The f-wave method for nonlinear conservation laws with spatially varying flux

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Outline

1. Review
   - The wave-propagation form
   - Problems in variable coefficient cases
2. The f-wave method
   - Definition
   - Advantages
3. Nonlinear elastic waves in layered media
   - Model setup
   - Simulations

Review on the wave-propagation form

Given system of conservation laws

$$q_t + f(q)_x = 0,$$

we have the discretization using the wave-propagation form:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\sum (s^p)^+ \mathcal{W}^p_{i-1/2} + \sum (s^p)^- \mathcal{W}^p_{i+1/2}\right)$$

where $\mathcal{W}^p_{i-1/2}$ are the waves of discontinuities generated at the interface between cells $i-1$ and i

Review on the wave-propagation form

Generally, $\mathcal{W}^p_{i-1/2}$ is obtain by decomposing the jumps $Q^n_i - Q^n_{i-1}$ into the eigenvectors of some matrix A approximating the Jacobian matrix $\nabla_q f(q)$

$$Q^n_i - Q^n_{i-1} = \sum \mathcal{W}^p_{i-1/2} = \sum \alpha^p r^p$$

Problems in variable coefficient cases

Consider generalized Riemann problem

$$\begin{cases} q_t + (u_l q)_x = 0 & \text{if } x < 0 \\ q_t + (u_r q)_x = 0 & \text{if } x > 0 \end{cases}, \quad q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

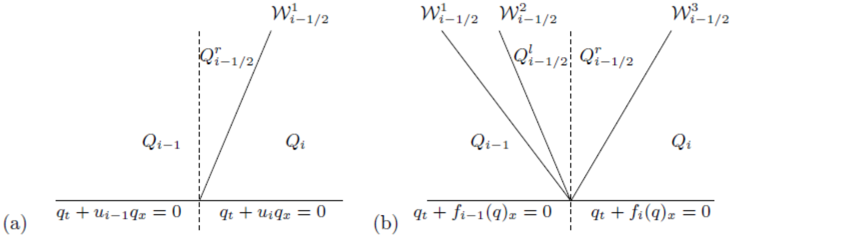


Fig. 1. (a) *Riemann solution for the variable-coefficient advection equation in the case* $u_{i-1} > 0$ *and* $u_i > 0$. (b) *Structure of the Riemann solution for a generalized Riemann problem with* $m = 3$.

[1, p. 960]

What is the f-wave method?

Flux differencing form:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(f(Q_{i+1/2}) - f(Q_{i-1/2})\right)$$

we call the wave $\mathcal{W}$ w-waves. If instead we decompose the difference of flux, we get f-waves $\mathcal{Z}_{i-1/2}^p$.

$$f(Q_i) - f(Q_{i-1}) = \sum \beta_{i-1/2}^p r_{i-1/2}^p \equiv \sum \mathcal{Z}_{i-1/2}^p.$$

Why f-waves?

- Easy to implement.
- Better conservation.
- Directly generalized to the case with source term.
- May be a more fundamental object.

Conservation: Roe condition

For the standard w-wave propagation form, an approximate Riemann solver is conservative iff

$$A_{i-1/2}(Q_i - Q_{i-1}) = f_i(Q_i) - f_{i-1}(Q_{i-1})$$

is satisfied. However, the conservation is a natural result in the f-wave method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum sign((\lambda^p)^-)\mathcal{Z}_{i+1/2}^p + \sum sign((\lambda^p)^+)\mathcal{Z}_{i-1/2}^p\right]$$

$$Q_{i+1}^{n+1} = Q_{i+1}^n - \frac{\Delta t}{\Delta x}\left[\sum sign((\lambda^p)^-)\mathcal{Z}_{i+3/2}^p + \sum sign((\lambda^p)^+)\mathcal{Z}_{i+1/2}^p\right]$$

1D Elastic wave equations

$$\epsilon(x,t)_t - u(x,t)_x = 0$$
$$\rho(x)u(x,t)_t - \sigma(\epsilon,x)_x = 0$$

The first equation is definition of strain. The second one is the the Newton's second law. We shall consider a nonlinear model with exponential relation

$$\sigma(x,t) = \exp(K(x)\epsilon(x,t)) - 1$$

in (Periodically) layered media

$$K(x) = \begin{cases} K_A & \text{if } j\delta < x < (j+\alpha)\delta \text{ for some integer } j, \\ K_B & \text{otherwise.} \end{cases}$$

Structure of solution to the Riemann problem

The Jacobian

$$f_q(q,x) = \begin{pmatrix} 0 & -\frac{1}{\rho} \\ -\sigma_\epsilon(\epsilon,x) & 0 \end{pmatrix}.$$

Sound speed (eigenvalues)

$$c(q,x) = \sqrt{\frac{\sigma_\epsilon(\epsilon,x)}{\rho(x)}} \implies \text{No transonic rarefaction wave}$$

Therefore no entropy fix is required and we can choose the solution to the Riemann problem to be all shock solution.

