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Traces on group C^*-algebras, sofic groups and Lück's conjecture

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Abstract

We give a reformulation of the proof of a Theorem of Elek and Szabó establishing Lück's determinant conjecture for sofic groups. It is based on traces on free group C^*-algebras. We briefly discuss the relation with Atiyah's problem on the integrality of L^2-Betti numbers.

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0. Introduction

In [7], B. Fuglede and R.V. Kadison introduce a determinant function $\Delta : M \to \mathbb{R}_+$ where M is a II_1 factor, by setting $\Delta(x) = \exp(\tau(\ln |x|))$ (where τ is the trace of M). This is a well defined function (see Section 1.2) which satisfies many of the usual properties of a determinant.

In connexion with many problems and conjectures concerning discrete groups, W. Lück (see [10]) introduced a modified determinant by setting $\Delta_+(x) = \exp(\tau(\ln_+ |x|))$ where $\ln_+(t) = 0$ if $t = 0$ and $\ln_+(t) = \ln t$ for $t > 0$ (see Section 1.3). He conjectured that for any group G and any $x \in M_n(\mathbb{Z}G)$ we have $\Delta_+(|x|) \geq 1$.

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Lück's determinant conjecture is related with many interesting problems (*cf.* [11,10]). In particular:

- Let us recall a problem stated first by Atiyah in the torsion free case, and extended by various authors (*cf.* [10,17,16] for more details): investigate the possible values of von Neumann dimensions of the kernels of elements in $M_n(\mathbb{Z}G)$. Validity of Lück's conjecture ensures a kind of stability of the von Neumann dimension of the kernel of an element in $M_n(\mathbb{Z}G)$ and allows its computation in some cases.
- For CW complexes whose fundamental group satisfies this conjecture, one can define L^2 torsion [10]. In [12], Lück, Sauer and Wegner prove that the L^2-torsion is invariant under uniform measure equivalence.

In [15,17], Schick shows that amenable groups satisfy Lück's conjecture. He shows that the class of groups satisfying Lück's conjecture is closed under taking subgroups, direct limits and inverse limits.

In [5], Elek and Szabó generalize Schick's results by proving that sofic groups satisfy Lück's conjecture. Sofic groups were introduced by Gromov [9] as a generalization of both amenable and residually finite groups. In a sense, sofic groups are the groups that can be well approximated by finite groups, *i.e.* that can be almost embedded into permutation groups.

In this paper we give a reformulation of Lück's conjecture in terms of traces on $C^*(\mathbb{F}_\infty)$ and use it to reformulate the proof of [5] in a somewhat more conceptual way. We hope that it may help understanding the proofs of [15] and [5].

We say that a trace τ on $C^*(\mathbb{F}_\infty)$ satisfies Lück's condition if for all $f \in M_n(\mathbb{Z}\mathbb{F}_\infty)$, we have $\tau(\ln_+(|f|)) \geq 0$. In this sense, a group G satisfies Lück's conjecture if and only if the trace $\tau_G \circ \pi$ satisfies Lück's condition, where τ_G is the canonical trace on G and π a surjective morphism $\mathbb{F}_\infty \to G$ (*i.e.* a generating system of G). It is then easily seen (a detailed account is given in the sequel) that:

Fact 1. (Proposition 2.3) A permutational trace *i.e.* a trace of the form $tr \circ \sigma$ satisfies Lück's condition where σ is a finite dimensional representation of $\mathbb{F}_\infty$ by permutation matrices and tr is the normalized trace on matrices.

Fact 2. (Proposition 2.6) The set of traces satisfying Lück's condition is closed (for the weak topology).

Fact 3. (Proposition 3.4) A group G is sofic if and only if the associated trace $\tau_G \circ \pi$ as above is in the closure of permutational traces.

Moreover, we notice that the stability condition of the von Neumann dimension of the kernel of matrices over $\mathbb{Z}[G]$ established in [15,16] is a consequence of the following fact (see Section 4.1 for the definition of this von Neumann dimension $\dim_\tau$):

Fact 4. (Proposition 4.1) For $a \in M_n(\mathbb{Z}\mathbb{F}_\infty)$, the map $\tau \mapsto \dim_\tau \ker a$ is continuous on the set of traces satisfying Lück's condition.

Finally, we extend this result to $a \in M_n(\overline{\mathbb{Q}}\mathbb{F}_\infty)$. We moreover prove that, for any trace of $C^*(\mathbb{F}_\infty)$ in the closure of permutational traces, the dimension $\dim_\tau(\ker a)$ does not depend on the embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$. We deduce the following extension of a result of [4] to sofic groups which is proved by A. Thom in (the proof of) [18, Theorem 4.3].

