Infinite Interval Problems Modeling the Flow of a Gas Through a Semi-Infinite Porous Medium

By Ravi P. Agarwal and Donal O'Regan

Various existence results are presented for boundary-value problems on the infinite interval. In particular, our theory includes a discussion of a problem arising in the unsteady flow of a gas through a semi-infinite porous medium.

1. Introduction

In the study of the unsteady flow of a gas through a semi-infinite porous medium [1, 2] initially filled with gas at a uniform pressure $p_0 > 0$, at time $t = 0$, the pressure at the outflow face is suddenly reduced from p_0 to $p_1 \geq 0$ ($p_1 = 0$ is the case of diffusion into a vacuum) and is, thereafter, maintained at this lower pressure. The unsteady isothermal flow of gas is described by the nonlinear partial differential equation

$$\nabla^2(p^2) = 2A\frac{\partial p}{\partial t}$$

where the constant A is given by the properties of the medium. In the one-dimensional medium extending from $x = 0$ to $x = \infty$, this reduces to

$$\frac{\partial}{\partial x}\left(p\frac{\partial p}{\partial x}\right) = A\frac{\partial p}{\partial t}$$

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with the boundary conditions

$$\begin{cases} p(x,0) = p_0, 0 < x < \infty \\ p(0,t) \ = p_1(< p_0), 0 \le t < \infty. \end{cases}$$

To obtain a similarity solution, we introduce the new independent variable

$$z = \frac{x}{\sqrt{t}} \left(\frac{A}{4p_0} \right)^{\frac{1}{2}}$$

and the dimension-free dependent variable w, defined by

$$w(z) = \alpha^{-1} \left(1 - \frac{p^2(x)}{p_0^2} \right), \qquad \text{where} \qquad \alpha = 1 - \frac{p_1^2}{p_0^2}.$$

In terms of the new variable, the problem takes the form

$$\begin{cases} w''(z) + \frac{2z}{(1-\alpha w(z))^{\frac{1}{2}}} w'(z) = 0 \\ w(0) = 1, \qquad w(\infty) = 0. \end{cases}$$

Making the change of variable $u(z) = 1 - \alpha w(z)$, we have the boundary-value problem

$$\begin{cases} u''(z) + \frac{2z}{[u(z)]^{\frac{1}{2}}} u'(z) = 0, \qquad 0 < z < \infty \\ u(0) = 1 - \alpha\,, \qquad 0 < \alpha \le 1 \\ u(\infty) = 1. \end{cases} \tag{1}$$

Motivated by the above example, we establish in this article a general existence theory for the boundary-value problem

$$\begin{cases} y'' + \phi(t)\, f(t,y,y') = 0, \qquad 0 < t < \infty \\ y(0) = a_0, \qquad 0 \le a_0 < 1 \\ \lim_{t\to\infty}\, y(t) = 1. \end{cases} \tag{2}$$

Our theory, which complements the work in [3–5], automatically produces the existence of a solution for (1). The technique we use to establish existence for (2) is based on a diagonalization argument and a well-known existence principle [6] for boundary value problems on finite intervals. For the convenience of the reader we state the existence principle from [6] that we use in this article.

THEOREM 1. *Let* $n \in \{1, 2, \ldots\}$ *be fixed. Suppose* $f : [0, n] \times \mathbf{R}^2 \to \mathbf{R}$ *is continuous, and* $\phi \in C[0, n]$ *with* $\phi > 0$ *on* $(0, n]$. *In addition, assume there is a constant* $M \geq \max\{|a|, |b|\}$, *independent of* λ, *with*

$$\max\left\{\sup_{t\in[0,n]} |y(t)|, \sup_{t\in[0,n]} |y'(t)|\right\} \leq M$$

for any solution $y \in C^2[0, n]$ *to*

$$\begin{cases} y'' + \lambda\phi(t)f(t,y,y') = 0, & 0 < t < n \\ y(0) = a, y(n) = b \end{cases} \tag{3$_\lambda$}$$

for each $\lambda \in (0,1)$. *Then* $(3)_1$ *has at least one solution in* $C^2[0, n]$.

