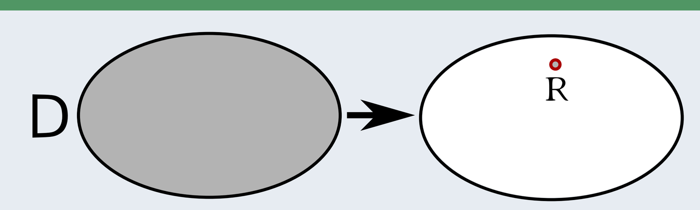
Local-Access Generators

Amartya Shankha Biswas, Ronitt Rubinfeld, Anak Yodpinyanee
CSAIL, MIT

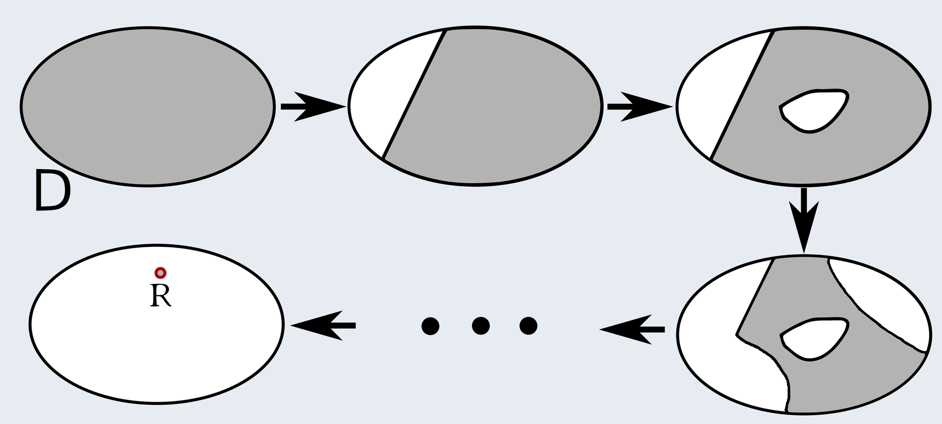
Partial Sampling from a Distribution

Full Sampling $R \sim \mathsf{D}$ in $\mathcal{O}(T)$ time



Do we need to spend $\mathcal{O}(T)$ upfront?

Partial sampling: each step should take $\tilde{\mathcal{O}}(T/N)$ time.



Problem Statement

A local-access generator of a random object $R \sim \mathsf{D}$, provides indirect access to R' with a *query oracle* s.t.

- All query responses (*partial samples*) are **consistent**
- The **distribution** of R' is ϵ-close to D in L_1 distance

Sampling $G(n, p)$: Vertex-Pair queries

Vertex-Pair: Given vertices u, v, decide whether $(u, v) \in E$.
Trivial: just a collection of $\binom{n}{2}$ Bernoulli RVs with bias p.

Next-Neighbor queries (skip-sampling)

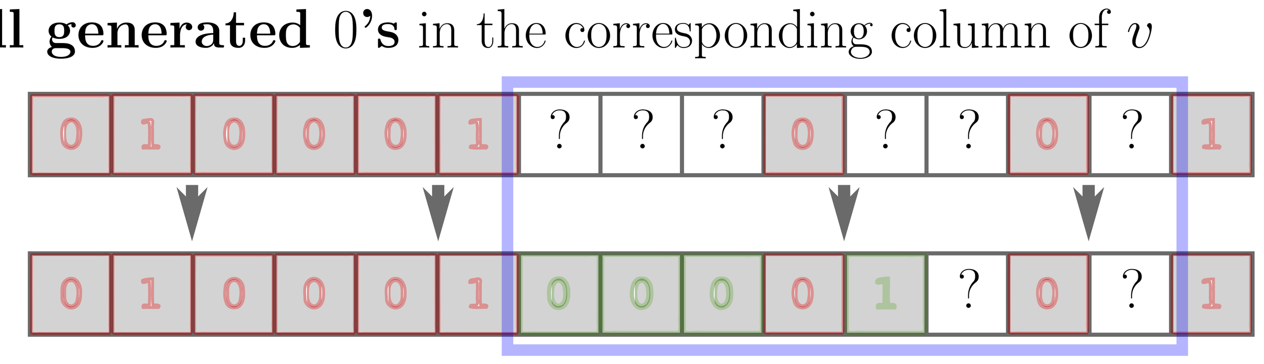
Next-Neighbor: Return neighbors of v in order.

