WATER RESOURCES RESEARCH, VOL. 18, NO. 4, PAGES 1049–1052, AUGUST 1982

Nonlinear Equation Governing Flow in a Saturated Porous Medium

D. D. JOSEPH

Department of Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis, Minnesota 55455

D. A. NIELD

Department of Mathematics, University of Auckland, New Zealand

G. PAPANICOLAOU

Courant Institute of Mathematical Sciences, New York University, New York, New York 10012

It is argued that the appropriate generalization of Darcy's law when inertia effects are included takes the form $\nabla p = -(\mu/k)\,\mathbf{V} - (\rho c/k^{1/2})|\mathbf{V}|\mathbf{V}$, div $\mathbf{V} = 0$, where k is the permeability of the medium and the 'form drag constant' c is a coefficient which is independent of the pressure p, the seepage velocity $\mathbf{V}$, and the density ρ and viscosity μ of the fluid but which is dependent on the geometry of the medium. We formulate a nonlinear extension of Brinkman's self-consistent theory for the flow of a viscous fluid through a swarm of spherical particles. We equate the drag per unit volume given by the right hand side of the first of the above equations to the total drag ND on the N particles contained within that unit volume, in an infinite region Ω, where D is the drag on a single particle placed in a velocity field $\mathbf{v}$ subject to $\rho(\mathbf{v}\cdot\nabla)\mathbf{v}$ + grad $p = \mu\nabla^2\mathbf{v} - \mu/k\ \mathbf{v} - (c\rho/k^{1/2})|\mathbf{v}|\mathbf{v}$, div $\mathbf{v} = 0$, $\mathbf{v}|_{\partial\Omega}$ is a prescribed constant, where μ is the viscosity. Without solving these equations, we obtain an estimate for c from the known experimental drag law for a solid sphere placed in a uniform stream.

INTRODUCTION

It is generally accepted that the appropriate form of the momentum equation for steady slow flow through a porous medium is Darcy's law, which is a linear relationship between p and V:

$$\nabla p = -\frac{\mu}{k}\mathbf{V} \tag{1}$$

which neglects inertial effects. There are at present several different views as to how Darcy's law should be generalized to include inertial effects. We argue that the effects of inertia appear as a drag proportional to the square of the velocity. The idea is that the drag due to the 'dead water' region behind particles in a stream produces a quadratic drag in much the same way as the drag of one cyclist drafting another.

The suggestion that the one-dimensional form of (1) be modified by the addition to the right-hand side of a term proportional to ρV^2 dates back to *Dupuit* [1863], but the modified equation is usually associated with the name *Forchheimer* [1901]. Forchheimer and others have also included a term proportional to V^3, but this does not seem to be consistent with experiment. The increased resistance is due to a 'pressure drag,' like the aerodynamic drag behind blunt bodies, and is essentially quadratic in the velocity. It arises from the convective term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ in the Navier-Stokes equation for flow of a viscous fluid with velocity $\mathbf{u}$ as an average is taken over many pores in the medium, just as the linear Darcy term arises from the viscous term $(\mu/\rho)\nabla^2\mathbf{u}$ in the Navier-Stokes equation. The available experimental data correlates well with a quadratic drag law (see, for example, *MacDonald et al.* [1979]).

Some authors have attempted to account for inertial effects by writing $\mathbf{V} = \eta\mathbf{U}$, where η is the porosity of the medium (volume of fluid/total volume), and substituting in the Navier-Stokes equation with a Darcy viscous term to get

$$\frac{\rho}{\eta}\frac{\partial\mathbf{V}}{\partial t} + \frac{\rho}{\eta^2}(\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla p - \frac{\mu}{k}\mathbf{V} \tag{2}$$

