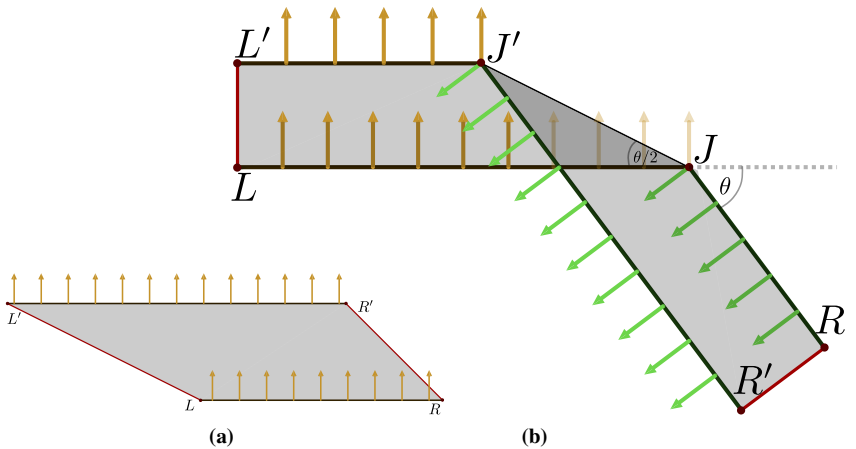


Figure 3

**Definition 15.** *The surface traced out by a segment  $s$  is a trapezoid  $Z_s$  (Figure 3a). Specifically, if  $L, R$  are the initial left and right endpoints of  $s$ , and  $L', R'$  are the final endpoints,  $Z_s = LL'R'R$  is the corresponding trapezoid (Figure 3a).*

### Define gluing

**Lemma 16.** *Consider a joint  $J$  with segments  $l$  and  $r$ , which has zero orthogonal joint velocity (i.e.  $\mathbf{v}_l^\perp = \mathbf{v}_r^\perp = 0$ ). The gluing of trapezoids  $Z_l$  and  $Z_r$  along the joint trajectory  $\mathcal{T}_J$  is isometric to a larger trapezoid. This joint trajectory is actually nothing but a crease in the folded state.*

**Lemma 17.**

**Theorem 18.** *The resulting gluing from any given cross section interval  $\mathcal{C}$  with evolution time  $T$  is isometric to a  $T$ -strip.*

## 3.4.2 Gluing the Interval Strips Together

## 4 Orthogonal Graphs

In this section, we outline a construction of orthogonal graphs with arbitrary rational extrusion heights. In our construction, the cross section at will always be on the  $x-z$  plane. This makes the analysis much simpler.

### 4.1 Grid Extrusion

To simplify the presentation, we will consider an uniform  $X-Y$  grid, with arbitrary rational extrusion heights corresponding to every grid square.

**Definition 19.** *An  $n \times m$  rational grid extrusion is a 3-dimensional structure, whose projection onto the  $x-y$  plane forms an unit grid of size  $n \times m$ . Additionally, the unit face corresponding to the location  $(i, j)$ , is extruded  $E_{ij}$  units in the  $z$  direction, where  $E_{ij}$  is a rational number.*

#### 4.1.1 Strip Extrusion

We also consider each "column" of a given grid extrusion separately as an individual *strip extrusion*. We will construct each of the  $n$  strips independently, and attach them together with *strip connectors*.

## 4.2 Optimality

Under some suitable assumptions, our construction can be made  $2 + \varepsilon$  optimal, for arbitrarily small  $\varepsilon$ .

We define the maximum deltas along the  $y$ -axis as  $D_j = \max_i \|E_{i,j} - E_{i,j+1}\|$ , and let  $Y = n + \sum_{i=1}^{n-1} D_j$ . We also define the lowest and highest points along  $x$ -axis as  $L_i = \max_j E_{ij}$  and  $H_i = \max_j E_{ij}$ , and let

$$X = n + \sum_{i=1}^n [(H_i - \min(L_i, L_{i+1})) + (H_i - \min(L_i, L_{i-1}))]$$

The terms in the summation account for the total length of all the necessary worst case vertical walls, and the  $n$  is for the top faces.

**Claim 20.** *The x-axis length of the strip of paper required to fold this shape can be made arbitrarily close to  $X$ .*

**Claim 21.** *The y-axis length of the strip of paper required to fold this shape will be exactly  $Y$ .*

First, we pick an  $\varepsilon$ , such that  $2\varepsilon$  divides all extrusion heights. The construction will require paper of size  $X' \times Y$ , where  $X' = X + 2\varepsilon(n - 1)$ .

