Paper Title

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Abstract: *The abstract must summarise the main content of the paper, and may be up to approximately 100 words in length. Unlike previous OSME publications, the abstract will be included in the proceedings. This abstract needs not be the same as that submitted for acceptance to the conference.*

1 Introduction

Outline

2 Extruded Surfaces (Preliminaries)

3 Cross Sections

We introduce a new method of origami construction, using cross section disgrams. Instead of beginning our construction from a 2-dimensional sheet of paper, we consider a 1-dimensional cross section moving forwards in time. A simple example is demonstrated in Figure 2.

Conservation of length

Definition 1. *A cross section is defined as an ordered list of line segments $\langle s_1, s_2, \cdots, s_n \rangle$, such that for every segment s_i (except the last), the right endpoint of s_i coincides with the left endpoint of s_{i+1}. Each line segment is associated with a velocity vector $\mathbf{v}_i$ of unit magnitude. Each segment is also associated with an unit orientation vector $\mathbf{o}_i$, which is the unit vector from the left endpint to the right endpoint.*

Lemma 2. *All points on a single segment move with the same velocity v_i.*

3.1 Joints

Definition 3. *A cross section with n segments is also associated with a list of joints $\langle J_1, \cdots J_{n-1} \rangle$, where J_i corresponds to the right endpoint of s_i. The velocity of a joint J_i, is $\mathbf{J}_v = \mathbf{v}_i + \mathbf{v}_{i+1}$ (i.e. the sum of the velocities of the corresponding segments).*

Definition 4. *Note that the trajectory of the joints form a crease ... angle is angle between direction vectors. Creases are also created when a segment changes direction... angle is change in direction vector.*

Definition 5. *A joint plane is the plane that coincides with both segments l and r associated with a particular joint J.*

Definition 6. *Consider a joint J associated with segments l and r, and joint plane $\mathscr{P}$, where $\mathbf{v}_l$ and $\mathbf{v}_r$ are the velocities of segments l and r. We define $\mathbf{v}_l^{\shortparallel}$ and $\mathbf{v}_l^{\perp}$ as the components of $\mathbf{v}_l$ coinciding with, and orthogonal to the joint plane respectively. Similarly, we define $\mathbf{v}_r^{\shortparallel}$ and $\mathbf{v}_r^{\perp}$, as the components of $\mathbf{v}_r$. Further, note that $\mathbf{v}_l^{\shortparallel}$ and $\mathbf{v}_r^{\shortparallel}$ have to be orthogonal to $\mathbf{o}_l$ and $\mathbf{o}_r$ respectively.*

Claim 7. *For a joint J associated with segments l and r, $\mathbf{v}_l^{\perp} = \mathbf{v}_r^{\perp}$.*

Corollary 8. *For a joint J associated with segments l and r, $\left\|\mathbf{v}_l^{\shortparallel}\right\| = \|\mathbf{v}_r^{\shortparallel}\|$.*

Claim 9. *A joint J corresponding to segments s and t is* valid *if the resulting evolution of the joint preserves distances.*

3.2 Time Travel

In the process of time travel, the lengths of segments may change. This can also be visualized as movement of the corresponding joint along one of the segments. The movement of a joint increases the length of one of its associated segments, and decreases the length of the other segment by the same amount (this ensures that the total length is preserved). Most velocity sequences are invalid. For example...

We will use some abuuse of notation, by using $\mathbf{v}_i, \mathbf{o}_i$ etc. to refer to the corresponsing parameters of s_i.

Definition 10. *For ever segment s, we associate a left velocity L_s, which indicates the rate at which s grows from its left endpoint. Similarly, we define a right velocity $R_s \ldots$*

Claim 11. *The right velocity of s_i is the negation of the left velocity of s_{i+1} i.e. $R_i = -L_{i+1}$. Furthermore, the left velocity of the first segment, and the right velocity of the last segment should be zero, i.e. $L_0 = R_n = 0$ This ensures that the total length of the cross section does not change.*

