AMATH574 - Final Project

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1 Introduction

The pseudo-1D Euler equations will be explored for the case of an unsteady nozzle problem with fixed geometry. The simulations of the nozzle will be run until the steady state is reached for various pressure ratios that will result in purely subsonic flow, choked subsonic flow, and purely supersonic flow. In each case, the steady state results will be compared to the theoretical quasi-1D nozzle equations for the accuracy of the different flux methods used. Because the pseudo-1D Euler equations inherently contains a source term, fractional splitting methods will be used to account for the source term. This will require the implementation of a Runge-Kutta method in order to solve the ODE when the solution is reacted according to the source term.

2 Pseudo-1D Euler Equations

Pseudo-1D Euler eqns

$$\frac{d}{dt}\begin{bmatrix} \rho A \\ \rho u A \\ eA \end{bmatrix} + \frac{d}{dx}\begin{bmatrix} \rho u A \\ (\rho u^2 + p)A \\ u(e+p)A \end{bmatrix} = \begin{bmatrix} 0 \\ p\frac{dA}{dx} \\ 0 \end{bmatrix} \tag{2.1}$$

The flux Jacobian of the system is given by

$$f'(q) = \begin{bmatrix} 0 & A & 0 \\ A(p_\rho - u^2 & Au(2-p_e)) & Ap_e \\ Au(p_\rho - H) & A(H - u^2 p_e) & Au(1+p_e) \end{bmatrix} \tag{2.2}$$

where $H = \frac{e+p}{\rho}$, $p_\rho = \frac{\gamma-1}{2}u^2$, and $p_e = \gamma - 1$. The eigenvalues may be solved by taking the determinant and setting it equal to zero

$$det\begin{bmatrix} -\lambda & A & 0 \\ A(p_\rho - u^2 & Au(2-p_e)) - \lambda & Ap_e \\ Au(p_\rho - H) & A(H - u^2 p_e) & Au(1+p_e) - \lambda \end{bmatrix} = 0 \tag{2.3}$$

where the eigenvalues are then given by

$$\lambda^1 = A(u-c), \quad \lambda^2 = uA, \quad \lambda^3 = A(u+c) \tag{2.4}$$

Solving for the eigenvectors, r, results in

$$\begin{gathered} (f'(q) - \lambda I)r = 0 \\ R = \left[\; r^1 \;\middle|\; r^2 \;\middle|\; r^3 \;\right] = \begin{bmatrix} 1 & 1 & 1 \\ u-c & u & u+c \\ H - uc & H - \frac{c^2}{\gamma-1} & H + uc \end{bmatrix} \end{gathered} \tag{2.5}$$

3 Euler Equation Riemann Problem

To solve the Euler equation Riemann problem, the waves are split into the left going and right going waves according to the eigenvalues. This means that if the flow is subsonic then λ^1 is a left going wave and λ^2 and λ^3 are right going waves, while if the flow is supersonic then all waves are right going waves. The coefficients of these eigenvectors, α, may be determined by the following which involves taking the inverse of the eigenvector matrix, R.

$$q_r - q_l = R\alpha \to R^{-1}(q_r - q_l) = \alpha \tag{3.1}$$

where

$$R^{-1} = \begin{bmatrix} \frac{c^2+H(\gamma-1)+cu+(\gamma-1)u^2}{2c^2} & -\frac{c+(\gamma-1)u}{2c^2} & \frac{\gamma-1}{2c^2} \\ \frac{(\gamma-1)(H-u^2)}{c^2} & \frac{(\gamma-1)u}{c^2} & -\frac{\gamma-1}{c^2} \\ \frac{c^2-H(\gamma-1)-cu+(\gamma-1)u^2}{2c^2} & \frac{c-(\gamma-1)u}{2c^2} & \frac{\gamma-1}{2c^2} \end{bmatrix} \tag{3.2}$$

