

The f-wave method for nonlinear conservation laws with spatially varying flux

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Outline

- 1 Review
 - The wave-propagation form
 - Problems in variable coefficient cases
- 2 The f-wave method
 - Definition
 - Advantages
- 3 Nonlinear elastic waves in layered media
 - Model setup
 - Simulations

Review on the wave-propagation form

Given system of conservation laws

$$q_t + f(q)_x = 0,$$

we have the discretization using the wave-propagation form:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left(\sum (s^p)^+ \mathcal{W}_{i-1/2}^p + \sum (s^p)^- \mathcal{W}_{i+1/2}^p \right)$$

where $\mathcal{W}_{i-1/2}^p$ are the waves of discontinuities generated at the interface between cells $i-1$ and i

Review on the wave-propagation form

Generally, $\mathcal{W}_{i-1/2}^p$ is obtain by decomposing the jumps $Q_i^n - Q_{i-1}^n$ into the eigenvectors of some matrix A approximating the Jacobian matrix $\nabla_q f(q)$

$$Q_i^n - Q_{i-1}^n = \sum \mathcal{W}_{i-1/2}^p = \sum \alpha^p r^p$$

Consider generalized Riemann problem

(a) Q_{i-1} Q_i $Q_{i-1/2}^r$ $Q_{i-1/2}^l$ $W_{i-1/2}^1$ $W_{i-1/2}^2$ $W_{i-1/2}^3$ $q_t + u_{i-1}q_x = 0$ $q_t + u_i q_x = 0$

(b) Q_{i-1} Q_i $Q_{i-1/2}^1$ $Q_{i-1/2}^2$ $Q_{i-1/2}^3$ $Q_{i-1/2}^r$ $Q_{i-1/2}^l$ $W_{i-1/2}^1$ $W_{i-1/2}^2$ $W_{i-1/2}^3$ $q_t + f_{i-1}(q)x = 0$ $q_t + f_i(q)x = 0$

[1, p. 960]

What is the f-wave method?

Flux differencing form:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} (f(Q_{i+1/2}) - f(Q_{i-1/2}))$$

we call the wave \mathcal{W} w-waves. If instead we decompose the difference of flux, we get f-waves $\mathcal{Z}_{i-1/2}^p$.

$$f(Q_i) - f(Q_{i-1}) = \sum \beta_{i-1/2}^p r_{i-1/2}^p \equiv \sum \mathcal{Z}_{i-1/2}^p.$$

Why f-waves?

- Easy to implement.
- Better conservation.
- Directly generalized to the case with source term.
- May be a more fundamental object.

Conservation: Roe condition

For the standard w-wave propagation form, an approximate Riemann solver is conservative iff

$$A_{i-1/2}(Q_i - Q_{i-1}) = f_i(Q_i) - f_{i-1}(Q_{i-1})$$

is satisfied. However, the conservation is a natural result in the f-wave method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x} \left[\sum \text{sign}((\lambda^p)^-) \mathcal{Z}_{i+1/2}^p + \sum \text{sign}((\lambda^p)^+) \mathcal{Z}_{i-1/2}^p \right]$$

$$Q_{i+1}^{n+1} = Q_{i+1}^n - \frac{\Delta t}{\Delta x} \left[\sum \text{sign}((\lambda^p)^-) \mathcal{Z}_{i+3/2}^p + \sum \text{sign}((\lambda^p)^+) \mathcal{Z}_{i+1/2}^p \right]$$

Convergence study

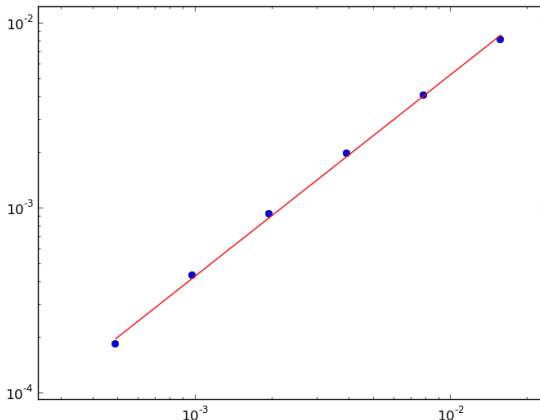


Figure: L_1 error versus cell length. Second order accuracy is obtained for smooth solution.

1D Elastic wave equations

$$\begin{aligned}\epsilon(x, t)_t - u(x, t)_x &= 0 \\ \rho(x)u(x, t)_t - \sigma(\epsilon, x)_x &= 0\end{aligned}$$

The first equation is definition of strain. The second one is the the Newton's second law. We shall consider a nonlinear model with exponential relation

$$\sigma(x, t) = \exp(K(x)\epsilon(x, t)) - 1$$

in (Periodically) layered media

$$K(x) = \begin{cases} K_A & \text{if } j\delta < x < (j + \alpha)\delta \text{ for some integer } j, \\ K_B & \text{otherwise.} \end{cases}$$

Structure of solution to the Riemann problem

The Jacobian

$$f_q(q, x) = \begin{pmatrix} 0 & -\frac{1}{\rho} \\ -\sigma_\epsilon(\epsilon, x) & 0 \end{pmatrix}.$$

Sound speed (eigenvalues)

$$c(q, x) = \sqrt{\frac{\sigma_\epsilon(\epsilon, x)}{\rho(x)}} \implies \text{No transonic rarefaction wave}$$

Therefore no entropy fix is required and we can choose the solution to the Riemann problem to be all shock solution.

