Augmented approximate Riemann solvers for the shallow water equations with varying bathymetry

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Outline

The shallow water equations

The shallow water equations are given by

$$
\begin{aligned}
h_t + (hu)_x &= 0 \\
(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x &= -ghb_x
\end{aligned}
$$

where $h(x,t)$ is the depth of the water (from the free surface to the sea floor), $u(x,t)$ is the velocity of the water and $b(x)$ is the fixed underwater topography.

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- Preservation of Non-Negativity of depth solutions.
- Prevention of non-physical oscillations along steep interfaces of shorelines.
- Preservation of Steady States.
  Since tsunami waves are small perturbations of an ocean at rest over variable bathymetry, it is critical that the Riemann solver be able to preserve steady states involving a flat surface even with large variations in the underlying topography.

The wave propagation algorithm

- The first order method

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(\mathcal{A}^+\Delta Q_{i-\frac{1}{2}}^n + \mathcal{A}^-\Delta Q_{i+\frac{1}{2}}^n).$$

Where the jump is decomposed into separate waves

$$Q_i^n - Q_{i-1}^n = \sum_{p=1}^{M_w} \mathcal{W}_{i-\frac{1}{2}}^p$$

which propagate from the discontinuity at speeds $s_{i-\frac{1}{2}}^p$. The fluctuations then used to update the cell values are

$$\mathcal{A}^-\Delta Q_{i-\frac{1}{2}} = \sum_{\{p:s_{i-\frac{1}{2}}^p<0\}} s_{i-\frac{1}{2}}^p \mathcal{W}_{i-\frac{1}{2}}^p$$

$$\mathcal{A}^+\Delta Q_{i-\frac{1}{2}} = \sum_{\{p:s_{i-\frac{1}{2}}^p>0\}} s_{i-\frac{1}{2}}^p \mathcal{W}_{i-\frac{1}{2}}^p.$$

The f-wave method

The f-wave method decomposes the jump in the flux instead of the actual cell average, thus the name.

$$f(Q_i) - f(Q_{i-1}) = \sum_{p=1}^{M_w} \mathcal{Z}^p_{i-\frac{1}{2}}$$

and we may define our updating fluctuations as

$$\mathcal{A}^- \Delta Q_{i-1/2} = \sum_{\{p: s^p_{i-1/2} < 0\}} \mathcal{Z}^p_{i-1/2}$$

$$\mathcal{A}^+ \Delta Q_{i-1/2} = \sum_{\{p: s^p_{i-1/2} > 0\}} \mathcal{Z}^p_{i-1/2}$$

The Roe solver

It is an approximate Riemann solver based on a linearization about a specific mean state $(\bar{h}, \hat{u})$ where $\bar{h}$ is the arithmetic mean of the depths on either side of the interface and the speed is given by

$$\hat{u} = \frac{\sqrt{h_{i-1}}u_{i-1} + \sqrt{h_i}u_i}{\sqrt{h_{i-1}} + \sqrt{h_i}}. \tag{1}$$

The eigenvalues (known as the Roe wave speeds) of the linearized Jacobian $\hat{A}$ are $\hat{\lambda}^1 = \hat{u} - \hat{c}$, $\hat{\lambda}^2 = \hat{u} + \hat{c}$ where $\hat{c} = \sqrt{g\bar{h}}$. The corresponding eigenvectors are

$$\hat{r}^1 = \begin{pmatrix} 1 \\ \hat{u} - \hat{c} \end{pmatrix}, \hat{r}^2 = \begin{pmatrix} 1 \\ \hat{u} + \hat{c} \end{pmatrix}. \tag{2}$$

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$$\check{s}^1_{i-\frac{1}{2}} = \min(\lambda^-(Q^n_{i-1}), \hat{\lambda}^-_{i-\frac{1}{2}}), \quad \check{s}^2_{i-\frac{1}{2}} = \max(\lambda^+(Q^n_i), \hat{\lambda}^+_{i-\frac{1}{2}})$$

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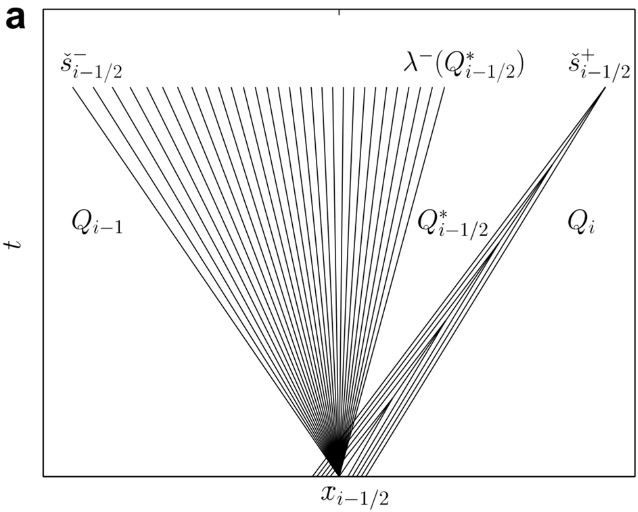
- The corresponding waves are given by $w^p_{i-\frac{1}{2}} = (1, \check{s}_{i-\frac{1}{2}^p})^T$
- Fails to capture large transonic rarefactions and does not preserve steady states with variable topography.

