Paper Title

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Abstract: *The abstract must summarise the main content of the paper, and may be up to approximately 100 words in length. Unlike previous OSME publications, the abstract will be included in the proceedings. This abstract needs not be the same as that submitted for acceptance to the conference.*

1 Introduction

Outline

2 Extruded Surfaces (Preliminaries)

3 Cross Sections

We introduce a new method of origami construction, using cross section disgrams. Instead of beginning our construction from a 2-dimensional sheet of paper, we consider a 1-dimensional cross section moving forwards in time. A simple example is demonstrated in Figure 4.

Conservation of length

4 Segments and Cross Sections

Definition 1. *A segment s is an oriented line segment with left and right endpoints $\mathbf{s}_l$ and $\mathbf{s}_r$. Each segment is also associated with an orientation vector $\hat{\mathbf{o}}_\mathbf{s} \equiv \frac{\mathbf{s}_r - \mathbf{s}_l}{\|\mathbf{s}_r - \mathbf{s}_l\|}$. This vector serves to disambiguate the orientation of zero length segments.*

Definition 2. *A cross section C is defined as an ordered list of line segments $\langle s_1, s_2, \cdots, s_n \rangle$, such that for every segment s_i (except the last), the right endpoint of s_i coincides with the left endpoint of s_{i+1}. Each segment s is also associated with a velocity vector $\hat{\mathbf{v}}_\mathbf{s}$ of unit magnitude. For a segment $s \in C$ we will also denote this velocity as $\hat{\mathbf{v}}_\mathbf{i}$.*

Definition 3. *Given a cross section $C = \langle s_1, s_2, \cdots s_n \rangle$, a node x denotes a point on one of the segments s_i. A joint node is a node that resides on the endpoint of a segment. The distance between two nodes on a cross section is defined as the overall length f cross section between the two nodes.*

Distance

Property 1. *All nodes on a segment s except the joint nodes move with the same velocity $\hat{\mathbf{v}}_\mathbf{s}$.*

Property 2. *The velocity $\hat{\mathbf{v}}_\mathbf{s}$ of segment s is orthogonal to its orientation $\hat{\mathbf{o}}_\mathbf{s}$.*

4.1 Joints

Definition 4. *A cross section with n segments is also associated with a list of joints $\langle J_1, \cdots J_{n-1} \rangle$, where J_i corresponds to the right endpoint of s_i (same as left endpoint of s_{i+1}. A particular joint $\mathbf{J}_i$ is associated with a left segment s_i, a right segment s_{i+1}, and a velocity $\mathbf{J}_v$.*

Definition 5. *A joint plane is the plane that coincides with both segments l and r associated with a particular joint J.*

Definition 6. *Consider a joint J associated with segments l, r, and joint plane $\mathscr{P}$, where $\mathbf{v}_l$ and $\mathbf{v}_r$ are the velocities of segments l and r. We define $\mathbf{v}_l^{\shortparallel}$ and $\mathbf{v}_l^{\perp}$ as the components of $\mathbf{v}_l$ coinciding with, and orthogonal to $\mathscr{P}$ respectively. Similarly, we define $\mathbf{v}_r^{\shortparallel}$ and $\mathbf{v}_r^{\perp}$, as the components of $\mathbf{v}_r$. By Property 2, $\mathbf{v}_l^{\shortparallel}$ and $\mathbf{v}_r^{\shortparallel}$ have to be orthogonal to $\hat{\mathbf{o}}_\mathbf{l}$ and $\hat{\mathbf{o}}_\mathbf{r}$ respectively.*

Property 3. *For a joint J associated with segments l and r, $\mathbf{v}_l^{\perp} = \mathbf{v}_r^{\perp}$. As a corollary, $\left\|\mathbf{v}_l^{\shortparallel}\right\| = \|\mathbf{v}_r^{\shortparallel}\|$.*

4.2 Time Travel

In the process of time travel, all the nodes on a segment s, except the *joint nodes* move with velocity $\hat{\mathbf{v}}_\mathbf{s}$. The joint node velocity $\mathbf{J}_v$ may have a component along $\hat{\mathbf{o}}_\mathbf{s}$. As a result, the lengths of segments may change (Figure 2b). This can be visualized as movement of the corresponding joint along one of the segments.

Definition 7. *For ever segment s in a cross section C, we associate a left pace L_s, which indicates the rate at which s shrinks from its left endpoint. Similarly, we define a right pace R_s grows from its right endpoint. Note that both these quantities can be negative.*

After time T, the length of a segment changes by $T \cdot (R_s - L_s)$. For a segment s_i with left joint J^L and right joint J^R, we obtain the relations

$$\mathbf{J}_v^L - \hat{\mathbf{v}}_{\mathbf{s_i}} = L_{s_i} \cdot \hat{\mathbf{o}}_{\mathbf{s_i}} \qquad\qquad \mathbf{J}_v^R - \hat{\mathbf{v}}_{\mathbf{s_i}} = R_{s_i} \cdot \hat{\mathbf{o}}_{\mathbf{s_i}} \tag{1}$$

Proposition 1. *A joint J corresponding to segments s and t is* valid *if and only if the evolution resulting from the velocities $\hat{\mathbf{v}}_\mathbf{l}$, $\hat{\mathbf{v}}_\mathbf{r}$, and $\mathbf{J}_v$ preserves distances between nodes.*

The movement of a joint increases the length of one of its associated segments, and decreases the length of the other segment by the same amount (this ensures that the total length is preserved). This puts some constraints on the possible velocities of adjacent segments.

Note that the trajectory of the joints form a crease ... angle is angle between direction vectors. Creases are also created when a segment changes direction... angle is change in direction vector.

Property 4. *The right pace of s_i is equal to the left pace of s_{i+1}. This is to preserve the overall length of the cross section, and the distance between any two nodes. Furthermore, the left pace of the first segment, and the right pace of the last segment should be zero, i.e. $L_0 = R_n = 0$ This ensures that the total length of the cross section does not change.*

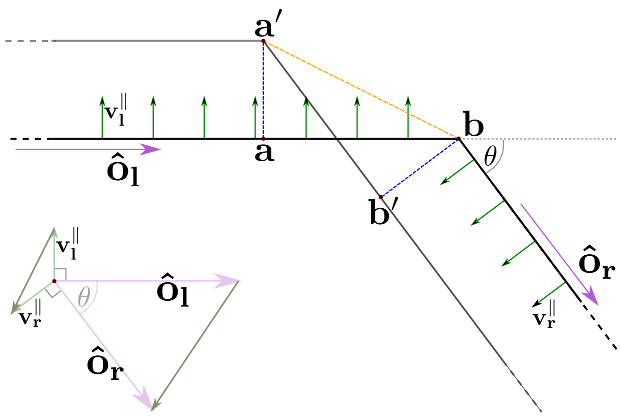


Figure 1: *A joint with segments l and r. The trajectory of the joint is shown in orange. The trajectories of a and b are shown in blue. The green arrows indicate* $\mathbf{v}^{||}$

