Local-Access Generators

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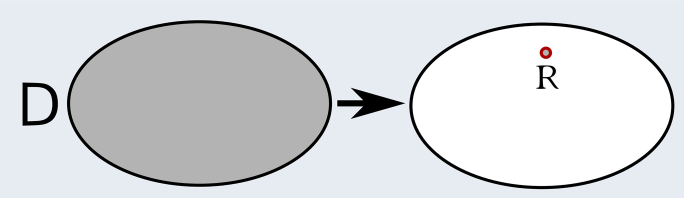
Partial Sampling from a Distribution

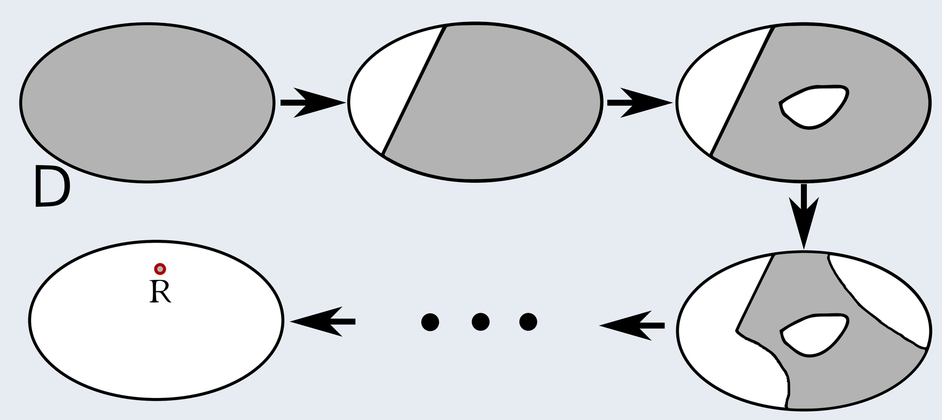
Full Sampling $R \sim \mathsf{D}$ in $\mathcal{O}(T)$ time



Do we need to spend $\mathcal{O}(T)$ upfront?

N steps of *Partial Sampling*

Each partial step should take $\tilde{\mathcal{O}}(T/N)$ time.



Problem Statement

A local-access generator of a random object $R \sim \mathsf{D}$, provides indirect access to R' with a *query oracle* s.t.

- All query responses (*partial samples*) are **consistent**
- The **distribution** of R' is ϵ-close to D in L_1 distance

Trivial Example - Sampling $G(n,p)$

Model: N vertex undirected graph: edge probability p

Query Model: Given vertices u, v, is $(u,v) \in E$?

- Just a collection of $\frac{N(N-1)}{2}$ Bernoulli RVs with bias p.

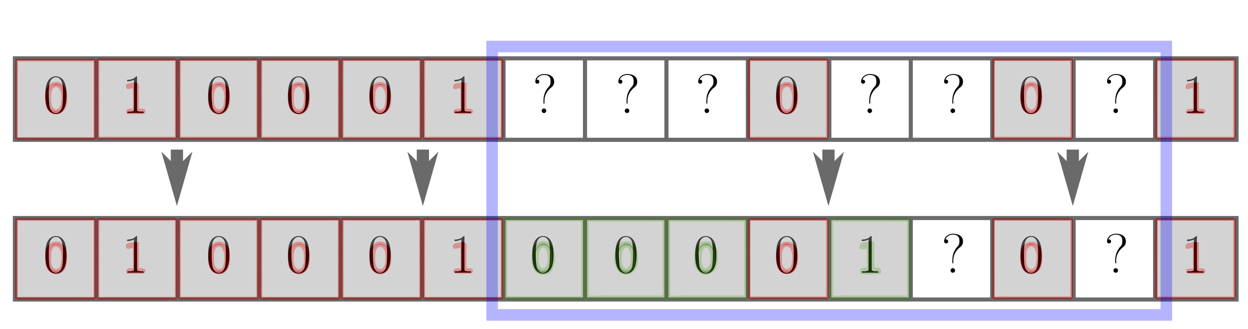
Find Next-Neighbor (*skip-sampling*)

Adjacency List query: Return neighbors of v in order.

$$\mathbb{P}[k \text{ non-neighbors before next-neighbor}] = p(1-p)^k$$

- Can sample from this distribution in $\tilde{\mathcal{O}}(1)$ time [ELMR17]
- Avoid sampling each 0 separately

Issue: Adjacency matrix is symmetric So, each zero must also appear in the corresponding column of v



If the sampled neighbor is a 0, discard and resample.

Cannot afford too many re-samplings.