Fact 5. (Corollary 4.6) Let Γ be a sofic group and $a \in M_n(\overline{\mathbb{Q}}\Gamma)$. The von Neumann dimension with respect to the group trace of Γ of the kernel of a does not depend on the embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$.

This paper is organized as follows: In the first section, we fix notation and recall definitions of the determinant of Fuglede–Kadison and the modified determinant of Lück.

In the second section, we define Lück's condition for a trace and establish facts 1 and 2 above.

In the third section we recall the definition of a sofic group and establish fact 3.

In Section 4, we explain the relation with Atiyah's problem and establish facts 4 and 5.

All traces that we consider throughout the paper are positive finite traces.

1. Positive traces and determinants

1.1. Traces and semi-continuous functions

Let A be a unital C^*-algebra. We endow the set $\mathcal{T}_A$ of (finite positive) traces on A with the pointwise convergence.

Let $\tau \in \mathcal{T}_A$ be a trace on A and a a self-adjoint element of A, and $\mu_{\tau,a}$ the corresponding spectral measure. If $f : \mathrm{Sp}\, a \to \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous function, we may write $f = \sup f_n$ where f_n is an increasing sequence of continuous functions. Since f is bounded below, we may define $\mu_{\tau,a}(f) \in \mathbb{R} \cup \{+\infty\}$ and we have, by the monotone convergence theorem, $\mu_{\tau,a}(f) = \sup \mu_{\tau,a}(f_n) = \sup \tau(f_n(a))$. In the sequel, this "number" will be denoted by $\tau(f(a))$.

In the same way, we define $\tau(f(a)) \in \mathbb{R} \cup \{-\infty\}$ for $f : \mathrm{Sp}\, a \to \mathbb{R} \cup \{-\infty\}$ upper semi continuous.

We obviously have:

Proposition 1.1. *If $f : \mathrm{Sp}\, a \to \mathbb{R} \cup \{+\infty\}$ is lower (*resp. *upper) semi-continuous, then the map $\tau \mapsto \tau(f(a))$ is lower (*resp. *upper) semi-continuous.*

Proof. Assume $f = \sup f_n$ is lower semi-continuous where (f_n) is an increasing sequence of continuous functions. Then $\tau \mapsto \tau(f(a))$ is the supremum of the continuous functions $\tau \mapsto \tau(f_n(a))$ and is therefore lower semi-continuous.

The upper semi-continuous case is obtained by replacing f by $-f$. $\square$

Remark 1.2. Let $\varphi : A \to B$ be a unital morphism of C^*-algebras and τ a trace on B. Then, for every self-adjoint element $a \in A$ and every lower semi-continuous function $f : \mathrm{Sp}\, a \to \mathbb{R} \cup \{+\infty\}$, writing $f = \sup f_n$ with continuous f_n, we find $f_n(\varphi(a)) = \varphi(f_n(a))$; hence passing to the supremum, one gets $\tau(f(\varphi(a))) = \tau \circ \varphi(f(a))$. The same equality holds of course for upper semi-continuous f.

1.2. The Fuglede–Kadison determinant [7]

The function $\ln : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is upper semi-continuous on $\mathbb{R}_+$.

Let A be a unital C^*-algebra and $\tau \in \mathcal{T}_A$ a (finite, positive) trace on A.

The Fuglede–Kadison determinant of $x \in A$ is $\Delta_\tau(x) = \exp(\tau(\ln(|x|)))$.

Recall from [7] that, for $x, y \in A$, we have $\Delta_\tau(xy) = \Delta_\tau(x)\Delta_\tau(y)$.

1.3. Lück's modified determinant [10]

For $t \in \mathbb{R}_+$, set

$$\ln_+(t) = \begin{cases} 0 & \text{if } t = 0 \\ \ln(t) & \text{if } t \neq 0. \end{cases}$$

Note that $\ln_+$ is also upper semi-continuous.

Lück's modified determinant of $x \in A$ is $\Delta^+_\tau(x) = \exp(\tau(\ln_+(|x|)))$.

If Tr_n is the unnormalized trace of $M_n(\mathbb{C})$, then $\Delta_{\mathrm{Tr}_n}(a) = |\mathrm{Det}(a)| = \mathrm{Det}(|a|)$ where Det is the usual determinant, and $\Delta^+_{\mathrm{Tr}_n}(a)$ is the product of nonzero eigenvalues of $|a|$ (counted with their multiplicity).

2. Traces satisfying Lück's condition

If G is a (discrete) group, we denote by $C^*(G)$ the full group C^*-algebra of G.

A trace on $C^*(G)$ is determined by its values on the dense subalgebra $\mathbb{C}G$, whence by linearity, by its value on the group elements. We will identify the set of traces on $C^*(G)$ with the set $\mathcal{T}_G$ of maps $G \to \mathbb{C}$ which are of positive type and constant on conjugacy classes.