Consider a joint $\mathbf{J}_i$ corresponding to segments $l = s_i$ and $r = s_{i+1}$, at time $t = 0$. Henceforth, we will refer to L_r as L, and R_l as R. Without loss of geneneralit y, we assume that $L < 0$. At a later time t, let the new joint position be $\mathbf{J}'_i$. We define nodes a and b corresponding to $\mathbf{J}_i$ and $\mathbf{J}_{i+1}$ respectively. We also define the initial and final positions of a as $\mathbf{a}$, and $\mathbf{a}'$, and similarly for b, we define $\mathbf{b}$ and $\mathbf{b}'$. Let d be the *separation* between nodes a and b. this setup is shown in Figure 1

Orthogonal velocity

First, note that $\mathbf{a}, \mathbf{b}$ lie on the segment l, and $\mathbf{a}', \mathbf{b}'$ lie on the segment r, which implies that $\mathbf{b}-\mathbf{a}=d\cdot\hat{\mathbf{o}}_\mathbf{l}$, and $\mathbf{b}'-\mathbf{a}'=d\cdot\hat{\mathbf{o}}_\mathbf{r}$.

$$\mathbf{b}'-\mathbf{a}' = (\mathbf{b}+t\cdot\mathbf{v}_r^{\shortparallel})-(\mathbf{a}+t\cdot\mathbf{v}_l^{\shortparallel}) \tag{2}$$

$$\implies \mathbf{b}'-\mathbf{a}' = (\mathbf{b}-\mathbf{a})+t\cdot(\mathbf{v}_r^{\shortparallel}-\mathbf{v}_l^{\shortparallel}) \tag{3}$$

$$\implies d\cdot\hat{\mathbf{o}}_\mathbf{r} = d\cdot\hat{\mathbf{o}}_\mathbf{l}+t\cdot(\mathbf{v}_r^{\shortparallel}-\mathbf{v}_l^{\shortparallel}) \tag{4}$$

$$\implies \mathbf{v}_r^{\shortparallel}-\mathbf{v}_l^{\shortparallel} = \frac{d}{t}\cdot(\hat{\mathbf{o}}_\mathbf{r}-\hat{\mathbf{o}}_\mathbf{l}) \tag{5}$$

$$= -R\cdot(\hat{\mathbf{o}}_\mathbf{r}-\hat{\mathbf{o}}_\mathbf{l}) = -L\cdot(\hat{\mathbf{o}}_\mathbf{r}-\hat{\mathbf{o}}_\mathbf{l}) \tag{6}$$

This is only possible if $\hat{\mathbf{o}}_\mathbf{l}\times\hat{\mathbf{o}}_\mathbf{r}$ is oriented opposite to $\mathbf{v}_l^{\shortparallel}\times\mathbf{v}_r^{\shortparallel}$.

Property 5. *Given two adjacent segments l and r in a cross section C, the vector $\hat{\mathbf{o}}_\mathbf{l}\times\hat{\mathbf{o}}_\mathbf{r}$ must be oriented opposite to $\mathbf{v}_l^{\shortparallel}\times\mathbf{v}_r^{\shortparallel}$.*

If the angle between the segments (between $\hat{\mathbf{o}}_\mathbf{l}$ and $\hat{\mathbf{o}}_\mathbf{r}$) is θ, the magnitude of $\hat{\mathbf{o}}_\mathbf{r}-\hat{\mathbf{o}}_\mathbf{l}$ is $\sqrt{2-2\cos(\theta)}$, and $\left\|\mathbf{v}_r^{\shortparallel}-\mathbf{v}_l^{\shortparallel}\right\| = v\cdot\sqrt{2-2\cos(\pi-\theta)}$. Here, v is magnitude of the plane velocity of J_i.

$$-L=-R=\frac{d}{t}=\hat{\mathbf{v}}_\mathbf{l}-\hat{\mathbf{o}}_\mathbf{l}\frac{\left\|\mathbf{v}_r^{\shortparallel}-\mathbf{v}_l^{\shortparallel}\right\|}{\left\|\hat{\mathbf{o}}_\mathbf{r}-\hat{\mathbf{o}}_\mathbf{l}\right\|} \tag{7}$$

$$= v\cdot\frac{\sqrt{\sin^2(\pi/2-\theta/2)}}{\sqrt{\sin^2\theta/2}} \tag{8}$$

$$= v\cdot\cot\left(\frac{\theta}{2}\right) = v\cdot\cot(\phi) \tag{9}$$

Here, we set $\phi=\theta/2$.

Property 6. *The velocity of a joint J associated with segments l and r, is a constant vector $\mathbf{J}_v=\hat{\mathbf{v}}_\mathbf{l}-\hat{\mathbf{o}}_\mathbf{l}\frac{\left\|\mathbf{v}_r^{\shortparallel}-\mathbf{v}_l^{\shortparallel}\right\|}{\left\|\hat{\mathbf{o}}_\mathbf{r}-\hat{\mathbf{o}}_\mathbf{l}\right\|}=\mathbf{v}_l-\left\|\mathbf{v}_l^{\shortparallel}\right\|\cdot\hat{\mathbf{o}}_\mathbf{l}\cdot\cot(\phi)$*

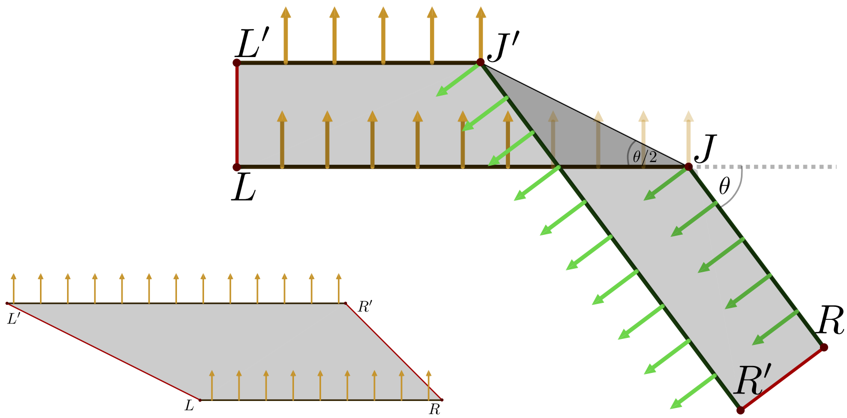
This can result in a segment length becoming zero then delete. We may create a new segment at any point of zero length.

Note that time evolution is reversible.

