Introduction to WENO Methods

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Outline

Summary: We will introduce the WENO methodology, specifically interpolation and reconstruction. We conclude by presenting WENO methods for the solution of hyperbolic conservation laws.

1.1 WENO Interpolation

Weighted essentially non oscillatory (WENO) methods preserve high-order accuracy in smooth regions and prevent oscillations near discontinuities.

The Interpolation Problem

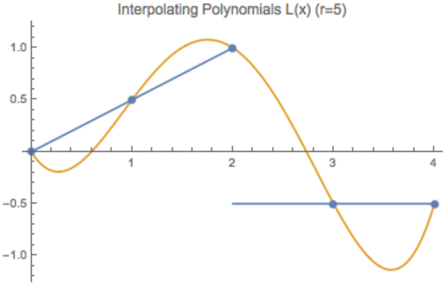
Given a set of distinct points $\{\mathbf{x}_j\}_{j=1}^{N}$, and data $\{\mathbf{f}_j\}_{j=1}$, how do we form an interpolating function?

If $f(x)$ is a **smooth function** and $f_j = f(x_j)$ than we can use

- Polynomial Interpolation.
- Rational Interpolation.
- Radial Basis Functions.
- Many Others.

If $f(x)$ is **discontinuous**, these interpolation methods can introduce unwanted oscillations.

Non-Smooth Interpolation

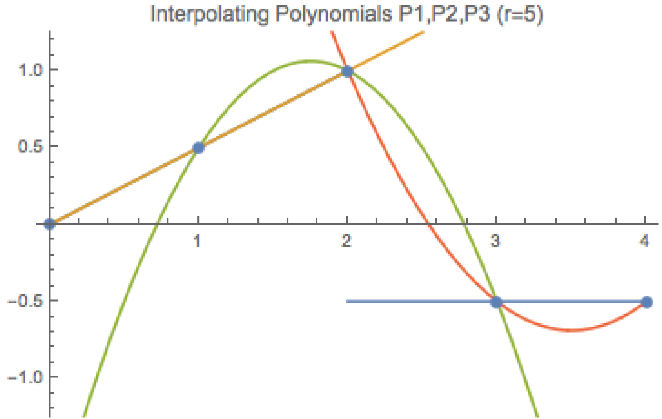


$$f(x) = \begin{cases} x/2 & x \geq 2 \\ -1/2 & x < 2 \end{cases}$$

The 5th order order polynomial interpolant introduces unwanted oscillations.

- We could consider lower-order polynomials and select the least oscillatory in the interval of interest (ENO Method).

Non-Smooth Interpolation



The 3rd order interpolating polynomials $P_j(x)$ each pass through the points x_j, x_{j+1}, x_{j+2} for $j = 1 \dots 3$.

The Smoothness Measure

- We require a measure of smoothness to select between Polynomials $P_1(x)$, $P_2(x)$ and $P_3(x)$.

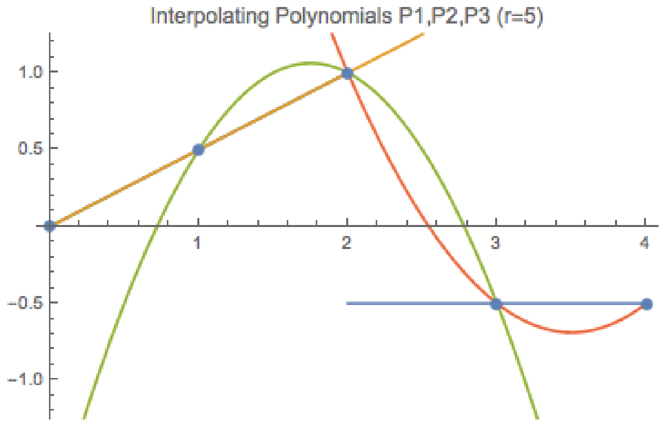
- First Proposed by Shu (1996)

$$S_k(x) = \sum_{i=1}^{r} (\Delta x)^{2i-1} \int_{x-\Delta x/2}^{x+\Delta x/2} \left(\frac{d^i P_k(x)}{dx^i} \right)^2 dx$$

  Sum over two-norms of derivatives, independent of the gridpspacing Δx.