2. Nonsingular problems

The example in Section 1 (when $0 < \alpha < 1$) motivates our discussion of the second-order nonsingular boundary-value problem

$$\begin{cases} y'' + \phi(t)f(t,y,y') = 0, & 0 < t < \infty \\ y(0) = a_0, & 0 < a_0 < 1 \\ \lim_{t\to\infty} y(t) = 1. \end{cases} \tag{4}$$

The goal of this section is to present a result that offers easily verifiable (see Example 1) criteria for (4). The technique we use to establish the existence of a positive solution to (4) is based on: (1) proving results in the finite interval $[0, n]$ for each $n \in \{1, 2, \ldots\}$; and (2) using a diagonalization argument.

THEOREM 2. *Suppose* $f : [0, \infty) \times [a_0, \infty) \times \mathbf{R} \to \mathbf{R}$ *is continuous,* $\phi \in C[0, \infty)$ *with* $\phi > 0$ *on* $(0, \infty)$. *In addition, assume the following conditions hold:*

$$v\, f(t,y,v) \geq 0 \text{ on } [0,\infty) \times [a_0,\infty) \times \mathbf{R} \tag{5}$$

$$\begin{cases} \text{there exists a continuous function } \tau : [0,\infty) \to (0,\infty) \text{ with} \\ f(t,y,v) \leq \tau(v) \text{ on } [0,\infty) \times [a_0,1] \times [0,\infty) \end{cases} \tag{6}$$

$$\int_0^\infty \frac{du}{\tau(u)} > \int_0^{1-a_0} \frac{du}{\tau(u)} + \int_0^1 \phi(s) \quad ds \tag{7}$$

$$\phi \quad is \ nondecreasing \ on \ (0,\infty) \tag{8}$$

and

$$\begin{cases} for \ constants \ H > a_0, \ K > 0 \ there \ exists \ a \ function \ \psi_{H,K} \ continuous \\ on \ [0,\infty) \ and \ positive \ and \ nondecreasing \ on \ (0,\infty), \ and \ a \ constant \\ 1 \le r < 2 \ with \ f(t,y,v) \ge \psi_{H,K}(t)v^r \ on \ [0,\infty) \times [a_0,H] \times [0,K]. \end{cases} \tag{9}$$

Then (4) *has a solution* $y \in C^2[0, \infty)$ *with* $1 \ge y(t) \ge a_0$ *for* $t \in [0, \infty)$.

Proof: Fix $n \in \{1, 2, \ldots\}$. We first show

$$\begin{cases} y'' + \phi(t) \quad f(t,y,y') = 0, & 0 < t < n \\ y(0) = a_0, & ty(n) = 1 \end{cases} \qquad (10)^n$$

has a solution $y_n \in C^2[0, n]$. To establish this the idea is to apply Theorem 1 to the modified problem

$$\begin{cases} y'' + \lambda\phi(t)f^\star(t,y,y') = 0, & 0 < t < n \\ y(0) = a_0, & y(n) = 1 \end{cases} \qquad (11)^n_\lambda$$

where $\lambda \in (0,1)$ and

$$f^\star(t,y,v) = \begin{cases} f(t,y,v), & y \ge a_0 \\ f(t,a_0,v), & y \le a_0. \end{cases}$$