Naïve solution: Toss $1/p$ coins until a neighbor is found.
Idea: can compute Next-Neighbor's distribution's CDF from
$\mathbb{P}[k \text{ non-neighbors before next-neighbor}] = p(1-p)^k$

Skip-sampling: Draw from Next-Neighbor distribution

- Can sample from this distribution in $\tilde{\mathcal{O}}(1)$ time [ELMR17]
- Further analysis required for finite-precision arithmetic

Issue: Adjacency matrix is symmetric; we need to **record all generated 0's** in the corresponding column of v



Issue: if the sampled neighbor is already 0, must re-sample
$\Rightarrow$ may hit 0's many times – **too many re-samplings**

Random-Neighbor queries (Bucketing-Generator)

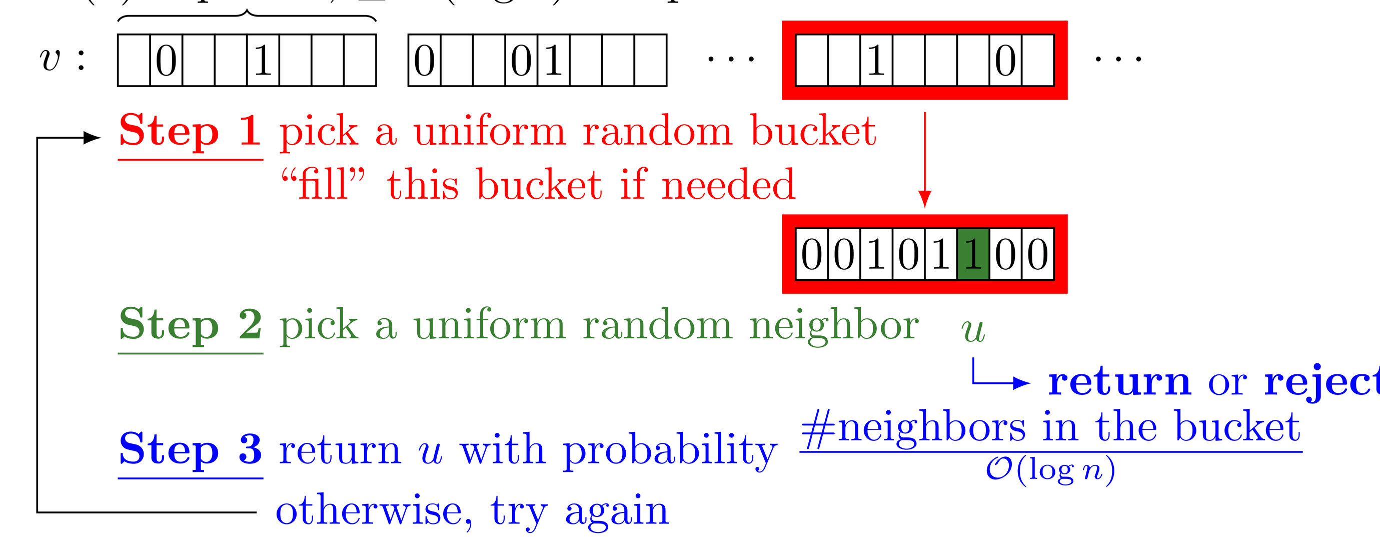
Random-Neighbor: Return a neighbor of v uniformly at random.