The weak topology on the set $\mathcal{T}_{C^*(G)} = \mathcal{T}_G$ of traces on $C^*(G)$ coincides with the topology of pointwise convergence on G.

We denote by τ_G the canonical trace on G, *i.e.* the one given by $\tau_G(x) = 1$ if $x = 1_G$—the unit of G and $\tau_G(x) = 0$ for $x \in G \setminus \{1_G\}$.

Definition 2.1. Let G be a countable group. We say that a trace τ on $C^*(G)$ satisfies Lück's condition if for $n \in \mathbb{N}$ and $a \in M_n(\mathbb{Z}G) \subset C^*(G) \otimes M_n(\mathbb{C})$, we have $(\tau \otimes \mathrm{Tr}_n)(\ln_+ |a|) \geq 0$ where Tr_n is the unnormalized trace on $M_n(\mathbb{C})$. We denote by $\Lambda_G \subset \mathcal{T}_G$ the set of traces on $C^*(G)$ satisfying Lück's condition

The group G is said to satisfy Lück's conjecture if the canonical trace τ_G of $C^*(G)$ satisfies Lück's condition.

Proposition 2.2. *Let G and H be groups and $\psi : G \to H$ a group homomorphism. We still denote by $\psi : C^*(G) \to C^*(H)$ its extension to group C^*-algebras. Let τ be a trace on $C^*(H)$.*

1. *If $\tau \in \Lambda_H$ (i.e. τ satisfies Lück's condition), $\tau \circ \psi \in \Lambda_G$.*
2. *If ψ is onto, the converse is true, i.e. if $\tau \circ \psi \in \Lambda_G$, then $\tau \in \Lambda_H$.*

Proof. 1. Suppose that τ satisfies Lück's condition.
We denote by $\psi : M_n(C^*(G)) \to M_n(C^*(H))$ the extension of ψ to matrices.
For all $a \in M_n(\mathbb{Z}G)$, since $\psi(a) \in M_n(\mathbb{Z}H)$, we have, by Remark 1.2,

$$(\tau \otimes \mathrm{Tr}_n)(\psi(\ln_+ |a|)) = (\tau \otimes \mathrm{Tr}_n)(\ln_+ |\psi(a)|) \geq 0.$$

2. If ψ is surjective, for all $a \in M_n(\mathbb{Z}H)$ there exists $b \in M_n(\mathbb{Z}G)$ such that $a = \psi(b)$. If $\tau \circ \psi$ satisfies Lück's condition then $(\tau \otimes \mathrm{Tr}_n)(\ln_+(|a|)) = (\tau \otimes \mathrm{Tr}_n) \circ \psi(\ln_+(|b|)) \geq 0$. $\square$

Let $\alpha_k : \mathfrak{S}_k \to \mathcal{U}_k$ be the representation of the symmetric group $\mathfrak{S}_k$ by permutation matrices. Denote also by α_k the associated morphism $C^*(\mathfrak{S}_k) \to M_k(\mathbb{C})$. We denote by

τ_k the trace on $C^*(\mathfrak{S}_k)$ defined by $\tau_k(f) = \frac{1}{k}\mathrm{Tr}_k(\alpha_k(f))$ for all $f \in C^*(\mathfrak{S}_k)$, where Tr_k is the unnormalized trace on $M_k(\mathbb{C})$.

Then, for $\sigma \in \mathfrak{S}_k$, we have

$$\tau_k(\sigma) = \frac{\mathrm{card}\,\{x|\sigma(x) = x\}}{k}.$$

Proposition 2.3 (*cf.* [10]). *Let $n \in \mathbb{N}^*$. The trace τ_k on $\mathfrak{S}_k$, satisfies Lück's condition.*

Proof. *We include the proof of* [10] *for reader's convenience.* Let Tr_{kn} be the unnormalized trace on $M_{kn}(\mathbb{Z})$. For $a \in M_n(\mathbb{Z}\mathfrak{S}_k)$ we have

$$\tau_k \otimes \mathrm{Tr}_n(a) = \frac{1}{k}\mathrm{Tr}_{kn}(\alpha_k(a)).$$

Now, $\Delta^+_{\mathrm{Tr}_{kn}}(\alpha_k(|a|))^2 = \exp(2k(\tau_k \otimes \mathrm{Tr}_n)(\ln_+(|a|)))$ is the product of non-zero eigenvalues of $\alpha_k(a^*a)$ (counted with multiplicity): it is the modulus of the non-zero coefficient of lowest degree of the characteristic polynomial of $\alpha_k(a^*a)$.

