Paper Title

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Abstract: *The abstract must summarise the main content of the paper, and may be up to approximately 100 words in length. Unlike previous OSME publications, the abstract will be included in the proceedings. This abstract needs not be the same as that submitted for acceptance to the conference.*

1 Introduction

Outline

2 Extruded Surfaces (Preliminaries)

3 Cross Sections

We introduce a new method of origami construction, using cross section disgrams. Instead of beginning our construction from a 2-dimensional sheet of paper, we consider a 1-dimensional cross section moving forwards in time. A simple example is demonstrated in Figure 4.

Conservation of length

4 Segments and Cross Sections

Definition 1. *A segment s is an oriented line segment with left and right endpoints $\mathbf{s}_l$ and $\mathbf{s}_r$. Each segment is also associated with an orientation vector $\hat{\mathbf{o}}_\mathbf{s} = \frac{\mathbf{s}_r - \mathbf{s}_l}{\|\mathbf{s}_r - \mathbf{s}_l\|}$. This vector serves to disambiguate the orientation of zero length segments.*

Definition 2. *A cross section C is defined as an ordered list of line segments $\langle s_1, s_2, \cdots, s_n \rangle$, such that for every segment s_i (except the last), the right endpoint of s_i coincides with the left endpoint of s_{i+1}. Each segment s is also associated with a velocity vector $\hat{\mathbf{v}}_\mathbf{s}$ of unit magnitude. For a segment $s \in C$ we will also denote this velocity as $\hat{\mathbf{v}}_\mathbf{i}$.*

Property 1. *The velocity $\hat{\mathbf{v}}_\mathbf{s}$ of segment s is orthogonal to its orientation $\hat{\mathbf{o}}_\mathbf{s}$.*

4.1 Joints

Definition 3. *A cross section with n segments is also associated with a list of joints* $\langle J_1, \cdots J_{n-1} \rangle$, *where* J_i *corresponds to the right endpoint of* s_i *(same as left endpoint of* s_{i+1}. *A particular joint* $\mathbf{J}_i$ *is associated with a left segment* $J_l = s_i$, *and a right segment* $J_r = s_{i+1}$.

Definition 4. *A joint plane is the plane that coincides with both segments l and r associated with a particular joint J.*

Definition 5. *Consider a joint J associated with segments* $l = J_l$ *and* $r = J_r$ *with joint plane* $\mathscr{P}$, *where* $\mathbf{v}_l$ *and* $\mathbf{v}_r$ *are the velocities of segments l and r. We define* $\mathbf{v}_l^{\shortparallel}$ *and* $\mathbf{v}_l^{\perp}$ *as the components of* $\mathbf{v}_l$ *coinciding with, and orthogonal to* $\mathscr{P}$ *respectively. Similarly, we define* $\mathbf{v}_r^{\shortparallel}$ *and* $\mathbf{v}_r^{\perp}$, *as the components of* $\mathbf{v}_r$. *By Property 1,* $\mathbf{v}_l^{\shortparallel}$ *and* $\mathbf{v}_r^{\shortparallel}$ *have to be orthogonal to* $\hat{\mathbf{o}}_\mathbf{l}$ *and* $\hat{\mathbf{o}}_\mathbf{r}$ *respectively.*

Property 2. *For a joint J associated with segments l and r,* $\mathbf{v}_l^{\perp} = \mathbf{v}_r^{\perp}$. *As a corollary,* $\left\|\mathbf{v}_l^{\shortparallel}\right\| = \|\mathbf{v}_r^{\shortparallel}\|$.

4.2 Time Travel

Lemma 1. *All points on a single segment s move with the same velocity* v_i.

In the process of time travel, the lengths of segments may change. This can also be visualized as movement of the corresponding joint along one of the segments. The movement of a joint increases the length of one of its associated segments, and decreases the length of the other segment by the same amount (this ensures that the total length is preserved). Most velocity sequences are invalid. For example...

We will use some abuse of notation, by using $\mathbf{v}_i, \mathbf{o}_i$ etc. to refer to the corresponding parameters of s_i.

Note that the trajectory of the joints form a crease ... angle is angle between direction vectors. Creases are also created when a segment changes direction... angle is change in direction vector.

Proposition 1. *A joint J corresponding to segments s and t is* valid *if and only if the resulting evolution of the joint preserves distances.*

Definition 6. *For ever segment s in a cross section C, we associate a left velocity* L_s, *which indicates the rate at which s grows from its left endpoint. Similarly, we define a right velocity* R_s ...