- **ENO Strategy**:
  Interpolate using $P_k(x)$ where $\min_{k\in[1,2,3]} |S_k(x)|$

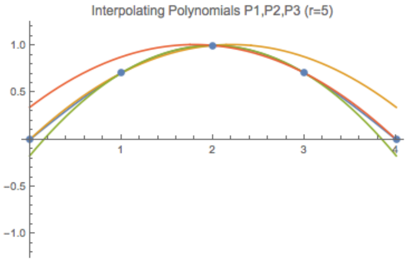
Non-Smooth Interpolation

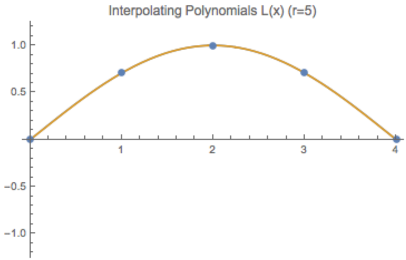


Yellow Polynomial is the smoothest in 3rd cell and will be selected by ENO.

Smooth Interpolation

Consider the smooth function $f(x) = \sin(4\pi x/n)$





- 5th order polynomial accurately reproduces $f(x)$.
- 3rd order polynomial perform poorly in comparison.

- **WENO** combines the advantages of high-order interpolation on smooth data, and low-order, oscillation minimizing interpolation on discontinuous data.

WENO Interpolation

The 5th order polynomial $L(x)$ can be formed as a convex combination (in P_k) of the lower order polynomials

$$L(x) = \sum_{i=1}^{3} C_k(x) P_k(x) \qquad \text{where} \qquad \sum_{i=1}^{3} C_k = 1,\ C_k \geq 0$$

WENO Strategy : Form a convex combination of $P_k(x)$

$$f(x) \approx \sum_{k=1}^{3} w_k(x) P_k(x).$$

- $w_1 + w_2 + w_3 = 1$ where $w_j \geq 0$
- $w_k(x) \approx C_k(x)$ in intervals where $f(x)$ is smooth.
- $w_k(x) \approx 0$ if polynomial $P_k(x)$ interpolates a discontinuity.

Solving for $C_k(x)$

The $(2r-1)$th order polynomial $L(x)$ can be formed as a linear combination (in P_k) of the *rth* order interpolating Polynomials

$$L(x) = \sum_{i=1}^{r} C_k^r(x) P_k(x)$$

To solve for $C_k^r(x)$ let $C_k^r(x) = \sum_{i=1}^{r} \alpha_{k,i} x^{i-1}$ $\qquad (r^2 \text{ conditions})$

Match coefficients $[f_j \cdot x^k]$ for $j, k = 1, \ldots r$ $\qquad (r^2 \text{ conditions})$

$r = 2$ (Convex for $x \in [0,2]$)

$$C_1^2(x) = 1 - \tfrac{x}{2} \qquad P_1^2(x) = f_2 + (-f_2 + f_3)(-1 + x)$$

$$C_2^2(x) = \tfrac{x}{2} \qquad P_2^2(x) = f_1 - (f_1 + f_2)x$$

WENO Weights

For $2r - 1$ data points, the WENO interpolant is of the form

$$f(x) \approx \sum_{k=1}^{r} w_k^r(x) P_k(x).$$

where the nonlinear weights are defined as

$$w_k(x) = \frac{\alpha_k}{\sum_{i=1}^{r} \alpha_i}, \qquad \text{and} \qquad \alpha_j(x) = \frac{C_k^r(x)}{(\epsilon + S_j(x))^2} \qquad (\epsilon \ll 1)$$

$S_j(x)$ denotes the usual smoothness measure

$$S_k(x) = \sum_{i=1}^{r} (\Delta x)^{2i-1} \int_{x-\Delta x/2}^{x+\Delta x/2} \left(\frac{d^i P_k(x)}{dx^i} \right)^2 dx$$

From Taylor analysis, we require that $S_j(x) = \alpha(x) + O(h^{r-1})$ to ensure $O(h^{2r-1})$ accuracy in smooth regions.

WENO Weights (Smooth Function)

For a smooth function, we expect that the smoothness measures

$$S_1 \approx S_2 \approx ... \approx S_r \approx \alpha$$

Therefore,

$$\alpha_j(x) = \frac{C_k^r(x)}{(\epsilon + S_j(x))^2} \approx \frac{C_k^r(x)}{\hat{\alpha}}$$

Since the weights C_k^r are convex, then

$$w_k(x) = \frac{\alpha_k}{\sum_{i=1}^r \alpha_i} = \frac{C_k^r(x)}{\sum_{i=1}^r C_i^r} = C_k^r$$