From (5), we note that $v f^\star(t, y, v) \ge 0$ for $(t, y, v) \in [0, \infty) \times \mathbf{R} \times \mathbf{R}$. Let $y \in C^2[0, n]$ be any solution of $(11)^n_\lambda$ for $0 < \lambda < 1$. We first show

$$y'(t) \ge 0 \qquad \text{for} \qquad t \in [0,n]. \tag{12}$$

To see this note for any $\eta \in [0, n)$ and $t > \eta$, we have

$$-[y'(t)]^2 + [y'(\eta)]^2 = 2\lambda\int_\eta^t \phi(s)y'(s)\ f^\star[s,y(s),y'(s)]ds \ge 0,$$

and so

$$[y'(t)]^2 \le [y'(\eta)]^2 \qquad \text{for} \qquad t \ge \eta. \tag{13}$$

If $y'(0) = 0$, then (13) with $\eta = 0$ implies $y'(t) = 0$ for $t \in [0, n]$, a contradiction, because $1 > a_0$. Thus, $y'(0) \neq 0$. Then, either $y'(t) \neq 0$ for $t \in [0, n]$ or there exists $\delta \in (0, n)$ with $y'(t) \neq 0$ for $t \in [0, \delta)$ and $y'(\delta) = 0$.

Case 1. $y' \neq 0$ on $[0, n]$. If $y'(t) < 0$ for $t \in [0, n]$ then $a_0 = y(0) > y(n) = 1$, a contradiction. Thus, $y'(t) > 0$ for $t \in [0, n]$.

Case 2. There exists $\delta \in (0, n)$ with $y'(t) \neq 0$ for $t \in [0, \delta)$ and $y'(\delta) = 0$. Now (13) with $\eta = \delta$ implies $y'(t) = 0$ for $t \in [\delta, n]$, so $y(t) = 1$ for $t \in [\delta, n]$. If $y'(t) < 0$ for $t \in [0, \delta)$ then $y(\delta) < a_0$, a contradiction. Hence, $y'(t) > 0$ for $t \in [0, \delta)$.

Consequently (12) is true. Now, the definition of $f^{\star}$ together with (5) implies $y''(t) \leq 0$ for $t \in (0, n)$ and also

$$a_0 \leq y(t) \leq 1 \qquad \textit{for} \qquad t \in [0, n]. \tag{14}$$

In addition $\sup_{t \in [0, n]} y'(t) = y'(0)$. The mean value theorem guarantees that there exists $\xi \in (0, 1)$ with $y'(\xi) = y(1) - y(0) \leq y(1) - a_0 \leq 1 - a_0$. Now, from (6), we have

$$\frac{-y''(t)}{\tau(y'(t))} \leq \phi(t) \qquad \text{for} \qquad t \in (0, n),$$

and so integrating from 0 to ξ yields

$$\int_{y'(\xi)}^{y'(0)} \frac{du}{\tau(u)} \leq \int_0^{\xi} \phi(s)ds.$$

Consequently,

$$\int_0^{y'(0)} \frac{du}{\tau(u)} \leq \int_0^{y'(\xi)} \frac{du}{\tau(u)} + \int_0^1 \phi(s)ds \leq \int_0^{1-a_0} \frac{du}{\tau(u)} + \int_0^1 \phi(s)ds.$$

Let

$$J(z) = \int_0^z \frac{du}{\tau(u)},$$

and so we have [note (7)],

$$y'(0) \leq J^{-1}\left(\int_0^{1-a_0} \frac{du}{\tau(u)} + \int_0^1 \phi(s)ds\right) \equiv K. \tag{15}$$

Combining (12) and (15) gives

$$0 \leq y'(t) \leq K \qquad \text{for} \qquad t \in [0, n]. \tag{16}$$

Theorem 1 implies that $(11)_1^n$ [and consequently $(10)^n$] has a solution $y_n \in C^2[0, n]$ with

$$a_0 \leq y_n(t) \leq 1 \qquad \text{and} \qquad 0 \leq y_n'(t) \leq K \qquad \text{for} \qquad t \in [0,n]. \tag{17}$$

Next, we obtain a sharper lower bound on $y_n(t)$. Note assumption (9) guarantees that there is a continuous function $\psi_{1,K}$ positive and nondecreasing on $(0,\infty)$, and a constant $1 \leq r < 2$ with $f(t,y,v) \geq \psi_{1,K}(t)\, v^r$ for $(t, y, v) \in [0, \infty) \times [a_0, 1] \times [0, K]$. Thus,

$$-y_n''(t) \geq \phi(t)\psi_{1,K}(t)[y_n'(t)]^r \qquad \text{for} \qquad t \in (0,n). \tag{18}$$