This law is clearly false since it fails to give any inertial drag effect for the case of steady unidirectional flow, since the left-hand side of (2) is then identically zero. *Kitaev et al.* [1975] have gone a step further and adopted a steady flow equation equivalent to

$$(\rho/\eta^2)(\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla p - A\mathbf{V} - B|\mathbf{V}|\mathbf{V} \tag{3}$$

where A and B are coefficients based on an equation obtained by *Ergun* [1952] (5) below, with $\alpha = 1.75$ and $\beta = 150$). Kitaev et al. claimed that the Ergun equation, being based on experiments with unidirectional flow, 'takes into account gravitational force and forces of viscous and surface friction but not inertia effects (convective acceleration) [*Kitaev et al.*, 1975, p. 184]. We disagree with Kitaev et al. Not only does the last term in (3) take inertial effects into account, but the term on the left-hand side of that equation should be left out. An essential feature of flow in porous media is that the term which is quadratic in the velocity is associated with a pressure drag of the type which always exists in flows around blunt bodies and hence acts in the direction of $-\mathbf{V}$.

We note in passing the works of *Yamamoto and Yoshida* [1974], *Vaisman and Gol'dshtik* [1978], and others, who have taken into account the role of inertial effects by using a boundary layer approach. The problem of what happens near the boundary of a porous medium is an important one,

Paper number 2W0806.
0043-1397/82/002W-0806$05.00

but it is not considered here. Likewise, we do not consider the effect of hydrodynamic dispersion. Having explored these inappropriate leads, we now indicate what we consider to be an appropriate description of the non-Darcy flow when inertia effects are significant.

THE NONLINEAR EQUATION

The experimental work discussed by *Ward* [1964] and *Beavers et al.* [1973] indicates that the appropriate form for the dynamic equation governing one-dimensional flow of a fluid in a saturated porous medium is

$$-\frac{dp}{dx} = \frac{\mu V}{k} + \frac{c\rho V^2}{k^{1/2}} \tag{4}$$

where p is the pressure, V is the mean filter velocity, μ and ρ are the viscosity and density of the fluid, k is the permeability of the medium, and c is a nondimensional coefficient. Ward concluded from his data that c was a universal constant, equal to 0.55, for all permeable materials, but later experiments reported by *Beavers and Sparrow* [1969] and *Schwartz and Probstein* [1969] indicate that c has a more or less universal value for a particular family of materials. For example, c is 0.1 approximately for foamed metal fibers and 0.26 for compacted polyethylene particles of random shape. Furthermore, *Beavers et al.* [1973] showed that bounding walls could have a substantial effect on the value of c. They found that their data correlated fairly well with the expression

$$c = 0.55(1 - 5.5d/D_e)$$

where d is the diameter of their spheres and D_e is the equivalent diameter of the bed defined as

$$D_e = 2wh/(w + h)$$

where h and w are the height and width of the bed.

Several attempts have been made to give a theoretical justification for the linear version of (3) (Darcy's law) but comparatively little theoretical work has been published concerning the Forchheimer quadratic term involving V^2. The most noteworthy effort is that of *Irmay* [1958], who gave a nonrigorous derivation of the Forchheimer law, obtaining an expression which can be written in the form

$$-\frac{dp}{dx} = \frac{\beta\mu(1-\eta)^2 V}{d^2\eta^3} + \frac{\alpha\rho(1-\eta)V^2}{d\eta^3} \tag{5}$$

where d is the mean particle diameter and α and β are shape factors which need to be determined empirically. Equations (4) and (5) are identical if we make

$$k = \frac{d^2\eta^3}{\beta(1-\eta)^2} \tag{6a}$$

$$\frac{c}{k^{1/2}} = \frac{\alpha(1-\eta)}{d\eta^3} \tag{6b}$$

so that $c = \alpha(\beta\eta^3)^{-1/2}$. According to *Bear* [1972], Irmay later adopted the values $\alpha = 0.6$ and $\beta = 180$, which make $c = 0.45\eta^{-3/2}$. This reasoning makes c depend rather strongly on the porosity, η. For $\eta = 0.5$, it predicts $c = 0.13$. The agreement with the experimental work is not as good as one would hope for, but at least Irmay's expression gives c to the correct order of magnitude.