4.3 Cross Section Interval Folding

Definition 8. *We consider a cross section composed of segments $\langle s_1,s_2,\cdots,s_n\rangle$ with total length X (i.e. $\sum|s_i|=X$). If we allow this cross section to evolve for time T, we obtain a new cross section $\langle r_1,r_2,\cdots,r_n\rangle$. The evolution forms a cross section interval $\mathscr{C}$ of length T. The initial cross section is denoted as $\mathscr{C}_I=\langle s_1,s_2,\cdots s_n\rangle$, and the final cross section is denoted as $\mathscr{C}_F=\langle r_1,r_2,\cdots r_n\rangle$.*

In this section, we focus on a single cross section interval $\mathscr{C}$ with segments $\langle s_1,s_2,\cdots s_n\rangle$ evolving over time T. First consider the surface traced out by an individual segment s_i. Since the endpoints of s_i move in a straight line, each segment traces a trapezoid.



(a) **(b)**

Figure 2

What is a trace?

We define the *height* of a trapezoid to be the distance between its parallel sides.

Lemma 1. *The surface traced out by a segment s in time T is a trapezoid Z_s of height T (Figure 2a). Specifically, if L,R are the initial left and right endpoints of s, and L',R' are the final endpoints, $Z_s = LL'R'R$ is the corresponding trapezoid (Figure 2a).*

Proof. By Property 6, we know that the joint trajectories are straight lines, which form the non-parallel sides of the trapezoid. Since the non-joint nodes on a segment all have the same velocity, the initial and final segment positions form the parallel sides. □

Define gluing

Define joint trajectory

Lemma 2. *Consider a joint J with segments l and r, which has zero orthogonal joint velocity (i.e. $\mathbf{v}_l^{\perp} = \mathbf{v}_r^{\perp} = \mathbf{0}$). The gluing of trapezoids Z_l and Z_r along the joint trajectory $\mathscr{T}_J$ is isometric to a larger trapezoid. This joint trajectory is actually nothing but a crease in the folded state.*

Proof. We consider the evolution of joint J for time T (Figure 2b). Since, $\hat{\mathbf{v}}_\mathbf{l} = \mathbf{v}_l^{\shortparallel}$, from Property 6, we know that $-R_l = \left\|\mathbf{v}_l^{\shortparallel}\right\| \cdot \hat{\mathbf{o}}_\mathbf{v} \cdot \cot(\phi) = \hat{\mathbf{o}}_\mathbf{v} \cdot \cot(\phi)$. So, $\mathbf{J}_v = \hat{\mathbf{v}}_\mathbf{l} - \hat{\mathbf{o}}_\mathbf{v} \cdot \cot(\phi)$. Therefore,

$$\cot(\angle LJJ') = \frac{\|T \cdot R_l\|}{\|T \cdot (\mathbf{J}_v - R_l)\|} = \cot(\phi) \tag{10}$$

$$\implies \angle LJJ' = \phi \tag{11}$$

finish ...

□

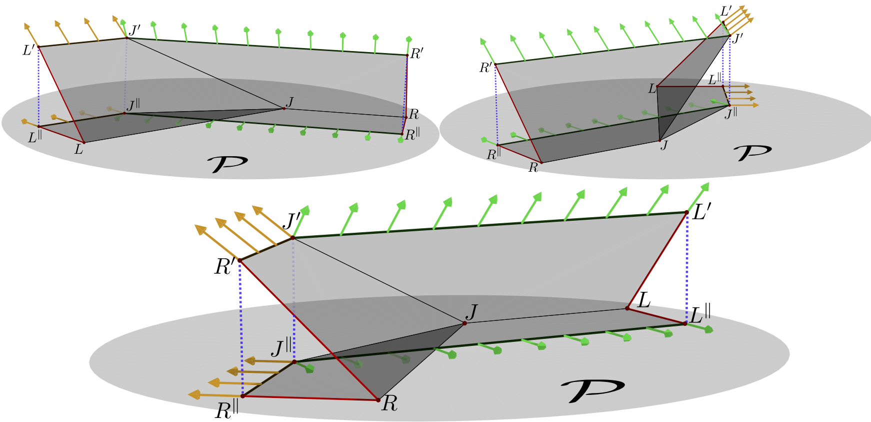


Figure 3: *Evolution of a joint with non-zero orthogonal velocity from LJR to $L'J'R'$. The blue dotted lines represent the projection of the final state to the joint plane $\mathscr{P}$.*

Lemma 3. *Consider a joint J with segments l and r with non-zero orthogonal velocity. The gluing of trapezoids Z_l and Z_r along the joint trajectory $\mathscr{T}_J$ is isometric to a larger trapezoid.*

Proof. As before, let LJR and $L'J'R'$ represent the initial and final positions of the segments respectively. We also construct the projection of $L'J'R'$ to the joint plane $\mathscr{P}$ as $L^{\parallel}J^{\parallel}R^{\parallel}$. The evolution of the projection is analogous to the setting in Lemma 2. Therefore, $\angle LJJ^{\parallel} = \phi = \theta/2$ and $\angle RJJ^{\parallel} = \pi - \phi$.

Consider the positive z-axis along the joint orthogonal velocity (i.e. normal to the joint plane $\mathscr{P}$). We define the orthogonal diaplacement vector as $\overrightarrow{JJ'} = z \cdot \hat{\mathbf{k}}$. Let the positive x-axis be along $JJ^{\parallel}$. So, the unit vector along $\overrightarrow{JJ'}$ is $\frac{1}{\sqrt{1+z^2}}(1,0,z)$.

Since $\angle J^{\parallel}JR = \phi$, the unit vector along $\overrightarrow{J}R$ is $(\cos\phi, \sin\phi, 0)$. So, we compute $\cos\angle RJJ' = \cos\phi/\sqrt{1+z^2}$. Similarly, since $\angle J^{\parallel}JL = \pi - \phi$, the unit vector along $\overrightarrow{J}L$ is $(-\cos\phi, \sin\phi, 0)$, which implies that $\cos\angle LJJ' = -\cos\phi/\sqrt{1+z^2}$. Finally, since both $\angle LJJ$ and $\angle RJJ'$ are less than π, and $\cos(\angle LJJ') = -\cos(\angle RJJ')$, we conclude that $\angle LJJ' + \angle RJJ' = \pi$.