Approximate Riemann solver

Eigenvectors are

$$r^1(q,x) = \begin{pmatrix} 1 \\ Z(q,x) \end{pmatrix}, \quad \text{for } s^1(q,x) = -\sqrt{\frac{\sigma_\epsilon(\epsilon,x)}{\rho(x)}}$$

$$r^2(q,x) = \begin{pmatrix} 1 \\ -Z(q,x) \end{pmatrix}, \quad \text{for } s^2(q,x) = \sqrt{\frac{\sigma_\epsilon(\epsilon,x)}{\rho(x)}}$$

where $Z(q,x) = \rho(x)c(q,x)$ is the impedance.

Approximate Riemann solver

Instead of giving the approximate Jacobian $A_{i-1/2}$ directly, we specify the eigenvalues and eigenvectors of $A_{i-1/2}$:

$$r^1_{i-1/2} = r^1_{i-1} = \begin{bmatrix} 1 \\ Z_{i-1} \end{bmatrix}, \quad s^1_{i-1/2} = -\sqrt{\frac{\sigma'_{i-1}(\epsilon_{i-1})}{\rho_{i-1}}} \tag{1}$$

$$r^2_{i-1/2} = r^2_{i} = \begin{bmatrix} 1 \\ -Z_{i} \end{bmatrix}, \quad s^2_{i-1/2} = -\sqrt{\frac{\sigma'_{i}(\epsilon_{i})}{\rho_{i}}} \tag{2}$$

Here we are using eigenvalues and eigenvectors according to the cells where they are.

Convergence study

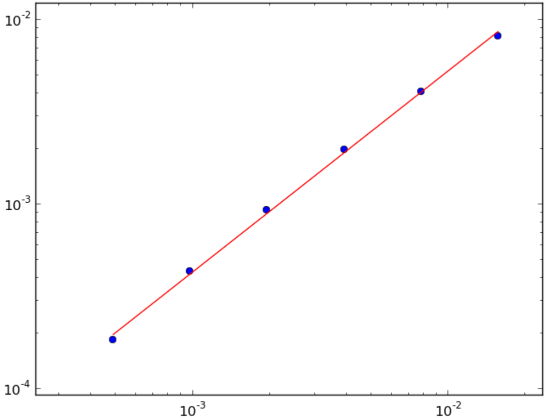
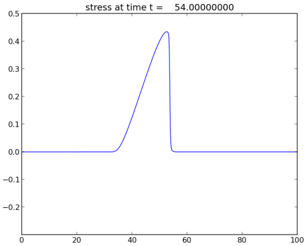


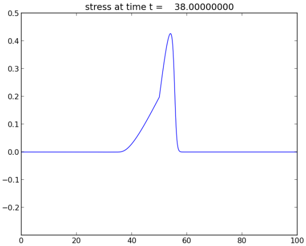
Figure: L_1 error versus cell length. Second order accuracy is obtained for smooth solution.

Stress waves pass through interface of 2 media

Homogeneous:

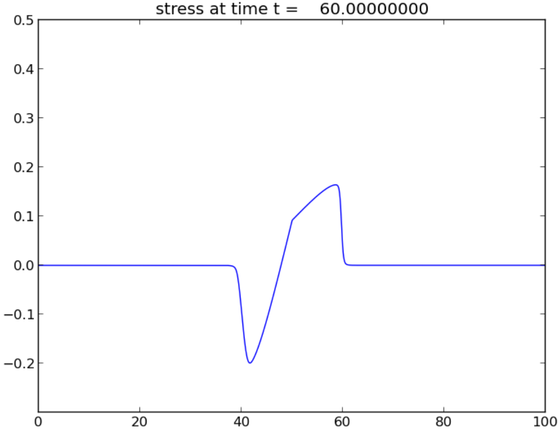


Speed mismatch:

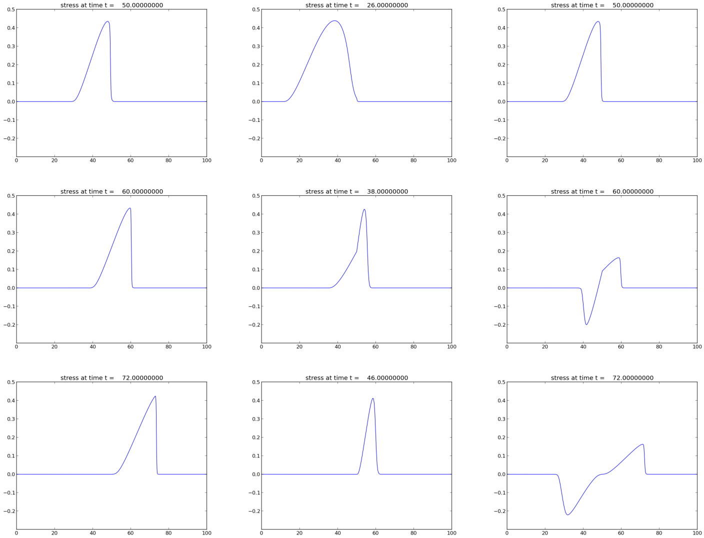


Stress wave pass through interface of 2 media

Impedance mismatch:

