Numerical Study

0.1 Kuramoto-Shinomoto-Sakaguchi MV-SDE

Let us first write the MV-SDE of the model we are going to analyse, namely the Kuramoto-Shinomoto-Sakaguchi model:

$$dX_t = (\mathbb{E}[\sin(X_t)]\cos(X_t) - \mathbb{E}[\cos(X_t)]\sin(X_t))\,dt + \sigma dW_t, \quad X_0 = x_0,$$

where, in our case, $\sigma = 0.5$ and $x_0 = 0.5$. From this differential equation it is easily seen that:

- K = 3, d = 1 e q = 1,
- $\varphi(x) = (1, \sin x, \cos x)$,
- $\alpha(t, x) = (0, \cos x, -\sin x)^T$,
- $\beta(t, x) = (\sigma, 0, 0)^T$.

In our numerical study, the results and the observations are obtained from the work of two macro functions, both aimed at the approximate calculation of the values of $\mathbb{E}[\sin(X_t)]$ and $\mathbb{E}[\cos(X_t)]$.

- The first one, which we use to obtain the solution that will play the role of benchmark, is the well-known Euler - Monte Carlo method. It consists in approximating the two expected values with a direct average of $M_1 = 10^6$ simulations of processes $(X_t)_{t=0,\cdots T}$ obtained by dividing the interval $[0, T]$ into N_1 time steps and applying Euler's method to each of them. Specifically, Euler's Method calculates the $N + 1$ values of a discretized process by obtaining, at each step, the value at time t from the value of the process at the previous time and the value of the time step h, according to the formula:

  $$X_{t+1} = X_t + \text{drift}(t) \cdot h + \text{diffusion}(t) \cdot \sqrt{h} \cdot W,$$

  where W is the sample of a Standard Normal. In our case the time steps are all equivalent since we divide the interval $[0, T]$ into equispaced points. We observe that the function is structured in such a way

that it can also be applied to McKean-Vlasov SDEs, for which Euler's method at step $t+1$ requires that the expected values sought, at time t, are known within the drift and diffusion parts. To obtain these we again use an average over the M_1 simulations of the realisations of the processes at time t.

- The second one is the Stochastic Gradient Descent method. This returns us as output the polynomials $(\mathcal{L}(a))_1(t)$ and $(\mathcal{L}(a))_2(t)$ which will be the approximations of $\mathbb{E}[\sin(X_t)]$ and $\mathbb{E}[\cos(X_t)]$ respectively. As for the choice of the space of polynomials, the two outputs are calculated in the basis of orthogonal Lagrange polynomials. In the test we will change the dimension n of the space of polynomial. Each element of the basis is of the form:

$$g_i(t) := \prod_{j\leq n \text{ and } j\neq n} \left(\frac{t-t_j}{t_i-t_j}\right), \text{ with Chebyshev knots } \frac{T}{2}+\frac{T}{2}\cos\left(\frac{2k+1}{2n+2}\pi\right).$$

Again, we will use to the N Euler steps to find the solution of the SDEs. This time, Euler's method for simulating $Z(\xi, W)$ and $\left(Z^a(\tilde{\xi},\tilde{W}), \partial_{a_{h,j}} Z^a(\tilde{\xi},\tilde{W})\right)$ will have to carry out 4 processes simultaneously: $(X_t)_{t=0,\cdots T}$ and $(Z_t)_{t=0,\cdots T}$ one-dimensional and the two $(Y_t)^{(1,2)}_{t=0,\cdots T}$ $n+1$ dimensional. Furthermore, it will be necessary to use at each step the value obtained for the process X in order to calculate the two $Y^{(1,2)}$. We note that X and Z implement the Euler step in the same way as the previous function, but with two different samples of the Brownian. In this simplified algorithm the maps $\mathbf{h}$ and H are taken as the identity and the null function, respectively. By taking the values of the coefficient functions for the MV-SDE relative to the Kuramoto-Shninomoto-Sakaguchi model, we obtain that specifically the equations become:

$$dZ_t = ((\mathcal{L}a)_1(t)\cos(Z_t) - (\mathcal{L}a)_2(t)\sin(Z_t))\,dt + \sigma dW_t,$$

$$dY^{j,1}_t = \left(g_j(t)\cos(Z_t) - Y^{j,1}_t\left((\mathcal{L}a)_1(t)\sin(Z_t) + (\mathcal{L}a)_2(t)\cos(Z_t)\right)\right)dt,$$