Since $LJJ'L'$ and $RJJ'R'$ are both trapezoids, this implies that the resulting gluing along JJ' is a larger trapezoid. □

Definition 9. *Consider a cross section interval $\mathscr{C}$ formed from a cross section C evolving over time T. By Lemma 1, the segments $\langle s_1, s_2, \cdots s_n\rangle$ form trapezoids $\langle Z_1, Z_2, \cdots Z_n\rangle$ each of height T. The folding $\mathscr{F}_C^T$ corresponding to $\mathscr{C}$ is formed* *by successively gluing the trapezoids Z_i to Z_{i+1} along the trajectory of joint J_i (for $1 \leq i < n$) to form a connected shape.*

Definition 10. *Given a cross section interval $\mathscr{C}$ with folding $\mathscr{F}_C^T$, the initial-boundary of a folding $\mathscr{F}_C^T$ is defined as the union of the initial cross section segments in $\mathscr{C}_I$, where the right endpoint of the i^{th} segment is attached to the left endpoint of the $(i+1)^{th}$ segment. Similarly, the final-boundary of $\mathscr{F}_C^T$ is defined as the union of the final segments in $\mathscr{C}_F$.*

Theorem 1. *Consider a cross section interval $\mathscr{C}$ formed from a cross section C evolving over time T to form a folding $\mathscr{F}_C^T$. Further assume that the total length of cross section C is X units. Then, $\mathscr{F}_C^T$ is isometric to a $X \times T$ strip of paper.*

Proof. By repeated use of Lemma 3, we know that $\mathscr{F}_C^T$ is isometric to a trapezoid. Let L, L' be the initial and final positions of the left (non-parallel) edge of the trapezoid, and let R, R' be the initial and final positions of the right edge of the trapezoid. Say that C comprises of segments $\langle s_1, s_2, \cdots s_n \rangle$. From Property 4, we know that the left pace of s_0 is zero. So, the line LL' follows the trajectory of $\hat{\mathbf{v}}_\mathbf{0}$, which is orthogonal to the segment s_0. In other words, the left edge of the trapezoid has length T, and is orthogonal to the parallel edges. Similarly, because the right pace of s_n is zero, the right edge of the trapezoid is also orthogonal. Therefore, $\mathscr{F}_C^T$ is isometric to a right angled trapezoid (i.e. a strip) of length X and width T. $\square$

property for NO zero length segments

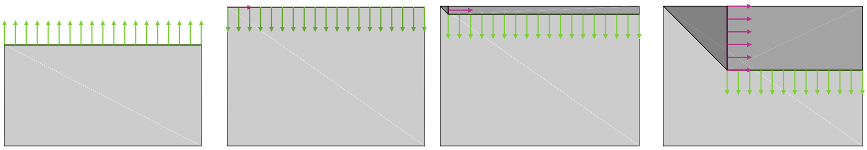
4.4 Multiple Cross Sections

Definition 11. *Given two cross section intervals $\mathscr{C}$ and $\mathscr{D}$, such that $\mathscr{C}_F$ and $\mathscr{D}_I$ are equivalent, we say that $\mathscr{D}$ is next-compatible with $\mathscr{C}$ and $\mathscr{C}$ is previous-compatible with $\mathscr{D}$. Two cross sections $C = \langle s_1, s_2, \cdots s_n \rangle$ and $D = \langle r_1, r_2, \cdots r_m \rangle$ are equivalent if and only if C and D correspond to the same sequence of segments after the deletion of all zero length segments.*

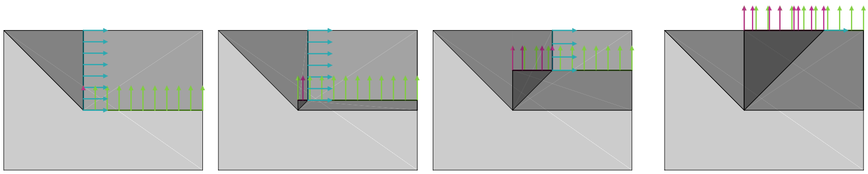
Definition 12. *A cross section sequence is a sequence is an ordered list of cross section intervals $\langle \mathscr{C}_1, \mathscr{C}_2, \cdots \mathscr{C}_n \rangle$, such that C_i is next-compatible with C_{i-1} for all $i \in [n-1]$. This is equivalent to stating that C_i is previous-compatible with C_{i+1} for all $i \in [n-1]$. Note that we do not care about the directions of the segments.*

We will represent our full construction as a valid *cross section sequence*. Given a cross section sequence $\langle \mathscr{C}_1, \mathscr{C}_2, \cdots \mathscr{C}_n \rangle$, the transition from $\mathscr{C}_i$ to $\mathscr{C}_{i+1}$ corresponds to the deletion of one or more length zero segments from $\mathscr{C}_i$, and the addition of one or more zero length segments to obtain $\mathscr{C}_{i+1}$. One simple example is shown in Figure 4.

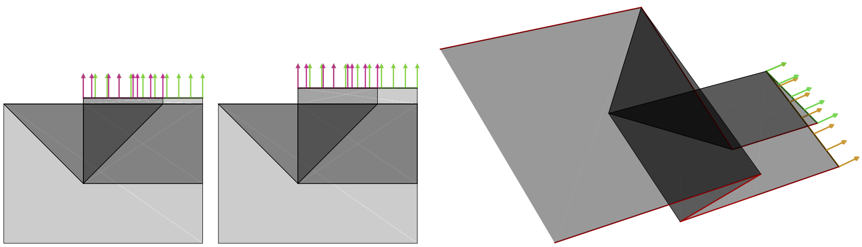
Definition 13. *Given a cross section sequence $\langle \mathscr{C}_1, \mathscr{C}_2, \cdots \mathscr{C}_n \rangle$, We obtain $\mathscr{F}_i^T$ as the folding of cross section $\mathscr{C}_i$. For every joint J_i, we attach the corresponding trapezoids for s_i and s_{i+1} along the joint trajectory to obtain the folding of $\mathscr{C}$.*



(a) *The trivial cross section.* **(b)** *New zero length segment inserted at left endpoint. Existing segment reverses direction.* **(c)** *Evolution continues*



(d) *New zero length segment (red) inserted between two existing segments with the same velocity as the rightmost segment. Existing segments retain their direction.* **(e)** *Length of leftmost (blue) segment becomes zero.*



(f) *Leftmost (zero length) segment is deleted. The remaining two segments continue in the same direction. Overall strip width has been reduced* **(g)** *Side view of strip narrowing gadget, with layers separated. The red lines denote the boundaries of the cross section.*

Figure 4: *Cross section evolution of a strip narrowing gadget.*

cite

4.4.1 Evolution Corresponds to Flat Paper

In this section we will demonstrate that the folding formed by cross section evolution is realizable from a sheet of flat paper. We note here that our construction may still result in self intersections.

We consider a cross section sequence $\langle \mathscr{C}_1, \mathscr{C}_2, \cdots \mathscr{C}_n \rangle$, where each cross section interval $\mathscr{C}_i$ has evolution time T_i.

We will then use Theorem 12 to attach the sequence of $X \times T_i$ strips, to form a complete $X \times T$ sheet of paper, where $T = \sum T_i$.

Theorem 1. *Consider a cross section sequence $\mathscr{C} = \langle \mathscr{C}_1, \mathscr{C}_2, \cdots \mathscr{C}_m \rangle$ where each cross section interval $\mathscr{C}_i$ evolves over time T_i to form a folding $\mathscr{F}_i$ such that the following properties hold for all segments and joints in each of the cross sections involved.*

Property 1. *All nodes on a segment s except the joint nodes move with the same velocity* $\hat{\mathbf{v}}_\mathbf{s}$.