Property 3. *The right velocity of* s_i *is the negation of the left velocity of* s_{i+1}. *This is to ensure the preservation of distance between any two nodes. Furthermore, the left velocity of the first segment, and the right velocity of the last segment should be zero, i.e.* $L_0 = R_n = 0$ *This ensures that the total length of the cross section does not change.*

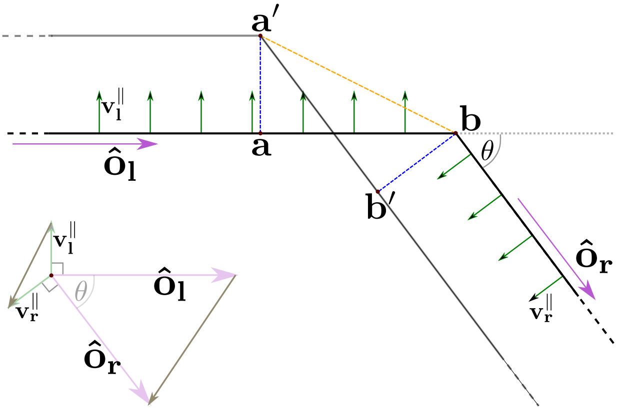


Figure 1: *A joint with segments l and r. The trajectory of the joint is shown in orange. The trajectories of a and b are shown in blue. The green arrows indicate* $\mathbf{v}^{||}$

Define Nodes

Consider a joint $\mathbf{J}_i$ corresponding to segments $l = s_i$ and $r = s_{i+1}$, at time $t = 0$. Henceforth, we will refer to L_l as L, and R_r as R. Without loss of genenerality, we assume that $L < 0$. At a later time t, let the new joint position be $\mathbf{J}'_i$. We define nodes a and b corresponding to $\mathbf{J}_i$ and $\mathbf{J}_{i+1}$ respectively. We also define the initial and final positions of a as $\mathbf{a}$, and $\mathbf{a}'$, and similarly for b, we define $\mathbf{b}$ and $\mathbf{b}'$. Let d be the *separation* between nodes a and b. this setup is shown in Figure 1

First, note that $\mathbf{a}, \mathbf{b}$ lie on the segment l, and $\mathbf{a}', \mathbf{b}'$ lie on the segment r, which implies that $\mathbf{b} - \mathbf{a} = d \cdot \hat{\mathbf{o}}_\mathbf{l}$, and $\mathbf{b}' - \mathbf{a}' = d \cdot \hat{\mathbf{o}}_\mathbf{r}$.

$$\mathbf{b}' - \mathbf{a}' = (\mathbf{b} + t \cdot \mathbf{v}_r^{||}) - (\mathbf{a} + t \cdot \mathbf{v}_l^{||}) \tag{1}$$

$$\implies \mathbf{b}' - \mathbf{a}' = (\mathbf{b} - \mathbf{a}) + t \cdot (\mathbf{v}_r^{||} - \mathbf{v}_l^{||}) \tag{2}$$

$$\implies d \cdot \hat{\mathbf{o}}_\mathbf{r} = d \cdot \hat{\mathbf{o}}_\mathbf{l} + t \cdot (\mathbf{v}_r^{||} - \mathbf{v}_l^{||}) \tag{3}$$

$$\implies \mathbf{v}_r^{||} - \mathbf{v}_l^{||} = \frac{d}{t} \cdot (\hat{\mathbf{o}}_\mathbf{r} - \hat{\mathbf{o}}_\mathbf{l}) \tag{4}$$

$$= R \cdot (\hat{\mathbf{o}}_\mathbf{r} - \hat{\mathbf{o}}_\mathbf{l}) = -L \cdot (\hat{\mathbf{o}}_\mathbf{r} - \hat{\mathbf{o}}_\mathbf{l}) \tag{5}$$

This implies that $\hat{\mathbf{o}}_\mathbf{l} \times \hat{\mathbf{o}}_\mathbf{r}$ is oriented opposite to $\mathbf{v}_l^{||} \times \mathbf{v}_r^{||}$. If the angle between the segments (between $\hat{\mathbf{o}}_\mathbf{l}$ and $\hat{\mathbf{o}}_\mathbf{r}$) is θ, the magnitude of $\hat{\mathbf{o}}_\mathbf{r} - \hat{\mathbf{o}}_\mathbf{l}$ is $\sqrt{2 - 2\cos(\theta)}$, and

