

# Efficient Origami Construction of Orthogonal Terrains using Cross-Section Evolution

*Amartya Shankha Biswas, Erik D. Demaine, Jason S. Ku*

## 1 Introduction

Many algorithms and universality results exist for producing parameterized families of origami structures, but few are provably efficient, i.e. provide constructions from a paper having dimensions within a low constant factor of an optimal construction. At 5OSME, [Demaine et al. 10] presented a efficient construction for folding orthogonal mazes which is computable in polynomial time. Origamizer, presented in [Demaine and Tachi 17] constructs foldings corresponding to general polyhedral surfaces, but does not provide any bound on the efficiency of the constructions. On the other hand, Treemaker from [Lang 96] produces efficient crease patterns to fold uniaxial bases, but may require exponential time to find an efficient solution.

In this paper, we present an algorithm for efficiently producing an origami folding that corresponds to an input **orthogonal terrain** with arbitrary rational extrusion heights. A folding corresponds to an orthogonal terrain if the folding covers every point on the terrain, but no point on the folding exists above the terrain. This result improves an algorithm, [Benbernou et al. 10] also presented at 5OSME, applicable to a more general class of inputs, providing a universal construction to fold general orthogonal polyhedra, though the construction is less inefficient than our construction applied to orthogonal terrains. Our construction approach follows three steps:

1. Decompose the orthogonal terrain into strips, constant along one dimension.
2. Cover the strips efficiently using rectangular strips of paper.
3. Stitch the strips together along matching boundaries.

In order to better communicate the algorithm and the final folded state produced, we also introduce a new **cross-section evolution** representation of a folded isometry: a straight line is swept across the crease pattern of a folded surface, and we keep track of how the folding of the line evolves as a cross-section of the folded surface. The propagation of the cross-section between crease pattern vertices is uniquely determined by the initial orientation of the cross-section, so the folded isometry can be constructed by sweeping the line and locally modifying the cross-section when crossing crease pattern vertices during propagation. This representation not only simplifies the description of the 3D folded isometries constructed, but also provides a simpler framework to argue that the folded state does not self intersect, by propagating planar cross-sections monotonically along a single direction.

We then show that our construction's efficiency is within a small constant factor of any folding with optimal efficiency.

## 2 Cross Section Evolution

We introduce a new method of origami construction, using cross section diagrams. Instead of beginning our construction from a 2-dimensional sheet of paper, we consider a 1-dimensional cross section moving forwards in time. A simple example using strip narrowing [Demaine et al. 00] is demonstrated in Figure 4.

### 2.1 Segments and Cross Sections

**Definition 1.** A segment  $s$  is an oriented line segment with left and right endpoints  $s_l$  and  $s_r$ . Each segment is also associated with an orientation vector  $\hat{\mathbf{o}}_s \equiv \frac{s_r - s_l}{\|s_r - s_l\|}$ . This vector serves to disambiguate the orientation of zero length segments.

**Definition 2.** A cross section  $C$  is defined as an ordered list of line segments  $\langle s_1, s_2, \dots, s_n \rangle$ , such that for every segment  $s_i$  (except the last), the right endpoint of  $s_i$  coincides with the left endpoint of  $s_{i+1}$ . Each segment  $s$  is also associated with a velocity vector  $\hat{\mathbf{v}}_s$  of unit magnitude. For a segment  $s \in C$  we will also denote this velocity as  $\hat{\mathbf{v}}_i$ .

**Definition 3.** Given a cross section  $C = \langle s_1, s_2, \dots, s_n \rangle$ , a node  $x$  denotes a point on one of the segments  $s_i$ . A joint node is a node that resides on the endpoint of a segment. The distance between two nodes on a cross section is defined as the overall length of cross section between the two nodes.

**Property 1.** All non-joint nodes on a segment  $s$  move with the same velocity  $\hat{\mathbf{v}}_s$ .

**Property 2.** The velocity  $\hat{\mathbf{v}}_s$  of segment  $s$  is orthogonal to its orientation  $\hat{\mathbf{o}}_s$ .

### 2.2 Joints

**Definition 4.** A cross section with  $n$  segments is also associated with a list of joints  $\langle J_1, \dots, J_{n-1} \rangle$ , where  $J_i$  corresponds to the right endpoint of  $s_i$  (same as left endpoint of  $s_{i+1}$ ). A particular joint  $J_i$  is associated with a left segment  $s_i$ , a right segment  $s_{i+1}$ , and a velocity  $\mathbf{J}_i$ .

**Definition 5.** A joint plane is the plane that coincides with both segments  $l$  and  $r$  associated with a particular joint  $J$ . In case the two segments are coincident (i.e.  $\hat{\mathbf{o}}_l = \pm \hat{\mathbf{o}}_r$ ), we define the joint plane to be orthogonal to  $\hat{\mathbf{v}}_l = \hat{\mathbf{v}}_r$ .

**Definition 6.** Consider a joint  $J$  associated with segments  $l$ ,  $r$ , and joint plane  $\mathcal{P}$ , where  $\hat{\mathbf{v}}_l$  and  $\hat{\mathbf{v}}_r$  are the velocities of segments  $l$  and  $r$ . We define  $\hat{\mathbf{v}}_l^{\parallel}$  and  $\hat{\mathbf{v}}_l^{\perp}$  as the components of  $\hat{\mathbf{v}}_l$  coinciding with, and orthogonal to  $\mathcal{P}$  respectively. Similarly, we define  $\hat{\mathbf{v}}_r^{\parallel}$  and  $\hat{\mathbf{v}}_r^{\perp}$ , as the components of  $\hat{\mathbf{v}}_r$ . By Property 2,  $\hat{\mathbf{v}}_l^{\parallel}$  and  $\hat{\mathbf{v}}_r^{\parallel}$  are orthogonal to  $\hat{\mathbf{o}}_l$  and  $\hat{\mathbf{o}}_r$  respectively. This uniquely determines the direction of all velocity components.

Henceforth, we will refer to the  $\hat{\mathbf{v}}^{\parallel}$  component as the *joint plane velocity*, and the  $\hat{\mathbf{v}}^{\perp}$  component as the *joint orthogonal velocity*,

**Property 3.** For a joint  $J$  associated with segments  $l$  and  $r$ ,  $\hat{\mathbf{v}}_l^{\perp} = \hat{\mathbf{v}}_r^{\perp}$  i.e. the joint orthogonal velocities have to be equal, such that the joint plane moves with a fixed velocity along it's normal. As a corollary,  $\|\hat{\mathbf{v}}_l^{\parallel}\| = \|\hat{\mathbf{v}}_r^{\parallel}\|$ .