Case 1. $r = 1$. Integrating (18), with $r = 1$, from t to n yields

$$y_n'(t) \geq \phi(t)\psi_{1,K}(t)\int_t^n y_n'(s)ds = \phi(t)\psi_{1,K}(t)[1 - y_n(t)].$$

Integration from 0 to t now yields

$$y_n(t) \geq 1-(1-a_0)\exp\left[-\int_0^t \phi(s)\psi_{1,K}(s)ds\right] \equiv \Phi_1(t) \qquad \text{for} \qquad t \in [0,n]. \tag{19}$$

Note $\Phi_1(t) \to 1$ as $t \to \infty$.

Case 2. $1 < r < 2$. We know either $y_n'(t) > 0$ for $t \in [0, n]$, or there exists $\delta \in (0, n)$ with $y_n'(t) > 0$ for $t \in [0, \delta)$ and $y_n'(t) = 0$ for $t \in [\delta, n]$, so in both cases, there exists $\delta \in (0, n)$ with $y_n'(t) > 0$ for $t \in [0, \delta)$. Multiply (18) by $y_n'(t)^{1-r}$ and integrate from t to δ to obtain [note ϕ and $\psi_{1,K}$ are nondecreasing on $(0, \infty)$],

$$y_n'(t) \geq \left\{(2-r)\phi(t)\psi_{1,K}(t)[1-y_n(t)]\right\}^{\frac{1}{2-r}}.$$

As a result

$$[1-y_n(t)]^{\frac{-1}{2-r}} y_n'(t) \geq [(2-r)\phi(t)\psi_{1,K}(t)]^{\frac{1}{2-r}},$$

and so integration from 0 to t yields

$$[1-y_n(t)]^{\frac{1-r}{2-r}} \geq (1-a_0)^{\frac{1-r}{2-r}} + \left(\frac{2-r}{1-r}\right)\int_0^t \left[(2-r)\phi(s)\psi_{1,K}(s)\right]^{\frac{1}{2-r}} ds. \tag{20}$$

Let

$$\Phi_r(t) = 1 - \frac{1}{\left\{(1-a_0)^{\frac{1-r}{2-r}} + \left(\frac{2-r}{r-1}\right)\int_0^t \left[(2-r)\phi(s)\psi_{1,K}(s)\right]^{\frac{1}{2-r}} ds\right\}^{\frac{2-r}{r-1}}}$$

and notice (20) implies

$$y_n(t) \geq \Phi_r(t) \qquad \text{for} \qquad t \in [0, n]. \tag{21}$$

Note $\Phi_r(t) \to 1$ as $t \to \infty$.

As a result, we have

$$\Phi_r(t) \leq y_n(t) \leq 1 \qquad \text{with} \qquad 0 \leq y_n'(t) \leq K \qquad \text{for} \qquad t \in [0, n], \tag{22}$$

and

$$-y_n''(t) \leq \phi(t)\tau\left[y_n'(t)\right] \leq \phi(t) \qquad \sup_{v\in[0,K]} \quad \tau(v) \qquad \text{for} \qquad t \in (0, n). \tag{23}$$

For $n \geq 1$, an integer, let

$$u_n(x) = \begin{cases} y_n(x), \ x \in [0, n] \\ 1, \ x \in (n, \infty). \end{cases}$$