Since $\alpha_k(a^*a) \in M_{kn}(\mathbb{Z})$, it follows that its characteristic polynomial has integer coefficients so that $\Delta_+^{\mathrm{Tr}_{kn}}(a^*a) \in \mathbb{N}^*$. $\square$

Definition 2.4. We call permutational trace on a group G, a trace of the form $\tau_k \circ f$ where f is a group morphism from G to $\mathfrak{S}_k$.

It follows from 2.2 to 2.3 that permutational traces satisfy Lück's condition.

We will also use the following more general sets of traces.

Notation 2.5. *Let A be a unital C^*-algebra, $d \in \mathbb{N}$ and $a \in M_d(A)$. Let $s \in \mathbb{R}$. Denote by $\Lambda_{a,s} \subset \mathcal{T}_A$ the set of traces on A such that $(\tau \otimes \mathrm{Tr}_d)(\ln_+(|a|)) \geq s$.*

Proposition 2.6. 1. *Let A be a unital C^*-algebra, $d \in \mathbb{N}$ and $a \in M_d(A)$. Let $s \in \mathbb{R}$. The set $\Lambda_{a,s}$ is closed in $\mathcal{T}_A$ (for the pointwise topology on traces).*
2. *Let G be a group. The set Λ_G is closed in the set $\mathcal{T}_G$ of traces on $C^*(G)$ for the pointwise topology.*

Proof. 1. Since $\ln_+$ is upper semi-continuous, the map $\tau \mapsto (\tau \otimes \mathrm{Tr}_d)(\ln_+|a|)$ is upper semi-continuous by Proposition 1.1, therefore the set $\Lambda_{a,s}$ is closed.
2. The set Λ_G is the intersection over all $d \in \mathbb{N}$ and $a \in M_d(\mathbb{Z}G)$ of the closed subsets $\Lambda_{a,0}$. It is closed. $\square$

Note that the set $\Lambda_{a,s}$ only depends on the abelian C^*-subalgebra of $M_d(A)$ generated by a^*a and that the Proposition 2.6 can immediately be extended to all positive forms.

3. Sofic groups and traces

Denote by δ_k the Hamming distance on $\mathfrak{S}_k$ defined by

$$\delta_k(\sigma_1, \sigma_2) = \frac{1}{k}\mathrm{card}\,\{x \in \{1, \ldots, k\}\,|\sigma_1(x) \neq \sigma_2(x)\}$$

with $\sigma_1, \sigma_2 \in \mathfrak{S}_k$.

Remarks 3.1. 1. The distance δ_k is left and right-invariant, *i.e.* for $\alpha, \beta, \sigma_1, \sigma_2 \in \mathfrak{S}_{k_n}$, we have

$$\delta_k(\alpha \circ \sigma_1 \circ \beta, \alpha \circ \sigma_2 \circ \beta) = \delta_k(\sigma_1, \sigma_2).$$

2. For $\sigma \in \mathfrak{S}_k$, we have $\delta_k(\sigma, Id_k) = 1 - \tau_k(\sigma)$. In particular, τ_k is 1-lipschitz for δ_k.

Let $(k_n)_{n\in\mathbb{N}}$ be a sequence of integers. Put

$$\bigoplus_{n\in\mathbb{N}}^{c_0} \mathfrak{S}_{k_n} = \left\{ (\sigma_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathfrak{S}_{k_n} | \delta_{k_n}(\sigma_n, Id_{k_n}) \to 0 \right\} \subset \prod_{n\in\mathbb{N}} \mathfrak{S}_{k_n}.$$

By the invariance property of δ_k, it is a normal subgroup of $\prod_{n\in\mathbb{N}} \mathfrak{S}_{k_n}$.

Recall Gromov's definition of a sofic group (*cf.* [9]; see also [6,13] for a very nice presentation of sofic groups).

Definition 3.2. A countable group Γ is said to be *sofic* if there exists a sequence of maps $(f_n)_{n\in\mathbb{N}} : \Gamma \to \mathfrak{S}_{k_n}$ such that:

(a) For all $x, y \in \Gamma$, $\delta_{k_n}(f_n(x)f_n(y), f_n(xy)) \to 0$,
(b) For all $x \in \Gamma$, $\tau_{k_n}(f_n(x)) \to \tau_\Gamma(x)$.

Such a sequence $(f_n)_{n\in\mathbb{N}}$ is called a *sofic approximation* of Γ.

Remark 3.3. Let $q : \prod_{n\in\mathbb{N}} \mathfrak{S}_{k_n} \to \prod_{n\in\mathbb{N}} \mathfrak{S}_{k_n} / \bigoplus_{n\in\mathbb{N}}^{c_0} \mathfrak{S}_{k_n}$ be the quotient map. Property (a) of Definition 3.2, means that $g = q \circ (f_n)_{n\in\mathbb{N}} : \Gamma \to \prod_{n\in\mathbb{N}} \mathfrak{S}_{k_n} / \bigoplus_{n\in\mathbb{N}}^{c_0} \mathfrak{S}_{k_n}$ is a morphism.

