Project Proposal

WENO High-Resolution Finite Volume Methods

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1 Introduction

Essentially non-oscillatory or ENO methods have historically proven successful at capturing shock discontinuities by choosing the smoothest interpolating polynomial at several neighboring stencils. Weighted ENO or WENO methods use a convex combination of the interpolating polynomial at all stencils and hence preserve both the high-resolution for shocks and high-order accuracy for smooth data.

We propose the implementation of finite-volume and finite-difference WENO methods as investigated in [1, 2, 3]. WENO methods preserve high-order accuracy in regions where the solution is smooth and resolve discontinuities without introducing spurious oscillations. We will implement and test these schemes on one-dimensional hyperbolic equations such as the advection and burgers equations with shocks. We will also include comparisons with `clawpack`'s results for non-WENO schemes. We also plan to investigate the benefits of using Total Variation Diminishing (TVD) Runge-Kutta schemes versus traditional non-TVD integrators. If there is time, we will extend our analysis to more complicated systems of hyberbolic equations including shallow water wave equations.

2 Interpolation and Reconstruction

Consider the problem of constructing the function $L(x)$ that passes though a set of data $\{f_j\}_{j=0}^{N}$ at distinct node locations $\{\mathbf{x}_j\}_{j=0}^{N}$. High-order polynomial interpolation, radial basis functions, or moving least-squares perform excellent when the data f_j originates from a smooth function $f(x)$ such that $f_j = f(x_j)$. However, these traditional methods often introduce unwanted oscillations when applied to discontinuous data. Weighted essentially non-oscillatory (WENO) methods can be used to overcome this difficulty and provide high-order accuracy in smooth regions, as well as "oscillation free" approximations near jumps or higher-order discontinuities. The WENO methodology can be applied to the interpolation problem, or more generally to any function reconstruction problem. We describe these two approaches in the following sections.

2.1 Interpolation

We seek an approximation to the function $f(x)$, given $2r-1$ equispaced data points $\{(x_j, f_j)\}_{j=0}^{2r-1}$. Consider the set of interpolating polynomials $\{P_k(x)\}_{k=1}^{r}$, where the kth polynomial passes through the points $\{(x_j, f_j)\}_{j=k}^{k+r-1}$ (See Figure 1). The polynomials $P_k(x)$ each provide an $\mathrm{O}(\Delta x^r)$ approximation to $f(x)$ on the interval $[x_1, x_{2r-1}]$. In cases where the data f_j is discontinuous, we should select the polynomial which introduces the fewest oscillations. Conversely, when the data f_j is smooth, we should instead consider the interpolating polynomial $L(x)$ which passes through all $2r-1$ points and provides $O(\Delta x^{2r-1})$ accuracy.

We begin our discussion of WENO by noting that the polynomial $L(x)$ can be written as a linear combination of the $P_k(x)$ such that

$$L(x) = \sum_{i=0}^{r} C_k^r(x) P_k(x)$$

Therefore, it is sensible to choose a weighted combinations of the polynomials $P_j(x)$ so that

$$f(x) \approx \sum_{k=1}^{r} w_k^r(x) P_k(x).$$

The weights $w_k(x)$ should be chosen so that when $f(x)$ is smooth everywhere, $w_k(x) \approx C_k^r(x)$, and when $f(x)$ is discontinuous inside the interval $[x_k, x_k + r]$, $w(x) \approx 0$. In order to determine $w(x)$, we must define a smoothness measure for each of the Polynomials $P_k(x)$. As first proposed in [], we consider the smoothness measure

$$S_k(x) = \sum_{i=1}^{r} (\Delta x)^{2i-1} \int_{x-\Delta x/2}^{x+\Delta x/2} \left(\frac{dP_k(x)}{dl} \right)^2 dx \tag{1}$$

which is a sum over two-norms of derivatives, independent of the gridpspacing Δx. We then use the constants S_j to determine the nonlinear WENO weights

$$w_k(x) = \frac{\alpha_k}{\sum_{i=1}^{r} \alpha_i}, \qquad \text{and} \qquad \alpha_j(x) = \frac{C_k^r(x)}{(\epsilon + S_j(x))^p} \tag{2}$$

where ϵ is commonly taken to be 10^{-6} and $p = 2$. From Taylor analysis, it follows that $S_j(x) = \alpha(x) + O(h^{r-1})$ (where α does not depend on k) ensures that the approximation will be $2r-1$ accurate in smooth regions. Though we do not demostrate this here, it has been shown for Eq. (1) in []. The weights and smoothness indicators are not unique, and other choices may be beneficial for certain applications []. Moreover, the weights $w_k(x)$ must be positive and convex so that $\sum_{k=1}^{r} w_k = 1$. Several modifications of the WENO procedure have been proposed to deal with negative weights []. We provide coefficients C_k and measures $S_k(x)$ for $r = 2$ and $r = 3$ in Appendix A.