$$dY^{j,2}_t = \left(-g_j(t)\sin(Z_t) - Y^{j,2}_t\left((\mathcal{L}a)_1(t)\sin(Z_t) + (\mathcal{L}a)_2(t)\cos(Z_t)\right)\right)dt,$$

with $Z_0 = X_0$, $Y^{j,1}_0 = 0$ and $Y^{j,2}_0 = 0$ and for $j = 0, \cdots, n$. These processes are necessary to calculate the realisation of the gradient for the Stochastic Descent, i.e. the random variable v. Having divided time into N time steps and approximated the integral with a summation

the writing of v, component by component, is as follows:

$$v_{j,1}(a;W;\tilde{W}) = \\ 2h\sum_{t=0}^{N}\Big[(\sin(Z_t^a(W)) - (\mathcal{L}a)_1(t))\cdot\Big(\cos(Z_t^a(\tilde{W}))Y_t^{a;j,1}(\tilde{W}) - g_j(t)\Big) \\ + (\cos(Z_t^a(W)) - (\mathcal{L}a)_2(t))\cdot\Big(-\sin(Z_t^a(\tilde{W}))Y_t^{a;j,1}(\tilde{W})\Big)\Big],$$

$$v_{j,2}(a;W;\tilde{W}) = \\ 2h\sum_{t=0}^{N}\Big[(\sin(Z_t^a(W)) - (\mathcal{L}a)_1(t))\cdot\Big(\cos(Z_t^a(\tilde{W}))Y_t^{a;j,2}(\tilde{W})\Big) \\ + (\cos(Z_t^a(W)) - (\mathcal{L}a)_2(t))\cdot\Big(-\sin(Z_t^a(\tilde{W}))Y_t^{a;j,2}(\tilde{W}) - g_j(t)\Big)\Big],$$

with $j = 0,\cdots,n$. We conclude highlighting that, before returning the value v, this function averages M realisations obtained corresponding to as many independent simulations of Brownian motions. If this parameter is 1, the method is a classical SGD method, but if taken to ∞ it leads to a GD method, i.e. deterministic descent. This strategy, called Mini Batch, is the core of the numerical analyses we have done. Finally, regarding the choice of *learing rates* necessary for the calculation of v at each iteration, we choose $\eta_m = \dfrac{r_0}{(m+1)^\rho}$, where the factors $r_0 \in (0,+\infty)$ and $\dfrac{1}{2} < \rho \leq 1$ will change in the tests.

0.2 Tables and Graphs

The aim of our analysis is to find the number of iterations required for the Gradient Descent method to converge. We therefore explicate the stopping criterion for the iteration of a_m: fixed $\gamma_{1,bench}$ and $\gamma_{2,bench}$ obtained from the first function, the iterations stops when:

$$\frac{\|\sum_{i=0}^{n}(a_1)_i g_i - \gamma_{1,bench}\|_{L_2}}{\|\gamma_{1,bench}\|_{L_2}} < 1\% \quad \text{and} \quad \frac{\|\sum_{i=0}^{n}(a_2)_i g_i - \gamma_{2,bench}\|_{L_2}}{\|\gamma_{2,bench}\|_{L_2}} < 1\%.$$

Therefore, the second function, described in the previous section, will produce as output a solution with a relative error of 1%, in norm L_2, with respect to the benchmark solution. In order to make these results as general and correct as possible, we repeat the same test by varying: final instant $T = 0.5, 1, 2, 4$; dimension of the space of polynomials $n = 3, 4, 5, 6$; and values of the parameters $\rho = 0.6, 0.7, 0.8, 0.9$ and $r_0 = 1, 5, 10$ of the *learning rates*. We specify that, in order to obtain the same error order in Euler's

method, we subdivide the interval $[0, T]$ into $N_1 = 50, 100, 200, 400$ steps, respectively at the values of T just listed. In particular, we repeat for each case the same test 10 times and show: the average convergence times, a table that for each combination of ρ and r_0 shows the minimum, maximum and average number of convergence iterations and the graphs of the approximate solutions. The graphs we will show are those with the values of ρ and r_0 which in the tables have the smallest number of average iterations, for each combination of the parameters T, n and M. This is done by varying the value of the Mini Batch $M = 1, 10, 10^2, 10^3, 10^4$ and the degree of the polynomials n. We group these tables and graphs into sections according to the value of T.