Property 2. *The velocity* $\hat{\mathbf{v}}_\mathbf{s}$ *of segment s is orthogonal to its orientation* $\hat{\mathbf{o}}_\mathbf{s}$.

Property 3. *For a joint J associated with segments l and r,* $\mathbf{v}_l^\perp = \mathbf{v}_r^\perp$. *As a corollary,* $\left\|\mathbf{v}_l^{\parallel}\right\| = \left\|\mathbf{v}_r^{\parallel}\right\|$.

Property 4. *The right pace of* s_i *is equal to the left pace of* s_{i+1}. *This is to preserve the overall length of the cross section, and the distance between any two nodes. Furthermore, the left pace of the first segment, and the right pace of the last segment should be zero, i.e.* $L_0 = R_n = 0$ *This ensures that the total length of the cross section does not change.*

Property 5. *Given two adjacent segments l and r in a cross section C, the vector* $\hat{\mathbf{o}}_\mathbf{l} \times \hat{\mathbf{o}}_\mathbf{r}$ *must be oriented opposite to* $\mathbf{v}_l^{\parallel} \times \mathbf{v}_r^{\parallel}$.

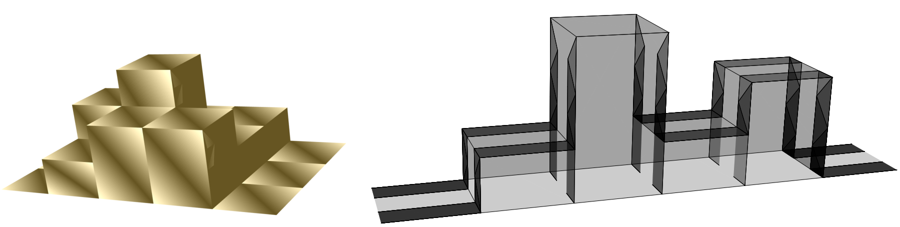
Property 6. *The velocity of a joint J associated with segments l and r, is a constant vector* $\mathbf{J}_v = \hat{\mathbf{v}}_\mathbf{l} - \hat{\mathbf{o}}_\mathbf{l} \frac{\left\|\mathbf{v}_r^{\parallel} - \mathbf{v}_l^{\parallel}\right\|}{\left\|\hat{\mathbf{o}}_\mathbf{r} - \hat{\mathbf{o}}_\mathbf{l}\right\|} = \mathbf{v}_l - \left\|\mathbf{v}_l^{\parallel}\right\| \cdot \hat{\mathbf{o}}_\mathbf{l} \cdot \cot(\phi)$

Then, the folding $\mathscr{F}_\mathscr{C}$ *obtained by successively gluing the final boundary of* $\mathscr{F}_i$ *to the initial boundary of* $\mathscr{F}_{i+1}$ *(for each* $1 \le i < m$*), is isometric to a* $X \times T$ *strip of paper, where* $T = \sum T_i$.

Proof. □

5 Orthogonal Terrains

In this section, we outline a construction of orthogonal terrains with arbitrary rational extrusion heights. In our construction, the cross section at will always be on the $x - z$ plane. This makes the analysis much simpler.



(a) *Orthogonal terrain.* **(b)** *A column extrusion with heights* $\{0,1,3,1,2,0\}$

Figure 5

To simplify the presentation, we will consider an uniform $X - Y$ grid, with arbitrary rational extrusion heights corresponding to every grid square (Figure 5a).

Definition 14. *An $n \times m$ rational grid extrusion is a 3-dimensional structure, whose projection onto the $x-y$ plane forms an unit grid of size $n \times m$. In the 3-dimensional structure, the unit face corresponding to the location $(i.j)$, exists at height $E_{i,j}$ (in the z direction), where $E_{i,j}$ is a rational number.*

We consider each "column" of a given grid extrusion separately as an individual *column extrusion*. We will construct each of the n columns independently (Figure 7), and attach them together with *column connectors* (Figure 11).

We begin by choosing an $2\varepsilon = 1/K$ where K is a positive integer. This is chosen such that $E_{i,j}$ is an integral multiple of 2ε for all i, j. Then we scale the entire construction up by a factor of K.

Given a strip of size $X \times T$, we will consider the evolution of a cross section of length X evolving for time T.

5.1 Construction of Column Extrusion by Level Shifting

First, we consider a single column of the orthogonal terrain $\{E_{i1}, E_{i2}, \cdots, E_{i,n}\}$. We denote the column extrusion heights as $\{H_1, H_2, \cdots H_n\}$, where $H_j = E_{i,j}$. Consider the decomposition of T into the following time intervals.

$$T = 1 + D_1 + 1 + D_2 + 1 + D_3 + \cdots\cdots + 1 + D_{m-1} + 1 \tag{12}$$

Here, the times corresponding to

- the i^{th} 1 is realized as the surface at height H_i.
- $D_i = |H_i - H_{i+1}|$ is realized as the transition between H_i and H_{i+1}.

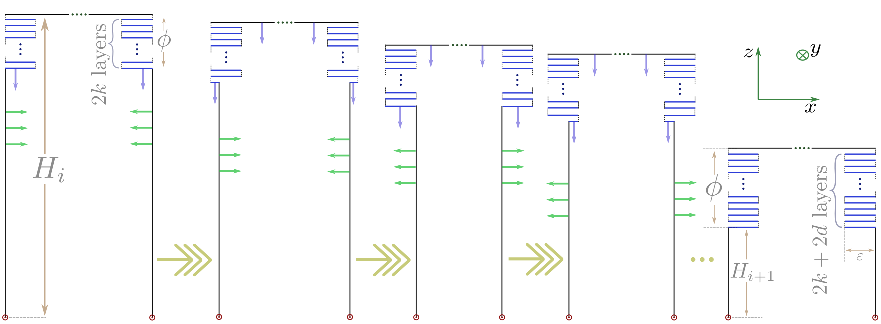
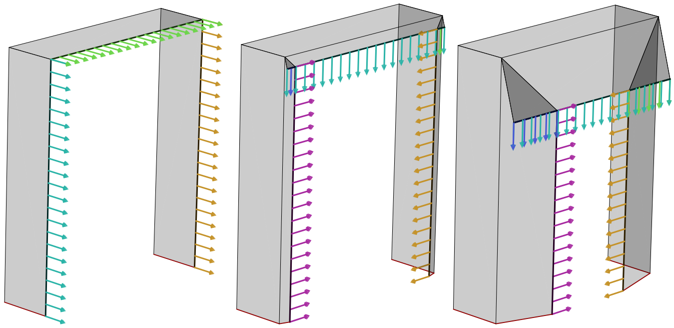