2.2 Conservation Laws and Reconstruction

2.2.1 Solving Conservation Laws

We seek to numerically integrate the conservation law

$$q_t + f(q)_x = 0.$$

Integrating once in time, once in space, and appropriately exchanging the order of integration leads to

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \left[q(x, t^{n+1}) - q(x, t^n) \right] dt = - \int_{t^n}^{t^{n+1}} \left[f(q(x_{i+1/2}, t)) - f(q(x_{i-1/2}, t)) \right] dt.$$

Dividing through by $\Delta x = (x_i - x_{i-1})$, and letting Q_i denote the ith average of Q, and $q_{i-1/2}$ denote $q(x_{i-1/2})$, this becomes

$$Q_i^{n+1} = Q_{i-1}^n + \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \left[f(q_{i+1/2}(t)) - f(q_{i-1/2}(t)) \right] dt$$

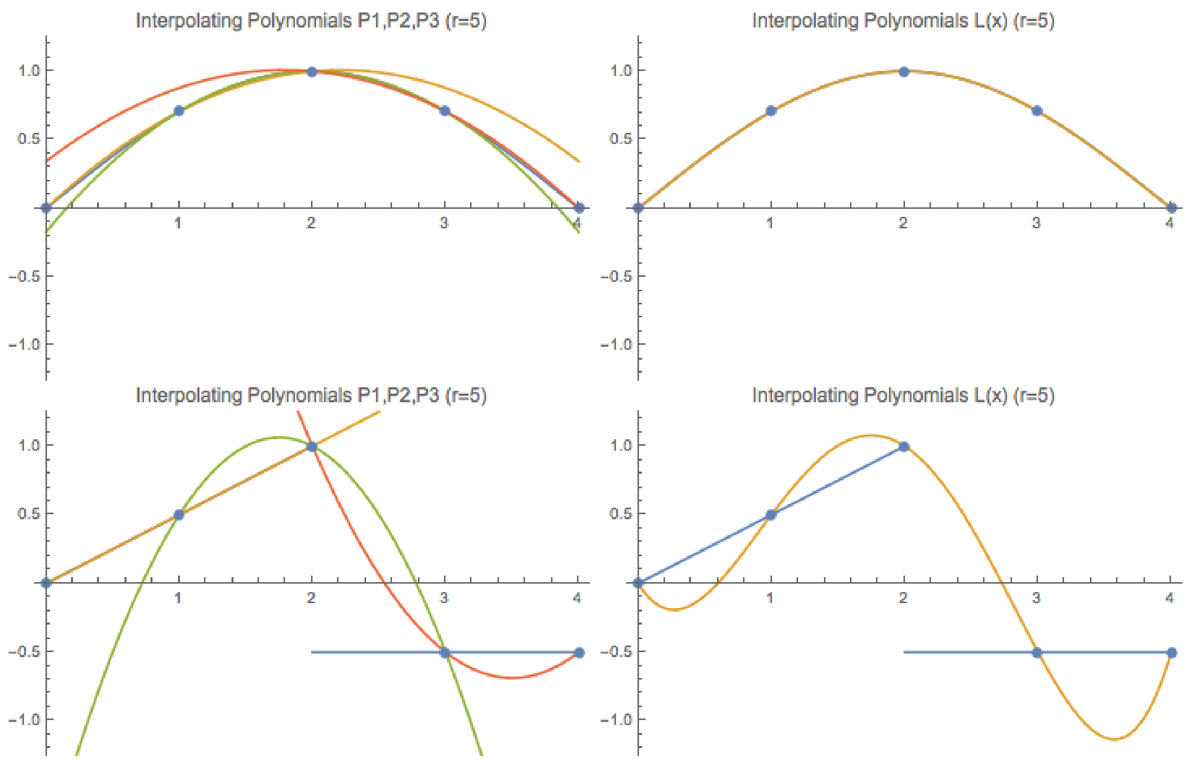


Figure 1: We consider a smooth function $f_s(x) = \sin(\pi x/4)$ (Blue curve in top 2 figures) and the discontinuous function

$$f_d(x) = \begin{cases} \frac{x}{2} & x \leq 2 \\ -\frac{1}{2} & x > 1 \end{cases}$$