For flow through a porous bed of spherical particles, we would like to relate k and c to the size of the spheres and the way they are packed. For the situation where the spheres are not closely packed, such a relationship for k has been found by *Brinkman* [1947], who considered the situation when the porosity was sufficiently large for one to take the equation for flow past an individual sphere to be

$$\text{grad } p = -\frac{\mu}{k}\mathbf{v} + \mu\nabla^2\mathbf{v} \tag{7}$$

where $\mathbf{v}$ and p are the fluid velocity and pressure and μ is the viscosity of the fluid. It is not appropriate to replace μ in (7) by an effective viscosity μ', as one does with suspensions, since the fluid in the porous medium retains its bulk properties. For an incompressible fluid, $\mathbf{v}$ satisfies

$$\text{div } \mathbf{v} = 0 \tag{8}$$

Brinkman solved equations (7) and (8) subject to the appropriate boundary conditions ($\mathbf{v} = 0$ on the surface of the sphere, and $\mathbf{v} = \mathbf{v}_0$ at large distances from the sphere). He calculated the drag on the sphere to be mD_s, where $D_s = 6\pi\mu v_0 a$ is the Stokes drag on a sphere of radius a and $m = 1 + \lambda a + \lambda^2 a^2/3$, where $\lambda = k^{-1/2}$. He then identified v_0 with a unidirectional mean filter velocity and equated the total force on the spheres contained in a column of the medium to the Darcy drag on that column. He thereby obtained a relationship between a and the porosity η and hence an expression for the multiplication factor m, which can be written as

$$m = \left\{1 + \frac{3(1-\eta)}{4}\left[1 - \left(\frac{8}{1-\eta} - 3\right)^{1/2}\right]\right\}^{-1} \tag{9}$$

This requires that the permeability be given by

$$k = k_0/m \tag{10}$$

where k_0 is the value of k in the limit as $\eta \to 1$. According to (9), m becomes unbounded as $\eta \to \frac{1}{3}$, and hence we must assume that $\frac{1}{3} < \eta < 1$. Brinkman showed that (10), with m given by (9), was in qualitative agreement with an experimental relation formulated by Carman which is now widely accepted ((6a) of this paper, with $\beta = 180$).

It is desirable to repeat Brinkman's analysis for the nonlinear problem in which the Navier-Stokes equations, rather than the linearized (Stokes) equations govern the flow around the spheres. In this problem the effects of the presence of a swarm of other spheres is accounted for by the nonlinear 'Forchheimer' drag law rather than the linearized (Darcy) drag law. This leads us directly to the following self-consistent problem of the Brinkman type:

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \text{grad } p = \mu\nabla^2\mathbf{v} - \frac{\mu}{k}\mathbf{v} - \frac{c\rho}{k^{1/2}}|\mathbf{v}|\mathbf{v} \tag{11}$$

$$\text{div } \mathbf{v} = 0 \qquad \mathbf{v}|_{\partial\Omega} = \text{prescribed constant}$$

Pending analysis of (11), we shall proceed in a speculative manner and investigate the implications of making the assumption that the quadratic drag acting on an array of spheres is related to the quadratic drag on a single one in the same manner as the respective linear drags are related. In other words, we shall suppose that the Forchheimer quadrat-

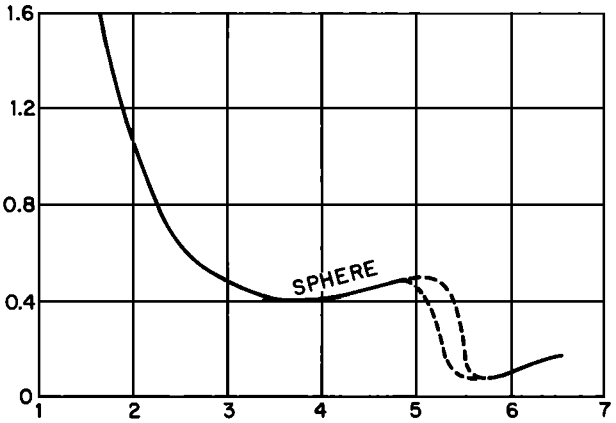

Fig. 1. Measured drag coefficient $C_D = D/(\frac{1}{2}\rho U_0^2 \pi a^2)$ versus $\log_{10} R$ where $R = 2aU_0/\nu$ for a sphere of radius a. We may represent this graph by (13).