The coefficients may then be solved to be

$$\begin{aligned} \alpha &= R^{-1}(qr - ql) = R^{-1}\delta \\ &= \begin{bmatrix} \frac{c^2+H(\gamma-1)+cu+(\gamma-1)u^2}{2c^2}\delta^1 - \frac{c+(\gamma-1)u}{2c^2}\delta^2 + \frac{\gamma-1}{2c^2}\delta^3 \\ \frac{(\gamma-1)(H-u^2)}{c^2}\delta^1 + \frac{(\gamma-1)u}{c^2}\delta^2 - \frac{\gamma-1}{c^2}\delta^3 \\ \frac{c^2-H(\gamma-1)-cu+(\gamma-1)u^2}{2c^2}\delta^1 + \frac{c-(\gamma-1)u}{2c^2}\delta^2 + \frac{\gamma-1}{2c^2}\delta^3 \end{bmatrix} \\ &= \begin{bmatrix} (\gamma-1)\frac{(H+u^2)\delta^1-u\delta^2+\delta^3}{2c^2} + \frac{(c^2+cu)\delta^1-c\delta^2}{2c^2} \\ (\gamma-1)\frac{(H-u^2)\delta^1+u\delta^2-\delta^3}{c^2} \\ (\gamma-1)\frac{(u^2-H)\delta^1-u\delta^2+\delta^3}{2c^2} + \frac{(c^2-cu)\delta^1+c\delta^2}{2c^2} \end{bmatrix} \end{aligned} \tag{3.3}$$

where δ^i refers to the ith element of the δ vector. Because solving the exact Riemann solution is very expensive, the roe approximation is used instead which estimates velocity, $\hat{u}$, total specific enthalpy, $\hat{H}$, and sound speed, $\hat{c}$, by

$$\begin{aligned} \hat{u} &= \frac{\sqrt{\rho_{i-1}}u_{i-1} + \sqrt{\rho_i}u_i}{\sqrt{\rho_{i-1}} + \sqrt{\rho_i}} \\ \hat{H} = \frac{\sqrt{\rho_{i-1}}H_{i-1} + \sqrt{\rho_i}H_i}{\sqrt{\rho_{i-1}} + \sqrt{\rho_i}} &= \frac{(E_{i-1}+p_{i-1})/\sqrt{\rho_{i-1}} + (E_i+p_i)/\sqrt{\rho_i}}{\sqrt{\rho_{i-1}} + \sqrt{\rho_i}} \\ \hat{c} &= \sqrt{(\gamma-1)\left(\hat{H} - \frac{1}{2}\hat{u}^2\right)} \end{aligned} \tag{3.4}$$

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These values are then plugged into Eqn. 2.4 and 3.3 to solve the Riemann problem with the approximated values.

4 Numerical Method

Because the psuedo-1D Euler equation given by Eqn. 2.1 contains a source term, $p\frac{dA}{dx}$, the typical Riemann solving method can't be applied. In order to account for this source term, a fractional splitting method is incorporated. Two different types of fractional splitting will be tested: Gudonov's splitting, 1st order accurate, and Strang splitting, 2nd order accurate.

Starting with Gudonov splitting, at each timestep, the solution will be advected and then reacted. This two step method takes the form of

$$\begin{aligned} \text{Step 1: } & q_t + \bar{u} q_x = 0 \\ \text{Step 2: } & q_t = -\beta(q) \end{aligned} \tag{4.1}$$

The first step will be solved first using various different limiter methods, upwind, Lax-Wendroff, superbee, and MC, to test the solutions sensitivity to these different methods. The second step will be solved by using a two stage Runge-Kutta method to solve the ODE. The two steps when specified to the psuedo-1D Euler equations is given by

$$\begin{aligned} \text{Step 1: } & \begin{bmatrix} \rho A \\ \rho u A \\ e A \end{bmatrix}_t + \begin{bmatrix} \rho u A \\ (\rho u^2 + p) A \\ u(e+p) A \end{bmatrix}_x = 0 \\ \text{Step 2: } & \begin{bmatrix} \rho A \\ \rho u A \\ e A \end{bmatrix} = \begin{bmatrix} 0 \\ p \frac{dA}{dx} \\ 0 \end{bmatrix} \end{aligned} \tag{4.2}$$