(blue curve bottom two figures). We take $r = 3$ and $\Delta x = 1$ and compare the interpolating polynomials $S_1(x)$, $S_2(x)$, $S_3(x)$ (Respectively Colored in Yellow, Green, Red) to the interpolant $L(x)$ which passes through all five points. Notice that for the smooth function, the interpolant $L(x)$ provides a visibly better approximation, whereas $S_1(x)$, $S_2(x)$ and $S_3(x)$ provide better approximations in the appropriate interval.

We can define $\mathbf{Q} = [Q_1, \ldots Q_n]^T$, and rewrite this system compactly as

$$\begin{cases} \frac{d}{dt}\mathbf{Q} = L(\mathbf{Q}) \\ \mathbf{Q}(x, t=0) = Q_0 \end{cases} \qquad \text{where,} \qquad L(Q)_i = f(q_{i-1}) - f(q_i). \tag{3}$$

In order to integrate this Eq. (3), we require a method for estimating $f(q_{i-1/2})$ and $f(q_{i+1/2})$ from the cell averages Q_i. We proceed by describing general polynomial reconstruction in the next section, followed by the WENO methodology.

2.2.2 Polynomial Reconstruction

In the reconstruction problem, we are given function averages a_j at grid points x_j. The function averages are taken between cells such that

$$a_j = \int_{x_{j-1/2}}^{x_{j+1/2}} f(x)dx$$

We can define the reconstructing function $L(x)$ by imposing the conditions

$$a_j = \int_{x_{j-1/2}}^{x_{j+1/2}} L(x)dx \qquad j = 1, \dots, N$$

For example, if we take $L(x) = \sum_{i=1}^{N} c_i x^i$, then we have the system $\mathbf{Ac} = \mathbf{b}$ where

$$\mathbf{A}_{ij} = \frac{x^j_{i+1/2} - x^j_{i-1/2}}{j}, \qquad \mathbf{b}_j = a_j.$$

The reconstructing function $L(x)$ can then be evaluated at the points x_j.

2.2.3 WENO for Reconstruction

We seek an approximation to the function $q(x)$, given $2r-1$ cell averages $\{(x_j, Q_j)\}_{j=1}^{2r-1}$. Let $\{R_k(x)\}_{k=1}^{r}$ be the set of reconstructing polynomials, where averages of kth polynomial over the cells $x_k, \dots x_{k+r}$ are given by Q_k, ... Q_{k+r}. Next, let $L(x)$ denote the reconstructing polynomial which satisfies all $2r-1$ cell averages. It is again possible to form weighted linear combinations of reconstruction polynomials.

$$L(x) = \sum_{k=0}^{r} C_k^r(x) R_k(x)$$

As before, we approximate $q(x)$ with a linear combinations of $R_j(x)$ so that

$$q(x) \approx \sum_{k=1}^{r} w_k^r(x) R_k(x).$$

As we have shown in Section 2.2.1, we are solely interested evaluating $q(x)$ at the cell midpoints $x = x_{j+1/2}$ given the averages $U_{j-r}, \dots U_{j+r-1}$. Thus we neglect x dependence simply write

$$q_{j+1/2} \approx \sum_{k=1}^{r} w_k^r R_k(x_{j+1/2}) \qquad \text{where} \qquad R_k(x_{j+1/2}) = \sum_{l=1}^{r} a_{j,l}^r Q_j$$

The WENO smoothness measures S_j are chosen by replacing $P_j(x)$ with $R_j(x)$ in Eq. (1), and the WENO weights are determined from from Eq. (2). We provide a table of coefficients C_k^r, and $a_{k,l}^r$ for $r = 2$, and $r = 3$ in Appendix B.