Define Nodes

Consider a joint $\mathbf{J}_i$ corresponding to segments $l = s_i$ and $r = s_{i+1}$, at time $t = 0$. Henceforth, we will refer to L_l as L, and R_r as R. Without loss of genenerality, we assume that $L < 0$. At a later time t, let the new joint position be $\mathbf{J}_i'$. We define nodes a and b corresponding to $\mathbf{J}_i$ and $\mathbf{J}_{i+1}$ respectively. We also define the initial and final positions of a as $\mathbf{a}$, and $\mathbf{a}'$, and similarly for b, we define $\mathbf{b}$ and $\mathbf{b}'$. Let d be the *separation* between nodes a and b. this setup is shown in Figure 1

First, note that $\mathbf{a}, \mathbf{b}$ lie on the segment l, and $\mathbf{a}', \mathbf{b}'$ lie on the segment r, which

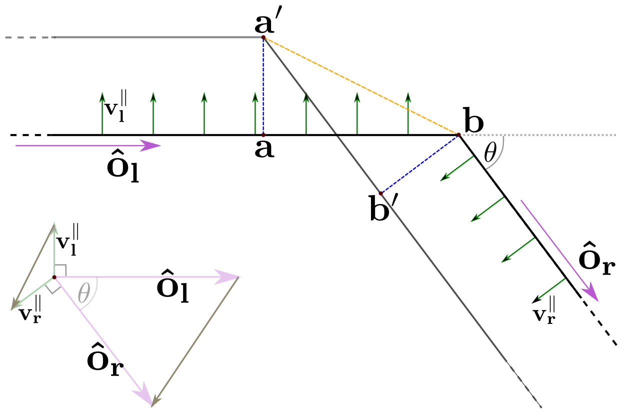


Figure 1: *A joint with segments l and r. The trajectory of the joint is shown in orange. The trajectories of a and b are shown in blue. The green arrows indicate* $\mathbf{v}^{||}$

implies that $\mathbf{b}-\mathbf{a}=d\cdot\hat{\mathbf{o}}_\mathbf{l}$, and $\mathbf{b}'-\mathbf{a}'=d\cdot\hat{\mathbf{o}}_\mathbf{r}$.

$$\mathbf{b}'-\mathbf{a}' = (\mathbf{b}+t\cdot\mathbf{v}_r^{||})-(\mathbf{a}+t\cdot\mathbf{v}_l^{||}) \tag{1}$$

$$\implies \mathbf{b}'-\mathbf{a}' = (\mathbf{b}-\mathbf{a})+t\cdot(\mathbf{v}_r^{||}-\mathbf{v}_l^{||}) \tag{2}$$

$$\implies d\cdot\hat{\mathbf{o}}_\mathbf{r} = d\cdot\hat{\mathbf{o}}_\mathbf{l}+t\cdot(\mathbf{v}_r^{||}-\mathbf{v}_l^{||}) \tag{3}$$

$$\implies \mathbf{v}_r^{||}-\mathbf{v}_l^{||} = \frac{d}{t}\cdot(\hat{\mathbf{o}}_\mathbf{r}-\hat{\mathbf{o}}_\mathbf{l}) \tag{4}$$

$$= R\cdot(\hat{\mathbf{o}}_\mathbf{r}-\hat{\mathbf{o}}_\mathbf{l}) = -L\cdot(\hat{\mathbf{o}}_\mathbf{r}-\hat{\mathbf{o}}_\mathbf{l}) \tag{5}$$

This implies that $\hat{\mathbf{o}}_\mathbf{l}\times\hat{\mathbf{o}}_\mathbf{r}$ is oriented opposite to $\mathbf{v}_l^{||}\times\mathbf{v}_r^{||}$. If the angle between the segments (between $\hat{\mathbf{o}}_\mathbf{l}$ and $\hat{\mathbf{o}}_\mathbf{r}$) is θ, the magnitude of $\hat{\mathbf{o}}_\mathbf{r}-\hat{\mathbf{o}}_\mathbf{l}$ is $\sqrt{2-2\cos(\theta)}$, and $|\mathbf{v}_r^{||}-\mathbf{v}_l^{||}| = v\cdot\sqrt{2-2\cos(\pi-\theta)}$. Here, v is magnitude of the plane velocity of J_i.