Issue: next-neighbor can't jump to a random potential neighbor of v
Bucketing: Divide each row of the adjacency matrix into contiguous buckets
$\Rightarrow$ random neighbor of $v \approx$ random neighbor in a random bucket of v

Issue: do **not** know $\deg(v)$; must return each neighbor with probability $\frac{1}{\deg(v)}$
Rejection Sampling: Return **any** neighbor with the **same** probability

Bucketing: expected #neighbors in a bucket $= \Theta(1)$ expected, $\leq \mathcal{O}(\log n)$ w.h.p. $\Rightarrow$ #neighbors $\approx$ #buckets

Algorithm:



Step 1 pick a uniform random bucket
"fill" this bucket if needed

Step 2 pick a uniform random neighbor u
$\hookrightarrow$ **return** or **reject**

Step 3 return u with probability $\frac{\#\text{neighbors in the bucket}}{\mathcal{O}(\log n)}$
otherwise, try again

$$\mathbb{P}[\text{return } u] = \frac{1}{\#\text{buckets}} \times \frac{1}{\#\text{neighbors in bucket}} \times \frac{\#\text{neighbors in bucket}}{\mathcal{O}(\log n)} \approx \frac{\Omega(1/\log n)}{\#\text{neighbors of } v}$$

$\mathbb{P}[\text{return any neighbor}] \approx \Omega(1/\log n) \Rightarrow \mathcal{O}(\log n)$ iterations suffice

Data Structure: bucket maintains its known neighbors and a **filled** marker

"fill" with expected $\mathcal{O}(1)$ Next-Neighbor queries
Random-Neighbor succeeds in $\mathcal{O}(\log n)$ tries
$\Big\}$ $\mathcal{O}(\log n)$ expected time
$\tilde{\mathcal{O}}(n+m)$ *total* space usage

Stochastic Block Model (Counting-Generator)

Model: Each v is assigned to a *random* community (color) from $C_1, \ldots, C_r$.
$\Rightarrow$ If $u \in C_i, v \in C_j$, then $\mathbb{P}[(u,v) \in E] = p_{ij}$. ($|C_i|$'s are given as input)

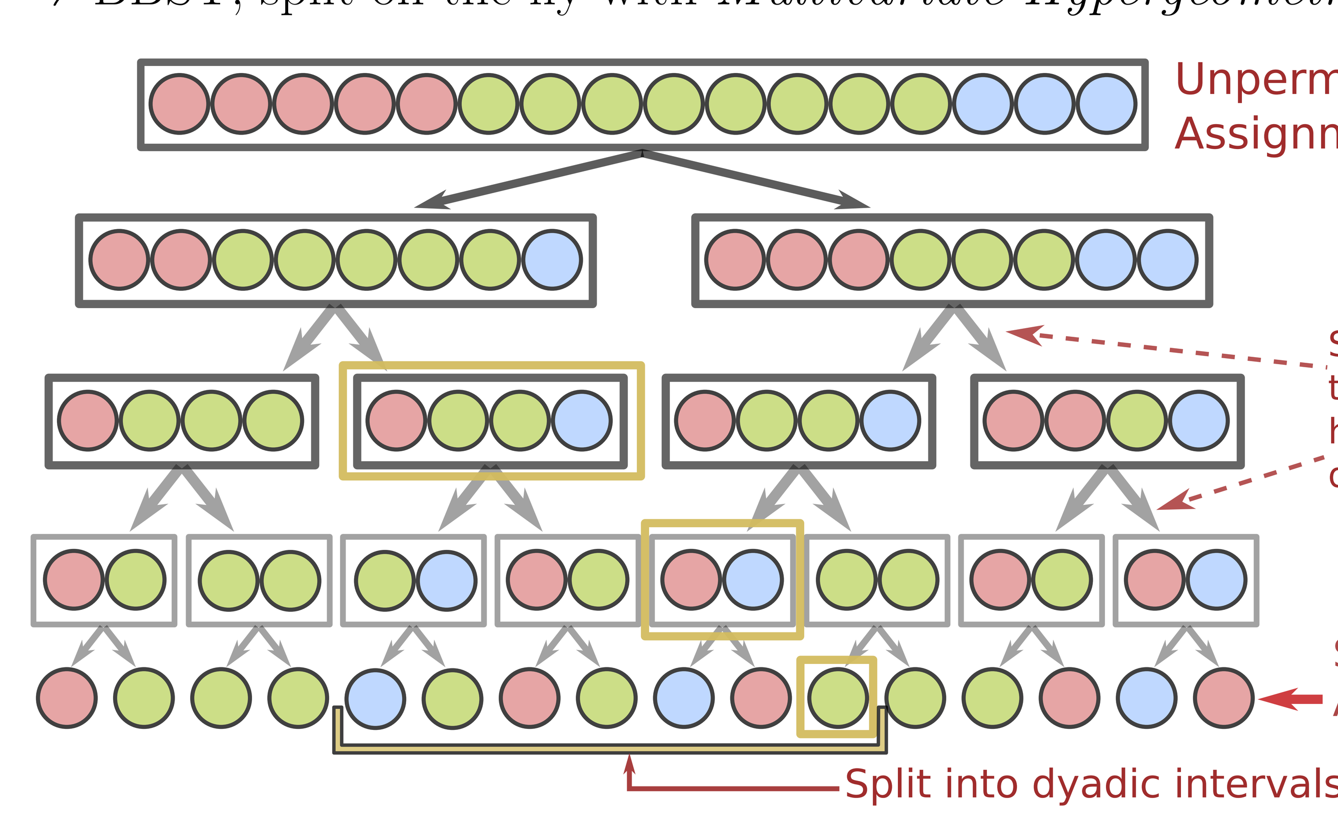
Idea #1 use #nodes of each color in *any* contiguous range to generate SBM