There are two components to the construction

- $n$  strips parallel to the y-axis. Each of these strips will fold to the corresponding strip in the extruded graph. The total area of these strips will be  $X \times Y$ .
- $n - 1$  Intermediate strips, each of size  $2\varepsilon \times Y$  to connect the main strips together.

The total area is therefore  $X \times Y + (n - 1)2\varepsilon \times Y = X' \times Y$ . This can of course be made arbitrarily close to  $X \times Y$ .

### 4.3 Construction of Grid Extrusion

Given a strip of size  $w \times Y$ , we will consider the evolution of the cross sections along the length  $Y$ . Imagine that there are  $T = \frac{Y}{\varepsilon}$  time steps over which the cross section evolves along the y-axis. Consider the decomposition of  $Y$  as

$$Y = (1 + D_1) + (1 + D_2) + (1 + D_3) + \cdots + (1 + D_{n-1})$$

Here, the times corresponding to

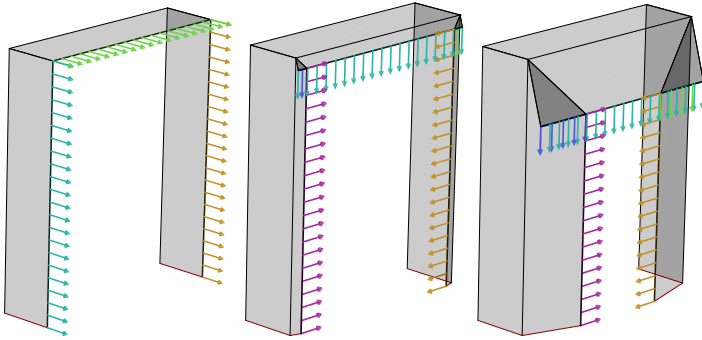
- the  $i^{th}$  "1" is realized as the strip  $i - 1 \leq y \leq i$
- $D_i$  is the transition between  $i - 1 \leq y \leq i$  and  $i \leq y \leq i + 1$

### 4.4 Construction of Strip Extrusion

#### 4.4.1 Level Shifting

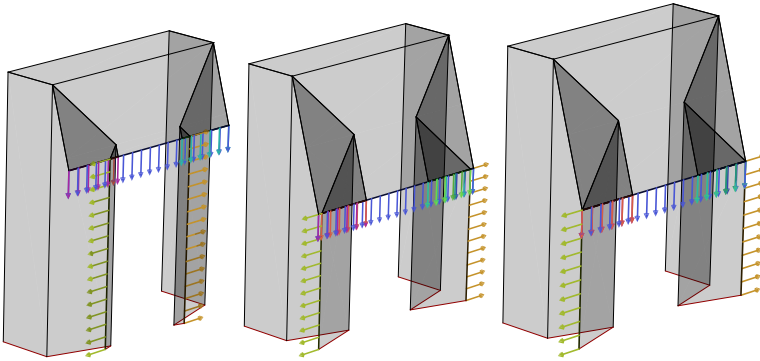
### 4.5 Construction of Strip Connectors