### 2.3 Time Travel

In the process of time travel, all the nodes on a segment  $s$ , except the *joint nodes* move with velocity  $\hat{\mathbf{v}}_s$  (orthogonal to  $\hat{\mathbf{o}}_s$ ). For example, if we allow a single segment of length  $X$  to evolve for time  $T$ , it will result in a  $X \times T$  strip of paper.

The joint node velocity  $\mathbf{J}_v$  may have a component along a corresponding segment  $s$  (along  $\hat{\mathbf{o}}_s$ ). As a result, the lengths of segments may change (Figure 2b). This can be visualized as movement of the corresponding joint along one of the segments.

**Definition 7.** For ever segment  $s$  in a cross section  $C$ , we associate a left pace  $L_s$ , which indicates the rate at which  $s$  shrinks from its left endpoint. Similarly, we define a right pace  $R_s$  grows from its right endpoint. Note that both these quantities can be negative.

After time  $T$ , the length of a segment changes by  $T \cdot (R_s - L_s)$ . The length of a segment is not allowed to become negative. For a segment  $s_i$  with left joint  $J^L$  and right joint  $J^R$ , we obtain the relations

$$\mathbf{J}_v^L - \hat{\mathbf{v}}_{s_i} = L_{s_i} \cdot \hat{\mathbf{o}}_{s_i} \quad \mathbf{J}_v^R - \hat{\mathbf{v}}_{s_i} = R_{s_i} \cdot \hat{\mathbf{o}}_{s_i} \quad (1)$$

**Proposition 1.** A joint  $J$  corresponding to segments  $s$  and  $t$  is valid if and only if the evolution resulting from the velocities  $\hat{\mathbf{v}}_l$ ,  $\hat{\mathbf{v}}_r$ , and  $\mathbf{J}_v$  preserves distances between nodes.

**Property 4.** The right pace of  $s_i$  is equal to the left pace of  $s_{i+1}$ . This is to preserve the overall length of the cross section, and the distance between any two nodes. Furthermore, the left pace of the first segment, and the right pace of the last segment should be zero, i.e.  $L_0 = R_n = 0$ . This ensures that the total length of the cross section does not change.

The movement of a joint increases the length of one of its associated segments, and decreases the length of the other segment by the same amount (this ensures that the total length is preserved). This puts some constraints on the possible velocities of adjacent segments.

Consider a joint  $\mathbf{J}_i$  corresponding to segments  $l = s_i$  and  $r = s_{i+1}$ , at time  $t = 0$ . Henceforth, we will refer to  $L_r$  as  $L$ , and  $R_l$  as  $R$ . Without loss of generality, we assume that  $L < 0$ . At a later time  $t$ , let the new joint position

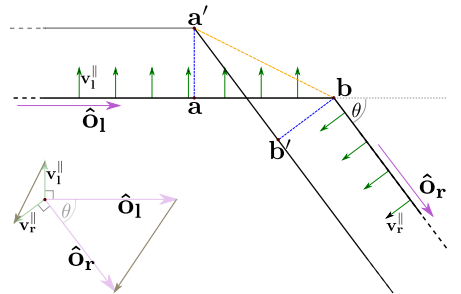


Figure 1: A joint  $J_i$  moving with velocity  $\mathbf{J}_v$  along the normal of segment  $l$ .

be  $\mathbf{J}'_i$ . We define nodes  $a$  and  $b$  corresponding to  $\mathbf{J}_i$  and  $\mathbf{J}_{i+1}$  respectively. We also define the initial and final positions of  $a$  as  $\mathbf{a}$ , and  $\mathbf{a}'$ , and similarly for  $b$ , we define  $\mathbf{b}$  and  $\mathbf{b}'$ . Let  $d$  be the separation between nodes  $a$  and  $b$ . this setup is shown in Figure 1

First, note that  $\mathbf{a}, \mathbf{b}$  lie on the segment  $l$ , and  $\mathbf{a}', \mathbf{b}'$  lie on the segment  $r$ , which implies that  $\mathbf{b} - \mathbf{a} = d \cdot \hat{\mathbf{o}}_l$ , and  $\mathbf{b}' - \mathbf{a}' = d \cdot \hat{\mathbf{o}}_r$ .

$$\mathbf{b}' - \mathbf{a}' = (\mathbf{b} + t \cdot \hat{\mathbf{v}}_r^{\parallel}) - (\mathbf{a} + t \cdot \hat{\mathbf{v}}_l^{\parallel}) \quad (2)$$

$$\implies \mathbf{b}' - \mathbf{a}' = (\mathbf{b} - \mathbf{a}) + t \cdot (\hat{\mathbf{v}}_r^{\parallel} - \hat{\mathbf{v}}_l^{\parallel}) \quad (3)$$

$$\implies d \cdot \hat{\mathbf{o}}_r = d \cdot \hat{\mathbf{o}}_l + t \cdot (\hat{\mathbf{v}}_r^{\parallel} - \hat{\mathbf{v}}_l^{\parallel}) \quad (4)$$

$$\implies \hat{\mathbf{v}}_r^{\parallel} - \hat{\mathbf{v}}_l^{\parallel} = \frac{d}{t} \cdot (\hat{\mathbf{o}}_r - \hat{\mathbf{o}}_l) = -R \cdot (\hat{\mathbf{o}}_r - \hat{\mathbf{o}}_l) = -L \cdot (\hat{\mathbf{o}}_r - \hat{\mathbf{o}}_l) \quad (5)$$

This is only possible if  $\hat{\mathbf{o}}_l \times \hat{\mathbf{o}}_r$  is oriented opposite to  $\hat{\mathbf{v}}_l^{\parallel} \times \hat{\mathbf{v}}_r^{\parallel}$ .

**Property 5.** *Given two adjacent segments  $l$  and  $r$  in a cross section  $C$ , the vector  $\hat{\mathbf{o}}_l \times \hat{\mathbf{o}}_r$  must be oriented opposite to  $\hat{\mathbf{v}}_l^{\parallel} \times \hat{\mathbf{v}}_r^{\parallel}$ .*

If the angle between the segments (between  $\hat{\mathbf{o}}_l$  and  $\hat{\mathbf{o}}_r$ ) is  $\theta$ , the magnitude of  $\hat{\mathbf{o}}_r - \hat{\mathbf{o}}_l$  is  $\sqrt{2 - 2\cos(\theta)}$ , and  $\|\hat{\mathbf{v}}_r^{\parallel} - \hat{\mathbf{v}}_l^{\parallel}\| = v \cdot \sqrt{2 - 2\cos(\pi - \theta)}$ . Here,  $v$  is magnitude of the plane velocity of  $J_i$ . Given  $\phi = \theta/2$ , we get –

$$-L = -R = \frac{d}{t} = \hat{\mathbf{v}}_l - \hat{\mathbf{o}}_l \frac{\|\hat{\mathbf{v}}_r^{\parallel} - \hat{\mathbf{v}}_l^{\parallel}\|}{\|\hat{\mathbf{o}}_r - \hat{\mathbf{o}}_l\|} = v \cdot \frac{\sqrt{\sin^2(\pi/2 - \theta/2)}}{\sqrt{\sin^2 \theta/2}} = v \cdot \cot(\phi) \quad (6)$$