Bucketing-Generator & Random-Neighbor Queries

Problem: next-neighbor cannot "jump" to a random potential neighbor of v

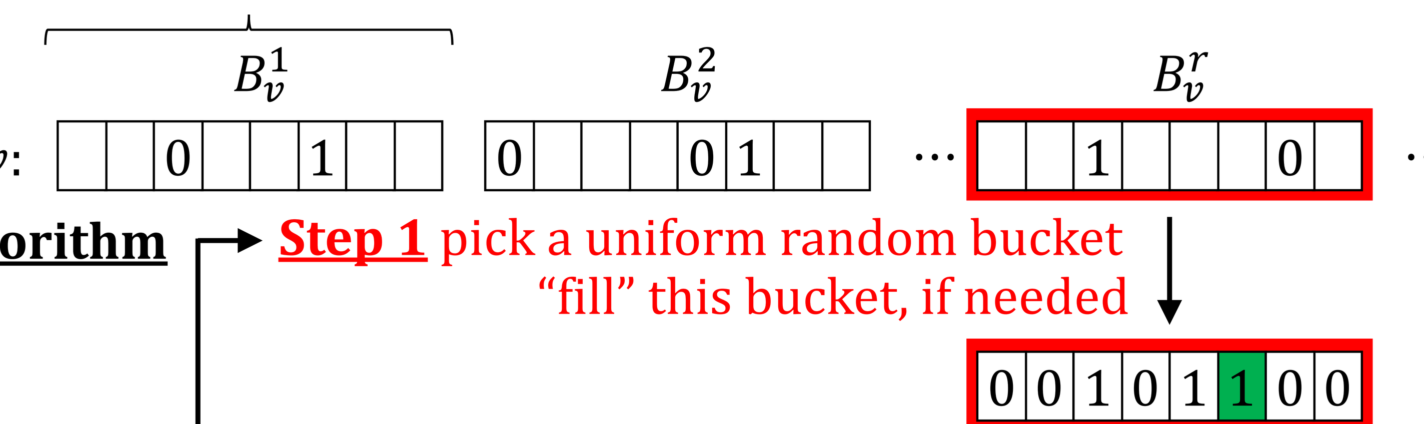
Bucketing Divide each row of the adjacency matrix into contiguous buckets
⇒ random neighbor of $v \approx$ random neighbor in a random bucket of v

Problem: Do NOT know $\deg(v)$: Must return each neighbor with prob. $1/\deg(v)$

Rejection Sampling Normalize probability of returning any specific neighbor

Problem: next-neighbor cannot "jump" to a random potential neighbor of v
⇒ suffice to show that **any neighbor** is returned with the **equal** probability

#neighbors in each bucket
$\sim \Theta(1)$ in expectation, $O(\log n)$ max w.h.p. ⇒ #buckets ~ #neighbors



Algorithm

Step 1 pick a uniform random bucket
"fill" this bucket, if needed

Step 2 pick a uniform random neighbor u → return or reject

Step 3 return u with probability $\frac{\#\text{neighbors in bucket}}{O(\log n)}$
otherwise, try again

$$\Pr[u \text{ returned}] = \frac{1}{\#\text{buckets}} \times \frac{1}{\#\text{neighbors in bucket}} \times \frac{\#\text{neighbors in bucket}}{O(\log n)} \sim \frac{\Omega(1/\log n)}{\#\text{neighbors}}$$

$\Pr[\text{some neighbor returned}] \sim \Omega(1/\log n) \Rightarrow O(\log n)$ tries suffices

Data Structure Buckets contains set of known neighbors, and **"filled"** marker
⇒ "fill" with expected $\Theta(1)$ next-neighbor queries
⇒ random-neighbor succeeds in $O(\log n)$ tries
$O(\log n)$ time *per query*
$\tilde{O}(m+n)$ space usage

