Comparison of Two Second Order Traffic Flow Models

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Group 6

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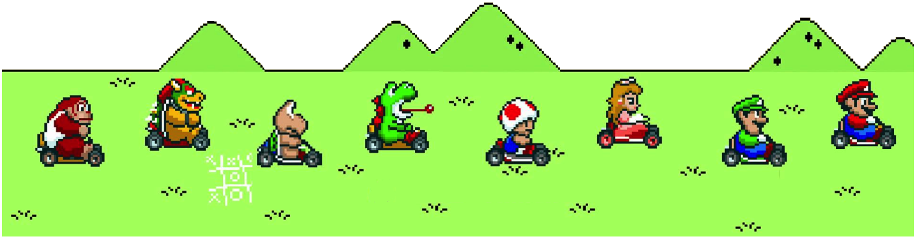


Introduction

First and second order traffic flow models:

LWR: $\rho_t + \Big(\rho V(\rho)\Big)_x = 0$

PW: $\begin{cases} \rho_t + (\rho v)_x = 0 \\ v_t + v v_x + \dfrac{p'(\rho)}{\rho} \rho_x = \dfrac{V(\rho) - v}{\tau} + \mu v_{xx} \end{cases}$

AR: $\begin{cases} \rho_t + (\rho v)_x = 0 \\ v_t + \Big(v - \rho p'(\rho)\Big)_x = 0 \end{cases}$

Introduction

- Are second-order models always better than first-order models?
- Not always! The PW model for traffic flow, meant to improve the LWR model's behavior near shocks, actually performs worse than the original in some cases.
- The PW model is based on compressible fluid dynamics, which ignores some fundamental differences between fluids and cars. Because of this, it can sometimes predict negative velocities.
- The AR model corrects the problem of negative velocities by utilizing a convective derivative.

Lighthill-Whitham-Richards (LWR) Model

Proposed by Lighthill and Whitham (1955) and Richards (1956), the LWR model describes traffic flow on a single one-way road without entrances or exits:

$$\rho_t + \Big(\rho V(\rho)\Big)_x = 0$$

- $\rho =$ density
- $V(\rho) =$ preferred velocity, a given nonincreasing function of ρ, nonnegative for ρ between 0 and ρ_m (the"jam" density)
- Predicts piece-wise smooth density, with transitions between regions approximated by shocks
- Problem: Doesn't adequately describe the motion of cars passing through shocks (cars change velocity instantaneously)

Payne-Whitham (PW) Model

Viewing traffic as a compressible fluid, Payne (1971) and Whitham (1974) introduced a second equation analogous to the conservation of momentum, ρv, in fluids:

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ (\rho v)_t + \left(\rho v^2 + p(\rho)\right)_x = 0 \end{cases}$$

- ρ = density
- v = velocity
- $p(\rho)$ = "anticipation factor", describes how a driver reacts to variations in density with respect to space

Payne-Whitham (PW) Model

Rewriting the second equation in terms of velocity and adding relaxation and viscosity terms, we get the full PW Model:

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ v_t + v v_x + \dfrac{p'(\rho)}{\rho}\rho_x = \dfrac{V(\rho) - v}{\tau} + \mu v_{xx} \end{cases}$$

- τ and μ are nonnegative constants
- Relaxation term: represents driver's reaction time
- Diffusion term: represents the driver's awareness of conditions ahead (and behind)

Payne-Whitham (PW) Model

A fundamental difference between fluids and traffic:

"A fluid particle responds to stimuli from the front and from behind, but a car is an anisotropic particle that mostly responds to frontal stimuli." [2]

Payne-Whitham (PW) Model

Homogeneous model:

$$\begin{pmatrix} \rho \\ v \end{pmatrix}_t + \begin{pmatrix} v & \rho \\ \dfrac{p'(\rho)}{\rho} & v \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_x = 0$$

Eigenvalues:

$$\lambda = v \pm \sqrt{p'(\rho)}$$

From the eigenvalues we see that one characteristic speed is greater than the velocity, which means future conditions are partly determined by conditions behind. This produces an undesirable effect: negative velocities!