#nodes of each color in any $[a, b]$ $\to$ probability that v has a neighbor in any $[a, b]$ $\to$ CDF of distribution of v's Next-Neighbor $\to$ Next-Neighbor implementation $\to$ Random-Neighbor implementation

#nodes of each color in any $[a, b]$ $\to$ expected #neighbors of v in any $[a, b]$ $\to$ bucketing of v's adjacency matrix row $\to$ Random-Neighbor implementation

suffice for generating *any* random graphs with *independent edge probabilities*

Idea #2 implement a **Counting-Generator** to answer counting queries
$\Rightarrow$ BBST, split on-the-fly with *Multivariate Hypergeometric Distribution*



Multivariate Hypergeometric Distribution

[GGN10] solves the special case of $r = 2$ and $B = 2\ell$.

Counting-Generator

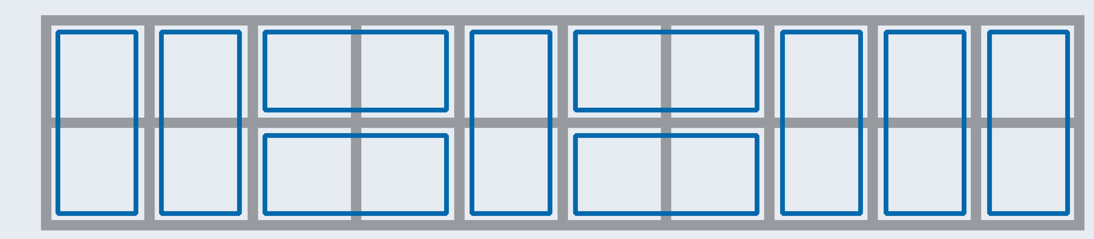
- **Extending to** $B \neq 2\ell$: Divide ℓ into dyadic segments.
- **Extending to** $r > 2$: Make a tree with a leaf for each C_i Every branch in the tree is equivalent to a 2-splitting

- Use Counting-Generator to sample community counts
- Run the Bucketing-Generator as before