$|\mathbf{v}_r^{\shortparallel} - \mathbf{v}_l^{\shortparallel}| = v \cdot \sqrt{2 - 2\cos(\pi - \theta)}$. Here, v is magnitude of the plane velocity of J_i.

$$R = \frac{d}{t} = \frac{|\mathbf{v}_r^{\shortparallel} - \mathbf{v}_l^{\shortparallel}|}{|\hat{\mathbf{o}}_\mathbf{r} - \hat{\mathbf{o}}_\mathbf{l}|} \tag{6}$$

$$= v \cdot \frac{\sqrt{\sin^2(\pi/2 - \theta/2)}}{\sqrt{\sin^2 \theta/2}} \tag{7}$$

$$= v \cdot \cot\left(\frac{\theta}{2}\right) \tag{8}$$

Property 4. *The velocity of a joint J associated with segments l and r, is a constant vector* $\mathbf{J}_v = \hat{\mathbf{v}}_\mathbf{l} + \hat{\mathbf{o}}_\mathbf{l} \frac{|\mathbf{v}_r^{\shortparallel} - \mathbf{v}_l^{\shortparallel}|}{|\hat{\mathbf{o}}_\mathbf{r} - \hat{\mathbf{o}}_\mathbf{l}|}$

Surface traced out by cross section

This can result in a segment length becoming zero then delete. We may create a new segment at any point of zero length.

Note that time evolution is reversible.

4.3 Cross Section Interval Folding

Definition 7. *We consider a cross section C composed of segments $\langle s_1, s_2, \cdots, s_n\rangle$ with total length X (i.e. $\sum |s_i| = X$). If we allow this cross section to evolve for time T, we obtain a new cross section $C_T = \langle r_1, r_2, \cdots, r_n\rangle$. The evolution from C to C_T forms a cross section interval $\mathscr{C}$ of length T.*

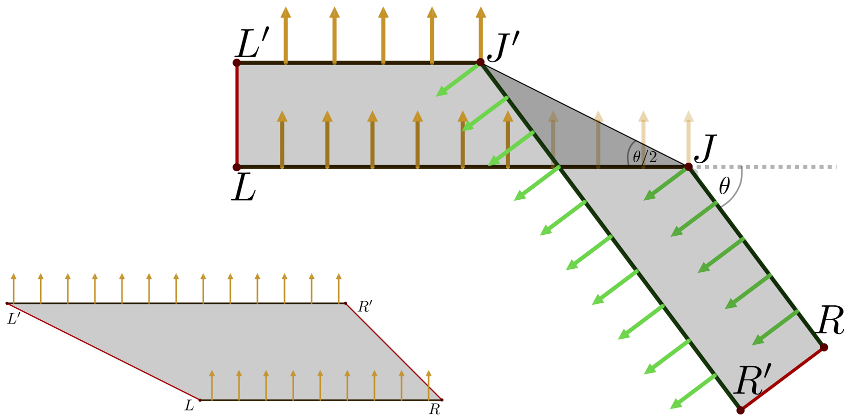
Definition 8. *Say that the total length of the cross sections in $\mathscr{C}$ is X (we showed in ... that length is conserved). We will denote a strip of paper of size $X \times L$ as an L-strip.*

4.3.1 Interval Folding results in Flat Strip

In this section, we focus on a single cross section interval $\mathscr{C}$ with segments $\langle s_1, s_2, \cdots s_n\rangle$ evolving over time T. First consider the surface traced out by an individual segment s_i. Since the endpoints of s_i move in a straight line, each segment traces a trapezoid.

Definition 9. *Given a cross section interval $\mathscr{C}$ formed from a cross section C evolving over time T, we define the folding of $\mathscr{C}$ as the surface swept out by C from over time T. Precisely*

1. *The folding of a the i^{th} segment is the trapezoid formed by the initial and final state of the segment (s_i and r_i). This forms a trapezoid of height T (Figure 2a). The non-parallel sides are straight lines due to Property 4. In case either the initial or final segment has length, the resulting trapezoid will be a triangle.*



(a) **(b)**

Figure 2

2. *For every joint J_i, we attach the corresponding trapezoids for s_i and s_{i+1} along the joint trajectory $\mathscr{T}_{J_i}$ to obtain the folding of $\mathscr{C}$.*

First, we will show that each $\mathscr{C}_i$ corresponds to a T_i-strip.