$$R = \frac{d}{t} = \frac{|\mathbf{v}_r^{||}-\mathbf{v}_l^{||}|}{|\hat{\mathbf{o}}_\mathbf{r}-\hat{\mathbf{o}}_\mathbf{l}|} \tag{6}$$

$$= v\cdot\frac{\sqrt{\sin^2(\pi/2-\theta/2)}}{\sqrt{\sin^2\theta/2}} \tag{7}$$

$$= v\cdot\cot\left(\frac{\theta}{2}\right) \tag{8}$$

This can result in a segment length becoming zero then delete. We may create a new segment at any point of zero length.

Note that time evolution is reversible.

3.3 Multiple Cross Sections

Definition 12. *We consider a cross section C composed of segments $\langle s_1, s_2, \cdots, s_n\rangle$ with total length X (i.e. $\sum |s_i| = X$). If we allow this cross section to evolve for time T, we obtain a cross section interval $\mathscr{C}$ of length T. Let $\mathscr{C}^I$ denote the initial cross section $C = \langle s_1, s_2, \cdots, s_m\rangle$, and $\mathscr{C}^F$ denote the final cross section $\langle r_1, r_2, \cdots, r_m\rangle$.*

Definition 13. *Given two cross section intervals $\mathscr{C}$ and $\mathscr{D}$, such that $\mathscr{C}^F$ and $\mathscr{D}^I$ are equivalent, we say that $\mathscr{D}$ is next-compatible with $\mathscr{C}$ and $\mathscr{C}$ is previous-compatible with $\mathscr{D}$. Two cross sections $C = \langle s_1, s_2, \cdots s_n\rangle$ and $D = \langle r_1, r_2, \cdots r_m\rangle$ are equivalent if and only if C and D correspond to the same sequence of segments after the deletion of all zero length segments.*

Definition 14. *A cross section sequence is a sequence is an ordered list of cross section intervals $\langle \mathscr{C}_1, \mathscr{C}_2, \cdots \mathscr{C}_n\rangle$, such that C_i is next-compatible with C_{i-1} for all $i \in [n-1]$. This is equivalent to stating that C_i is previous-compatible with C_{i+1} for all $i \in [n-1]$. Note that wedo not care about the directions of the segments.*

We will represent our full construction as a valid *cross section sequence*. Given a sross section sequence $\langle \mathscr{C}_1, \mathscr{C}_2, \cdots \mathscr{C}_n\rangle$, the transition from $\mathscr{C}_i$ to $\mathscr{C}_{i+1}$ corresponds to the deletion of one or more length zero segments from $\mathscr{C}_i$, and the addition of one or more zero length segments to obtain $\mathscr{C}_{i+1}$. One simple example is shown in Figure 2.

3.4 Evolution Corresponds to Flat Paper

In this section we wil demonstrate that the folding formed by cross section evolution is realizable from a sheet of flat paper. We note here that our construction may still result in self intersections.

We consider a cross section sequence $\langle \mathscr{C}_1, \mathscr{C}_2, \cdots \mathscr{C}_n\rangle$, where each cross section interval $\mathscr{C}_i$ has evolution time T_i. Say that the length ogf each cross section in the sequence is X. We will denote a strip of paper of size $X \times L$ as an L-strip.

First, we will show that each $\mathscr{C}_i$ corresponds to a T_i-strip. We will then use Theorem 14 to attach the sequence of T_i-strips, to form a complete $X \times T$ sheet of paper, where $T = \sum T_i$.

3.4.1 Cross Section Interval Forms a Strip from a Gluing of Trapezoids

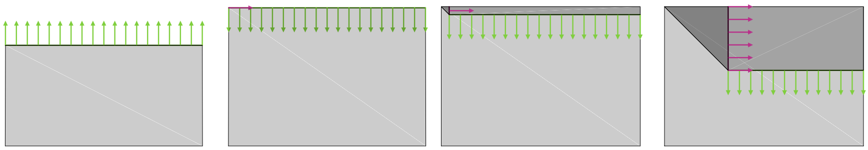
3.4.2 Interval Strip Gluing

4 Orthogonal Graphs

In this section, we outline a construction of orthogonal graphs with arbitrary rational extrusion heights. In our construction, the cross section at will always be on the $x-z$ plane. This makes the analysis much simpler.