In particular, if (a) is satisfied, $(f_n(1)) \in \bigoplus_{n\in\mathbb{N}}^{c_0} \mathfrak{S}_{k_n}$. Therefore conditions (a) and (b) are equivalent to (a) and

(b′) for all $x \in \Gamma$, $x \neq 1$, $\delta_{k_n}(f_n(x), Id) \to 1$.

We now prove the rather easy characterization of sofic groups in terms of traces:

Proposition 3.4. *Let Γ be a countable group. Denote by τ_Γ the canonical trace on $C^*(\Gamma)$. The following are equivalent:*

(i) *The group Γ is sofic.*
(ii) *There exist a group G and a surjective morphism $\varphi : G \to \Gamma$ such that $\tau_\Gamma \circ \varphi$ is in the closure of the permutational traces on G.*
(iii) *For every surjective morphism $\varphi : \mathbb{F}_\infty \to \Gamma$ (i.e. for every generating system of Γ), $\tau_\Gamma \circ \varphi$ is in the closure of permutational traces on $\mathbb{F}_\infty$.*

Proof. (iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) Assume that there is an onto morphism $\varphi : G \to \Gamma$ and a sequence $(\pi_n)_{n\in\mathbb{N}}$ of morphisms $\pi_n : G \to \mathfrak{S}_{k_n}$ such that for $y \in G$,

$$\tau_{k_n}(\pi_n(y)) \to \tau_\Gamma(\varphi(y)). \tag{3.1}$$

Let s be any set-theoretic section of φ and put $f_n = \pi_n \circ s$. We show that the sequence $f = (f_n)$ is a sofic approximation.

(b) Let $x \in \Gamma$. Applying (3.1) to $y = s(x)$, we get (b).
(a) The family $\pi = (\pi_n)$ determines a morphism $\pi : G \to \prod_{n\in\mathbb{N}} \mathfrak{S}_{k_n}$. For $y \in \ker\varphi$, since $\tau_\Gamma(\varphi(y)) = 1$, we find $\delta_{k_n}(1, \pi_n(y)) = 1 - \tau_{k_n}(\pi_n(y)) \to 0$; therefore $\pi(y) \in \bigoplus_{n\in\mathbb{N}}^{c_0} \mathfrak{S}_{k_n} = \ker q$. From the inclusion $\pi(\ker\varphi) \subset \ker q$, it follows that $q \circ f$ is a morphism, and (a) is satisfied.

(i) $\Rightarrow$ (iii) Let $f = (f_n)$ be a sofic approximation of Γ. Then $q \circ f : \Gamma \to \prod_{n\in\mathbb{N}} \mathfrak{S}_{k_n} / \bigoplus_{n\in\mathbb{N}}^{c_0} \mathfrak{S}_{k_n}$ is a morphism.

Let S be any generating subset of Γ.

Since $\mathbb{F}_S$ is free, the morphism $q\circ f\circ\varphi$ lifts to a morphism $\pi = (\pi_n) : \mathbb{F}_S \to \prod_{n\in\mathbb{N}} \mathfrak{S}_{k_n}$.

Now, for $y \in \mathbb{F}_S$, since $\pi(y)^{-1}(f \circ \varphi)(y) \in \ker q$, we find $\delta_{k_n}(\pi_n(y), f_n(\varphi(y))) \to 0$, whence (since τ_{k_n} is 1-lipschitz), $|\tau_{k_n}(\pi_n(y)) - \tau_{k_n}(f_n(\varphi(y)))| \to 0$. Property (b) of the sofic approximation f yields $\tau_{k_n}(\pi_n(y)) \to \tau_\Gamma \circ \varphi(y)$. $\square$

Theorem 3.5 (*cf.* [5]). *Every sofic group satisfies Lück's conjecture.*

Proof. Let Γ be a sofic group. Then, there exists a surjective morphism $\varphi : \mathbb{F}_\infty \to \Gamma$. By Proposition 3.4, the trace $\tau_\Gamma \circ \varphi$ is the limit of permutational traces on $C^*(\mathbb{F}_\infty)$.

By Proposition 2.3, the canonical trace on $\mathfrak{S}_n$ satisfies Lück's conjecture and the same is true for a permutational trace on $C^*(\mathbb{F}_\infty)$ by Proposition 2.2(i).