Definition 10. *The surface traced out by a segment s is a trapezoid Z_s (Figure 2a). Specifically, if L, R are the initial left and right endpoints of s, and L', R' are the final endpoints, $Z_s = LL'R'R$ is the corresponding trapezoid (Figure 2a).*

Define gluing

Lemma 2. *Consider a joint J with segments l and r, which has zero orthogonal joint velocity (i.e. $\mathbf{v}_l^{\parallel} = \mathbf{v}_r^{\parallel} = 0$. The gluing of trapezoids Z_l and Z_r along the joint trajectory $\mathscr{T}_J$ is isometric to a larger trapezoid. This joint trajectory is actually nothing but a crease in the folded state.*

Lemma 3. *Consider a joint J with segments l and r with non-zero orthogonal velocity. The gluing of trapezoids Z_l and Z_r along the joint trajectory $\mathscr{T}_J$ is isometric to a larger trapezoid.*

Proof. As before, let LJR and $L'J'R'$ represent the initial and final positions of the segments respectively. We also construct the projection of $L'J'R'$ to the joint plane $\mathscr{P}$ as $L^{\parallel}J^{\parallel}R^{\parallel}$. The evolution of the projection is analogous to the setting in Lemma 2. Therefore, $\angle LJJ^{\parallel} = \phi = \theta/2$ and $\angle RJJ^{\parallel} = \pi - \phi$.

Consider the positive z-axis along the joint orthogonal velocity (i.e. normal to the joint plane $\mathscr{P}$). We define the orthogonal diaplacement vector as $\overrightarrow{JJ'} = z \cdot \hat{\mathbf{k}}$. Let the positive x-axis be along $JJ^{\parallel}$. So, the unit vector along $\overrightarrow{JJ'}$ is $\frac{1}{\sqrt{1+z^2}}(1,0,z)$.

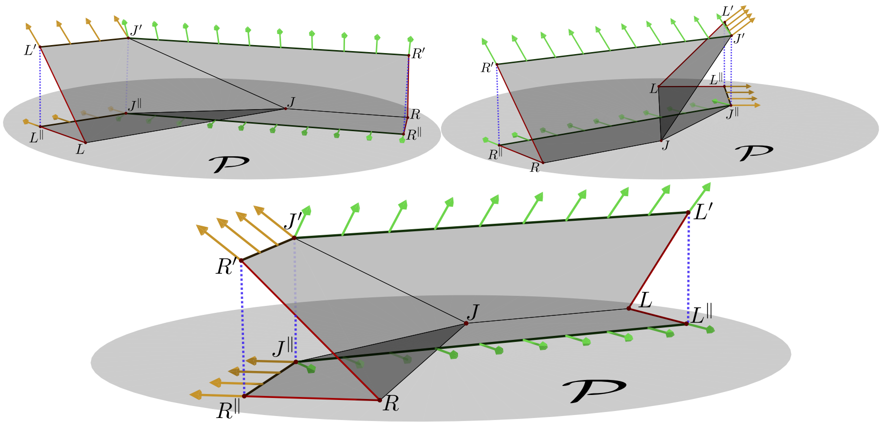


Figure 3: *Evolution of a joint with non-zero orthogonal velocity from LJR to $L'J'R'$. The blue dotted lines represent the projection of the final state to the joint plane $\mathscr{P}$.*

Since $\angle J^{\shortparallel}JR = \phi$, the unit vector along $\overrightarrow{J}R$ is $(\cos\phi, \sin\phi, 0)$. So, we compute $\cos\angle RJJ' = \cos\phi/\sqrt{1+z^2}$. Similarly, since $\angle J^{\shortparallel}JL = \pi - \phi$, the unit vector along $\overrightarrow{J}L$ is $(-\cos\phi, \sin\phi, 0)$, which implies that $\cos\angle LJJ' = -\cos\phi/\sqrt{1+z^2}$. Finally, since both $\angle LJJ$ and $\angle RJJ'$ are less than π, and $\cos(\angle LJJ') = -\cos(\angle RJJ')$, we conclude that $\angle LJJ' + \angle RJJ' = \pi$.