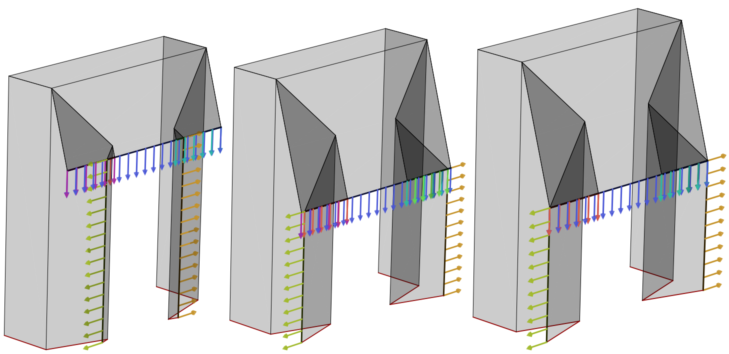
Figure 6: *Cross section change from level i to $i+1$. The* accordion segments *are separated for illustration. In reality, the accordion is folded flat. Zero distances are marked by ϕ. The direction of horizontal and vertical segments are shown by purple and green arrows respectively.*

To construct the column, we will present a cross section sequence. First, consider H_i, such that $H_{i+1} = H_i - 2\varepsilon \cdot d$. Figure 6 shows the cross section evolution. This cross section comprises of a two vertical lines separated by a top horizontal line. The vertical lines are connected to the top segment with a sequence of $2k$ horizontal segments that *accordion* back and forth. During the K-interval, all segments

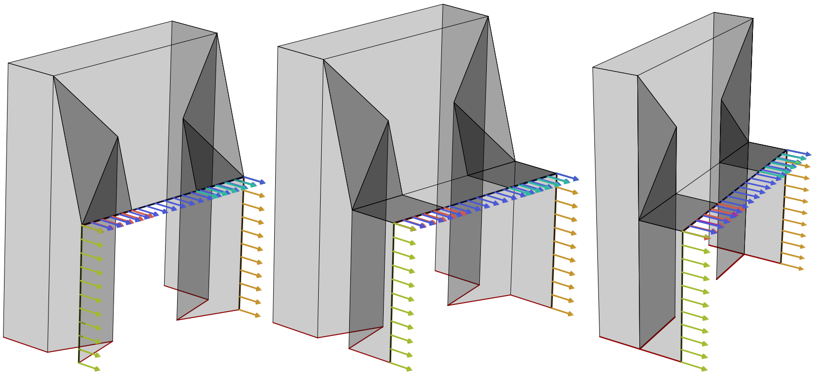


(a) *Higher initial level. Top segment moves down.*

(b) *Two segments created. New segments are aligned with top segment, and move down. Vertical segments move inwards.*



(c) *Two more segments created. Vertical segments reverse direction.*



(d) *Level shift completed with four new horizontal segments.*

(e) *Flat folded state.*

Figure 7: *Level Shifting Gadget. The separation along the Y direction illustrates the layering. The red line denotes the boundary of the cross section.*

move along the positive y direction (Figure 7a),to create the i^{th} level. Subsequently, during the level shift, all segments move in the $x-z$ plane (Figure 7b 7c).

y movement

Lemma 4. *The number of accordion folds during horizontal evolution (along the y axis) must be even.*

The top segment moves downwards in intervals of 2ε. During this process, the horizontal segments move downwards continuously (Figure 7b 7c). For the first ε time interval, a new horizontal downwards moving accordion segment of length zero is created at both accordions, and the vertical segments move towards each other along the x-axis (Figure 7b). For the next ε time interval, similar (oppositely oriented) accordion segments are created at the lowest position. This time, the vertical segments move outwards. Overall, two sets of accordion segments on either side are added, and the height of the top segment decreases by 2ε.

In the case that $H_{i+1} = H_i + 2\varepsilon \cdot d$, the level up-shift is simply the down-shift evolution in reverse (Figure 11). This transition is only possible if the initial number of accordion segments in the H_i cross section is at least $2d$. Specifically, assuming that the minimum height is zero, we have the following lemma.

Lemma 5. *If the number of accordion segments at level H_i is l_i, then the number of accordion segments after transitioning to level H_{i+1} is $l_{i+1} = l_i - (H_{i+1} - H_i)/\varepsilon$. Specifically, if the number of layers at level 0 is l, then the number of layers at level $\max\{H_i\}$ is $l - \max\{H_i\}/\varepsilon$.*

Corollary 1. *Since the number of accordion segments can never be negative, the minimum number of layers at at level zero is $L = \max\{H_i\}/\varepsilon$. This also ensures that every other level shift is also possible.*

Corollary 2. *The length of the cross section at a zero level is at least $1 + 2 \cdot L \cdot \varepsilon$. So, the minimum possible length of the cross section under our construction is $1 + 2 \cdot \max\{H_i\}$.*

This provides the minimum width of the strip required to construct a column extrusion. Now, we can also calculate the length of the strip as the total time evolution required

$$T = 1 + D_1 + 1 + D_2 + 1 + D_3 + \cdots\cdots + 1 + D_{m-1} + 1 = m + \sum_{i=1}^{m-1} |H_{i+1} - H_i|$$

Theorem 2. *A given column extrusion with heights $\{H_1, H_2, \cdots H_n\}$, can be constructed from a strip of paper with size $X \times T$, where*

$$X \geq 1 + 2 \cdot \max\{H_i\} \qquad\qquad T \geq \left(m + \sum_{i=1}^{m-1} |H_{i+1} - H_i| \right) \tag{13}$$

5.2 Multiple Column Extrusions form a Grid Extrusion

Now we consider mutiple column extrusions evolving in parallel. Henceforth, we will refer to the evolution of column cross sections along the y-axis (the "1"s in Equation 12 as *horizontal evoluton*. Meanwhile, vertical transition will refer to level shifting evolution in the $x-z$ plane. Let $\mathscr{C}^{(i)}$ be a valid cross section evolution corresponding to the i^{th} column in the grid extrusion (as defined in Section 5.1). As before, for simplicity, we will assume that the minimum height in each column is zero $\left(i.e.\ \min_j \{E_{i,j}\} = 0\right)$.