**Property 6.** *The velocity of a joint  $J$  associated with segments  $l$  and  $r$ , is a constant vector  $\mathbf{J}_v = \hat{\mathbf{v}}_l - \|\hat{\mathbf{v}}_l^{\parallel}\| \cdot \hat{\mathbf{o}}_l \cdot \cot(\theta/2)$ , where  $\theta$  is the angle between  $\hat{\mathbf{o}}_l$  and  $\hat{\mathbf{o}}_r$ .*

## 2.4 Cross Section Interval Folding

**Definition 8.** *We consider a cross section composed of segments  $\langle s_1, s_2, \dots, s_n \rangle$  with total length  $X$  (i.e.  $\sum |s_i| = X$ ). If we allow this cross section to evolve for time  $T$ , we obtain a new cross section  $\langle r_1, r_2, \dots, r_n \rangle$ . The evolution forms a cross section interval  $\mathcal{C}$  of length  $T$ . The initial cross section is denoted as  $\mathcal{C}_I = \langle s_1, s_2, \dots, s_n \rangle$ , and the final cross section is denoted as  $\mathcal{C}_F = \langle r_1, r_2, \dots, r_n \rangle$ .*

In this section, we focus on a single cross section interval  $\mathcal{C}$  with segments  $\langle s_1, s_2, \dots, s_n \rangle$  evolving over time  $T$ . First consider the surface traced out by an individual segment  $s_i$ . Since the endpoints of  $s_i$  move in a straight line, each segment traces a trapezoid. We define the *height* of a trapezoid to be the distance between its parallel sides.

**Lemma 1.** *The surface traced out by a segment  $s$  in time  $T$  is a trapezoid  $Z_s$  of height  $T$  (Figure 2a). Specifically, if  $L, R$  are the initial left and right endpoints of  $s$ , and  $L', R'$  are the final endpoints,  $Z_s = LL'R'R$  is the corresponding trapezoid (Figure 2a).*

*Proof.* By Property 6, we know that the joint trajectories are straight lines, which form the non-parallel sides. Since the non-joint nodes on a segment all have the same velocity, the initial and final segment positions form the parallel sides.  $\square$

**Lemma 2.** *Consider a joint  $J$  with segments  $l$  and  $r$ , which has zero orthogonal joint velocity ( i.e.  $\mathbf{v}_l^\perp = \mathbf{v}_r^\perp = \mathbf{0}$ ). The gluing of trapezoids  $Z_l$  and  $Z_r$  along the joint trajectory  $\mathcal{T}_J$  is isometric to a larger trapezoid.*

*Proof.* We consider the evolution of joint  $J$  for time  $T$  (Figure 2b). First consider a coordinate system with  $(\hat{\mathbf{v}}_l, \hat{\mathbf{o}}_l)$  as the basis. Since,  $\hat{\mathbf{v}}_l = \mathbf{v}_l^l$ , from Property 6, we know that  $-R_l = \|\mathbf{v}_l^l\| \cdot \hat{\mathbf{o}}_v \cdot \cot(\phi) = \hat{\mathbf{o}}_v \cdot \cot(\phi)$ . So,  $\mathbf{J}_v = \hat{\mathbf{v}}_l - \hat{\mathbf{o}}_v \cdot \cot(\phi)$ . Therefore,

$$\cot(\angle LJJ') = \frac{\|T \cdot R_l\|}{\|T \cdot (\mathbf{J}_v - R_l)\|} = \cot(\phi) \implies \angle LJJ' = \phi \quad (7)$$

Similarly, we consider a coordinate system with  $(\hat{\mathbf{v}}_r, \hat{\mathbf{o}}_r)$  as the basis. Since  $\mathbf{J}_v = \hat{\mathbf{v}}_r - \hat{\mathbf{o}}_r \cdot \cot(\phi)$ , we get

$$\cot(\angle RJJ') = \frac{-\|T \cdot L_r\|}{\|T \cdot (\mathbf{J}_v - L_r)\|} = -\cot(\phi) \implies \angle RJJ' = \pi - \phi$$

This implies that  $\cot(\angle RJJ') + \cot(\angle LJJ') = \pi$ . So, the gluing of  $Z_l$  and  $Z_r$  along the joint trajectory  $JJ'$  is isometric to a larger trapezoid.  $\square$

**Lemma 3.** *Consider a joint  $J$  with segments  $l$  and  $r$  with non-zero orthogonal velocity. The gluing of trapezoids  $Z_l$  and  $Z_r$  along the joint trajectory  $\mathcal{T}_J$  is isometric to a larger trapezoid.*



endpoint of the  $(i+1)^{\text{th}}$  segment. Similarly, the final-boundary of  $\mathcal{F}_C^T$  is defined as the union of the final segments in  $\mathcal{C}_F$ .

**Theorem 1.** *Consider a cross section interval  $\mathcal{C}$  formed from a cross section  $C$  evolving over time  $T$  to form a folding  $\mathcal{F}_C^T$ . Further assume that the total length of cross section  $C$  is  $X$  units. Then,  $\mathcal{F}_C^T$  is isometric to a  $X \times T$  strip of paper.*

*Proof.* By repeated use of Lemma 3, we know that  $\mathcal{F}_C^T$  is isometric to a trapezoid. Let  $L, L'$  be the initial and final positions of the left (non-parallel) edge of the trapezoid, and let  $R, R'$  be the initial and final positions of the right edge of the trapezoid. Say that  $C$  comprises of segments  $\langle s_1, s_2, \dots, s_n \rangle$ . From Property 4, we know that the left pace of  $s_0$  is zero. So, the line  $LL'$  follows the trajectory of  $\hat{\mathbf{v}}_0$ , which is orthogonal to the segment  $s_0$ . In other words, the left edge of the trapezoid has length  $T$ , and is orthogonal to the parallel edges. Similarly, because the right pace of  $s_n$  is zero, the right edge of the trapezoid is also orthogonal. Therefore,  $\mathcal{F}_C^T$  is isometric to a right angled trapezoid (i.e. a strip) of length  $X$  and width  $T$ .  $\square$