PW: Hugoniot Loci

From this system, the Rankine-Hugoniot condition

$$s(q_* - q) = f(q_*) - f(q)$$

gives us the two equations

$$\begin{aligned} s(\rho_* - \rho) &= \rho_* v_* - \rho v, \\ s(\rho_* v_* - \rho v) &= \rho_* v_*^2 + p(\rho) - \rho v^2 - p(\rho). \end{aligned}$$

From these two equations the equation for the Hugoniot loci can be found to be

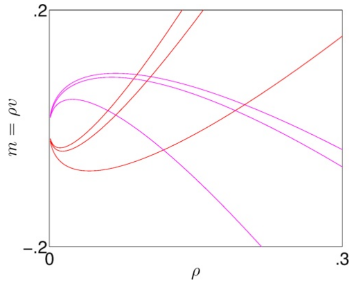
$$\rho v = \rho v_* \pm \rho \sqrt{(\rho - \rho_*)\,(p(\rho) - p(\rho_*))\,/\rho_* \rho}.$$

PW: Hugoniot Loci

Plotting this equation

$$\rho v = \rho v_* \pm \rho\sqrt{(\rho - \rho_*)\,(p(\rho) - p(\rho_*))\,/\rho_*\rho}.$$

gives



Here, λ_1 loci are shown in magenta and λ_2 loci are shown in red.

PW: Integral Curves

Using $p(\rho) = \rho^\gamma$ for $\gamma \neq 1$, since this is the value used in the AR model we will examine later, the equations for the integral curves can be found to be

$$\rho v = \rho v_* + \frac{2\rho}{\gamma - 1}\left(\sqrt{p'(\rho_*)} - \sqrt{p'(\rho)}\right)$$

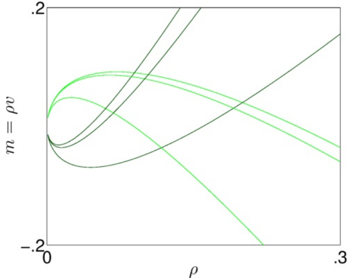
for λ_1, and

$$\rho v = \rho v_* + \frac{2\rho}{\gamma - 1}\left(\sqrt{p'(\rho)} - \sqrt{p'(\rho_*)}\right)$$

for λ_2.

PW: Integral Curves

Plotting these curves gives



Here, λ_1 integral curves are shown in light green and λ_2 integral curves are shown in dark green.

PW: Areas of Validity

Restrictions on the way that λ must vary across a wave allow us to identify the regions of the Hugoniot loci and integral curves that are valid for any given point.

Shocks:

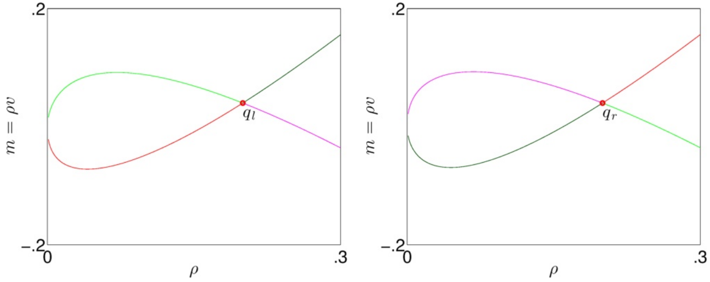
$$\begin{aligned} q_l \to q_r \quad &: \quad \lambda \text{ must decrease} \\ q_r \to q_l \quad &: \quad \lambda \text{ must increase} \end{aligned}$$

Rarefaction:

$$\begin{aligned} q_l \to q_r \quad &: \quad \lambda \text{ must increase} \\ q_r \to q_l \quad &: \quad \lambda \text{ must decrease} \end{aligned}$$

PW: Areas of Validity

Using this, we can determine the valid curves:

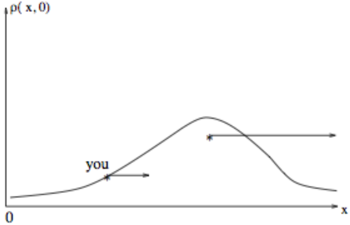


(a) Valid curves for a point q_l. (b) Valid curves for a point q_r.

Here, shocks are shown in red or magenta and rarefaction waves are shown in either light or dark green.

A thought experiment

Consider the situation where a driver is traveling with speed v. If the density ahead of him is increasing with respect to x but decreasing with respect to $x - vt$, will he speed up or slow down?