Then $\tau_\Gamma \circ \varphi \in \Lambda_{\mathbb{F}_\infty}$ because $\Lambda_{\mathbb{F}_\infty}$ is closed in the set of traces on $C^*(\mathbb{F}_\infty)$ by Proposition 2.6.

Since φ is surjective, we conclude by Proposition 2.2(ii), that τ_Γ satisfies Lück's conjecture. $\square$

Remark 3.6. A convenient way to state this result is to define the set of *sofic traces* on a group G as being the closure of permutational traces. Then

1. A group is sofic if and only if the trace it defines on $\mathbb{F}_\infty$ is sofic.
2. Every sofic trace satisfies Lück's condition.

On the other hand, not every trace on $\mathbb{F}_\infty$ satisfies Lück's condition. It becomes then a quite natural question to study the set of traces on $\mathbb{F}_\infty$ satisfying Lück's condition, the set of sofic traces, *etc.* Such a study is undertaken in [2].

Remark on hyperlinear groups and traces 3.7 (*See e.g. [14,13,3] For a Definition of Hyperlinear Groups*)**.** In the same way, we may define linear traces as the characters of finite dimensional representations and the set of *hyperlinear traces* as the closure of the set of linear traces. One then easily shows that a group is hyperlinear if and only if the trace it defines on $\mathbb{F}_\infty$ is hyperlinear.

One actually proves (see [2] for details):

Let Γ be a countable group. Denote by τ_Γ the canonical trace on $C^*(\Gamma)$. The following are equivalent:

(i) *The group Γ is hyperlinear.*
(ii) *There exist a group G and a surjective morphism $\varphi : G \to \Gamma$ such that the trace $\tau_\Gamma \circ \varphi$ is hyperlinear.*
(iii) *There exist a group G, a surjective morphism $\varphi : G \to \Gamma$ and a hyperlinear trace τ on G such that $\{g \in G;\ \tau(g) = 1\} = \ker\varphi$.*

(iv) *For every surjective morphism $\varphi : \mathbb{F}_\infty \to \Gamma$ (i.e. for every generating system of Γ), the trace $\tau_\Gamma \circ \varphi$ on $\mathbb{F}_\infty$ is hyperlinear.*

Moreover, Radulescu proved in [14] that a group is hyperlinear if and only if its group von Neumann algebra satisfies Connes embedding conjecture.

In fact, Connes' embedding conjecture is equivalent to the statement: "All the traces on $\mathbb{F}_\infty$ are hyperlinear". Indeed if M is a II_1 factor with trace τ acting on a separable Hilbert space, there is a morphism $\pi : C^*(\mathbb{F}_\infty) \to M$ with weakly dense range. Then M embeds in R^ω if and only if the trace $\tau \circ \pi$ is hyperlinear.

4. Relation with Atiyah's problem

Lück's conjecture implies a kind of stability of von Neumann dimension. This stability allows computing L^2-Betti numbers, and proving integrality in some cases [10,11,16], or actually disproving their rationality [8,1].

4.1. The method

Let A be a unital C^*-algebra, $k \in \mathbb{N}$ and $a \in A$. The characteristic function χ_0 of $\{0\}$ is upper semi-continuous. Given a trace τ on A, we put $\dim_\tau(\ker a) = \tau(\chi_0(|a|))$.

Note that if (h_n) is a decreasing sequence of continuous functions on $\mathbb{R}_+$ converging to the characteristic function of $\{0\}$, we have $\dim_\tau(\ker a) = \lim_n \tau(h_n(|a|))$.

Lück's method of handling Atiyah's problem can be understood in terms of traces via the following quite easy fact:

Proposition 4.1. *Let A be a unital C^*-algebra, $a \in A$ and $s \in \mathbb{R}$. The map $\tau \mapsto \dim_\tau(\ker a)$ is continuous on $\Lambda_{a,s}$.*

Proof. For $s' \in \mathbb{R}_+$, the set of traces $\Omega_{a,s'} = \{\tau;\ \tau(|a|) < s'\}$ is open. We only need to establish continuity on $\Lambda_{a,s} \cap \Omega_{a,s'}$. For $t \in \mathbb{R}_+$, put $\theta(t) = t - \ln_+(t)$. The function $\theta : \mathbb{R}_+ \to \mathbb{R}$ is continuous on $\mathbb{R}_+^*$, satisfies $\theta(t) > 0$ for $t \neq 0$ and $\lim_{t\to 0^+} \theta(t) = +\infty$. Moreover, for every $\tau \in \Lambda_{a,s} \cap \Omega_{a,s'}$, we have $\tau(\theta(|a|)) \leq s' - s$.