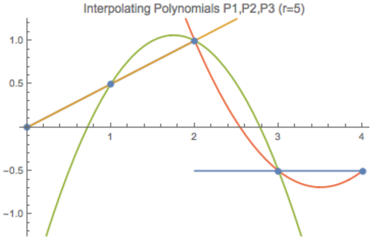
Thus the WENO interpolant will be of the form

$$\sum_{k=1}^{r} w_k^r(x) P_k(x) \approx \sum_{k=1}^{r} C_k^r(x) P_k(x) = L(x)$$

WENO Weights (Discontinous Function)

For a function with one discontinuity in the second to last cell $[x_{2j-3}, x_{2j-2}]$, we expect that the smoothness measures

$$S_j \approx \begin{cases} \alpha & j = 1 \\ \beta \gg 1 & \text{otherwise} \end{cases}$$



$$\alpha_j = \frac{C_j^r(x)}{(\epsilon + S_j(x))^2} \approx \begin{cases} \frac{C_k}{\alpha} & j = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$w_k = \frac{\alpha_k}{\sum_{i=1}^r \alpha_i} \approx \begin{cases} 1 & j = 1 \\ 0 & \text{otherwise} \end{cases}$$

WENO weights favor the smoothest stencil.

1.2 WENO Reconstruction

The WENO methodology can also be applied for function reconstruction from cell averages. This can be used to develop high-order solvers for conservation laws.

Polynomial Reconstruction From Cell Averages

Let a_j denote function averages taken between cells such that

$$a_j = \int_{x_{j-1/2}}^{x_{j+1/2}} f(x)dx$$

We can define the reconstructing polynomial (primitive function) $L(x)$ by imposing the conditions

$$a_j = \int_{x_{j-1/2}}^{x_{j+1/2}} L(x)dx \qquad j = 1, \ldots, N$$

If let $L(x) = \sum_{i=1}^{N} c_i x^i$, then we have the system $\mathbf{Ac} = \mathbf{b}$ where

$$\mathbf{A}_{ij} = \frac{x_{i+1/2}^{j} - x_{i-1/2}^{j}}{j}, \qquad \mathbf{b}_j = a_j.$$

We can now evaluate $L(x)$ at any points x_0.

WENO for Reconstruction I

Given $2r-1$ cell midpoints and averages $\{(x_j, Q_j)\}_{j=1}^{2r-1}$, Let

- $\{R_k(x)\}_{k=1}^{r}$ be the set of rth order reconstructing polynomials

$$Q_{j+k} = \int_{x_{j+k-1/2}}^{x_{j+k+1/2}} R_k(x)dx \quad j = 1, \dots, r$$

- Let $L(x)$ be the $(2r-1)$th order reconstructing polynomial

$$Q_j = \int_{x_{j-1/2}}^{x_{j+1/2}} L(x)dx \quad j = 1, \dots, n$$

As before, there exists for certain x, convex weights C_k^r such that

$$L(x) = \sum_{k=1}^{r} C_k^r(x) R_k(x)$$

WENO for Reconstruction II

The WENO reconstruction function $q(x)$ is given by

$$q(x) = \sum_{k=1}^{r} w_k^r(x) R_k(x).$$

where weights w_k^r are calculated as before:

$$\begin{aligned}
w_k &= \frac{\alpha_k}{\sum_{i=1}^{r} \alpha_i} \\
\alpha_j &= \frac{C_j^r}{(\epsilon + S_j(x))^2} \\
S_k(x) &= \sum_{i=1}^{r} (\Delta x)^{2i-1} \int_{x-\Delta x/2}^{x+\Delta x/2} \left(\frac{d^i R_k(x)}{dx^i} \right)^2 dx
\end{aligned}$$

Note: Smoothness indicators use reconstructing polynomials $R_k(x)$

2 Solving Conservation Laws

WENO reconstruction can be directly applied to conservation laws, for which we need high-order values at cell edges given only the cell averages.

Conservation Laws

Recall the conservation form of a hyperbolic equation

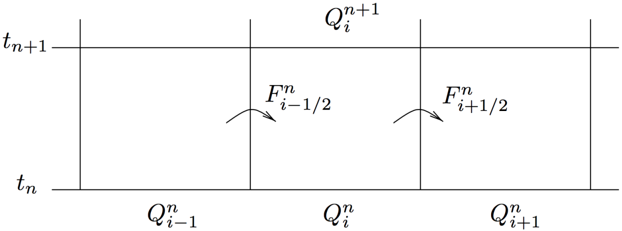
$$\frac{d}{dt}\int_{C_i} q(x,t)dx = f(q(x_{i-1/2}),t) - f(q(x_{i+1/2},t))$$

In terms of cell averages $Q_i^n = \frac{1}{\Delta x}\int_{C_i} q(x,t_n)dx$, the following ODE holds in each cell:

$$\frac{dQ_i^n}{dt} = \frac{1}{\Delta x}\left[f(q(x_{i-1/2}),t_n) - f(q(x_{i+1/2},t_n))\right]$$

We want to find highly accurate approximations of the fluxes at the cell boundaries given only the cell averages at the previous timestep.