Approximate Riemann solver

Eigenvectors are

$$r^1(q, x) = \begin{pmatrix} 1 \\ Z(q, x) \end{pmatrix}, \text{ for } s^1(q, x) = -\sqrt{\frac{\sigma_\epsilon(\epsilon, x)}{\rho(x)}}$$

$$r^2(q, x) = \begin{pmatrix} 1 \\ -Z(q, x) \end{pmatrix}, \text{ for } s^2(q, x) = \sqrt{\frac{\sigma_\epsilon(\epsilon, x)}{\rho(x)}}$$

where $Z(q, x) = \rho(x)c(q, x)$ is the impedance.

Approximate Riemann solver

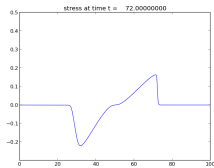
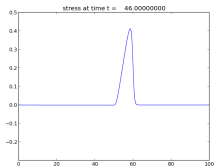
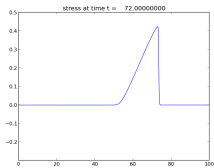
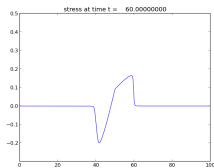
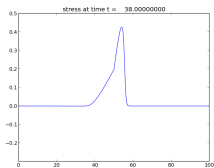
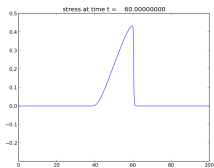
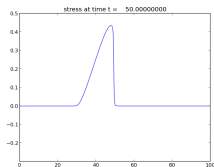
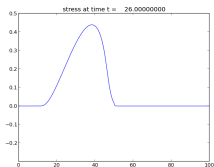
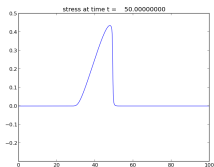
Instead of giving the approximate Jacobian $A_{i-1/2}$ directly, we specify the eigenvalues and eigenvectors of $A_{i-1/2}$:

$$r_{i-1/2}^1 = r_{i-1}^1 = \begin{bmatrix} 1 \\ Z_{i-1} \end{bmatrix}, \quad s_{i-1/2}^1 = -\sqrt{\frac{\sigma'_{i-1}(\epsilon_{i-1})}{\rho_{i-1}}} \quad (1)$$

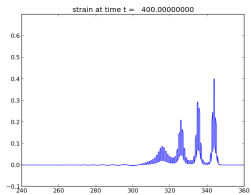
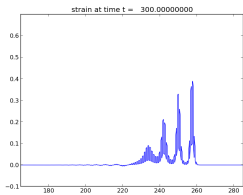
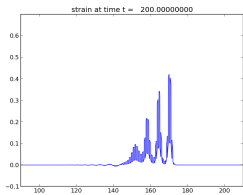
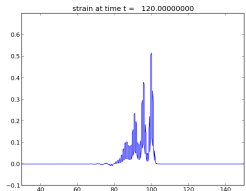
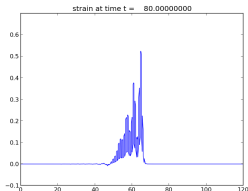
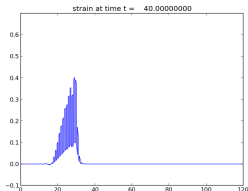
$$r_{i-1/2}^2 = r_i^2 = \begin{bmatrix} 1 \\ -Z_i \end{bmatrix}, \quad s_{i-1/2}^2 = -\sqrt{\frac{\sigma'_i(\epsilon_i)}{\rho_i}} \quad (2)$$

Here we are using eigenvalues and eigenvectors according to the cells where they are.

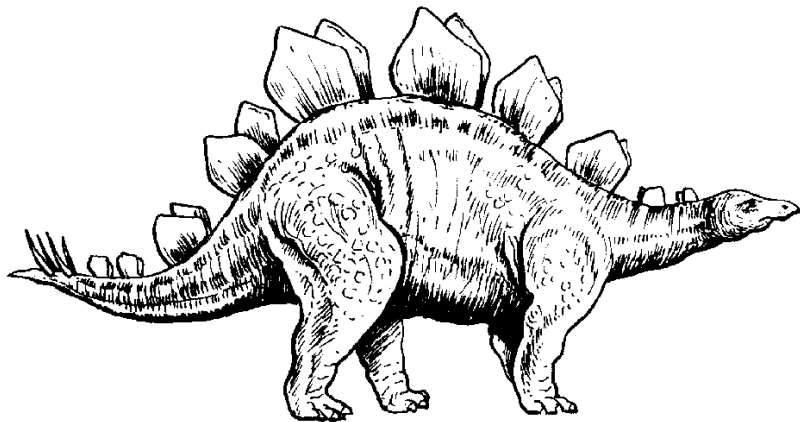
Homogeneous; speed, impedance mismatch



Generation of solitons in layered media



Stegosaurus



From <http://ancientadventurescambodia.com/tag/stegosaurus/>

Question session

Thanks for listening. Questions?

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