**Proposition 2.** *The trajectory of a joint forms a crease in the folded state.*

## 2.5 Multiple Cross Sections

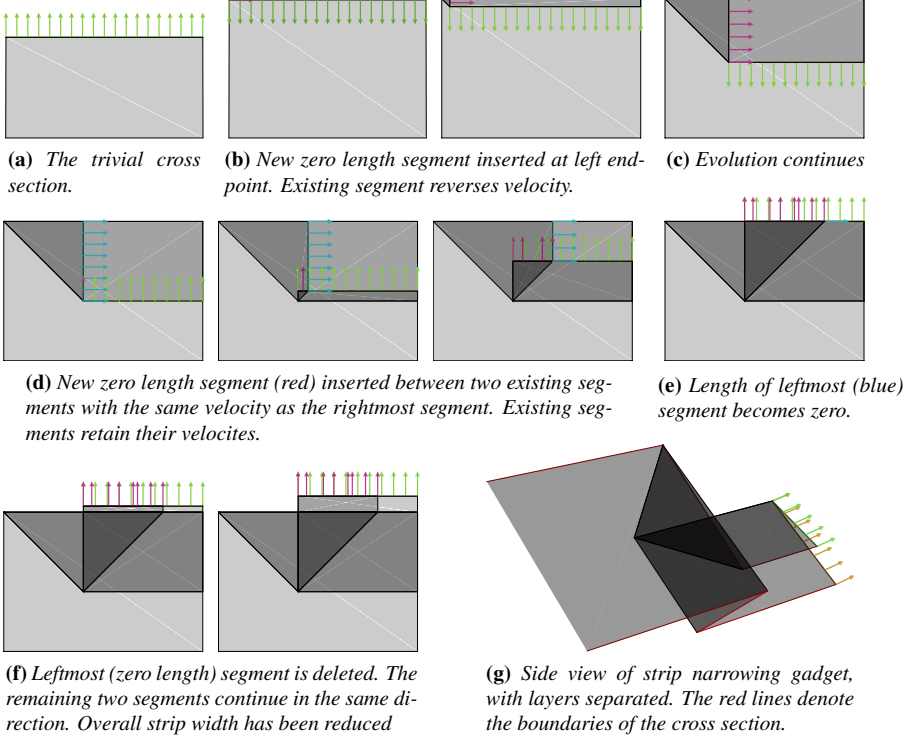
**Definition 11.** *Given two cross section intervals  $\mathcal{C}$  and  $\mathcal{D}$ , such that  $\mathcal{C}_F$  and  $\mathcal{D}_I$  are equivalent, we say that  $\mathcal{D}$  is next-compatible with  $\mathcal{C}$  and  $\mathcal{C}$  is previous-compatible with  $\mathcal{D}$ . Two cross sections  $C = \langle s_1, s_2, \dots, s_n \rangle$  and  $D = \langle r_1, r_2, \dots, r_m \rangle$  are equivalent if and only if  $C$  and  $D$  correspond to the same sequence of segments after the deletion of all zero length segments.*

**Definition 12.** *A cross section sequence is a sequence is an ordered list of cross section intervals  $\langle \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n \rangle$ , such that  $\mathcal{C}_i$  is next-compatible with  $\mathcal{C}_{i-1}$  for all  $i \in [n-1]$ . This is equivalent to stating that  $\mathcal{C}_i$  is previous-compatible with  $\mathcal{C}_{i+1}$  for all  $i \in [n-1]$ . Note that we do not care about the velocities of the segments.*

We will represent our full construction as a valid *cross section sequence*. Given a cross section sequence  $\langle \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n \rangle$ , the transition from  $\mathcal{C}_i$  to  $\mathcal{C}_{i+1}$  corresponds to the deletion of one or more length zero segments from  $\mathcal{C}_i$ , and the addition of one or more zero length segments to obtain  $\mathcal{C}_{i+1}$ . Figure 4 shows a simple example.

**Definition 13.** *Given a cross section sequence  $\langle \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n \rangle$ , We obtain  $\mathcal{F}_i^T$  as the folding of cross section  $\mathcal{C}_i$ . For each  $i \leq n-1$ , we attach  $\mathcal{F}_i^T$  to  $\mathcal{F}_{i+1}^T$  by gluing the final cross section of  $\mathcal{C}_i$  to the initial cross section of  $\mathcal{C}_{i+1}$ . This results in the final folding of the cross section sequence.*

**Proposition 3.** *In addition to the creases formed along the trajectory of joints (Proposition 2), creases are also created when a segment changes velocity. For instance, consider two adjacent cross sections  $\mathcal{C}$  and  $\mathcal{D}$ , with corresponding segments  $s_C \in \mathcal{C}_F$  and  $s_D \in \mathcal{D}_I$ , where  $s_C$  and  $s_D$  are identical except for their velocity direction. Then, the segment  $s_C$  (same as  $s_D$ ) forms a crease in the folded state.*



**Figure 4:** Cross section evolution of a strip narrowing gadget [Demaine et al. 00].

### 2.5.1 Evolution Corresponds to Flat Paper

In this section we will demonstrate that the folding formed by cross section evolution is realizable from a sheet of flat paper. We note here that our construction may still result in self intersections.

We consider a cross section sequence  $\langle \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n \rangle$ , where each cross section interval  $\mathcal{C}_i$  has evolution time  $T_i$ . We then use Theorem 12 to attach the sequence of  $X \times T_i$  strips, to form a complete  $X \times T$  sheet of paper, where  $T = \sum T_i$ .

**Theorem 1.** Consider a cross section sequence  $\mathcal{C} = \langle \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m \rangle$  where each cross section interval  $\mathcal{C}_i$  evolves over time  $T_i$  to form a folding  $\mathcal{F}_i$  such that the following properties hold for all segments and joints in each of the cross sections involved.

**Property 1.** All non-joint nodes on a segment  $s$  move with the same velocity  $\hat{\mathbf{v}}_s$ .

**Property 2.** The velocity  $\hat{\mathbf{v}}_s$  of segment  $s$  is orthogonal to its orientation  $\hat{\mathbf{o}}_s$ .

**Property 3.** For a joint  $J$  associated with segments  $l$  and  $r$ ,  $\hat{\mathbf{v}}_l^\perp = \hat{\mathbf{v}}_r^\perp$  i.e. the joint orthogonal velocities have to be equal, such that the joint plane moves with a fixed velocity along it's normal. As a corollary,  $\|\hat{\mathbf{v}}_l\| = \|\hat{\mathbf{v}}_r\|$ .