Conservation Laws



FVM Fig. 4.1 illustrating the conservation scheme [2]

WENO-5 Spatial Discretization

To reconstruct one boundary $f_{j+1/2}$, consider $r = 3$ stencils with 3 cell averages each:

- each quadratic interpolant is $O(\Delta x^3)$ at $P_k(f_{j+1/2})$
- "Optimal weights" $C_k = [\frac{1}{10}, \frac{6}{10}, \frac{3}{10}]$ satisfy

$$\sum_{k=1}^{3} C_k P_k(f_{j+1/2}) = O(\Delta x^5)$$

$$\sum_{k=1}^{3} C_k = 1$$

- Construct WENO weights w_k so $f_{j+1/2} = \sum_{k=1}^{r} w_k P_k(f_{j+1/2})$

$$\alpha_k = \frac{C_k}{(\epsilon + S_k(x))^p}, w_k = \frac{\alpha_k}{\sum_{j=1}^{r} \alpha_j}$$

Time Stepping

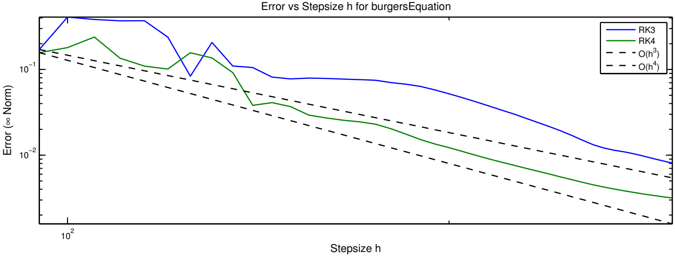


Figure : Order of accuracy in time for Burgers equation

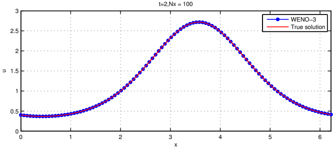
- Require high-order time-steppers to preserve fifth-order accurate spatial discretization
- Third order TVD Runge-Kutta requires $\Delta t \sim \Delta x^{5/3}$.
- Fourth order Runge-Kutta requires $\Delta t \sim \Delta x^{5/4}$

3 Numerical Experiments

We demonstrate the results of WENO reconstruction for scalar problems, for smooth and discontinuous solutions.

WENO5 fifth order accuracy

$$\begin{cases} u_t + u_x = 0 \\ u_0(x) = e^{\sin(x)} \end{cases}$$



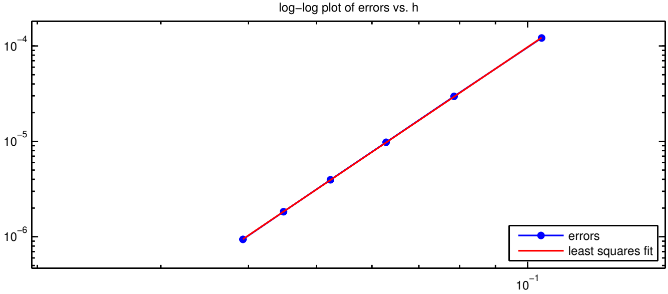
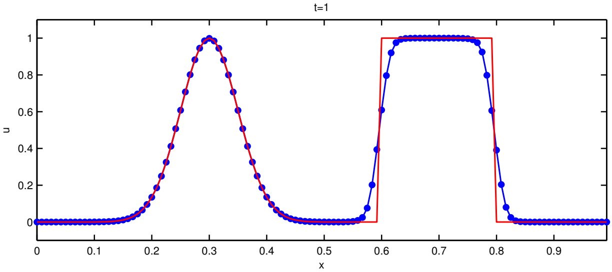
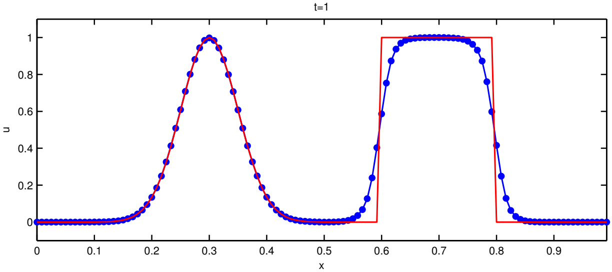