4.1 Grid Extrusion

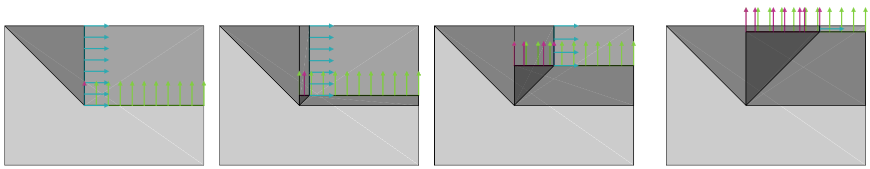
To simplify the presentation, we will consider an uniform $X-Y$ grid, with arbitrary rational extrusion heights corresponding to every grid square.



(a) *The trivial cross section.*

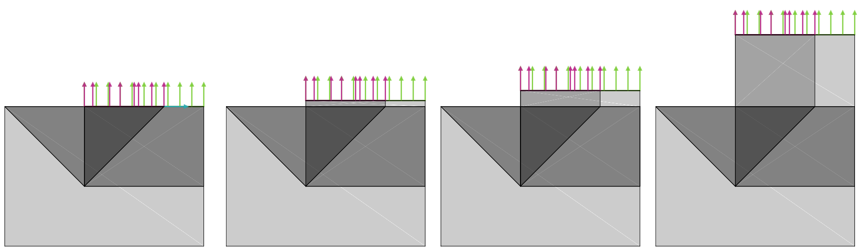
(b) *New zero length segment inserted at left endpoint. Existing segment reverses direction.*

(c) *Evolution continues*



(d) *New zero length segment (red) inserted between two existing segments with the same velocity as the rightmost segment. Existing segments retain their direction.*

(e) *Evolution continues*



(f) *Length of leftmost segment becomes zero.*

(g) *Leftmost (zero length) segment is deleted. The remaining two segments continue in the same direction. Overall strip width has been reduced*

Figure 2: *Cross Section evolution of a strip narrowing gadget.*

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Definition 15. *An $n \times m$ rational grid extrusion is a 3-dimensional structure, whose projection onto the $x - y$ plane forms an unit grid of size $n \times m$. Additionally, the unit face corresponding to the location $(i.j)$, is extruded E_{ij} units in the z direction, where E_{ij} is a rational number.*

4.1.1 Strip Extrusion

We also consider each "column" of a given grid extrusion separately as an indivisual *strip exrusion*. We will construct each of the n strips independently, and attachthem together with *strip connectors*.

4.2 Optimality

Under some suitable assumptions, our construction can be made $2 + \varepsilon$ optimal, for arbitrarily small ε.

We define the maximum deltas along the y-axis as $D_j = \max_i \|E_{i,j} - E_{i,j+1}\|$, and let $Y = n + \sum_{i=1}^{n-1} D_j$ We also define the lowest and highest points along x-axis as $L_i = \max_j E_{ij}$ and $H_i = \max_j E_{ij}$, and let

$$X = n + \sum_{i=1}^{n} [(H_i - \min(L_i, L_{i+1})) + (H_i - \min(L_i, L_{i-1}))]$$

The terms in the summation account for the total length of all the necessary worst case vertical walls, and the n is for the top faces.

Claim 16. *The x-axis length of the strip of paper required to fold this shape can be made arbitrarily close to X.*

Claim 17. *The y-axis length of the strip of paper required to fold this shape will be exactly Y.*

First, we pick an ε, such that 2ε divides all extrusion heights. The construction will require paper of size $X' \times Y$, where $X' = X + 2\varepsilon(n-1)$.

There are two components to the construction

- n strips parallel to the y-axis. Each of these strips will fold to the corresponding strip in the extruded graph. The total area of these strips will be $X \times Y$.
- $n-1$ Intermediate strips, each of size $2\varepsilon \times Y$ to connect the main strips together.