**Property 4.** *The right pace of  $s_i$  is equal to the left pace of  $s_{i+1}$ . This is to preserve the overall length of the cross section, and the distance between any two nodes. Furthermore, the left pace of the first segment, and the right pace of the last segment should be zero, i.e.  $L_0 = R_n = 0$ . This ensures that the total length of the cross section does not change.*

**Property 5.** *Given two adjacent segments  $l$  and  $r$  in a cross section  $C$ , the vector  $\hat{\mathbf{o}}_l \times \hat{\mathbf{o}}_r$  must be oriented opposite to  $\hat{\mathbf{v}}_l \times \hat{\mathbf{v}}_r$ .*

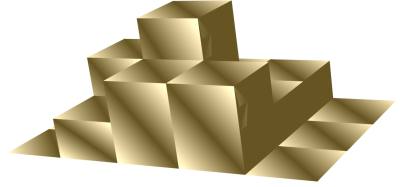
**Property 6.** *The velocity of a joint  $J$  associated with segments  $l$  and  $r$ , is a constant vector  $\mathbf{J}_v = \hat{\mathbf{v}}_l - \|\hat{\mathbf{v}}_l\| \cdot \hat{\mathbf{o}}_l \cdot \cot(\theta/2)$ , where  $\theta$  is the angle between  $\hat{\mathbf{o}}_l$  and  $\hat{\mathbf{o}}_r$ .*

Then, the folding  $\mathcal{F}_\mathcal{C}$  obtained by successively gluing the final boundary of  $\mathcal{F}_i$  to the initial boundary of  $\mathcal{F}_{i+1}$  (for each  $1 \leq i < m$ ), is isometric to a  $X \times T$  strip of paper, where  $T = \sum T_i$ .

*Proof.* From Theorem 1, we know that each  $\mathcal{F}_i$  is isometric to a strip  $Z_i$  of size  $X \times T_i$ . Further, the initial and final cross sections of  $\mathcal{C}_i$  form the parallel sides of  $Z_i$  (of length  $X$ ). The gluing of  $n$  strips of size  $X \times T_i$  forms a strip of size  $X \times T$ . So, the final folding is isometric to a sheet of paper with size  $X \times T$ .  $\square$

### 3 Orthogonal Terrains

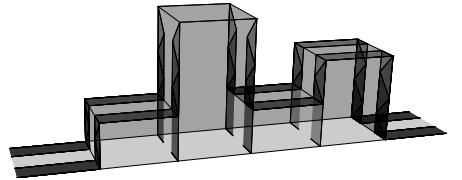
In this section, we outline a construction of orthogonal terrains with arbitrary rational extrusion heights. In our construction, the cross section at will always be on the  $x - z$  plane. To simplify the presentation, we will consider an uniform  $X - Y$  grid, with arbitrary rational extrusion heights corresponding to every grid square (Figure 5).



**Figure 5:** Example orthogonal terrain.

**Definition 14.** *An  $n \times m$  rational grid extrusion is a 3-dimensional structure, whose projection onto the  $x - y$  plane forms an unit grid of size  $n \times m$ . In the 3-dimensional structure, the unit face corresponding to the location  $(i, j)$ , exists at height  $E_{i,j}$  (in the  $z$  direction), where  $E_{i,j}$  is a rational number.*

We consider each "column" of a given grid extrusion separately as an individual *column extrusion* (Figure 6). We will construct each of the  $n$  columns independently (Figure 8), and attach them together with *column connectors* (Figure 12).



**Figure 6:** Column extrusion with heights  $\{0, 1, 3, 1, 2, 0\}$ .

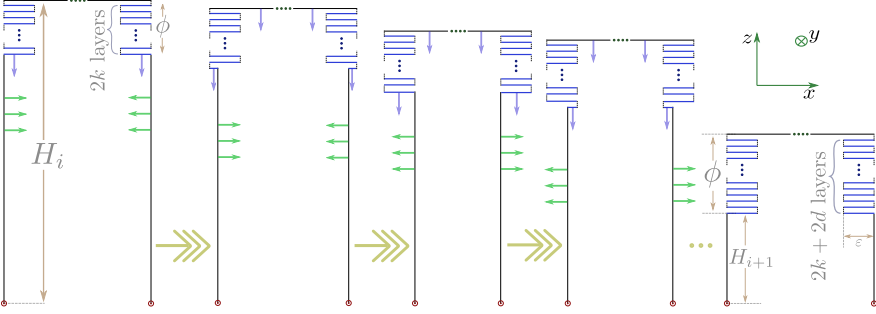
We begin by choosing an  $2\epsilon = 1/K$  where  $K$  is a positive integer. This is chosen such that  $E_{i,j}$  is an integral multiple of  $2\epsilon$  for all  $i, j$ .

### 3.1 Construction of Column Extrusion by Level Shifting

First, we consider a single column (Figure 6) of the orthogonal terrain  $\{E_{i1}, E_{i2}, \dots, E_{in}\}$ . We denote the column extrusion heights as  $\{H_1, H_2, \dots, H_n\}$ , where  $H_j = E_{i,j}$ . Consider the decomposition of  $T$  into the following time intervals.

$$T = 1 + D_1 + 1 + D_2 + 1 + D_3 + \dots + 1 + D_{m-1} + 1 \quad (8)$$

Here, the time corresponding to the  $i^{\text{th}}$  1 is realized as the surface at height  $H_i$ , and the time corresponding to  $D_i = |H_i - H_{i+1}|$  is the transition between  $H_i$  and  $H_{i+1}$ .