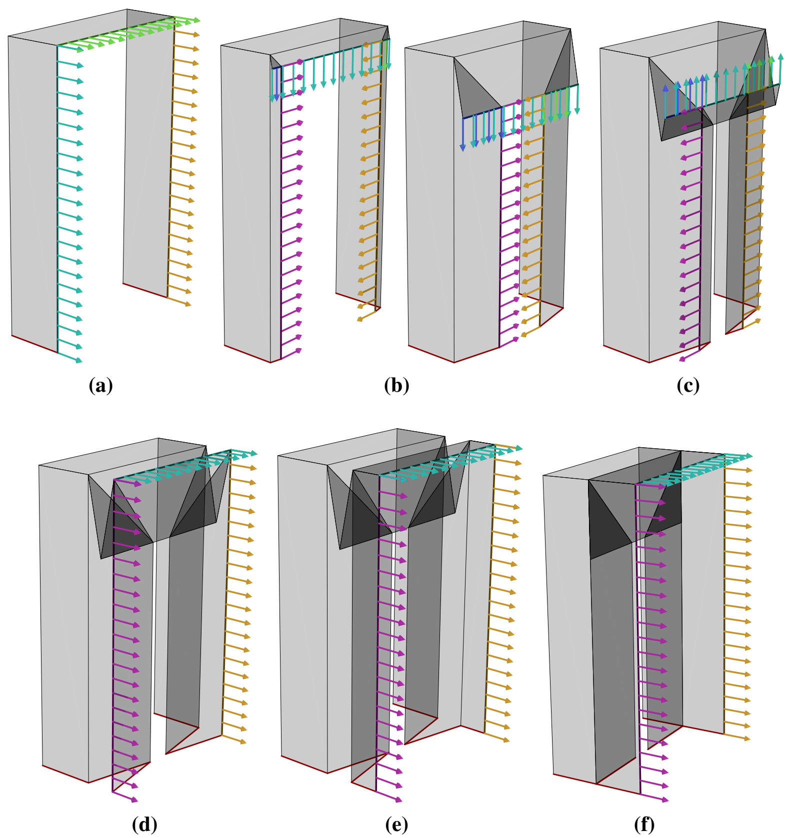


Figure 8: *Level Shifting Gadget. The separation along the Y direction illustrates the layering. The red line denotes the boundary of the cross section.*

We consider the parallel evolution of each column extrusion $\mathscr{C}^{(i)}$.

- During the horizontal evolution of row j, each $\mathscr{C}^{(i)}$ evolves along the positive y direction for time 1 at height $E_{i,j}$.
- During the vertical transition from row j to $j+1$, each $\mathscr{C}^{(i)}$ evolves for time $D_{ij} = \left|E_{i,j+1} - E_{i,j}\right|$.

Note that the vertical transition times are different for each $\mathscr{C}^{(i)}$. Since we want to glue the sequence of $\left\{\mathscr{C}^{(i)}\right\}$s, our constructions must have equal transition times.

Definition 15. *We define the common transition time from row j to row $j+1$ as*

$$D_j = \max_j \{D_{ij}\} = \max_j \{|E_{i,j+1} - E_{i,j}|\}$$

So, the slowest column dictates the transition time, and the faster columns have to *stall* for additional time. To achieve this, we define an *up-down* gadget, which is very similar to our original level shifting gadget. This up-down gadget (Figure 8) evolves for 2ε time, but the height of the corresponding column remains unchanged. This gadget starts at a height h, and for the first ε time interval (Figure 8b), evolves exactly the same way as the down-shift gadget (Figure 7b). For the second ε time interval, the cross section evolves in reverse (Figure 8b), back to it's original state (Figure 8c).

So, the vertical transition of $\mathscr{C}^{(i)}$ needs to use a total of $(D_j - D_{i,j})/(2\varepsilon)$ up-down gadgets (each gadget *stalls* for 2ε time). We obtain the following primitive, as a consequence of Theorem 2.

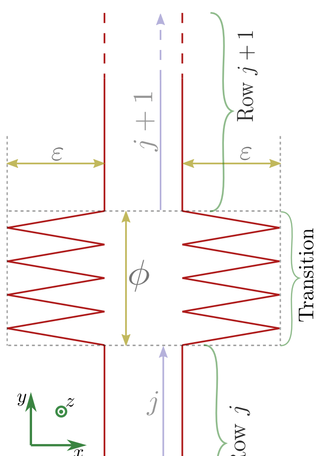
Proposition 2. *By Theorem 2, the extrusion for column i is constructed from a paper strip with size*

$$\left(1 + 2 \cdot \max_j \{E_{ij}\}\right) Xm + \sum_{j=1}^{m-1} D_j$$

5.3 Gluing Column Extrusions together with Strip Connectors

Now that all the column extrusions $\left\{\mathscr{C}^{(i)}\right\}$ have the same evolution time, we would like to glue them together into a continuous strip of paper. In order to achieve this, we consider the time-axis boundaries of $\mathscr{C}^{(i)}$ (red line in Figure 7 8). First, note that the boundaries always lie on the same plane, corresponding to the zero level. We will constrain this to be on the plane $z = 0$. Now, let us consider the motion of the boundary on this plane.

- During the horizontal evolution of any row j, both boundaries move along the positive y-axis with unit velocity (Figure 7a 7d 7e 8a 8d 8e).
- During the vertical transition from row j to $j+1$, both boundaries move back and forth along the x-axis with unit velocity. (Figure 7b 7c 8b 8b 8c). We can divide the vertial transition into $k = D_j/(2\varepsilon)$ intervals of length 2ε. Each of these interval segments is either a up-shift, a down-shift, or a up-down gadget.



- In the first half of each interval (ε time), the left and right boundaries move towards each other; i.e. the left boundary moves along the positive x-axis, and the right boundary moves along the negative x-axis for a distance of ε.
- In the second half of each interval, the left and right reverse directions, and return to their original positions.
- After D_j time, the boundaries return to their original positions, and resume their movement in the positive y direction.

We will attempt to construct a connector cross section sequence whose left and right boundaries line up with the adjacent column extrusion boundaries shown in Figure 9. Notice that the distance between *corresponding points* on the boundaries varies between 0 and 2ε. So, the connector strip must have width at least 2ε.