ic drag can be calculated by supposing that the quadratic pressure drag on a sphere is effectively multiplied by the above factor m due to the presence of the other spheres. We shall make use of experimental measurements of the total drag D on a sphere, which are displayed graphically in Figure 5.11.6 of *Batchelor* [1967] and in our Figure 1. This graph is a plot of the drag coefficient $C_D = D/(\frac{1}{2}\rho U_0^2 \pi a^2)$ versus the logarithm of the Reynolds number $R = 2aU_0/\nu$. Here a is the radius of the sphere, U_0 the uniform stream velocity, ρ and μ are the density and viscosity of the fluid, and $\nu = \mu/\rho$. For values of R up to 10^4, the curve can be fitted fairly well by

$$C_D = \frac{M(R)}{R} + 0.4 \tag{12}$$

where $M(R)$ is a slowly varying function of R. For example, $M(40) = 48$, $M(100) = 64$, $M(1000) = 80$, and for small values of R we expect that M would have the value 24, corresponding to Stokes flow. In terms of dimensional quantities, (12) can be written as

$$D = \frac{\pi}{4} M\mu a U_0 + 0.2\pi\rho a^2 U_0^2 \tag{13}$$

We now consider the porous medium formed by a distribution of such solid spheres, which occupy a fraction $1 - \eta$ of the volume, and so we expect that the drag force experienced by the solid material per unit volume of the medium will be

$$\hat{D} = m(1 - \eta)D/(\tfrac{4}{3}\pi a^3)$$

where D is given by (13).

We take U_0, the stream velocity experienced by a single sphere, to be V. Then we identify $\hat{D}$ with the right-hand side of (4) and get

$$k = \frac{a^2}{9m(1 - \eta)} \tag{14}$$

$$c = 0.05[m(1 - \eta)] \tag{15}$$

where m is given by (9).

These expressions may now be compared with *Irmay*'s [1958] formulas:

$$k = \frac{\eta^3 a^2}{45(1 - \eta)^2} \tag{16}$$

$$c = 0.045\eta^{-3/2} \tag{17}$$

Numerical values are given in Table 1.

From Table 1 we see that, using the *Brinkman* [1947] hypothesis, we get good agreement with Irmay. We get values of k/a^2 of the order of 50% of Irmay's values, while away from the singularity at $\eta = \frac{1}{3}$ our values for c are of the same order as Irmay's values for that parameter.

The available experiments most relevant to our work would appear to be these of *Beavers et al.* [1973], in which beds of randomly packed spheres of diameters 3, 6, and 14.3 mm in turn were used, with measured permeabilities 5.8×10^{-5}, 1.57×10^{-4}, and 1.02×10^{-3} cm^2, respectively, and thus k/a^2 values of 2.58×10^{-3}, 1.74×10^{-3}, and 2.00×10^{-3}, respectively. For large values of the bed size parameters d/D_e the measured values $\eta = 0.37$ and $c = 0.55$ were found. Unfortunately, this value of the porosity is uncomfortably close to the singularity at $\eta = \frac{1}{3}$ in the Brinkman theory, and a close match between theory and experiment should not be expected. For $\eta = 0.37$, we predict that $k/a^2 = 1.8 \times 10^{-4}$ and $c = 1.2$.

In fact, of course, the Brinkman approach is valid only when the spheres are not too closely packed, and in practice this seems to mean that the porosity must be at least larger than 0.6 and probably larger than 0.7. For comparison, the range of porosities of the materials for which data is given in *Ward*'s [1964] paper is 0.34 to 0.67, so the test between experiment and theory is not a very good one. Nevertheless, our predicted values of k/a^2 and c seem to be of the right order of magnitude and provide some ground for optimism that more precise analysis will give a better fit with experimental data.