The proposition is an immediate consequence of the following Lemma. □

Lemma 4.2. *Let $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous on $\mathbb{R}_+^*$, and satisfy $\theta(t) > 0$ for $t \neq 0$ and $\lim_{t\to 0^+} \theta(t) = +\infty$. Let $h_n : \mathbb{R}_+ \to [0,1]$ be a decreasing sequence of continuous functions converging (pointwise) to χ_0. Let $m \in \mathbb{R}_+$ and denote by $\Lambda_{a,m,\theta}$ the set of traces τ on A such that $\tau(\theta(|a|)) \leq m$. Then the sequence $\Big(\tau(h_n(|a|))\Big)$ converges to $\dim_\tau(\ker a)$ uniformly on $\Lambda_{a,m,\theta}$.*

Proof. The functions v_n defined on $\mathbb{R}_+$ by $v_n(t) = \begin{cases} \frac{h_n(t)}{\theta(t)} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$ are continuous. The sequence (v_n) decreases to 0; by Dini's theorem it converges uniformly to 0 on $\mathrm{Sp}\,|a|$. For $\tau \in \Lambda_{a,m,\theta}$, we have $0 \leq \tau(h_n(|a|)) - \dim_\tau(\ker a) = \tau(v_n\theta(|a|)) \leq m\|v_n\|_\infty$. □

It follows that, if the group Γ is either a direct limit, or a subgroup of an inverse limit (*e.g.* a residually finite group), or a sofic group, then for any $a \in M_k(\mathbb{Z}[\Gamma])$ the von Neumann dimension of $\ker a$ can be computed as a limit of simpler terms.

4.2. Algebraic coefficients

We now see that Proposition 4.1 applies also for $a \in M_n(\mathbb{C}\Gamma)$ with algebraic coefficients (see [4]).

If A is a unital C^*-algebra, $a \in A$ and τ is a trace on A, we define the rank $\mathrm{rk}_\tau(a)$ of a to be the von Neumann dimension of the closure of the image of a; we of course have $\mathrm{rk}_\tau(a) = \mathrm{codim}_\tau(\ker a) = \tau(1) - \dim_\tau(\ker a)$. We will use the following Lemma.

Lemma 4.3. *Let A be a unital C^*-algebra, $a, b, c \in A$ with a and c invertible. Then for any trace τ on A we have $\tau(\ln_+ |abc|) \le \ln(\|a\|\|c\|)\mathrm{rk}_\tau(b) + \tau(\ln_+ |b|)$.*

Proof. We first prove this inequality when $c = 1$: we prove, for invertible a,

$$\tau(\ln_+ |ab|) \le \ln(\|a\|)\mathrm{rk}_\tau(b) + \tau(\ln_+ |b|). \tag{$**$}$$

We may replace A by $\pi_\tau(A)''$ and thus assume A is a von Neumann algebra and τ a faithful normal trace. Let $b = u|b|$ be the polar decomposition of b. Let $p = u^*u$ be the domain projection of b and $A_p = pAp$. Let τ_p be the restriction of τ to A_p.

Now $|b|$ and $|au|$ are injective elements of A_p so that the Fuglede–Kadison determinant Δ_{τ_p} and Lück's modified determinant $\Delta^+_{\tau_p}$ coincide on them. We therefore have $\tau(\ln_+(|au||b|)) = \tau(\ln_+(|au|)) + \tau(\ln_+ |b|)$. Finally, $\ln_+(|au|) \le \ln \|a\| p$, and $\tau(p) = \mathrm{rk}_\tau(b)$.

To end, since $\tau(\ln_+ |x|) = \tau(\ln_+ |x^*|)$ we find (using $(**)$)

$$\tau(\ln_+ |bc|) \le \ln(\|c\|)\mathrm{rk}_\tau(b^*) + \tau(\ln_+ |b|).$$

Replacing b by bc in $(**)$ and noting that $\mathrm{rk}_\tau(b) = \mathrm{rk}_\tau(bc)$, we find:

$$\tau(\ln_+ |abc|) \le \ln(\|a\|)\mathrm{rk}_\tau(bc) + \tau(\ln_+ |bc|) \le \ln(\|a\|\|c\|)\mathrm{rk}_\tau(b) + \tau(\ln_+ |b|). \qquad \square$$

Proposition 4.4. *Let $a \in M_n(\overline{\mathbb{Q}}\mathbb{F}_\infty)$.*

(a) *There exists a constant m such that for any tracial state $\tau \in \Lambda_{\mathbb{F}_\infty}$ (i.e. a tracial state on $C^*(\mathbb{F}_\infty)$ satisfying Lück's condition) we have $(\tau \otimes \mathrm{Tr}_n)(\ln_+ |a|) \ge m$.*
(b) *The map $\tau \mapsto \dim_\tau(\ker a)$ is continuous on $\Lambda_{\mathbb{F}_\infty}$.*

Proof. Note that (b) is an immediate consequence of (a) and Proposition 4.1.

