Distributed Density Estimation

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1 Random Coloring of a Graph

We wish to locally sample an uniformly random coloring of a graph. A q-coloring of a graph $G = (V, E)$ is a function $\sigma : V \to [q]$, such that for all $(u, v) \in E$, $\sigma_u \neq \sigma_v$. We will consider only bounded degree graphs, i.e. graphs with max degree $\leq \Delta$. Otherwise, the coloring problem becomes NP-hard. cite

Using the technique of path-coupling, Vigoda showed that for $q > 2\Delta$, one can sample an uniformly cite random coloring by using a MCMC algorithm.

1.1 Glauber Dynamics

The Markov Chain proceeds in T steps. The state of the chain at time t is given by $\mathbf{X}^t \in [q]^{|V|}$. Specifically, the color of vertex v at step t is $\mathbf{X}_v^t$.

In each step of the Markov process, a pair $(v, c) \in V \times [q]$ is sampled uniformly at random. Subsequently, if the recoloring of vertex v with color c does not result in a conflict with v's neighbors, i.e. $c \notin \left\{X_u^t : u \in \Gamma(v)\right\}$, then the vertex is recolored i.e. $X_v^{t+1} \leftarrow c$.

After running the MC for $T = \mathcal{O}(n \log n)$ steps we reach the stationary distribution (ϵ close), and the coloring is an uniformly random one.

1.2 Local Coloring Algortihm

Given a vertex v, the local-access generator has to output the color of v after running $T = \mathcal{O}(n \log n) = k \cdot n \log n$ steps of Glauber Dynamics where k is a constant. For this algorithm to work, we will take $q > 2k\Delta \log n = \mathcal{O}(\Delta \log n)$.

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1.3 Modified Glauber Dynamics

Now we define a modified Markov Chain, with each step called an epoch. In the i^{th} epoch, denoted by $\mathcal{E}_i$,

- Pick a random permutation $\pi^{(i)}$ of the vertices V.
- Sample $n = |V|$ colors $\langle c_1, c_2, \cdots, c_n \rangle$ from $[q]$.
- Perform the standard update using the pairs $\left\langle (\pi_1^{(i)}, c_1), (\pi_2^{(i)}, c_2), \cdots, (\pi_n^{(i)}, c_n) \right\rangle$.

Theorem 1. *After $k \log n$ epochs, the Markov Chain is mixed.*

First we consider T iterations of the above MC, and the corresponding vertex and color samples.

$$\langle (v_1, c_1), (v_2, c_2), (v_3, c_3), \cdots, (v_T, c_T) \rangle \sim_{\mathcal{U}} (V \times [q])^T$$

A position i in the sequence is labeled *"ACCEPT"* if at the i^{th} step, v_i was recolored to c_i (no conflicts with neighbors). Otherwise, position i is marked *"REJECT"*.

Given a vertex v, we consider all instances of $(v, *)$, where $* \in [q]$. Let the last such occurrence be (v, c_t). We now need to compute whether position i was marked *"ACCEPT"* or *"REJECT"*.

References