A thought experiment

Since density ahead is increasing with respect to x, the PW model predicts that the driver will slow down (as we will see in our examples). However, since the traffic ahead is traveling faster, most drivers would actually accelerate.

Aw-Rascle (AR) Model

Aw and Rascle (2000) claim that instead of depending upon the derivative of pressure with respect to x, the anticipation factor should involve the convective derivative:

$$\partial_t + v\,\partial_x$$

This change reflects the fact that the driver's perspective is a moving frame of reference.

Aw-Rascle (AR) Model

Using the convective derivative, the proposed AR model is:

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ \Big(v + p(\rho)\Big)_t + v\Big(v + p(\rho)\Big)_x = 0 \end{cases}$$

Aw-Rascle (AR) Model

AR model in conservation form:

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ \Big(\rho(v + p(\rho))\Big)_t + \Big(\rho v(v + p(\rho))\Big)_x = 0 \end{cases}$$

Conserved quantities:

1. Mass (density): ρ
2. "Momentum": $y = \rho(v + p(\rho))$

Aw-Rascle (AR) Model

AR model rewritten in terms of density and velocity:

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ v_t + \Big(v - \rho p'(\rho)\Big) v_x = 0 \end{cases}$$

Linearized system:

$$\begin{pmatrix} \rho \\ v \end{pmatrix}_t + \begin{pmatrix} v & \rho \\ 0 & v - \rho p'(\rho) \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_x = 0$$

Eigenvalues:

$$\lambda_1 = v - \rho p'(\rho)$$
$$\lambda_2 = v$$

Note that neither characteristic speed is greater than the velocity!

Aw-Rascle (AR) Model

Aw and Rascle propose some conditions for traffic models [1]:

1. When solving the Riemann problem, the density and velocity must remain nonnegative and bounded from above.
2. When solving the Riemann problem, all waves connecting any state $Q = (\rho, v)$ to its left (behind it) must have a propagation speed (eigenvalue or shock speed) at most equal to the velocity v.
3. The solution to the Riemann problem must agree with the qualitative properties that each driver practically observes every day.

AR: Hugoniot Loci

In order to consider the Hugoniot loci for this system we must first write the system in conservation form. With some manipulation, we can get the system

$$\begin{aligned}
&\partial_t \rho + \partial_x(\rho v) = 0, \\
&\partial_t \left(\rho \left(v + p(\rho)\right)\right) + \partial_x \left(\rho v \left(v + p(\rho)\right)\right) = 0.
\end{aligned}$$

This can be rewritten as $q_t + f(q)_x = 0$ by defining

$$q = \begin{bmatrix} \rho \\ y \end{bmatrix}, \qquad f(q) = \begin{bmatrix} v\rho \\ vy \end{bmatrix}.$$

where $y = \rho\left(v + p(\rho)\right)$.

AR: Hugoniot Loci

From the Rankine-Hugoniot condition

$$s(q_* - q) = f(q_*) - f(q).$$

we get the two equations

$$\begin{aligned} s(\rho_* - \rho) &= v_*\rho_* - v\rho, \\ s(y_* - y) &= v_* y_* - vy. \end{aligned}$$

From these equations, the equations for the Hugoniot loci can be found to be

$$\begin{aligned} \rho v &= \rho\left(v_* + p(\rho_*)\right) - \rho p(\rho)), \\ \rho v &= \rho_* v_*, \end{aligned}$$

corresponding to λ_1 and λ_2, respectively.

AR: Integral Curves

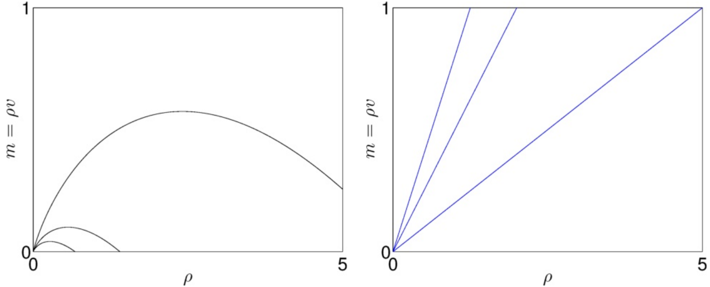
In order to consider the integral curves for this system [1] works from the system

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho v) = 0, \\ &\partial_t v + \left(v - p'(\rho)\rho\right) \partial_x v = 0. \end{aligned}$$

After working with these equations to find the integral curves, you find that they coincide with the Hugoniot loci.