Let $S = \{u_n\}_{n=1}^{\infty}$. The Arzela–Ascoli theorem guarantees a subsequence $N_1^\star$ of $\{1, 2, \ldots\}$ and functions $z_1^{(j)} \in C[0, 1]$, $j = 0, 1$, with $u_n^{(j)} \to z_1^{(j)}$, $j = 0, 1$, uniformly on $[0, 1]$ as $n \to \infty$ through $N_1^\star$. Also $\Phi_r(t) \leq z_1(t) \leq 1$ and $0 \leq z_1'(t) \leq K$ for $t \in [0, 1]$, and $z_1(0) = a_0$. Let $N_1 = N_1^\star \setminus \{1\}$. Again, the Arzela–Ascoli theorem guarantees a subsequence $N_2^\star$ of N_1 and functions $z_2^{(j)} \in C[0, 2]$, $j = 0, 1$, with $u_n^{(j)} \to z_2^{(j)}$, $j = 0, 1$, uniformly on $[0, 2]$ as $n \to \infty$ through $N_2^\star$. Note $z_2 = z_1$ on $[0, 1]$ because $N_2^\star \subseteq N_1$. Also $\Phi_r(t) \leq z_2(t) \leq 1$ and $0 \leq z_2'(t) \leq K$ for $t \in [0, 2]$, and $z_2(0) = a_0$. Let $N_2 = N_2^\star \setminus \{2\}$ and proceed inductively to obtain for $k = 2, \ldots$ a subsequence $N_k \subseteq \{1, 2, \ldots\}$ with $N_k \subseteq N_{k-1}$ and function $z_k^{(j)} \in C[0, k]$, $j = 0, 1$, with $u_n^{(j)} \to z_k^{(j)}$ uniformly on $[0, k]$ as $n \to \infty$ through N_k. Also $z_k = z_{k-1}$ on $[0, k-1]$ with $\Phi_r(t) \leq z_k(t) \leq 1$ and $0 \leq z_k'(t) \leq K$ for $t \in [0, k]$, and $z_k(0) = a_0$.

Define a function y as follows. Fix $x \in (0, \infty)$ and let $k \in \{1, 2, \ldots\}$ with $x \leq k$. Then define $y(x) = z_k(x)$. Then $y \in C^1[0, \infty)$ with $\Phi_r(t) \leq y(t) \leq 1$ and $0 \leq y'(t) \leq K$ for $t \in [0, \infty)$, and $y(0) = a_0$. Let x and k be as above. Also for $n \in N_k$ and $t \in (0, k]$ we have

$$y_n'(t) - y_n'(0) = -\int_0^t \phi(s) f\left[s, y_n(s), y_n'(s)\right] ds.$$

Let $n \to \infty$ through N_k so for $t \in (0, t_k]$, we have

$$z_k'(t) - z_k'(0) = -\int_0^t \phi(s) f\left[s, z_k(s), z_k'(s)\right] ds;$$

that is,

$$y'(t) - y'(0) = -\int_0^t \phi(s)f[s, y(s), y'(s)]ds.$$

Consequently, $y''(t) + \phi(t) f[t, y(t), y'(t)] = 0$ for $t \in (0, k]$ so in particular

$$y''(x) + \phi(x)f[x, y(x), y'(x)] = 0.$$

We can do this for every $x \in (0, \infty)$. Consequently, $y \in C^1[0, \infty)$ with $y''(t) + \phi(t) f[t, y(t), y'(t)] = 0$ for $t \in (0, \infty)$, $\Phi_r(t) \leq y(t) \leq 1$ and $0 \leq y'(t) \leq K$ for $t \in (0, \infty)$ and $y(0) = a_0$. Finally $\lim_{t\to\infty} y(t) = 1$, because $\lim_{t\to\infty} \Phi_r(t) = 1$, so y is a solution of (4). ■

Example 1: The boundary-value problem

$$\begin{cases} y'' + \frac{2t}{y^{\frac{1}{2}}}y' = 0, & 0 < t < \infty \\ y(0) = 1 - \alpha, 0 < \alpha < 1 \\ \lim_{t\to\infty} y(t) = 1 \end{cases} \tag{24}$$

has a solution $y \in C^2[0, \infty)$ with $y(t) \geq 1 - \alpha$ for $t \in [0, \infty)$.