Since $LJJ'L'$ and $RJJ'R'$ are both trapezoids, this implies that the resulting gluing along JJ' is a larger trapezoid. $\square$

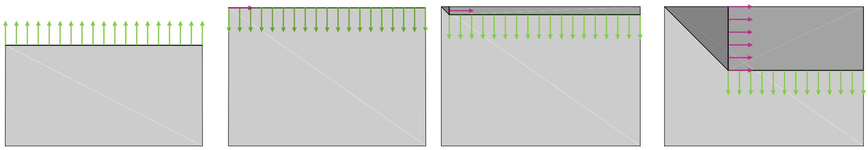
Theorem 1. *Consider a cross section interval $\mathscr{C}$ formed from a cross section C evolving over time T to form a folding F. Further assume that the total length of cross section C is X units. Then, F is isometric to a $X \times T$ strip of paper.*

Proof. From Lemma 3, we know that F is isometric to a trapezoid. Let L, L' be the initial and final positions of the left (non-parallel) edge of the trapezoid, and let R, R' be the initial and final positions of the right edge of the trapezoid. Say that C comprises of segments $\langle s_1, s_2, \cdots s_n\rangle$. From Property 3, we know that the left velocity of s_0 is zero. So, the line LL' follows the trajectory of $\hat{\mathbf{v}}_\mathbf{0}$, which is orthogonal to the segment s_0. In other words, the left edge of the trapezoid has length T, and is orthogonal to the parallel edges. Similarly, because the right velocity of s_n is zero, the right edge of the trapezoid is also orthogonal. Therefore, F is isometric to a right angled trapezoid (i.e. a strip) of length X and width T. $\square$

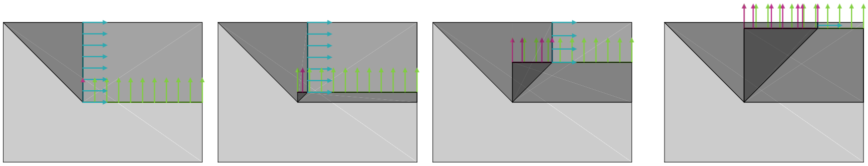
property for NO zero length segments

4.4 Multiple Cross Sections

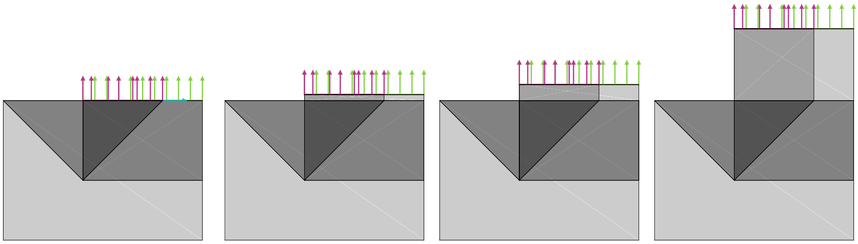
Definition 11. *Given two cross section intervals $\mathscr{C}$ and $\mathscr{D}$, such that $\mathscr{C}^F$ and $\mathscr{D}^I$ are equivalent, we say that $\mathscr{D}$ is next-compatible with $\mathscr{C}$ and $\mathscr{C}$ is previous-*



(a) *The trivial cross section.* **(b)** *New zero length segment inserted at left endpoint. Existing segment reverses direction.* **(c)** *Evolution continues*



(d) *New zero length segment (red) inserted between two existing segments with the same velocity as the rightmost segment. Existing segments retain their direction.* **(e)** *Evolution continues*



(f) *Length of leftmost segment becomes zero.* **(g)** *Leftmost (zero length) segment is deleted. The remaining two segments continue in the same direction. Overall strip width has been reduced*

Figure 4: *Cross Section evolution of a strip narrowing gadget.*

cite

compatible with $\mathscr{D}$. Two cross sections $C = \langle s_1, s_2, \cdots s_n\rangle$ and $D = \langle r_1, r_2, \cdots r_m\rangle$ are equivalent if and only if C and D correspond to the same sequence of segments after the deletion of all zero length segments.

Definition 12. *A cross section sequence is a sequence is an ordered list of cross section intervals $\langle \mathscr{C}_1, \mathscr{C}_2, \cdots \mathscr{C}_n\rangle$, such that C_i is next-compatible with C_{i-1} for all $i \in [n-1]$. This is equivalent to stating that C_i is previous-compatible with C_{i+1} for all $i \in [n-1]$. Note that wedo not care about the directions of the segments.*

We will represent our full construction as a valid *cross section sequence*. Given a cross section sequence $\langle \mathscr{C}_1, \mathscr{C}_2, \cdots \mathscr{C}_n\rangle$, the transition from $\mathscr{C}_i$ to $\mathscr{C}_{i+1}$ corresponds to the deletion of one or more length zero segments from $\mathscr{C}_i$, and the addition of one or more zero length segments to obtain $\mathscr{C}_{i+1}$. One simple example is shown in Figure 4.