transonic rarefaction



The augmented solver for homogeneous SWE

- Decomposing the jump into three waves

$$\begin{bmatrix} H_i - H_{i-1} \\ HU_i - HU_{i-1} \\ \varphi(Q_i) - \varphi(Q_{i-1}) \end{bmatrix} = \sum_{p=1}^{3} \alpha_{i-1/2}^{p} w_{i-1/2}^{p}.$$

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$$\mathcal{Z}_{i-1/2}^p = \begin{bmatrix} \mathbf{0}_{2\times 1} & \mathbf{I}_{2\times 2} \end{bmatrix} \alpha_{i-1/2}^p w_{i-1/2}^p$$

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- Update cell averages just like f-wave method

Determining the basis waves $w^p_{i-\frac{1}{2}}$

- To maintain non-negativity, we choose to include the Einfeldt speeds $\check{s}_{i-\frac{1}{2}}{}^{\pm}$ and their corresponding in our wave decomposition

$$\left\{w^1_{i-1/2}, s^1_{i-1/2}\right\} = \left\{\left(1, \check{s}^-_{i-1/2}, \left(\check{s}^-_{i-1/2}\right)^2\right)^T, \check{s}^-_{i-1/2}\right\}$$

$$\left\{w^3_{i-1/2}, s^3_{i-1/2}\right\} = \left\{\left(1, \check{s}^+_{i-1/2}, \left(\check{s}^+_{i-1/2}\right)^2\right)^T, \check{s}^+_{i-1/2}\right\}.$$

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- Choice of "corrector" wave

$$\left\{w^2_{i-1/2}, s^2_{i-1/2}\right\} = \left\{(0, 0, 1)^T, \frac{1}{2}\left(\check{s}^-_{i-1/2} + \check{s}^+_{i-1/2}\right)\right\},$$

The augmented solver for SWE with a source term

- One more component is added to the decomposition to account for the impact of the source term

$$\begin{bmatrix} H_i - H_{i-1} \\ HU_i - HU_{i-1} \\ \varphi(Q_i) - \varphi(Q_{i-1}) \\ B_i - B_{i-1} \end{bmatrix} = \sum_{p=0}^{3} \alpha^p_{i-1/2} w^p_{i-1/2},$$

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- Flux waves are given by

$$\mathcal{Z}^p_{i-1/2} = \left[\mathbf{0}_{2\times 1} \quad \mathbf{I}_{2\times 2} \quad \mathbf{0}_{2\times 1}\right] \alpha^p_{i-1/2} w^p_{i-1/2}$$

Determining w_0

From Theorem 1 in George 2008, steady states must satisfy

$$\tilde{q}(x_r,t) - \tilde{q}(x_l,t) = (b(x_r) - b(x_l)) \begin{bmatrix} \frac{g\bar{H}(q(x_l,t),q(x_r,t))}{\overline{\lambda^+\lambda^-}(q(x_l,t),q(x_r,t))} \\ 0 \\ -g\tilde{H}(q(x_l,t),q(x_r,t)) \\ 1 \end{bmatrix} \quad (3)$$

where

$$\overline{\lambda^+\lambda^-}(q(x_l,t),q(x_r,t)) = \left(\frac{u_l+u_r}{2}\right)^2 - g\left(\frac{h_l+h_r}{2}\right)$$

$$\bar{H}(q(x_l,t),q(x_r,t)) = \frac{h_l+h_r}{2}$$

$$\widetilde{\lambda^+\lambda^-}(q(x_l,t),q(x_r,t)) = \max(0,u_r u_l) - g\left(\frac{h_l+h_r}{2}\right)$$

$$\tilde{H}(q(x_l,t),q(x_r,t)) = \bar{H}(q(x_l,t),q(x_r,t))\frac{\widetilde{\lambda^+\lambda^-}(q(x_l,t),q(x_r,t))}{\overline{\lambda^+\lambda^-}(q(x_l,t),q(x_r,t))}.$$

- $w^0_{i-1/2}$ must capture the jump in steady states precisely and we choose

$$w^0_{i-1/2} = \begin{bmatrix} \frac{g\bar{H}(Q_{i-1},Q_i)}{\overline{\lambda^+\lambda^-}(Q_{i-1},Q_i)} \\ 0 \\ -g\tilde{H}(Q_{i-1},Q_i) \\ 1 \end{bmatrix}, \quad s^0_{i-1/2} = 0 \tag{4}$$

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- We use this new steady state wave along with the same waves from the previous Riemann solver:

$$\left\{w^1_{i-1/2}, s^1_{i-1/2}\right\} = \left\{\left(1, \check{s}^-_{i-1/2}, \left(\check{s}^-_{i-1/2}\right)^2, 0\right)^T, \check{s}^-_{i-1/2}\right\}$$