To see this, let

$$a_0 = 1 - \alpha, \phi(t) = 2t,\ f(t, y, v) = \frac{v}{y^{\frac{1}{2}}} \qquad \text{and} \qquad \tau(v) = \frac{v+1}{[a_0]^{\frac{1}{2}}}.$$

Clearly (5), (6), (7) (because

$$\left.\int_0^\infty \frac{du}{\tau(u)} = \infty\right)$$

and (8) are satisfied. Also (9) holds with $\psi_{H,K} = H^{-\frac{1}{2}}$ and $r = 1$. The existence of a solution to (24) follows from Theorem 1.

3. Singular problems

The example in Section 1 (when $\alpha = 1$) motivates our discussion of the singular boundary-value problem

$$\begin{cases} y'' + \phi(t)\ f(t, y, y') = 0, & 0 < t < \infty \\ y(0) = 0, \\ \lim_{t\to\infty} y(t) = 1. \end{cases} \tag{25}$$

THEOREM 3. *Suppose* f: $[0, \infty) \times (0, \infty) \times \mathbf{R} \to \mathbf{R}$ *is continuous,* $\phi \in C[0, \infty)$ *with* $\phi > 0$ *on* $(0, \infty)$. *In addition, assume the following conditions hold:*

$$v\, f(t, y, v) \geq 0 \qquad \text{on} \qquad [0, \infty) \times (0, \infty) \times \mathbf{R} \tag{26}$$

$$\begin{cases} \textit{there exists a continuous function } \tau : [0, \infty) \to (0, \infty) \\ \textit{a continuous nonincreasing function } g : (0, \infty) \to (0, \infty) \\ \textit{with } f(t, y, v) \leq g(y)\tau(v) \qquad \textit{on} \qquad [0, \infty) \times (0, 1] \times [0, \infty) \end{cases} \tag{27}$$

$$\int_0^1 g(u)du < \infty \tag{28}$$

$$\int_0^\infty \frac{du}{\tau(u)} > \int_0^1 \frac{du}{\tau(u)} + \phi(1) \int_0^1 g(w)dw \tag{29}$$

$$\phi \quad \textit{is nondecreasing on } (0, \infty) \tag{30}$$

and

$$\begin{cases} \textit{for constants } H > 0,\ K > 0, \textit{ there exists a function } \psi_{H,K} \textit{ continuous} \\ \textit{on } [0, \infty) \textit{ and positive and nondecreasing on } (0, \infty), \textit{ and a constant} \\ 1 \leq r < 2 \textit{ with } f(t, y, v) \geq \psi_{H,K}(t) v^r \textit{ on } [0, \infty) \times (0, H] \times [0, K]. \end{cases} \tag{31}$$

Then (25) *has a solution* $y \in C^1[0, \infty) \cap C^2(0, \infty)$ *with* $1 \geq y(t) > 0$ *for* $t \in (0, \infty)$.

Proof: Fix $n \in \{1, 2, \ldots\}$. The idea is to show again that

$$\begin{cases} y'' + \phi(t)\, f(t, y, y') = 0, \qquad 0 < t < n \\ y(0) = 0, \qquad y(n) = 1 \end{cases} \tag{32}$$

has a solution in $C^1[0, n] \cap C^2(0, n]$. To establish this, we first must discuss the boundary-value problem

$$\begin{cases} y'' + \phi(t)\, f(t, y, y') = 0, \qquad 0 < t < n \\ y(0) = \frac{1}{m}, \qquad y(n) = 1, \end{cases} \tag*{$(33)^m$}$$

for each $m \in \{2, 3, \ldots\}$. The technique we use to establish the existence of a solution to (32) is (1) prove $(33)^m$ has a solution for each $m \in \{2, 3, \ldots\}$; and