Definition 13. *Given a cross section sequence $\langle \mathscr{C}_1, \mathscr{C}_2, \cdots \mathscr{C}_n \rangle$, We obtain $\mathscr{F}_i$ as the folding of cross section $\mathscr{C}_i$. the transition from $\mathscr{C}_i$ to $\mathscr{C}_{i+1}$ corresponds to the deletion of one or more For every joint J_i, we attach the corresponding trapezoids for s_i and s_{i+1} along the joint trajectory to obtain the folding of $\mathscr{C}$.*

4.4.1 Evolution Corresponds to Flat Paper

In this section we will demonstrate that the folding formed by cross section evolution is realizable from a sheet of flat paper. We note here that our construction may still result in self intersections.

We consider a cross section sequence $\langle \mathscr{C}_1, \mathscr{C}_2, \cdots \mathscr{C}_n \rangle$, where each cross section interval $\mathscr{C}_i$ has evolution time T_i.

We will then use Theorem 12 to attach the sequence of T_i-strips, to form a complete $X \times T$ sheet of paper, where $T = \sum T_i$.

Theorem 1. *Consider a cross section sequence $\langle \mathscr{C}_1, \mathscr{C}_2, \cdots \mathscr{C}_n \rangle$ where each cross section interval $\mathscr{C}_i$ evolves over time T_i to form a folding F_i such that the following properties hold*

Property 1. *The velocity $\hat{\mathbf{v}}_\mathbf{s}$ of segment s is orthogonal to its orientation $\hat{\mathbf{o}}_\mathbf{s}$.*
Property 2. *For a joint J associated with segments l and r, $\mathbf{v}_l^\perp = \mathbf{v}_r^\perp$. As a corollary, $\left\|\mathbf{v}_l^{\shortparallel}\right\| = \|\mathbf{v}_r^{\shortparallel}\|$.*
Property 3. *The right velocity of s_i is the negation of the left velocity of s_{i+1}. This is to ensure the preservation of distance between any two nodes. Furthermore, the left velocity of the first segment, and the right velocity of the last segment should be zero, i.e. $L_0 = R_n = 0$ This ensures that the total length of the cross section does not change.*
Property 4. *The velocity of a joint J associated with segments l and r, is a constant vector $\mathbf{J}_v = \hat{\mathbf{v}}_\mathbf{l} + \hat{\mathbf{o}}_\mathbf{l} \frac{|\mathbf{v}_r^{\shortparallel} - \mathbf{v}_l^{\shortparallel}|}{|\hat{\mathbf{o}}_\mathbf{r} - \hat{\mathbf{o}}_\mathbf{l}|}$*

Then, F is isometric to a $X \times T$ strip strip of paper, where let $T = \sum T_i$.

Proof. □

5 Orthogonal Graphs

In this section, we outline a construction of orthogonal graphs with arbitrary rational extrusion heights. In our construction, the cross section at will always be on the $x - z$ plane. This makes the analysis much simpler.

5.1 Grid Extrusion

To simplify the presentation, we will consider an uniform $X - Y$ grid, with arbitrary rational extrusion heights corresponding to every grid square.

Definition 14. *An $n \times m$ rational grid extrusion is a 3-dimensional structure, whose projection onto the $x - y$ plane forms an unit grid of size $n \times m$. Additionally, the unit face corresponding to the location $(i.j)$, is extruded E_{ij} units in the z direction, where E_{ij} is a rational number.*

5.1.1 Strip Extrusion

We also consider each "column" of a given grid extrusion separately as an indi-visual *strip exrusion*. We will construct each of the n strips independently, and attachthem together with *strip connectors*.

5.2 Optimality

Under some suitable assumptions, our construction can be made $2+\varepsilon$ optimal, for arbitrarily small ε.

We define the maximum deltas along the y-axis as $D_j = \max_i \|E_{i,j} - E_{i,j+1}\|$, and let $Y = n + \sum_{i=1}^{n-1} D_j$ We also define the lowest and highest points along x-axis as $L_i = \max_j E_{ij}$ and $H_i = \max_j E_{ij}$, and let

$$X = n + \sum_{i=1}^{n} [(H_i - \min(L_i, L_{i+1})) + (H_i - \min(L_i, L_{i-1}))]$$

The terms in the summation account for the total length of all the necessary worst case vertical walls, and the n is for the top faces.