The total area is therefore $X \times Y + (n-1)2\varepsilon \times Y = X' \times Y$. This can of course be made arbitrarily close to $X \times Y$.

4.3 Construction of Grid Extrusion

Given a strip of size $w \times Y$, we will consider the evolution of the cross sections along the length Y. Imagine that there are $T = \frac{Y}{\varepsilon}$ time steps over which the cross section evolves along the y-axis. Consider the decomposition of Y as

$$Y = (1 + D_1) + (1 + D_2) + (1 + D_)) + \cdots + (1 + D_{n-1})$$

Here, the times corresponding to

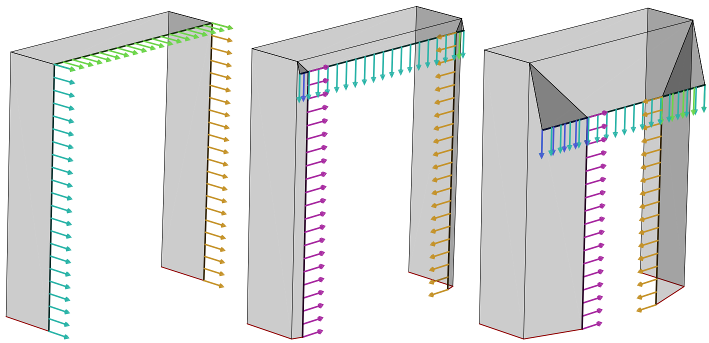
- the i^{th} "1" is realized as the strip $i - 1 \leq y \leq i$
- D_i is the transition between $i - 1 \leq y \leq i$ and $i \leq y \leq i + 1$

4.4 Construction of Strip Extrusion

4.4.1 Level Shifting

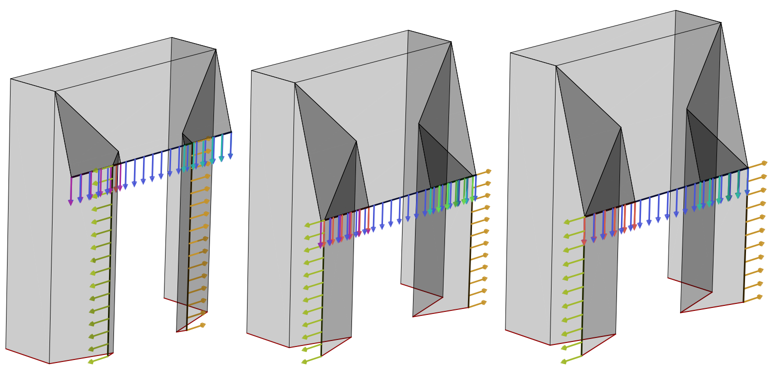
4.5 Construction of Strip Connectors

4.5.1 Alignment with Level Shifts

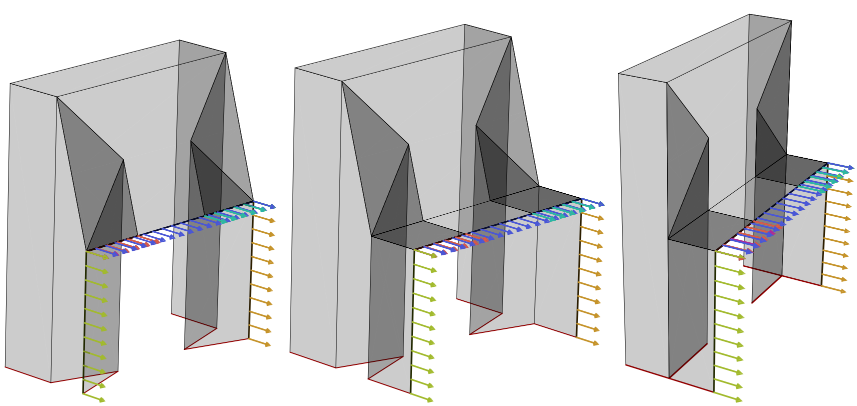


(a) *Higher initial level. Top segment moves down.*

(b) *Two segments created. New segments are aligned with top segment, and move down. Vertical segments move inwards.*