(2) apply the Arzela–Ascoli theorem. Fix $m \in \{2, 3, \ldots\}$. To show $(33)^m$ has a solution, we will apply Theorem 1 to the modified problem

$$\begin{cases} y'' + \lambda\phi(t)f^{\star}(t, y, y') = 0, & 0 < t < n \\ y(0) = \frac{1}{m}, \quad y(n) = 1 \end{cases} \qquad (34)^m_\lambda$$

where $\lambda \in (0,1)$ and

$$f^{\star}(t, y, v) = \begin{cases} f(t, y, v), & y \geq \frac{1}{m} \\ f(t, \frac{1}{m}, v), & y \leq \frac{1}{m}. \end{cases}$$

Let $y \in C^2[0, n]$ be any solution of $(34)^m_\lambda$ for $0 < \lambda < 1$. Essentially, the same reasoning as in Theorem 2 (here $a_0 = \frac{1}{m}$) establishes

$$y'(t) \geq 0 \quad \text{for} \quad t \in [0, n] \quad \text{and} \quad y'(0) > 0. \qquad (35)$$

As a result $y''(t) \leq 0$ for $t \in (0, n)$ and

$$\frac{1}{m} \leq y(t) \leq 1 \quad \text{for} \quad t \in [0, n]. \qquad (36)$$

In addition $\sup_{t\in[0,n]} y'(t) = y'(0)$, and there exists $\xi \in (0,1)$ with $y'(\xi) = y(1) - y(0) \leq 1$. Now, from (27), we have

$$\frac{-y'(t)y''(t)}{\tau(y'(t))} \leq \phi(t)g[y(t)]y'(t) \quad \text{for} \quad t \in (0, n),$$

and so, integrating from 0 to ξ yields

$$\int_0^{y'(0)} \frac{u\ du}{\tau(u)} \leq \int_0^{y'(\xi)} \frac{u\ du}{\tau(u)} + \phi(1)\int_{y(0)}^{y(\xi)} g(u)du \leq \int_0^1 \frac{u\ du}{\tau(u)} + \phi(1)\int_0^1 g(u)du.$$

Let

$$I(z) = \int_0^z \frac{u\,du}{\tau(u)},$$

and so we have

$$y'(0) \leq I^{-1}\left(\int_0^1 \frac{u\ du}{\tau(u)} + \phi(1)\int_0^1 g(u)du\right) \equiv V. \qquad (37)$$

Combining (35) and (37) gives

$$0 \le y'(t) \le V \qquad \text{for} \qquad t \in [0, n].$$

Theorem 1 implies that $(34)_1^m$ [and consequently $(33)^m$] has a solution $v_m \in C^2[0, n]$ with

$$\frac{1}{m} \le v_m(t) \le 1 \qquad \text{and} \qquad 0 \le v_m'(t) \le V \qquad \text{for} \qquad t \in [0, n]. \tag{38}$$

Next assumption (31) guarantees that there is a continuous function $\psi_{1,V}$ positive and nondecreasing on $(0, \infty)$, and a constant $1 \le r < 2$ with $f(t, y, v) \ge \psi_{1,V}(t)\, v^r$ for $(t, y, v) \in [0, \infty) \times (0, 1] \times [0, V]$. Thus,

$$-v_m''(t) \ge \phi(t)\psi_{1,V}(t)\, [v_m'(t)]^r \qquad \text{for} \qquad t \in (0, n). \tag{39}$$

Essentially the same reasoning as in Theorem 2 establishes

$$v_m(t) \ge \Psi_r(t) \qquad \text{for} \qquad t \in [0, n];$$

here

$$\Psi_1(t) = 1 - \exp\left[-\int_0^t \phi(s)\psi_{1,V}(s)ds\right]$$

and if $1 < r < 2$,

$$\Psi_r(t) = 1 - \frac{1}{\left\{1 + \left(\frac{2-r}{r-1}\right)\int_0^t \left[(2-r)\phi(s)\psi_{1,V}(s)\right]^{\frac{1}{2-r}} ds\right\}^{\frac{2-r}{r-1}}}.$$

Let

$$\Omega_r(t) = \begin{cases} \Psi_r(1)t, & t \in [0, 1] \\ \Psi_r(t), & t \ge 1. \end{cases}$$

