The f-wave method for nonlinear conservation laws with spatially varying flux

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Outline

1. Review
   - The wave-propagation form
   - Problems in variable coefficient cases
2. The f-wave method
   - Definition
   - Advantages
3. Nonlinear elastic waves in layered media
   - Model setup

Review on the wave-propagation form

Given system of conservation laws

$$q_t + f(q)_x = 0,$$

we have the discretization using the wave-propagation form:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(\sum (s^p)^+ \mathcal{W}_{i-1/2}^p + \sum (s^p)^- \mathcal{W}_{i+1/2}^p\right)$$

where $\mathcal{W}_{i-1/2}^p$ are the waves of discontinuities generated at the interface between cells x_{i-1} and x_i

Review on the wave-propagation form

Generally, $\mathcal{W}^p_{i-1/2}$ is obtain by decomposing the jumps $Q^n_i - Q^n_{i-1}$ into the eigenvectors of some matrix A approximating the Jacobian matrix $\nabla_q f(q)$

$$Q^n_i - Q^n_{i-1} = \sum \mathcal{W}^p_{i-1/2} = \sum \alpha^p r^p$$

Problems in variable coefficient cases

Consider generalized Riemann problem

$$\begin{cases} q_t + (u_l q)_x = 0 & \text{if } x < 0 \\ q_t + (u_r q)_x = 0 & \text{if } x > 0 \end{cases}, \quad q(x,0) = \begin{cases} q_l & \text{if } x < 0 \\ q_r & \text{if } x > 0 \end{cases}$$

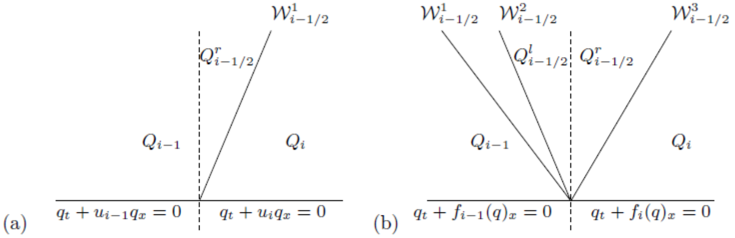


FIG. 1. (a) *Riemann solution for the variable-coefficient advection equation in the case* $u_{i-1} > 0$ *and* $u_i > 0$. (b) *Structure of the Riemann solution for a generalized Riemann problem with* $m = 3$.

[1, p. 960]

What is the f-wave method?

Flux differencing form:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(f(Q_{i+1/2}) - f(Q_{i-1/2})\right)$$

we call the wave $\mathcal{W}$ w-waves. If instead we decompose the difference of flux, we get f-waves $\mathcal{Z}^p_{i-1/2}$.

$$f(Q_i) - f(Q_{i-1}) = \sum \beta^p_{i-1/2} r^p_{i-1/2} \equiv \sum \mathcal{Z}^p_{i-1/2}.$$

Why f-waves?

- Easy to implement.
- Better conservation.
- Directly generalized to the case with source term.

Conservation: Roe condition

For the standard w-wave propagation form, an approximate Riemann solver is conservative iff

$$A_{i-1/2}(Q_i - Q_{i-1}) = f_i(Q_i) - f_{i-1}(Q_{i-1})$$

is satisfied. However, the conservation is a natural result in the f-wave method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\sum sign((\lambda^p)^-)\mathcal{Z}_{i+1/2}^p + \sum sign((\lambda^p)^+)\mathcal{Z}_{i-1/2}^p\right]$$

$$Q_{i+1}^{n+1} = Q_{i+1}^n - \frac{\Delta t}{\Delta x}\left[\sum sign((\lambda^p)^-)\mathcal{Z}_{i+3/2}^p + \sum sign((\lambda^p)^+)\mathcal{Z}_{i+1/2}^p\right]$$

1D Elastic wave equations

$$\epsilon(x,t)_t - u(x,t)_x = 0$$
$$\rho(x)u(x,t)_t - \sigma(\epsilon,x)_x = 0$$

The first equation is definition of strain. The second one is the the Newton's second law. We shall consider a nonlinear model with exponential relation

$$\sigma(x,t) = \exp(K(x)\epsilon(x,t)) - 1$$

in (Periodically) layered media

$$K(x) = \begin{cases} K_A & \text{if } j\delta < x < (j+\alpha)\delta \text{ for some integer } j, \\ K_B & \text{otherwise.} \end{cases}$$

Structure of solution to the Riemann problem

The Jacobian

$$f_q(q,x) = \begin{pmatrix} 0 & -\frac{1}{\rho} \\ -\sigma_\epsilon(\epsilon, x) & 0 \end{pmatrix}.$$

Sound speed (eigenvalues)

$$c(q,x) = \sqrt{\frac{\sigma_\epsilon(\epsilon,x)}{\rho(x)}} \Longrightarrow \text{No transonic rarefaction wave}$$

Therefore no entropy fix is required and we can choose the solution to the Riemann problem to be all shock solution.