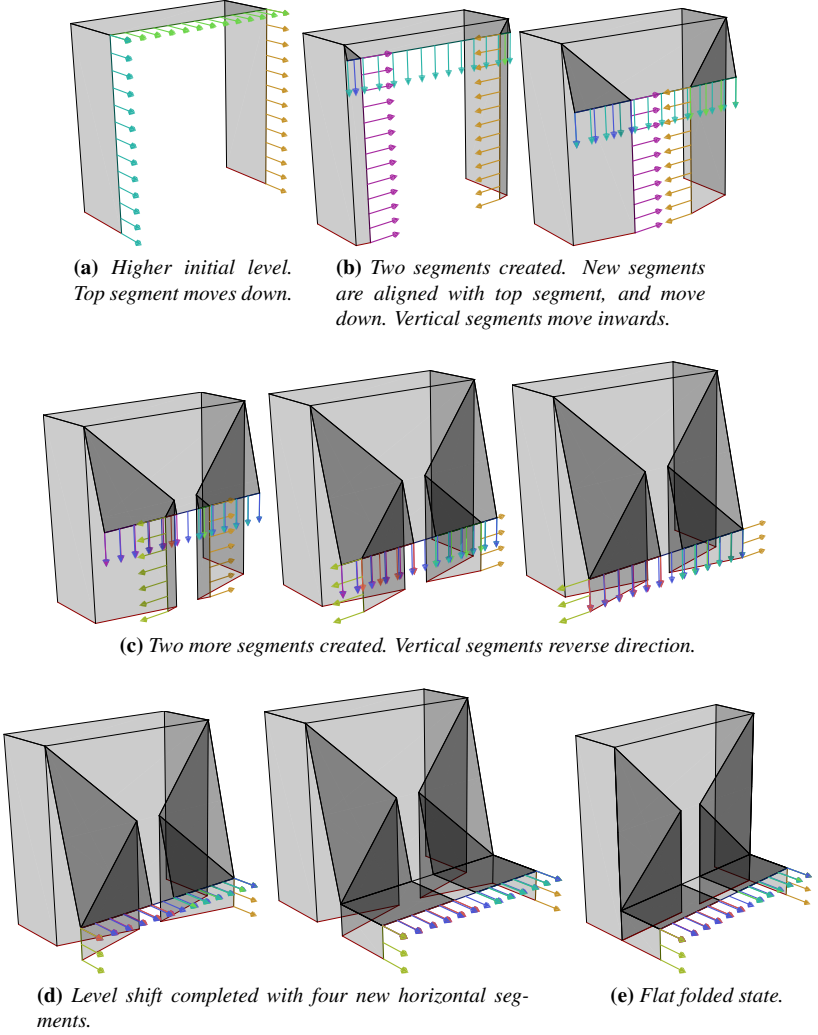


**Figure 7:** Cross section change from level  $i$  to  $i + 1$ . The accordion segments are separated for illustration. In reality, the accordion is folded flat. Zero distances are marked by  $\phi$ . The velocities of horizontal and vertical segments are shown by purple and green arrows respectively.

To construct the column, we will present a cross section sequence. First we describe a *down-shift* gadget. That is, consider  $i$ , such that  $H_{i+1} = H_i - 2\varepsilon \cdot d < H_i$ . Figure 7 shows the cross section evolution. This cross section comprises of a two vertical lines separated by a top horizontal line. The vertical lines are connected to the top segment with a sequence of  $2k$  horizontal segments that *accordion* back and forth. During each 1-interval, all segments move along the positive  $y$  direction (Figure 8a), to create the  $i^{\text{th}}$  level. Subsequently, during the level shift, all segments move in the  $x - z$  plane (Figure 8b, 8c).

**Property 1.** *The number of accordion folds during horizontal evolution (along the  $y$  axis) must be even.*

The top segment moves downwards in intervals of  $2\varepsilon$ . During this process, the horizontal segments move downwards continuously (Figure 8b, 8c). For the first  $\varepsilon$  time interval, a new horizontal downwards moving accordion segment of length zero is created at both accordions, and the vertical segments move towards each other along the  $x$ -axis (Figure 8b). For the next  $\varepsilon$  time interval, similar (oppositely oriented) accordion segments are created at the lowest position. This time, the vertical segments move outwards (Figure 8c), until they reach their original position (Figure 8d). Overall, two sets of accordion segments on either side are added, and the height of the top segment decreases by  $2\varepsilon$ .



**Figure 8:** Level shifting gadget. The separation along the  $Y$  direction illustrates the layering. The red line denotes the boundary of the cross section.

In the case that  $H_{i+1} = H_i + 2\varepsilon \cdot d$ , the level up-shift is simply the down-shift evolution in reverse (Figure 12). This transition is only possible if the initial number of accordion segments in the  $H_i$  cross section is at least  $2d$ . Specifically, assuming that the minimum height is zero, we have the following lemma.

**Lemma 4.** *If the number of accordion segments at level  $H_i$  is  $l_i$ , then the number of accordion segments after transitioning to level  $H_{i+1}$  is  $l_{i+1} = l_i - (H_{i+1} - H_i)/\varepsilon$ . Specifically, if the number of layers at level 0 is  $l$ , then the number of layers at level*

$\max \{H_i\}$  is  $l - \max \{H_i\} / \varepsilon$ .

**Corollary 1.** *Since the number of accordion segments can never be negative, the minimum number of layers at level zero is  $L = \max \{H_i\} / \varepsilon$ . This also ensures that every other level shift is also possible.*

**Corollary 2.** *The length of the cross section at a zero level is at least  $1 + 2 \cdot L \cdot \varepsilon$ . So, the minimum possible length of the cross section under our construction is  $1 + 2 \cdot \max \{H_i\}$ .*

This provides the minimum width of any strip required to construct a column extrusion. Also the minimum required strip length is given by

$$T = 1 + D_1 + 1 + D_2 + 1 + D_3 + \cdots + 1 + D_{m-1} + 1 = m + \sum_{i=1}^{m-1} |H_{i+1} - H_i|$$

**Theorem 2.** *A given column extrusion with heights  $\{H_1, H_2, \dots, H_n\}$ , can be constructed from a strip of paper with size  $X \times T$ , where*

$$X \geq 1 + 2 \cdot \max \{H_i\} \quad T \geq \left( m + \sum_{i=1}^{m-1} |H_{i+1} - H_i| \right) \quad (9)$$

### 3.2 Multiple Column Extrusions form a Grid Extrusion

Now, we consider mutiple column extrusions evolving in parallel. Henceforth, we will refer to the evolution of column cross sections along the y-axis (the "1"s in Equation 8 as *horizontal evolution*. Meanwhile, a *vertical transition* will refer to level shifting evolution in the  $x-z$  plane. Let  $\mathcal{C}^{(i)}$  be a valid cross section evolution corresponding to the  $i^{\text{th}}$  column in the grid extrusion (as defined in Section 3.1). As before, for simplicity, we will assume that the minimum height in each column is zero (i.e.  $\min_j \{E_{i,j}\} = 0$ ).

We consider the parallel evolution of each column extrusion  $\mathcal{C}^{(i)}$ .

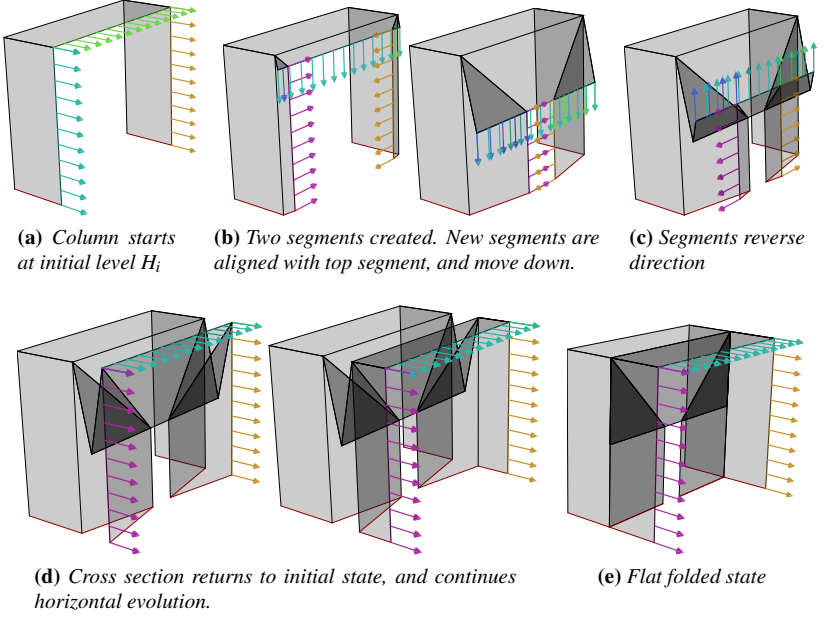
- During the horizontal evolution of row  $j$ , each  $\mathcal{C}^{(i)}$  evolves along the positive y direction for time 1 at height  $E_{i,j}$ .
- During the vertical transition from row  $j$  to  $j+1$ , each  $\mathcal{C}^{(i)}$  evolves for time  $D_{ij} = |E_{i,j+1} - E_{i,j}|$ .