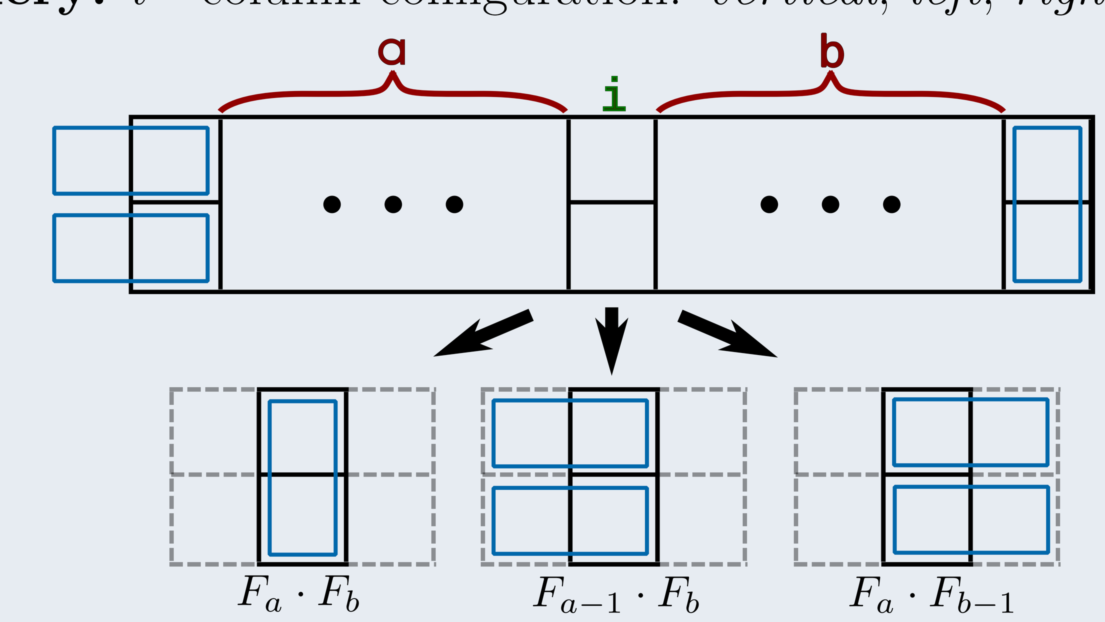
Work in Progress

Random Domino Tiling

A $2 \times n$ grid tiled with dominoes: F_n tilings possible.



Query: i^{th} column configuration: *vertical*, *left*, *right*?



Sufficent to approximate F_c/F_{c-1}: Use ϕ if $c = \Omega(\log(n))$
Open: $k \times n$ grid for $k = \omega(1)$ and Dimer model

Random Coloring: Glauber Dynamics

Consider a *uniform* random coloring of the input graph.
Query: What is v's color in the random coloring?

Global Algorithm (Glauber Dynamics) for $k > 2\Delta$

- Sample $r = \mathcal{O}(n \log n)$ (vertex, color) pairs $\langle (v_i, c_i) \rangle_{i \in [r]}$
- For steps $i = 1, \ldots, r$
  - If no neighbor of v_i has color c_i, set v_i's color to c_i
  - Else, do nothing

Local Algorithm for $k = \Omega(\Delta \log n)$

- Locally sample *all* occurences of $(v, \star)$: implemented effenciently with the proposed **Counting-Generator**
- Sample $(w, \star)$ if necessary, where w is neighbor of v
- Query tree is of size $\mathcal{O}(1)$ for $k = \Omega(\Delta \log n)$

Open: $k = o(\Delta \log n)$

[ELMR17] Guy Even, Reut Levi, Moti Medina, and Adi Rosen. Sublinear random access generators for preferential attachment graphs. In 44th International Colloquium on Automata, Languages, and Programming, ICALP 2017, July 10-14, 2017, Warsaw, Poland, pages 6:16:15, 2017.
[GGN10] Oded Goldreich, Shafi Goldwasser, and Asaf Nussboim. On the implementation of huge random objects. SIAM Journal on Computing, 39(7):27612822, 2010.