Claim 1. *The x-axis length of the strip of paper required to fold this shape can be made arbitrarily close to X.*

Claim 2. *The y-axis length of the strip of paper required to fold this shape will be exactly Y.*

First, we pick an ε, such that 2ε divides all extrusion heights. The construction will require paper of size $X' \times Y$, where $X' = X + 2\varepsilon(n-1)$.

There are two components to the construction

- n strips parallel to the y-axis. Each of these strips will fold to the corresponding strip in the extruded graph. The total area of these strips will be $X \times Y$.
- $n-1$ Intermediate strips, each of size $2\varepsilon \times Y$ to connect the main strips together.

The total area is therefore $X \times Y + (n-1)2\varepsilon \times Y = X' \times Y$. This can of course be made arbitrarily close to $X \times Y$.

5.3 Construction of Grid Extrusion

Given a strip of size $w \times Y$, we will consider the evolution of the cross sections along the length Y. Imagine that there are $T = \frac{Y}{\varepsilon}$ time steps over which the cross section evolves along the y-axis. Consider the decomposition of Y as

$$Y = (1 + D_1) + (1 + D_2) + (1 + D_) + \cdots + (1 + D_{n-1})$$

Here, the times corresponding to

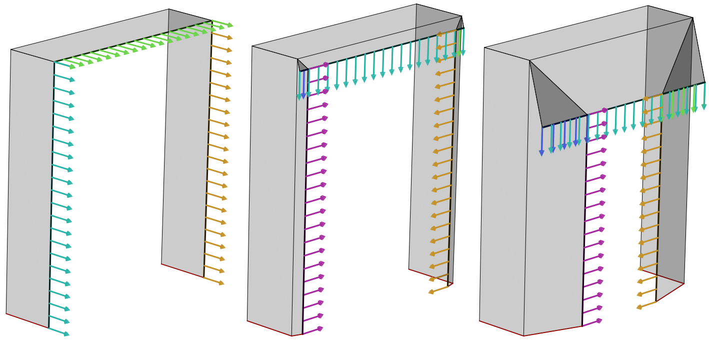
- the i^{th} "1" is realized as the strip $i-1 \leq y \leq i$
- D_i is the transition between $i-1 \leq y \leq i$ and $i \leq y \leq i+1$

5.4 Construction of Strip Extrusion

5.4.1 Level Shifting

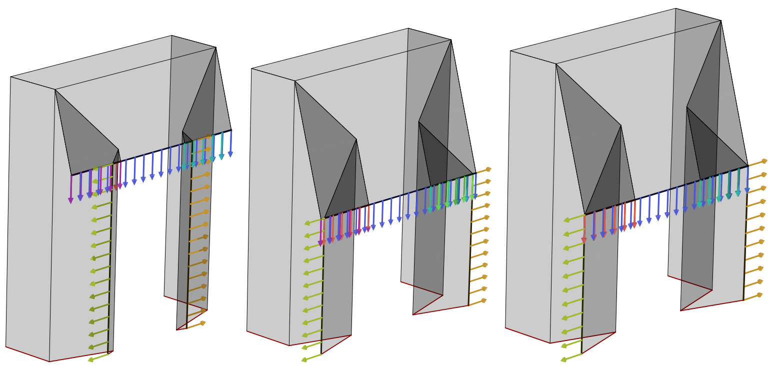
5.5 Construction of Strip Connectors

5.5.1 Alignment with Level Shifts

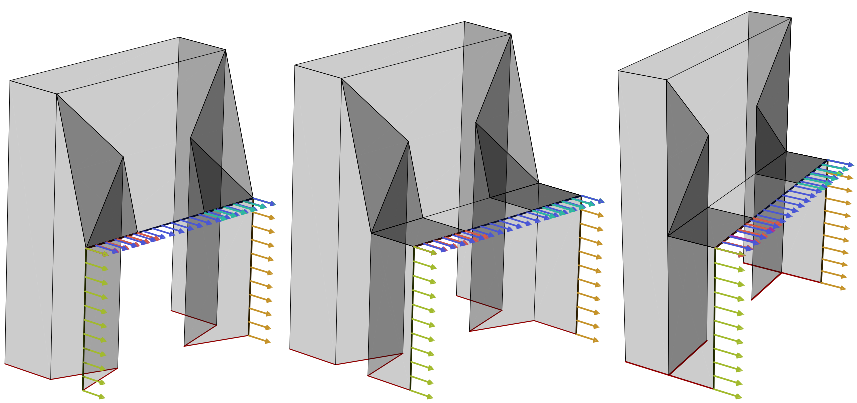


(a) *Higher initial level. Top segment moves down.*

(b) *Two segments created. New segments are aligned with top segment, and move down. Vertical segments move inwards.*