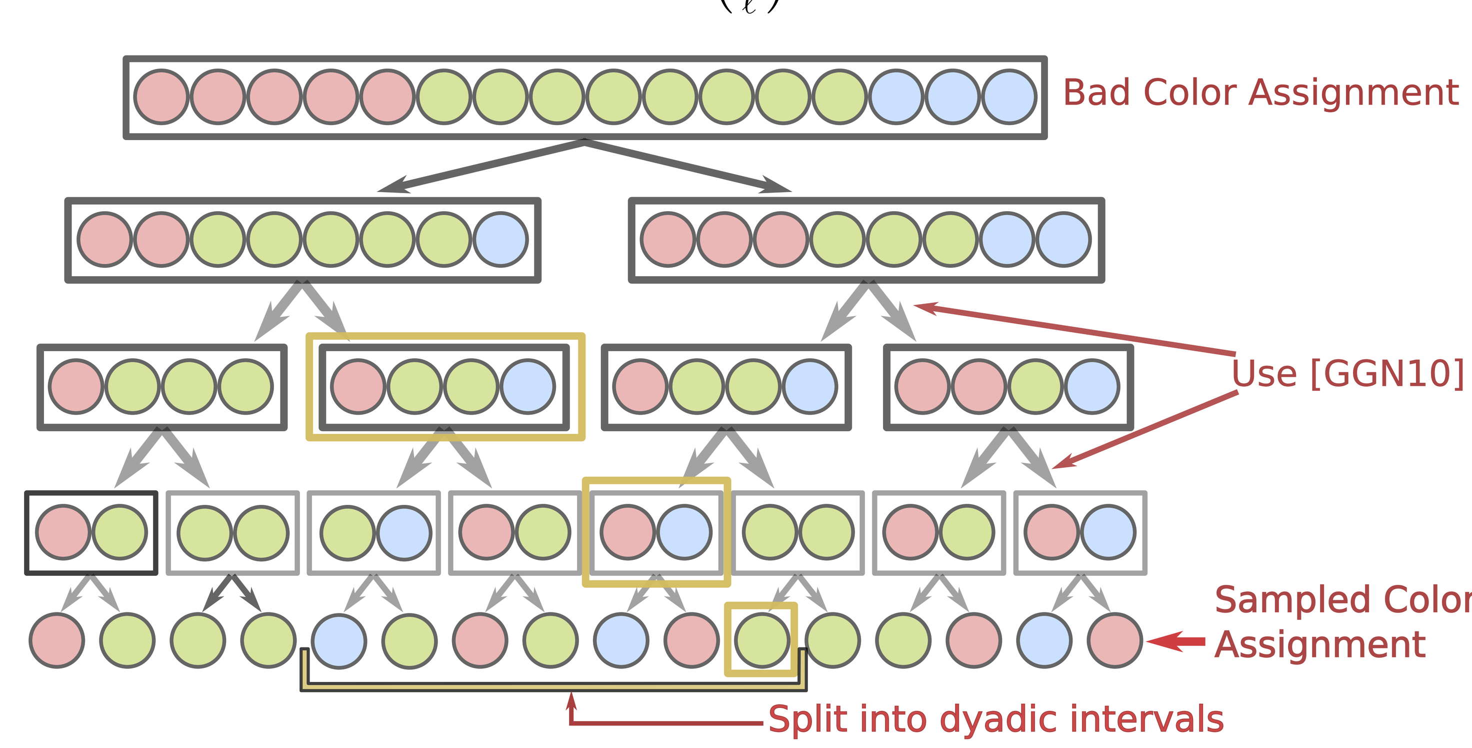
Stochastic Block Model

Communities $\{C_i\}_{i\in[r]}$ partition V: If $u \in C_i, v \in C_j$, then $\mathbb{P}_{(u,v)\in E} = p_{ij}$.

Given sizes of each comunity C_i and a range of length ℓ

- Count number of occurrences of each community in any contiguous range
- Sample from *Multivariate Hypergeometric Distribution*

$$\Pr[\mathbf{S}_\ell^{\mathbf{C}} = \langle s_1, \ldots, s_r \rangle] = \frac{\binom{C_1}{s_1} \cdot \binom{C_2}{s_2} \cdots \binom{C_r}{s_r}}{\binom{B}{\ell}} \quad \text{where } \ell = \sum_{i=1}^r s_i \text{ and } B = \sum_{i=1}^r C_i$$



Multivariate Hypergeometric Distribution

[GGN10] solves the special case of $r = 2$ and $B = 2\ell$.

Counting-Generator

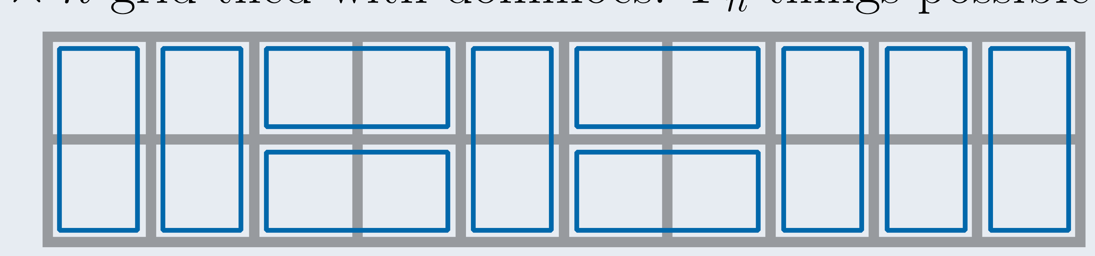
- **Extending to** $B \neq 2\ell$: Divide ℓ into dyadic segments.
- **Extending to** $r > 2$: Make a tree with a leaf for each C_i Every branch in the tree is equivalent to a 2-splitting