#### 4.5.1 Alignment with Level Shifts

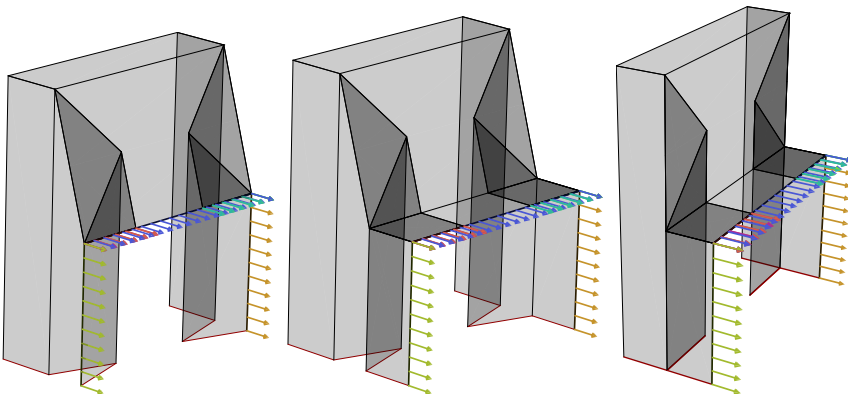


**(a)** Higher initial level. Top segment moves down.

**(b)** Two segments created. New segments are aligned with top segment, and move down. Vertical segments move inwards.



**(c)** Two more segments created. Vertical segments reverse direction.

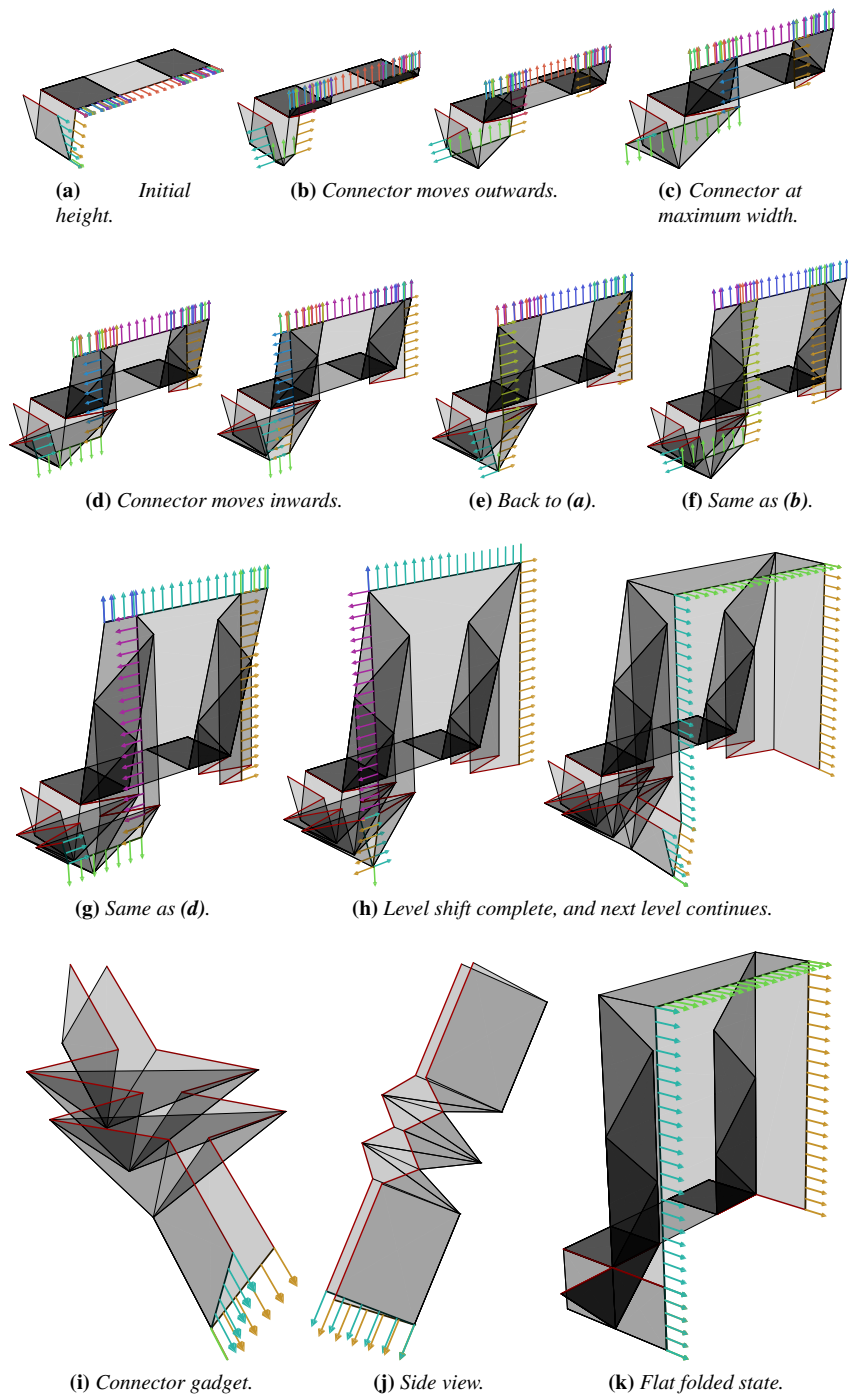


**(d)** Level shift completed with four new horizontal segments.

**(e)** Flat folded state.

**Figure 4:** Level Shifting Gadget. The separation along the  $Y$  direction serves to illustrate the layering.





**Figure 5:** Column gadget attached to a single column connector gadget. The red line demarcates the interface between the two gadgets.

## 5 Polygons

### References

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