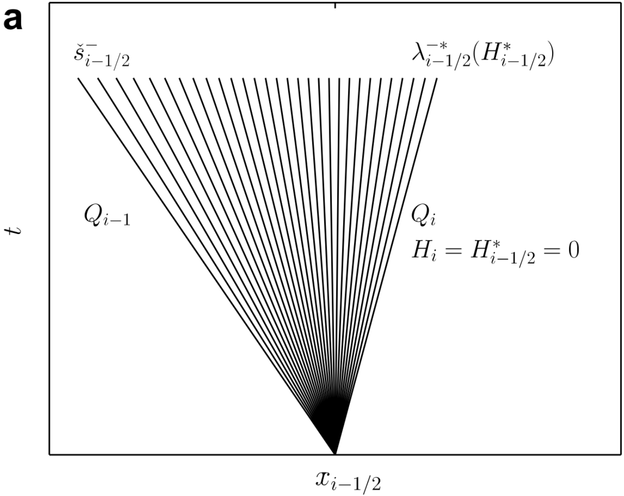
$$\left\{w^3_{i-1/2}, s^3_{i-1/2}\right\} = \left\{\left(1, \check{s}^+_{i-1/2}, \left(\check{s}^+_{i-1/2}\right)^2, 0\right)^T, \check{s}^+_{i-1/2}\right\}$$

$$\left\{w^2_{i-1/2}, s^2_{i-1/2}\right\} = \left\{(0,0,1,0)^T, \frac{1}{2}\left(\check{s}^-_{i-1/2} + \check{s}^+_{i-1/2}\right)\right\}.$$

Practical considerations for problems involving dry cells and shorelines

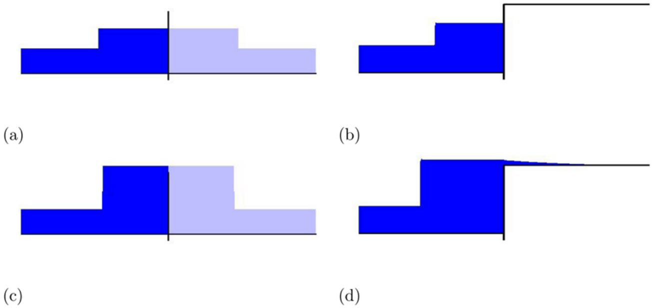
- Treatment for near-dry cells
  - When the depth of the cell is positive but very close to zero, numerical difficulties occur. Therefore, we consider cells with a depth smaller than a certain threshold a dry cell and set the depth and momentum to zero. In the simulation, the threshold is set to 10^{-3} meters.
- Treatment for neighboring wet and dry cells
- Treatment for steep shorelines
- Treatment for transonic rarefaction waves

Treatment for neighboring wet and dry cells



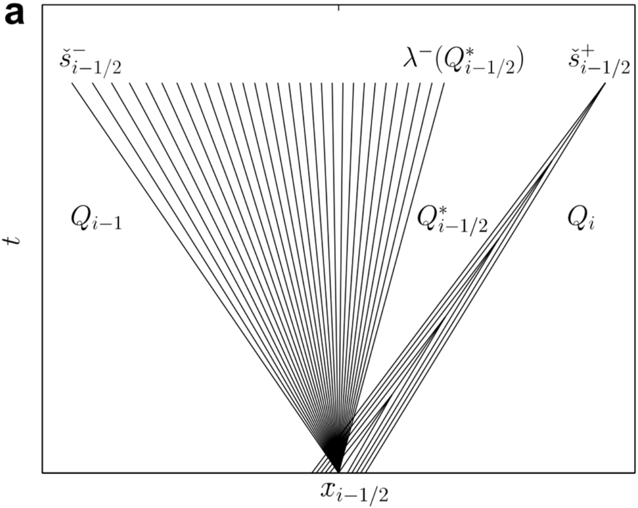
$$\check{s}^{-}_{i-\frac{1}{2}} = U_{i-1} - \sqrt{gH_{i-1}}, \quad \check{s}^{+}_{i-\frac{1}{2}} = U_{i-1} + 2\sqrt{gH_{i-1}}$$

Treatment for steep shorelines



$$\check{H}^{*} = \frac{HU_L - HU_R + \check{s}^{+}_{i-1/2}H_R - \check{s}^{-}_{i-1/2}H_L}{\check{s}^{+}_{i-1/2} - \check{s}^{-}_{i-1/2}} \qquad (5)$$

Treatment for transonic rarefaction waves



$$\{w^2_{i-\frac{1}{2}}, s^2_{i-\frac{1}{2}}\} = \{(1, \lambda^{-*}_{i-\frac{1}{2}}(H^*_{i-\frac{1}{2}}), (\lambda^{-*}_{i-\frac{1}{2}}(H^*_{i-\frac{1}{2}}))^2, 0), \lambda^{+*}_{i-\frac{1}{2}}(H^*_{i-\frac{1}{2}})\}.$$

Numerical Experiments

Numerical simulation of beach problem

Numerical Experiments

Numerical simulation of daybreak problem

Conclusions

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