3 Results

Our implementation uses WENO weights based on the smoothness measure introduced by Jiang & Shu in [1] that improves WENO accuracy for $r = 3$ from fourth order accurate to fifth order accurate. Denoting the interpolation polynomial on stencil $S_k, k \in [0, r-1]$ by $q_k(x)$, we have the smoothness measure

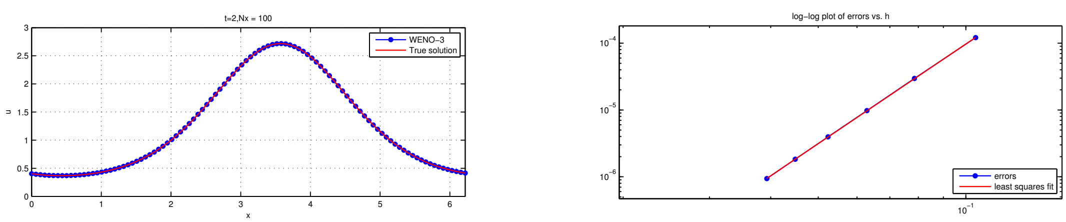
$$IS_k = \sum_{l=1}^{r-1} \int_{x_{j-1/2}}^{x_{j+1/2}} \Delta x^{2l-1} (q_k^{(l)})^2 dx,$$

Note the resemblance of this measure to the L^1 total variation measure as well as the Δx^{2l-1} scaling factor that eliminates dependency on the spatial resolution.

3.1 Convergence for Smooth Data

In our WENO implementation, we observe the desired fifth-order spatial accuracy in the L^2 norm for the advection equation with the smooth solution $u(x,t) = e^{\sin(x-t)}$:

$$u_t + u_x = 0, \quad u(x,0) = e^{\sin(x)}$$



(a) Solution at $t = 2$

(b) WENO5-RK4 $E(h) = O(h^{4.95856})$

Figure 2: WENO5-RK4 for smooth solutions

N	error	ratio	observed order
75	1.62482e-05	---	---
80	1.18145e-05	1.37528	4.93745
85	8.75308e-06	1.34975	4.94718
90	6.59404e-06	1.32742	4.95533
95	5.04236e-06	1.30773	4.96219
100	3.90809e-06	1.29024	4.96804

WENO5-RK4 stands for fifth-order WENO ($r = 3$) and RK4 denotes fourth-order Runge-Kutta time integration. Note we refined with $\Delta t \approx (\Delta x)^{5/4}$ so that the time-stepping is essentially fifth-order [1]. We calculate the least squares fit through the error on a log scale to observe the expected accuracy for both our WENO implementation and `clawpack`'s second-order accurate Lax-Wendroff scheme. Note that we observe fifth-order accuracy in as few as 75 grid points.

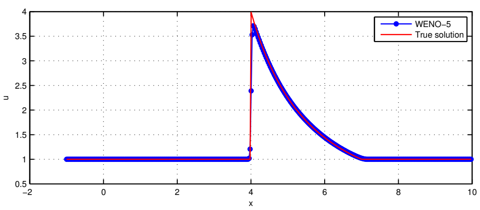
We briefly restate a result from [1]: in general, sufficiently smooth solutions using a WENO method of r candidate polynomials and a nth-order Runge-Kutta time integrator such that $n \geq \max(r, 3)$ will have accuracy $O(\Delta x^r)$.

3.2 Convergence away from discontinuities

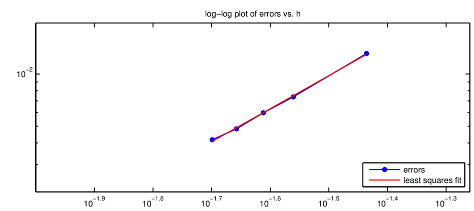
We now consider the following equation

$$u_t + (\sqrt{u})_x = 0, \quad u(x,0) = \begin{cases} 4 & \text{if } 0 < x < 1 \\ 1 & \text{otherwise} \end{cases}$$

For discontinuities we can only observe convergence in the typical sense in the continuous regions of the solution, as demonstrated below. The error from the WENO reconstruction at the discontinuity contaminates the fifth-order accuracy we would expect in the continuous regions to only second order accuracy in the following scenario. Note that in the region $6.5 < x < 7.5$ the true solution is continuous but not smooth.



(a) WENO5-RK5 approximation



(b) WENO5-RK4 error $E(h) = O(h^{1.90852})$

Figure 3: Error: WENO5-RK4 for shock

N	error	ratio	observed order
300	1.31431e-02	---	---
350	7.37517e-03	1.78207	2.00839
400	5.97375e-03	1.23460	1.78926
450	4.82424e-03	1.23828	2.02848
500	4.17701e-03	1.15495	1.51147

We observe that ordinary WENO-5 reconstruction requires an extremely fine grid to even obtain second-order accuracy away from the shock. In the following section we experiment with sharpening methods that resolve discontinuities by incorporating an augmented smoothness measure.