Figure : WENO5-RK4 $E(h) = O(h^{4.95856})$

WENO Weight Parameters p, ϵ

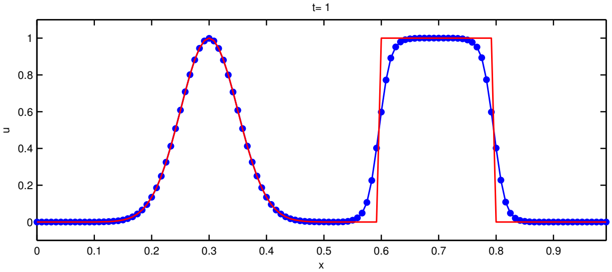


(a) $\epsilon = 1e-30, p = 1$

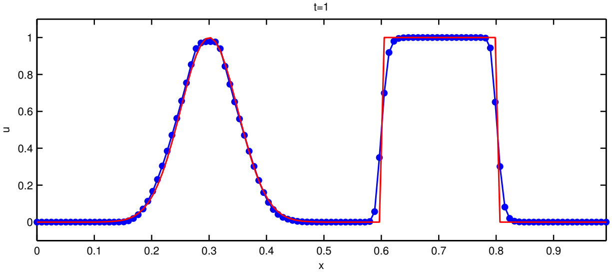


(b) $\epsilon = 1e-30, p = 3$

Discontinuities: Advection

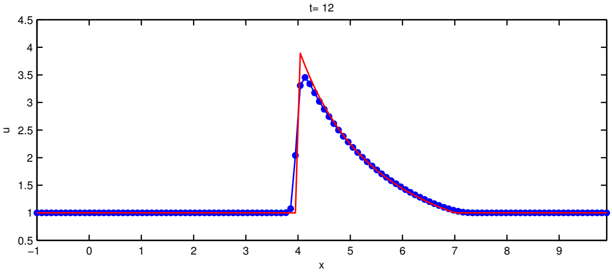


(a) WENO5-RK4 smooth

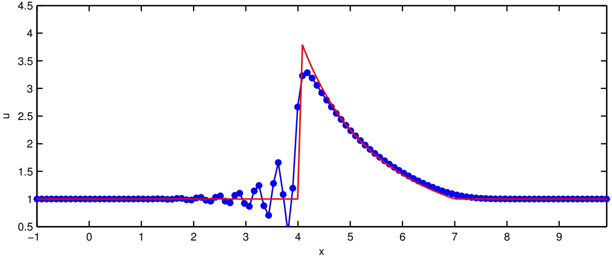


(b) CLAWPACK Superbee

Discontinuities: $u_t + (\sqrt{u})_x = 0$

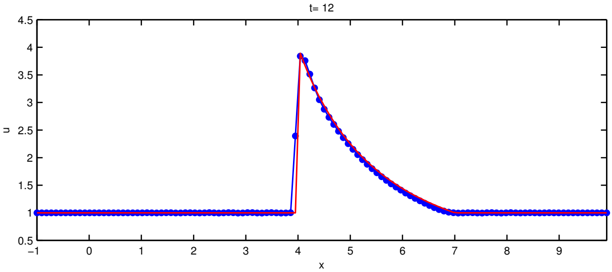


(a) WENO5-RK4

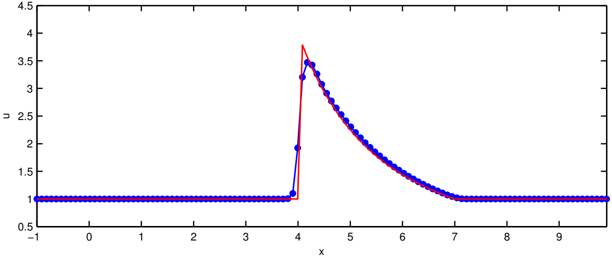


(b) CLAWPACK Lax-Wendroff, no limiter

Discontinuities: $u_t + (\sqrt{u})_x = 0$



(a) WENO5-RK4 with sharpening



(b) CLAWPACK mc limiter

Summary

WENO methods:

- excellent in smooth regions
- essentially non-oscillatory near jumps
- sharpening preserves accuracy in smooth regions

However,

- CLAWPACK's limiters are comparable to WENO near discontinuities
- WENO methods are computationally more expensive, especially with sharpening
- WENO-5 requires several ghost cells

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Thank you!