Note that the vertical transition times are different for each  $\mathcal{C}^{(i)}$ . Since we want to glue the sequence of  $\{\mathcal{C}^{(i)}\}$ s, our constructions must have equal transition times.

**Definition 15.** *We define the common transition time from row  $j$  to row  $j+1$  as*

$$D_j = \max_i \{D_{ij}\} = \max_i \{|E_{i,j+1} - E_{i,j}|\}$$

So, the slowest column dictates the transition time, and the faster columns have to *stall* for additional time. To achieve this, we define an *up-down* gadget, which



**Figure 9:** Up-down gadget. The separation along the  $Y$  direction illustrates the layering. The red line denotes the boundary of the cross section.

is very similar to our original level shifting gadget. This up-down gadget (Figure 9) evolves for  $2\epsilon$  time, but the height of the corresponding column remains unchanged. This gadget starts at a height  $H_i$ , and for the first  $\epsilon$  time interval (Figure 9b), evolves exactly the same way as the down-shift gadget (Figure 8b). For the second  $\epsilon$  time interval, the cross section evolves in reverse (Figure 9c), back to its original state at height  $H_i$  (Figure 9d).

So, the vertical transition of  $\mathcal{C}^{(i)}$  needs to use a total of  $(D_j - D_{i,j}) / (2\epsilon)$  up-down gadgets (each gadget stalls for  $2\epsilon$  time). We obtain the following primitive, as a consequence of Theorem 2.

**Proposition 4.** *By Theorem 2, the extrusion for column  $i$  is constructed from a paper strip with size*

$$\left(1 + 2 \cdot \max_j \{E_{ij}\}\right) \times \left(m + \sum_{j=1}^{m-1} D_j\right)$$

### 3.3 Gluing Column Extrusions together with Strip Connectors

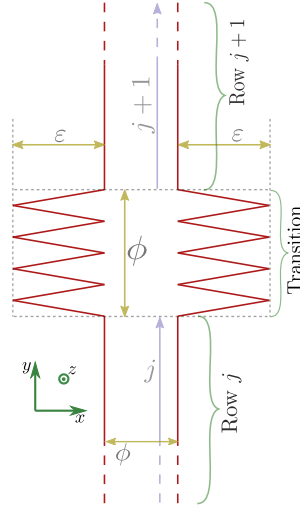
Now that all the column extrusions  $\{\mathcal{C}^{(i)}\}$  have the same evolution time, we would like to glue them together into a continuous strip of paper. In order to achieve this, we consider the time-axis boundaries of  $\mathcal{C}^{(i)}$  (red line in Figure 8, 9). First, note

that the boundaries always lie on the same plane, corresponding to the zero level. We will constrain this to be on the plane  $z = 0$ . Now, let us consider the motion of the boundary on this plane.

- During the horizontal evolution of any row  $j$ , both boundaries move along the positive  $y$ -axis with unit velocity (Figure 8a, 8d, 8e, 9a, 9d, 9e).
- During the vertical transition from row  $j$  to  $j + 1$ , both boundaries move back and forth along the  $x$ -axis with unit velocity. (Figure 8b, 8c, 9b, 9c). We can divide the vertical transition into  $k = D_j/(2\epsilon)$  intervals of length  $2\epsilon$ . Each of these interval segments is either a up-shift, a down-shift, or a up-down gadget.
  - In the first half of each interval ( $\epsilon$  time), the left and right boundaries move towards each other; i.e. the left boundary moves along the positive  $x$ -axis, and the right boundary moves along the negative  $x$ -axis for a distance of  $\epsilon$ .
  - In the second half of each interval, the left and right reverse velocities, and return to their original positions.
  - After  $D_j$  time, the boundaries return to their original positions, and resume their movement in the positive  $y$  direction.

We will attempt to construct a cross section sequence whose left and right boundaries line up with the adjacent column extrusion boundaries shown in Figure 10. Notice that the distance between *corresponding points* on the boundaries varies between 0 and  $2\epsilon$ . So, the connector strip must have width at least  $2\epsilon$ .

We outline a construction of a  $2\epsilon$  width strip, as shown in Figure 11. During horizontal evolution, the cross section comprises of two vertical segments, each of length  $\epsilon$ , which move along the same trajectory in the positive  $y$ -direction (during horizontal evolution Figure 12a, 12k). Now, consider a transition of length  $D_j$  divided into intervals of length  $2\epsilon$ , each of which evolves according to Figure 11.

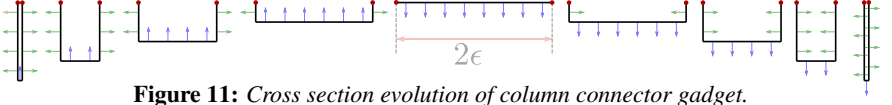


**Figure 10:** Boundaries of adjacent column extrusions ( $D_j = 8\epsilon$ ).  $\phi$  denotes a zero distance.

- The vertical segments move outwards with unit velocity to match the outward moving boundaries of the adjacent strips. An upwards moving horizontal segment of length zero is created between the two existing segments (Figure 12b).
- After time  $\epsilon$ , the length of the vertical segments become zero, and the horizontal segment spans the  $2\epsilon$  gap between the column boundaries (Figure 12c). Notice that that the boundary of the connector maintains its  $z$  coordinate (Figure 11).
- Then the connector segments reverse velocities, and retrace their path for  $\epsilon$  time (Figure 12d), until the vertical segments become length  $\epsilon$ , and the horizontal segment disappears (Figure 12e).

- The entire process repeats  $D_j/(2\epsilon)$  times (Figure 12f, 12g, 12h).

The completed connector gadget is shown in Figure 12i, 12j. We show the *connector gadget* attached to an *up-shift gadget* in Figure 12h, 12k.



**Figure 11:** Cross section evolution of column connector gadget.

### 3.4 Size of Construction

We define  $M_i = \max_j \{E_{ij}\}$ . From Proposition 4, we know that each column extrusion strip  $\mathcal{C}^{(i)}$  has width  $(1 + 2 \cdot \max_j \{E_{ij}\}) = 1 + 2 \cdot M_i$ . We can ignore the left, and right vertical strips for the leftmost, and rightmost columns. So, the leftmost and rightmost columns use strips of width  $(1 + M_0)$  and  $(1 + M_n)$  respectively.