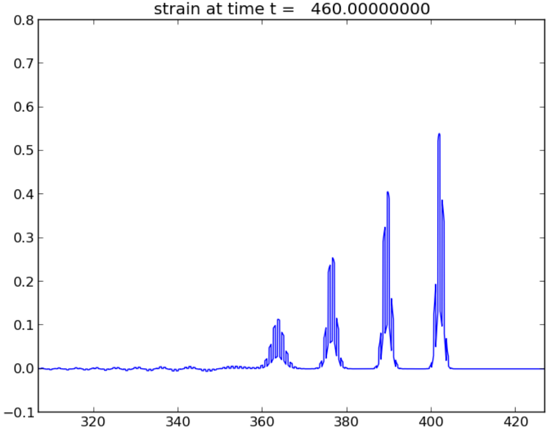


Homogeneous; speed, impedance mismatch



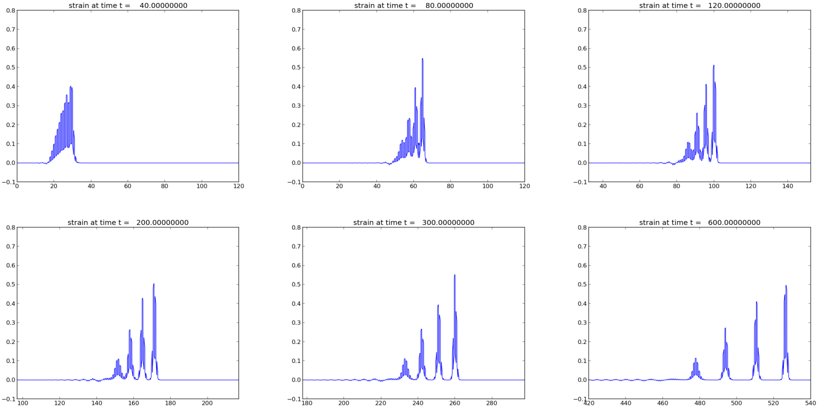
Generation of solitons in layered medium

Layered nonlinear medium:

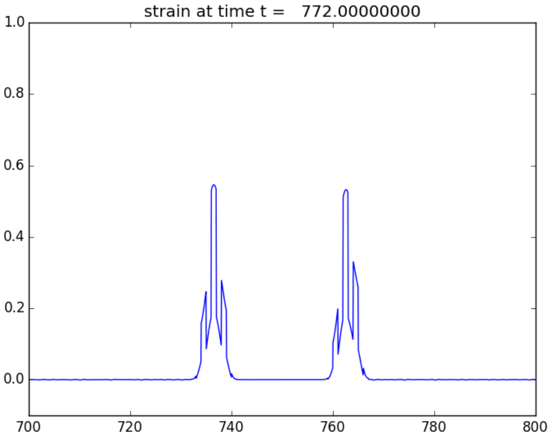


Generation of solitons in layered medium

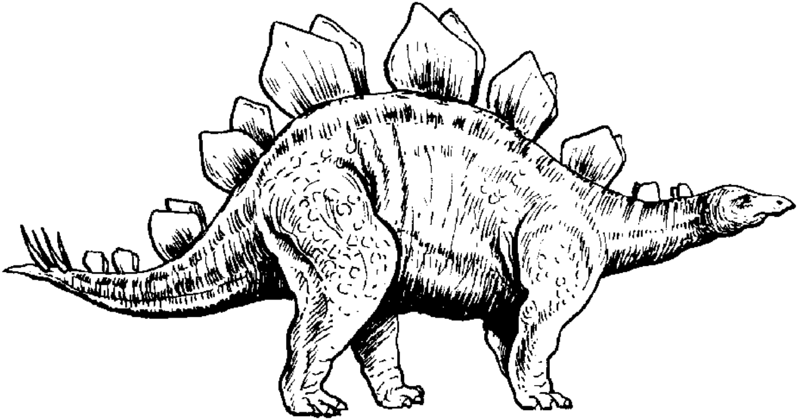
Strain and stress waves



Close-up view of 2 solitary waves interacting



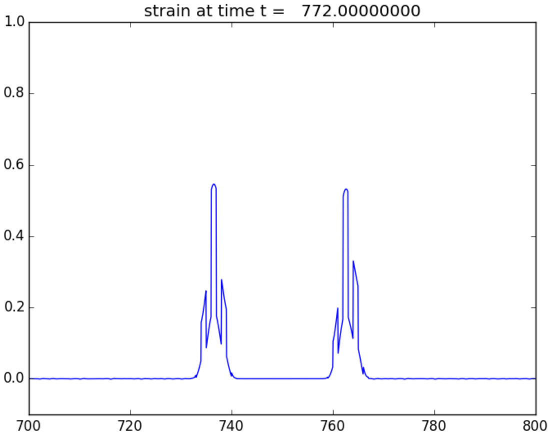
Stegosaurus



From http://ancientadventurescambodia.com /tag/stegosaurus/

Close-up view of 2 stegotons interacting

Collision:



Question session

Thanks for listening. Questions?

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