We claim

$$v_m(t) \ge \Omega_r(t) \qquad \text{for} \qquad t \in [0, n]. \tag{40}$$

To see this let

$$r(t) = v_m(t) - \left\{\frac{1}{m}(1-t) + v_m(1)t\right\}.$$

Then $r''(t) = v_m''(t) \le 0$ on $(0, n)$ with $r(0) = r(1) = 0$. Consequently, $r \ge 0$ on $[0, 1]$ so

$$v_m(t) \ge \frac{1}{m}(1-t) + v_m(1)t \ge v_m(1)t \ge \Psi_r(1)t \qquad \text{for} \qquad t \in [0, 1].$$

Hence (40) is true.

As a result we have

$$\Omega_r(t) \le v_m(t) \le 1 \qquad \text{with} \qquad 0 \le v_m'(t) \le V \qquad \text{for} \qquad t \in [0,n], \tag{41}$$

and

$$-v_m''(t) \le \phi(t) g[v_m(t)] \tau\left[v_m'(t)\right] \le \phi(t) g[\Omega_r(t)] \sup_{u\in[0,V]} \tau(u) \qquad \text{for} \qquad t \in (0,n). \tag{42}$$

Now the Arzela–Ascoli theorem guarantees that there is a subsequence N of $\{2, 3, \ldots\}$ and functions $y_n^{(j)} \in C\,[0, n]$, $j = 0, 1$, with $v_m^{(j)} \to y_n^{(j)}$, $j = 0, 1$, uniformly on $[0, n]$ as $m \to \infty$ through N. Also $y_n(0) = 0$, $y_n(n) = 1$ with

$$\Omega_r(t) \le y_n(t) \le 1 \qquad \text{and} \qquad 0 \le y_n'(t) \le V \qquad \text{for} \qquad t \in [0,n]. \tag{43}$$

Also

$$v_m'(t) - v_m'(0) = -\int_0^t \phi(s) f(s, v_m(s), v_m'(s))ds \qquad \text{for} \qquad t \in (0,n). \tag{44}$$

Let $m \to \infty$ through N [note (42)] in (44), and we obtain using the Lebesgue dominated convergence theorem that

$$y_n'(t) - y_n'(0) = -\int_0^t \phi(s)\, f[s, y_n(s), y_n'(s)]ds \qquad \text{for} \qquad t \in (0,n).$$

Consequently $y_n \in C^1[0, n] \cap C^2(0, n)$ is a solution of (32) with

$$\Omega_r(t) \le y_n(t) \le 1 \qquad \text{with} \qquad 0 \le y_n'(t) \le V \qquad \text{for} \qquad t \in [0,n], \tag{45}$$

and

$$-y_n''(t) \le \phi(t) g[\Omega_r(t)] \sup_{u\in[0,V]} \tau(u) \qquad \text{for} \qquad t \in (0,n). \tag{46}$$

Essentially the same reasoning as in Theorem 2 [from (23) onward] now establishes that (25) has the desired solution. ∎

Example 2: The boundary-value problem

$$\begin{cases} y'' + \frac{2t}{y^{\frac{1}{2}}} y' = 0, & 0 < t < \infty \\ y(0) = 0, \\ \lim_{t\to\infty} y(t) = 1 \end{cases} \tag{47}$$

has a solution $y \in C^1[0, \infty) \cap C^2(0, \infty)$ with $y(t) > 0$ for $t \in (0, \infty)$.

To see this let

$$\phi(t) = 2t, \quad f(t,y,v) = \frac{v}{y^{\frac{1}{2}}}, \qquad g(y) = \frac{1}{y^{\frac{1}{2}}} \qquad \text{and} \qquad \tau(v) = v + 1.$$

Clearly (26)–(30) are satisfied. Also, (31) holds with $\psi_{H,K} = H^{-\frac{1}{2}}$ and $r = 1$. The existence of a solution to (47) follows from Theorem 3.

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