- Use Counting-Generator to sample community counts
- Run the Bucketing-Generator as before

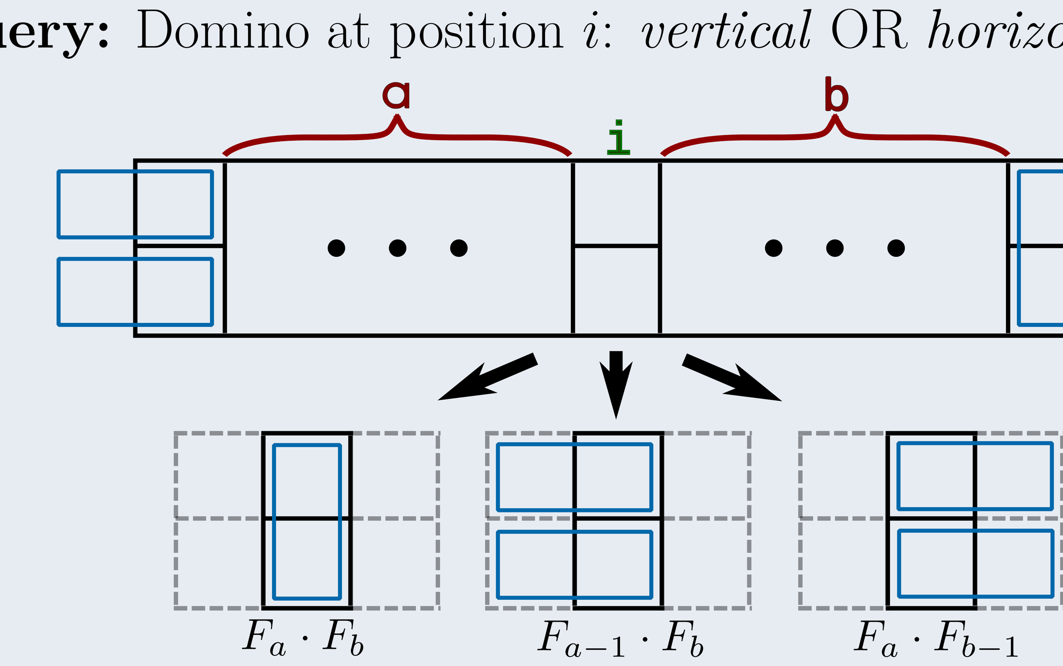
Work in Progress

Domino Tiling

A $2 \times n$ grid tiled with dominoes: F_n tilings possible.



Query: Domino at position i: *vertical* OR *horizontal*?



Sufficent to approximate F_c/F_{c-1}: Use ϕ if $c = \Omega(\log(n))$

Open: $k \times n$ grid for $k = \omega(1)$ and Dimer model.

Graph Coloring: Glauber Dynamics

Find random k-coloring for graph with max degree Δ

Global Algorithm (for $k > 2\Delta$)

- Sample $\mathcal{O}(n \log n)$ (vertex, color) pairs:
  $\{(v_1,c_1),(v_2,c_2),(v_3,c_3),\cdots,(v_r,c_r)\}$
- For steps $i \in [1 \cdots r]$
  - If no neighbor of v_i has color c_i set v_i's color to c_i.
  - Else, do nothing

Local Algorithm (for $k = \Theta(\Delta \log n)$)

- Given v, what is *color*(v) (in some random coloring)?
- Locally sample occurences of $(v, \star)$ using the *Count Splitting Generator*
- Sample $(w, \star)$ if necessary, where w is neighbor of v
- Query tree is bounded for $k = \Theta(\Delta \log n)$

[ELMR17] Guy Even, Reut Levi, Moti Medina, and Adi Rosen. Sublinear random access generators for preferential attachment graphs. In 44th International Colloquium on Automata, Languages, and Programming, ICALP 2017, July 10-14, 2017, Warsaw, Poland, pages 6:16:15, 2017.
[GGN10] Oded Goldreich, Shafi Goldwasser, and Asaf Nussboim. On the implementation of huge random objects. SIAM Journal on Computing, 39(7):27612822, 2010.