We prove (a). Let K be a finite extension of $\mathbb{Q}$ containing the coefficients of a, *i.e.* such that $a \in M_n(K\mathbb{F}_\infty)$.

Choosing a $\mathbb{Q}$ basis of K, we obtain an embedding $i : K \to M_d(\mathbb{Q})$ (where d is the dimension of K over $\mathbb{Q}$).

Giving all the embeddings of K to $\mathbb{C}$, we obtain an embedding $j = (j_1, \ldots, j_d) : K \to \mathbb{C}^d$. We will assume that the given embedding $K \subset \mathbb{C}$ is j_1.

These two embeddings are conjugate in $M_d(\mathbb{C})$: There exists an invertible matrix $c \in M_d(\mathbb{C})$ such that, for $x \in K$, the matrix $c\,i(x)c^{-1}$ is the diagonal matrix with coefficients $j_\ell(x)$.

Write then $i(a) = k^{-1}b$ where $b \in M_{dn}(\mathbb{Z}\mathbb{F}_\infty)$ and $k \in \mathbb{N}^*$.

For every $\tau \in \Lambda_{\mathbb{F}_\infty}$, we have

- $\tau \otimes Tr_{dn}(\ln_+ |b|) \ge 0$ (by definition of $\Lambda_{\mathbb{F}_\infty}$);
- it follows that $\tau \otimes Tr_{dn}(\ln_+ |i(a)|) \ge -nd \ln k$;

- using Lemma 4.3, we find $\sum_{\ell=1}^{d} \tau \otimes Tr_n(\ln_+ |j_\ell(a)|) \geq -nd \ln k - nd \ln(\|c\| \|c^{-1}\|)$.
- On the other hand for all ℓ, we have $\tau \otimes Tr_n(\ln_+ |j_\ell(a)|) \leq n \max(0, \ln \| j_\ell(a)\|)$;
- we find

$$\begin{aligned} \tau \otimes Tr_{dn}(\ln_+ |a|) &= \tau \otimes Tr_n(\ln_+ |j_1(a)|) \\ &\geq -nd \ln k - nd \ln(k\|c\| \|c^{-1}\|) \\ &\quad - n \sum_{\ell=2}^{d} \max(0, \ln \| j_\ell(a)\|). \quad \square \end{aligned}$$

Generalizing a result of [4], we find:

Proposition 4.5. *Let $a \in M_n(\overline{\mathbb{Q}}\mathbb{F}_\infty)$. For any sofic trace of $C^*(\mathbb{F}_\infty)$, the dimension $\dim_\tau(\ker a)$ does not depend on the embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$.*

Proof. Denote by Σ_a the set of tracial states on $C^*(\mathbb{F}_\infty)$ satisfying Lück's property and for which $\dim_\tau(\ker j(a))$ does not depend on the embedding $j : \overline{\mathbb{Q}} \to \mathbb{C}$. We wish to prove that every sofic trace is in Σ_a.

It follows from Proposition 4.4 (applied to a and $j(a)$ where j is another inclusion of $\overline{\mathbb{Q}}$ in $\mathbb{C}$) that the set Σ_a is closed in $\Lambda_{\mathbb{F}_\infty}$ and therefore in $\mathcal{T}_{\mathbb{F}_\infty}$.

Let $q : \mathbb{F}_\infty \to \mathfrak{S}_k$ be a morphism and τ_q the corresponding trace on $C^*(\mathbb{F}_\infty)$. Denote still by q the corresponding morphism $q : M_n(\overline{\mathbb{Q}}\mathbb{F}_\infty) \to M_{kn}(\overline{\mathbb{Q}})$. For any embedding $j : \overline{\mathbb{Q}} \to \mathbb{C}$, we have $j \circ q = q \circ j : M_n(\overline{\mathbb{Q}}\mathbb{F}_\infty) \to M_{kn}(\mathbb{C})$, so that we have

$$\dim_{\tau_q}(\ker j(a)) = \frac{1}{k} \dim_{\mathbb{C}} j \circ q(a) = \frac{1}{k} \dim_{\overline{\mathbb{Q}}} q(a).$$

It is independent of j.

In other words, Σ_a is closed and contains all permutational traces. Therefore Σ_a contains the closure of permutational traces: the sofic traces. $\square$

We immediately find the following:

Corollary 4.6. *Let Γ be a sofic group and $a \in M_n(\overline{\mathbb{Q}}\Gamma)$, then the von Neumann dimension (with respect to the group trace of Γ) of* ker *a does not depend on the embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$.* $\square$

A particular case of this corollary is proved by A. Thom [18, theorem 4.3.(ii)]. Note that Thom's proof actually establishes this result.

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