3.3 Sharpening Shocks

To obtain better resolution of contact discontinuities we follow the suggestion in [1] and implement Yang's artificial compression method [4]. This method has the additional benefit of retaining the original WENO accuracy because the magnitudes of the added terms are of the same order of the truncation error.

4 Summary

High-resolution WENO methods outperform standard ENO reconstruction particularly for equations with smooth solutions and rich nonlinear structure. They can be easily modified to yield good resolution at jump discontinuities without losing accuracy in smooth regions of the solution, a feature that makes WENO methods more attractive than standard second-order accurate wave-propagation solvers in practice.

A Interpolation Weights

We consider the interval $[1, 2r-1]$ with interpolation data $\{x_j, f_j\}_{j=1}^{2r-1}$ where $x_j = j$. We seek the polynomials P_k and the weights C_k^r such that

$$L(x) = \sum_{i=1}^{r} C_k^r(x) P_k^r(x)$$

For $r = 2$, we find that

$$C_1^2(x) = 1 - \tfrac{x}{2} \qquad P_1^2(x) = f_2 + (-f_2 + f_3)(-1 + x)$$

$$C_2^2(x) = \tfrac{x}{2} \qquad P_2^2(x) = f_1 - (f_1 + f_2)x$$

$$L(x) = f_1 + (-f_1 + f_2 + 1/2(f_1 - 2f_2 + f_3)(-1 + x))x$$

The smoothness measures are given by

$$S_1(x) = 4f_1^2 - 6f_1 f_2 + 2f_2^2 - 2(-f_1 + f_2)^2 - 2f_1^2 x + 4f_1 f_2 x - 2f_2^2 x$$
$$S_2(x) = 4f_2^2 - 6f_2 f_3 + 2f_3^2 - 2(-f_2 + f_3)^2 - 2f_2^2 x + 4f_2 f_3 x - 2f_3^2 x$$

B Reconstruction Weights

The polynomial coefficients for a general WENO method that uses r candidate polynomials P_k to approximate the flux, where

$$P_k^r(f_0, \dots, f_{r-1}) = \sum_{l=0}^{r-1} a_{k,l}^r f_l,$$

are given in the following table

Table 1: Coefficients $a_{k,l}^r$

r	k	$l=0$	$l=1$	$l=2$
2	0	-1/2	3/2	
	1	1/2	1/2	
3	0	1/3	-7/6	11/6
	1	-1/6	5/6	1/3
	2	1/3	5/6	-1/6

These polynomial coefficients were derived by solving a linear system for the coefficients a_i of $P(x) = a_0 + a_1 x + a_2 x^2$. For example, to obtain coefficients for the leftmost stencil $k = 0$ when $r = 3$, we solve

$$\int_{x_{j-5/2}}^{x_{j-3/2}} P(x)dx = u_{j-2}$$
$$\int_{x_{j-3/2}}^{x_{j-1/2}} P(x)dx = u_{j-1}$$
$$\int_{x_{j-1/2}}^{x_{j+1/2}} P(x)dx = u_j,$$

thus obtaining the coefficients a_i in terms of u_j, the values of u at the grid points. Evaluating $P(x)$ at $x_{j+1/2}$ yields

$$P(x_{j+1/2}) = \frac{1}{3}u_{j-2} - \frac{7}{6}u_{j-1} + \frac{11}{6}u_j,$$

which are exactly the coefficients given in row 4 of Table 1.

Table 2: Weights C_k^r

C_r^k	$k = 0$	$l = 1$	$l = 2$
$r = 2$	1/2	2/3	
$r = 3$	1/10	6/10	3/10

References

[1] Jiang, G.-S., and Shu, C.-W. Efficient implementation of weighted eno schemes. Tech. rep., DTIC Document, 1995.

[2] Ketcheson, D. I., Parsani, M., and LeVeque, R. J. High-order wave propagation algorithms for hyperbolic systems. *SIAM Journal on Scientific Computing 35*, 1 (2013), A351–A377.

[3] Shu, C.-W. High order weighted essentially nonoscillatory schemes for convection dominated problems. *SIAM review 51*, 1 (2009), 82–126.

[4] Yang, H. An artificial compression method for eno schemes: the slope modification method. *Journal of Computational Physics 89*, 1 (1990), 125–160.