Additionally, we have  $n - 1$  strip connectors, each of width  $2\epsilon$ . So, the total width of the orthogonal terrain construction is

$$X = 2(n - 1) \cdot \epsilon + n + 2 \cdot \sum_{i=1}^n M_i - M_0 - M_n$$

Now, note that our construction is also valid for any  $\epsilon' = \epsilon/(2k)$ , where  $k$  is an integer. In other words, we can make  $\epsilon$  arbitrarily small.

**Theorem 3.** *Using the time evolution (y dimension) from Proposition 4, we conclude that a grid extrusion can be folded from a strip of size  $X \times T$ , where*

$$X = n + 2 \cdot \sum_{i=1}^n M_i - M_0 - M_n + o(1) \quad T = m + \sum_{j=1}^{m-1} D_j \quad (10)$$

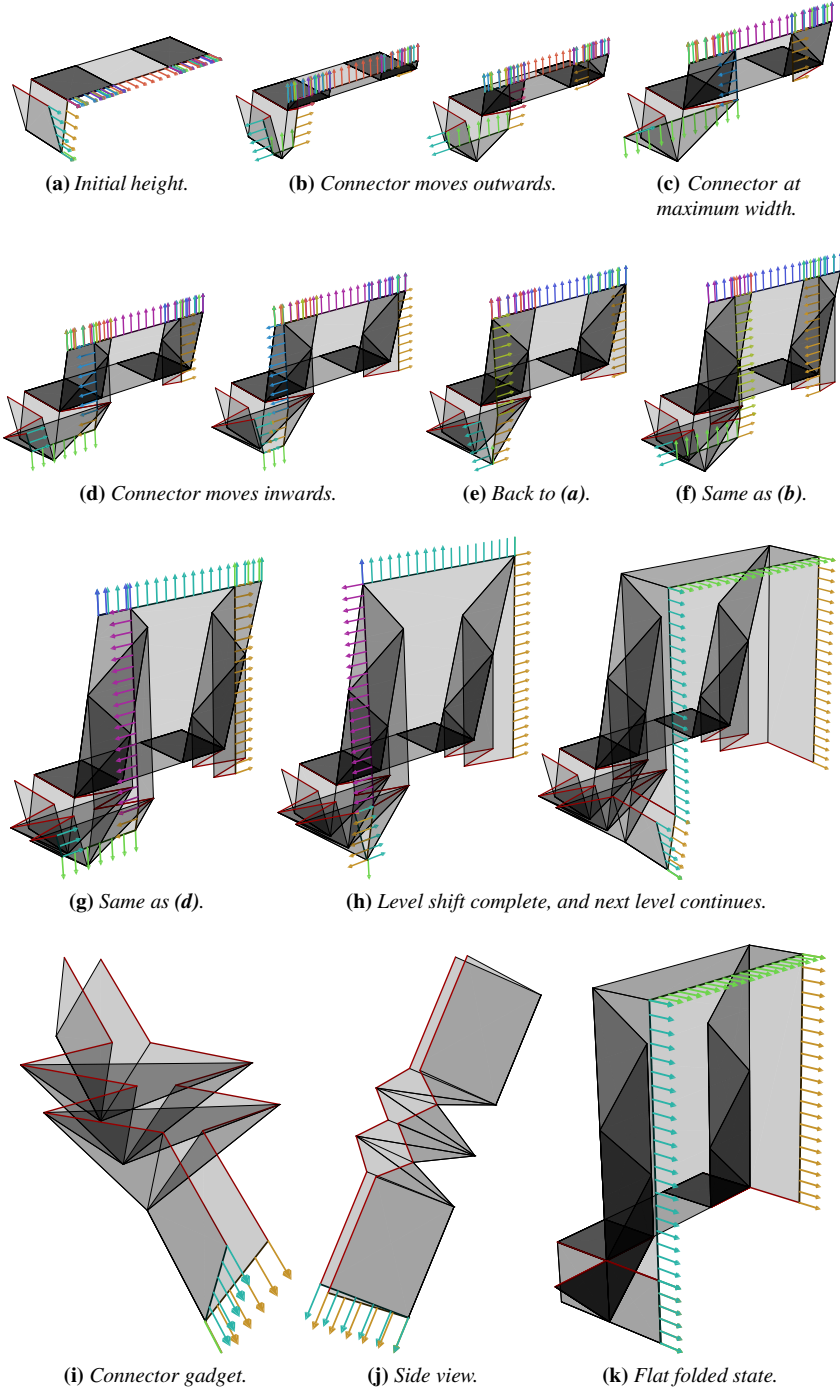
### 3.5 Optimality

Consider a  $n \times n$  orthogonal terrain where the highest points along  $x$ -axis are  $M = \max_j E_{ij}$ , and the maximum deltas along the  $y$ -axis are  $D_j = \max_i \|E_{i,j} - E_{i,j+1}\| = M$ . Now, we consider a sort of worst case orthogonal terrain defined as follows.

$$E_{ij} = \begin{cases} M, & \text{if } (i + j) \text{ is even} \\ 0, & \text{if } (i + j) \text{ is odd} \end{cases}$$

This forms a checkerboard alternating between the minimum and maximum heights. We assume that the top faces of the terrain form axis aligned squares in the unfolded state, The longest line formed along either axis is the total length of the top surfaces plus the sum of the height changes across the axis, which gives

$$L = n + \sum_{i=1}^n M = n + (n - 1) \cdot M$$



**Figure 12:** Column gadget attached to a single column connector gadget. The red line demarcates the interface between the two gadgets.



We conclude that the minimum required size of the folding is  $L \times L$ .

From Theorem 3, we know that the terrain can be folded from a  $X \times Y$  strip of paper, where  $X = n + 2(n - 1) \cdot M + o(1)$  and  $Y = n + (n - 1) \cdot M$ . Since  $Y = L$ , and  $X < 2L$  (assuming an appropriately small value of  $\epsilon$ ), our construction results in a folding is within a factor two of the optimal paper usage, under the aforementioned assumption.

## Acknowledgments

We thank Martin Demaine, Herng Yi Cheng, Aviv Ovadya, and Tomohiro Tachi for helpful early discussions about this family of problems at 5OSME.

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Amartya Shankha Biswas, Erik D. Demaine, and Jason S. Ku  
 Massachusetts Institute of Technology, 32 Vassar St., Cambridge, MA 02139  
 e-mail: [asbiswas@mit.edu](mailto:asbiswas@mit.edu), e-mail: [edemaine@mit.edu](mailto:edemaine@mit.edu), and e-mail: [jasonku@mit.edu](mailto:jasonku@mit.edu)