CONCLUSION

We have formulated a nonlinear theory which is consistent with the available experimental data for flow through porous material. Although our work is formulated for flow through a

TABLE 1. Values of k/a^2 and c Predicted From the Present Argument by Using the Brinkman Hypothesis for the Effective Stream Velocity Experienced by a Sphere and the Values Predicted by Irmay

	k/a^2		c	
η	Equation (15) [*Brinkman*, 1947]	Equation (17) [*Irmay*, 1958]	Equation (16) [*Brinkman*, 1947]	Equation (18) [*Irmay*, 1958]
0.35	0.000	0.002	2.672	0.217
0.40	0.001	0.004	0.659	0.178
0.45	0.002	0.007	0.356	0.149
0.50	0.005	0.011	0.234	0.127
0.55	0.010	0.018	0.168	0.110
0.60	0.018	0.030	0.126	0.097
0.65	0.029	0.050	0.097	0.086
0.70	0.048	0.085	0.076	0.077
0.75	0.079	0.150	0.059	0.069
0.80	0.132	0.284	0.046	0.063
0.85	0.233	0.606	0.035	0.057
0.90	0.463	1.620	0.024	0.053
0.95	1.261	7.620	0.015	0.049
1.00	∞	∞	0.000	0.045

fixed solid matrix, similar considerations apply to sedimentation problems in which solid particles fall through a viscous fluid.

Acknowledgments. The work of D. D. Joseph was supported by the U.S. Army Research Office and the National Science Foundation. D. A. Nield's contribution was made while he was enjoying the hospitality of the University of Minnesota while on study and research leave from the University of Auckland. The work of G. Papanicolaou was supported by the Office of Naval Research.

REFERENCES

Batchelor, G. K., *An Introduction to Fluid Dynamics*, Cambridge University Press, London, 1975.

Bear, J., *Dynamics of Fluids in Porous Media*, American Elsevier, New York, 1972.

Beavers, G. S., and E. M. Sparrow, Non-Darcy flow through fibrous porous media, *J. Appl. Mech.*, *36*, 711–714, 1969.

Beavers, G. S., E. M. Sparrow, and D. E. Rodenz, Influence of bed size on the flow characteristics and porosity of randomly packed beds of spheres, *J. Appl. Mech.*, *40*, 655–660, 1973.

Brinkman, H. C., A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles, *Appl. Sci. Res. Sec. A*, *1*, 27–34, 1947.

Dupuit, J., *Etudes Théoriques et Pratiques sur le Mouvement des Eaux*, Dunod, Paris, 1863.

Ergun, S., Fluid flow through packed columns, *Chem. Eng. Progr.*, *48*, 88–94, 1952.

Forchheimer, P., Wasserbewegung durch Boden, *VDI Z.*, *45*, 1782–1788, 1901.

Irmay, S., On the theoretical derivation of Darcy and Forchheimer formulas, *Eos Trans. AGU*, *39*, 702–707, 1958.

Kitaev, B. J., Yu. G. Yaroshenko, V. S. Shvydkii, Ya. M. Gordan, V. B. Shcherbatskii, F. R. Shklyar, and B. A. Bokovikov, Investigation of the aerodynamics in packed metallurgical furnaces with tuyere wind entry, paper presented at the Symposium on Blast Furnace Aerodynamics, Illawarra Branch of Aust. Inst. of Mining and Metal., Woolongong, Australia, 1975.

Macdonald, I. F., M. S. El-Sayed, K. Mow, and F. A. L. Dullien, Flow through porous media: The Ergun equation revisited, *Ind. Chem. Fundam.*, *18*, 199–208, 1979.

Schwartz, J., and R. F. Probstein, Experimental study of slurry separators for use in desalination, *Desalination*, *6*, 239–266, 1969.

Ward, J. C., Turbulent flow in porous media, *J. Hydraul. Div. Am. Soc. Civ. Eng.*, *90*(HY5), 1–12, 1964.

Vaisman, A. M., and M. A. Gol'dshtik, Dynamic model of liquid motion in a porous medium, *Fluid Dyn. Engl. Transl.*, *13*, 864–869, 1978.

Yamamoto, K., and Z. Yoshida, Flow through a porous wall with convective acceleration, *J. Phys. Soc. Jpn.*, *37*, 774–779, 1974.

(Received October 14, 1981;
revised April 27, 1982;
accepted May 24, 1982.)