AR: Curves

Plotting these curves gives:



(a) λ_1-curves in the M plane. (b) λ_2-curves in the M plane.

AR: Areas of Validity

Restrictions on the way that λ must vary across a wave allow us to identify the regions of the Hugoniot loci and integral curves that are valid for any given point.

Shocks:

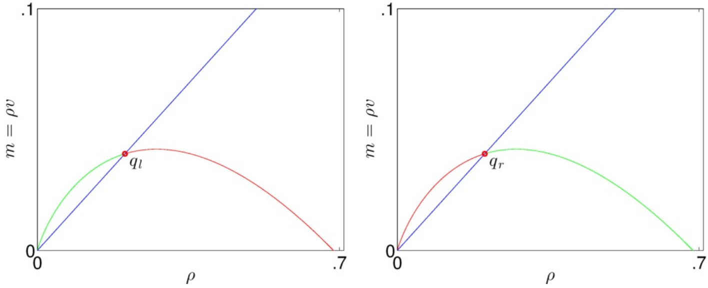
$$\begin{aligned} q_l \rightarrow q_r \quad &: \quad \lambda \text{ must decrease} \\ q_r \rightarrow q_l \quad &: \quad \lambda \text{ must increase} \end{aligned}$$

Rarefaction:

$$\begin{aligned} q_l \rightarrow q_r \quad &: \quad \lambda \text{ must increase} \\ q_r \rightarrow q_l \quad &: \quad \lambda \text{ must decrease} \end{aligned}$$

AR: Areas of Validity

Using this, we can determine the valid curves:



(a) Valid waves for the point q_l. (b) Valid waves for the point q_r.

Here, shocks are shown in red, rarefaction waves are shown in green, and the contact discontinuity is shown in blue.

Examples

To illustrate a couple of examples where the AR model preforms better than the PW model, consider two different examples:

Table : Initial values used in examples 1 and 2.

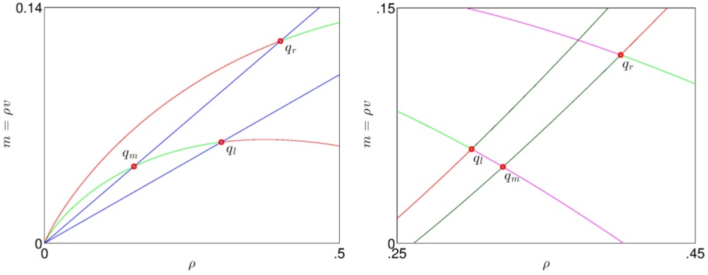
	v_l	ρ_l	v_r	ρ_r
Example 1	0.2	0.3	0.3	0.4
Example 2	0	0	0.3	0.4

Example 1

Initial Conditions:

v_l	ρ_l	v_r	ρ_r
0.2	0.3	0.3	0.4

The solution to the Riemann problem is:



(a) Solution from the AR model. (b) Solution from the PW model.

Example 1

Initial Conditions:

v_l	ρ_l	v_r	ρ_r
0.2	0.3	0.3	0.4

AR model:

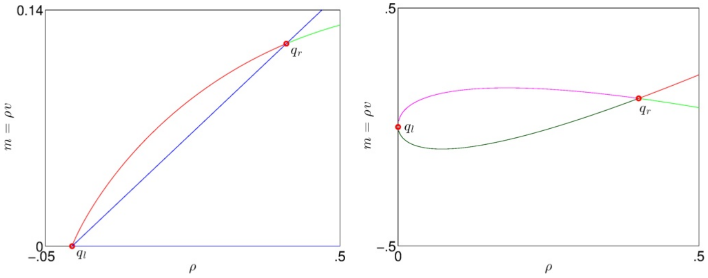
PW Model:

Example 2

Initial Conditions:

v_l	ρ_l	v_r	ρ_r
0	0	0.3	0.4

The solution to the Riemann problem is:



(a) Solution from the AR model. (b) Solution from the PW model.

Example 2

Initial Conditions:

v_l	ρ_l	v_r	ρ_r
0	0	0.3	0.4

AR model:

PW Model:

Questions?

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