Approximate Riemann solver

Eigenvectors are

$$r^1(q,x) = \begin{pmatrix} 1 \\ Z(q,x) \end{pmatrix}, \quad \text{for } s^1(q,x) = -\sqrt{\frac{\sigma_\epsilon(\epsilon,x)}{\rho(x)}}$$

$$r^2(q,x) = \begin{pmatrix} 1 \\ -Z(q,x) \end{pmatrix}, \quad \text{for } s^2(q,x) = \sqrt{\frac{\sigma_\epsilon(\epsilon,x)}{\rho(x)}}$$

where $Z(q,x) = \rho(x)c(q,x)$ is the impedance.

Approximate Riemann solver

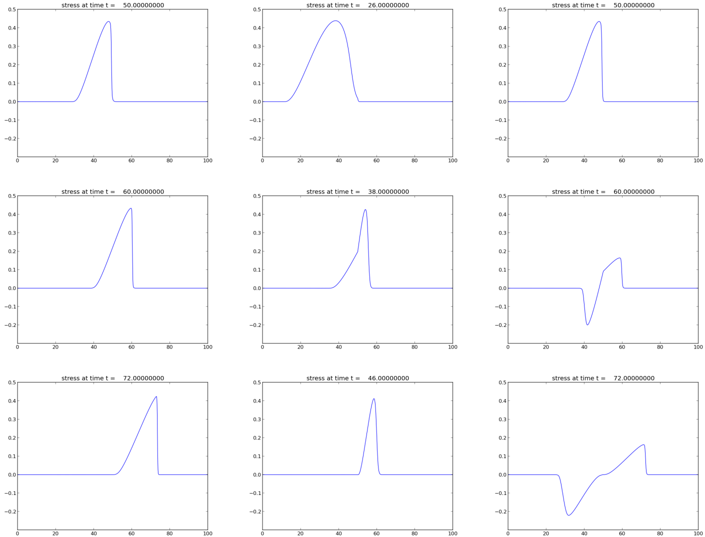
Instead of giving the approximate Jacobian $A_{i-1/2}$ directly, we specify the eigenvalues and eigenvectors of $A_{i-1/2}$:

$$r^1_{i-1/2} = r^1_{i-1} = \left[\begin{array}{c} 1 \\ Z_{i-1} \end{array} \right], \quad s^1_{i-1/2} = -\sqrt{\frac{\sigma'_{i-1}(\epsilon_{i-1})}{\rho_{i-1}}} \tag{1}$$

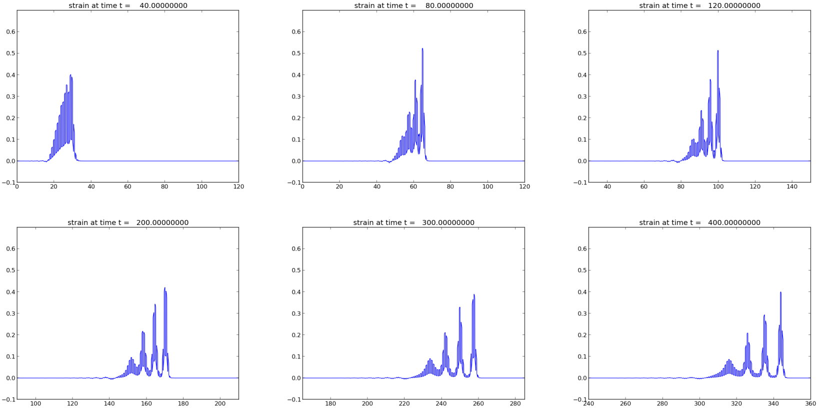
$$r^2_{i-1/2} = r^2_{i} = \left[\begin{array}{c} 1 \\ -Z_{i} \end{array} \right], \quad s^2_{i-1/2} = -\sqrt{\frac{\sigma'_{i}(\epsilon_{i})}{\rho_{i}}} \tag{2}$$

Here we are using eigenvalues and eigenvectors according to the cells where they are.

Effect of impedance across the interface



Generation of solitons in layered media



Question session

Thanks for listening. Questions?

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