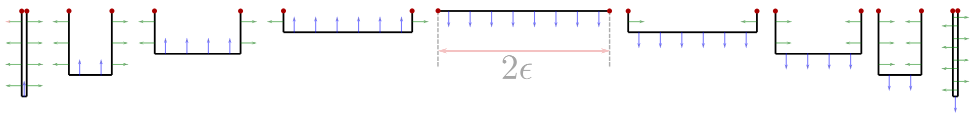
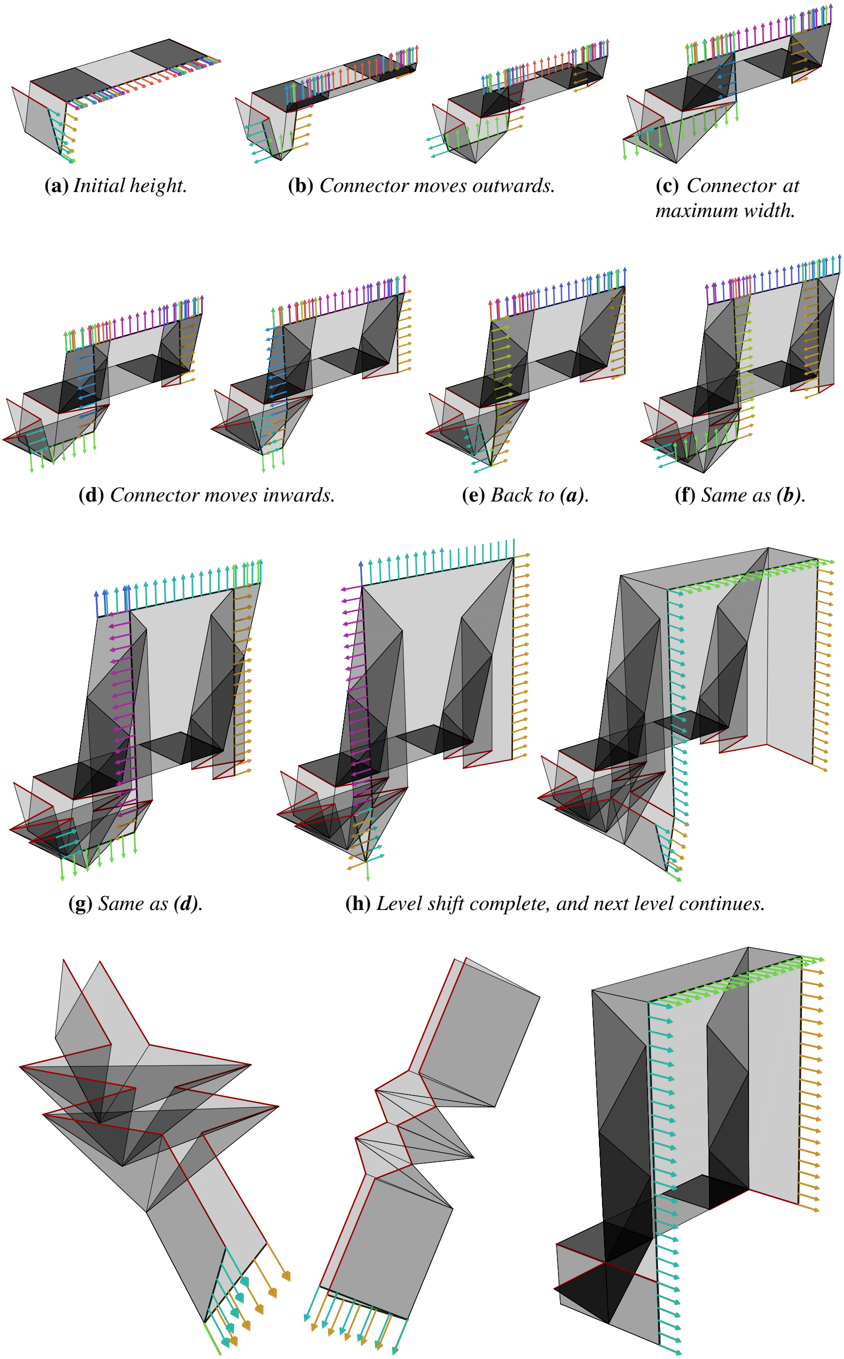


Figure 10: *Cross section evolution of column connector gadget.*

We outline a construction of a 2ε width strip, as shown in Figure 10. During horizontal evolution, the cross section comprises of two vertical segments, each of length ε, which move along the same trajectory in the positive y-direction (during horizontal evolution Figure 11a 11k). Now, consider a transition of length D_j divided into intervals of length 2ε, each of which evolves according to Figure 10.



(a) *Initial height.* **(b)** *Connector moves outwards.* **(c)** *Connector at maximum width.*

(d) *Connector moves inwards.* **(e)** *Back to **(a)**.* **(f)** *Same as **(b)**.*

(g) *Same as **(d)**.* **(h)** *Level shift complete, and next level continues.*

(i) *Connector gadget.* **(j)** *Side view.* **(k)** *Flat folded state.*

Figure 11: *Column gadget attached to a single column connector gadget. The red line demarcates the interface between the two gadgets.*

- The vertical segments move outwards with unit velocity to match the outward moving boundaries of the adjacent strips. An upwards moving horizontal segment of length zero is created between the two existing segments (Figure 11b).
- After time ε, the length of the vertical segments become zero, and the horizontal segment spans the 2ε gap between the column boundaries (Figure 11c). Notice that that the boundary of the connector maintains its z coordinate (Figure 10).
- Then the connector segments reverse direction, and retrace their path for ε time (Figure 11d), until the vertical segments become length ε, and the horizontal segment disappears (Figure 11e).
- The entire process repeats $D_j/(2\varepsilon)$ times (Figure 11f, 11g, 11h).

The completed connector gadget is shown in Figure 11i, 11j. We show the *connector gadget* attched to an *up-shift gadget* in Figure 11h, 11k.

5.4 Size of Construction

From Proposition 2, we know that each column extrusion strip $\mathscr{C}^{(i)}$ has width $\left(1+2\cdot\max_j\left\{E_{ij}\right\}\right)$. Additionally, we have $n-1$ strip connecors, each of width 2ε. We define $M_i=\max_j\left\{E_{ij}\right\}$. So, the total width of the orthogonal terrain construction is

$$X=2(n-1)\cdot\varepsilon+n+2\cdot\sum_{i=1}^{n}M_i$$

Now, note that our construction is also valid for any $\varepsilon'=\varepsilon/(2k)$, where k is an integer. In other words, we can make ε arbitrarily small.

Theorem 3. *Using the time evolution (y dimension) from Proposition 2, we conclude that a grid extrusion can be folded from a strip of size $X\times T$, where*

$$X=n+2\cdot\sum_{i=1}^{n}M_i+o(1) \qquad\qquad T=m+\sum_{j=1}^{m-1}D_j \tag{14}$$

5.5 Removing the Zero Level Assumption

5.6 Optimality

Under some suitable assumptions, our construction can be made $2+\varepsilon$ optimal, for arbitrarily small ε.

We define the maximum deltas along the y-axis as $D_j=\max_i\|E_{i,j}-E_{i,j+1}\|$, and let $Y=n+\sum_{i=1}^{n-1}D_j$ We also define the lowest and highest points along x-axis as $L_i=\max_j E_{ij}$ and $H_i=\max_j E_{ij}$, and let

$$X=n+\sum_{i=1}^{n}\left[(H_i-\min(L_i,L_{i+1}))+(H_i-\min(L_i,L_{i-1}))\right]$$

The terms in the summation account for the total length of all the necessary worst case vertical walls, and the n is for the top faces.

Claim 1. *The x-axis length of the strip of paper required to fold this shape can be made arbitrarily close to X.*

Claim 2. *The y-axis length of the strip of paper required to fold this shape will be exactly Y.*

First, we pick an ε, such that 2ε divides all extrusion heights. The construction will require paper of size $X' \times Y$, where $X' = X + 2\varepsilon(n-1)$.

There are two components to the construction

- n strips parallel to the y-axis. Each of these strips will fold to the corresponding strip in the extruded graph. The total area of these strips will be $X \times Y$.
- $n-1$ Intermediate strips, each of size $2\varepsilon \times Y$ to connect the main strips together.

The total area is therefore $X \times Y + (n-1)2\varepsilon \times Y = X' \times Y$. This can of course be made arbitrarily close to $X \times Y$.

References

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