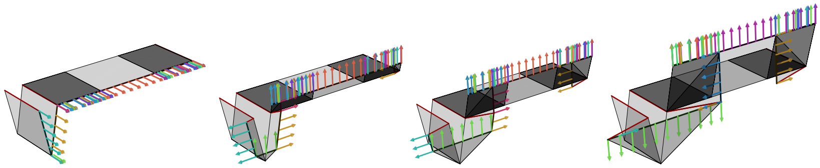
(c) *Two more segments created. Vertical segments reverse direction.*



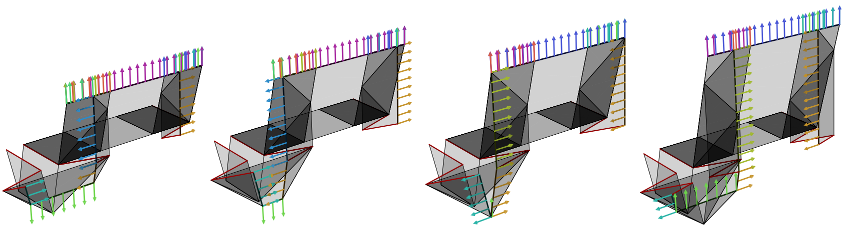
(d) *Level shift completed with four new horizontal segments.*

(e) *Flat folded state.*

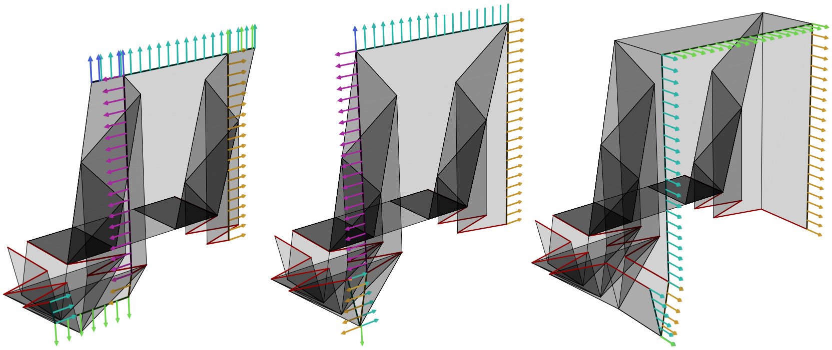
Figure 5: *Level Shifting Gadget. The separation along the Y direction serves to illustrate the layering.*



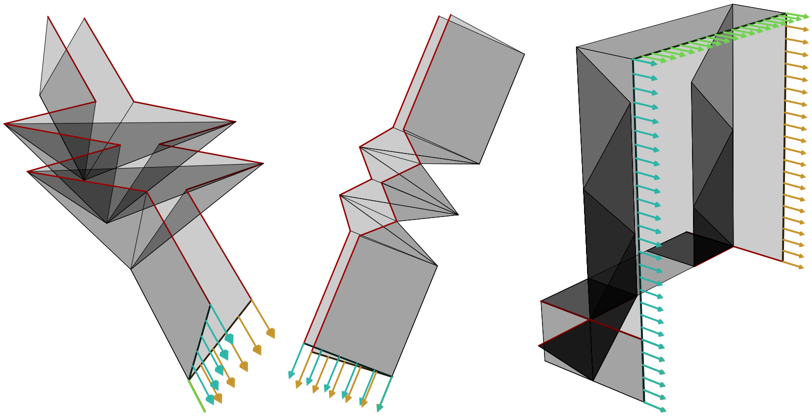
(a) *Initial height.* **(b)** *Connector moves outwards.* **(c)** *Connector at maximum width.*



(d) *Connector moves inwards.* **(e)** *Back to **(a)**.* **(f)** *Same as **(b)**.*



(g) *Same as **(d)**.* **(h)** *Level shift complete, and next level continues.*



(i) *Connector gadget.* **(j)** *Side view.* **(k)** *Flat folded state.*

Figure 6: *Column gadget attached to a single column connector gadget. The red line demarcates the interface between the two gadgets.*

6 Polygons

References

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