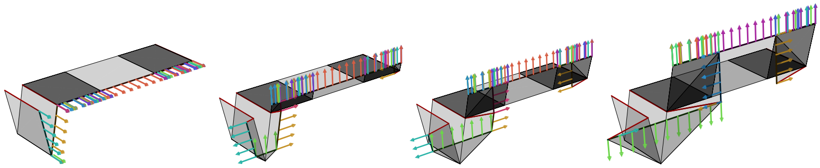
(c) *Two more segments created. Vertical segments reverse direction.*



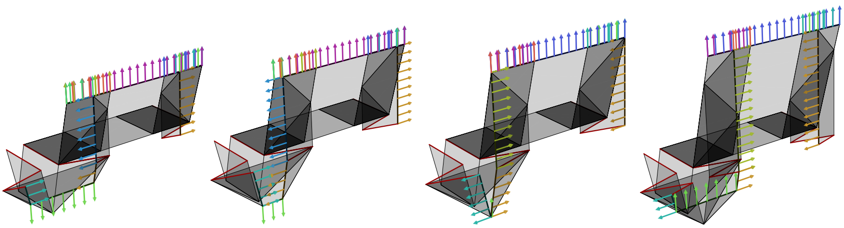
(d) *Level shift completed with four new horizontal segments.*

(e) *Flat folded state.*

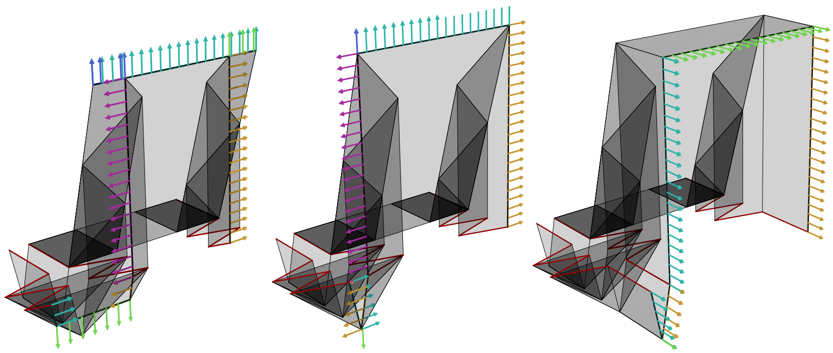
Figure 3: *Level Shifting Gadget. The separation along the Y direction serves to illustrate the layering.*



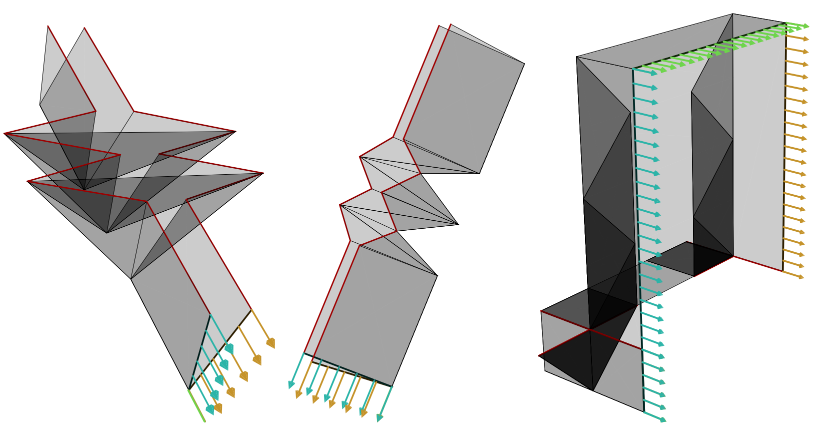
(a) *Initial height.* **(b)** *Connector moves outwards.* **(c)** *Connector at maximum width.*



(d) *Connector moves inwards.* **(e)** *Back to **(a)**.* **(f)** *Same as **(b)**.*



(g) *Same as **(d)**.* **(h)** *Level shift complete, and next level continues.*



(i) *Connector gadget.* **(j)** *Side view.* **(k)** *Flat folded state.*

Figure 4: *Column gadget attached to a single column connector gadget. The red line demarcates the interface between the two gadgets.*

5 Polygons

References

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