N.B. 0.2.1. *When values such as 'overflow' or* 49999 *appear in the tables, they respectively mean that an overflow occurred during the execution of the programme, i.e. the limit of the value storage capacity was reached (i.e. too large a number); while the second means that the algorithm did not reach convergence within the* 50000 *iterations imposed as a threshold.*

N.B. 0.2.2. *For the first three values of* M*, i.e.* $M = 1, 10, 100$*, we check the solution with the benchmark every ten iterations in order to keep execution times down. While for the last two cases, i.e.* $M = 1000, 10000$*, this check takes place every step.*

N.B. 0.2.3. *We specify that the algorithms were written in the Python programming language, specifically in the Jupyter Notebook web application for creating and sharing computational documents. With regard to the algorithm execution times, I write below the specifications of the machine where I ran the tests:*

Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz 1.80 GHz
Installed RAM 8.00 GB (7.90 GB usable)
System Type 64-bit operating system, x64-based processor

0.3 T = 0.5

Case n = 3

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.95	1.48	4.34
$\rho = 0.7$	2.17	1.5	1.80
$\rho = 0.8$	2.82	1.23	1.96
$\rho = 0.9$	19.22	1.14	1.35

Tabella 1: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	80	490	278	170	720	438	630	2500	1281
$\rho = 0.7$	30	2380	639	140	730	442	80	1050	532
$\rho = 0.8$	40	2070	831	100	1240	362	210	1170	581
$\rho = 0.9$	400	20930	5668	160	600	338	150	1020	403

Tabella 2: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.41	0.32	0.74
$\rho = 0.7$	0.48	0.38	0.44
$\rho = 0.8$	1.25	0.34	0.40
$\rho = 0.9$	2.46	0.35	0.41

Tabella 3: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	20	230	105	10	150	82	60	510	192
$\rho = 0.7$	30	400	124	30	190	99	30	270	113
$\rho = 0.8$	30	1160	322	30	180	88	60	170	104
$\rho = 0.9$	20	2660	635	20	220	89	30	290	106

Tabella 4: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.11	0.14	0.19
$\rho = 0.7$	0.18	0.12	0.17
$\rho = 0.8$	0.36	0.15	0.17
$\rho = 0.9$	0.53	0.14	0.12

Tabella 5: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	10	30	17	10	50	23	10	80	30
$\rho = 0.7$	10	60	29	10	40	19	10	70	27
$\rho = 0.8$	20	270	58	10	70	24	10	50	27
$\rho = 0.9$	20	470	86	10	50	22	10	40	20

Tabella 6: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.34	0.09	0.26
$\rho = 0.7$	0.42	0.09	0.20
$\rho = 0.8$	0.52	0.10	0.18
$\rho = 0.9$	0.84	0.09	0.16

Tabella 7: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	7	13	9.1	2	4	2.5	6	11	7.1
$\rho = 0.7$	9	15	11.4	2	5	2.4	5	8	5.6
$\rho = 0.8$	11	27	14.1	2	6	2.7	4	7	5
$\rho = 0.9$	14	35	23	2	5	2.4	4	6	4.4

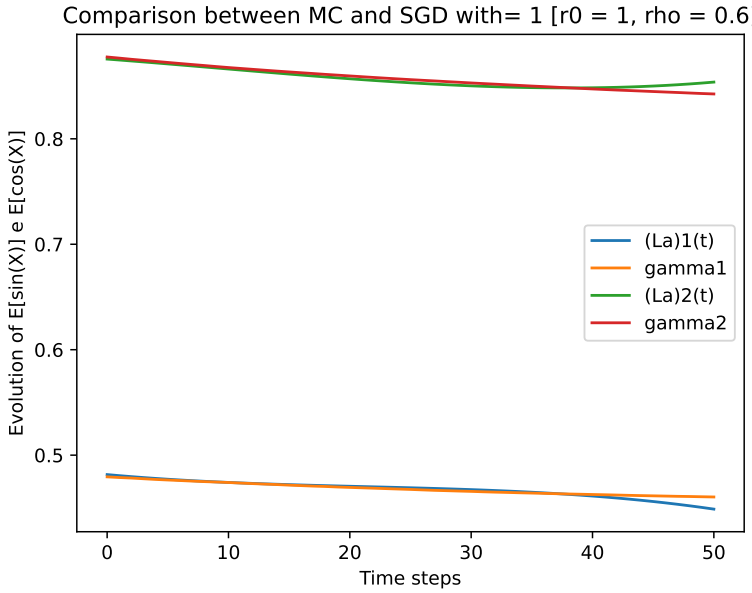
Tabella 8: Number of iterations m to achieve convergence with $M = 1000$

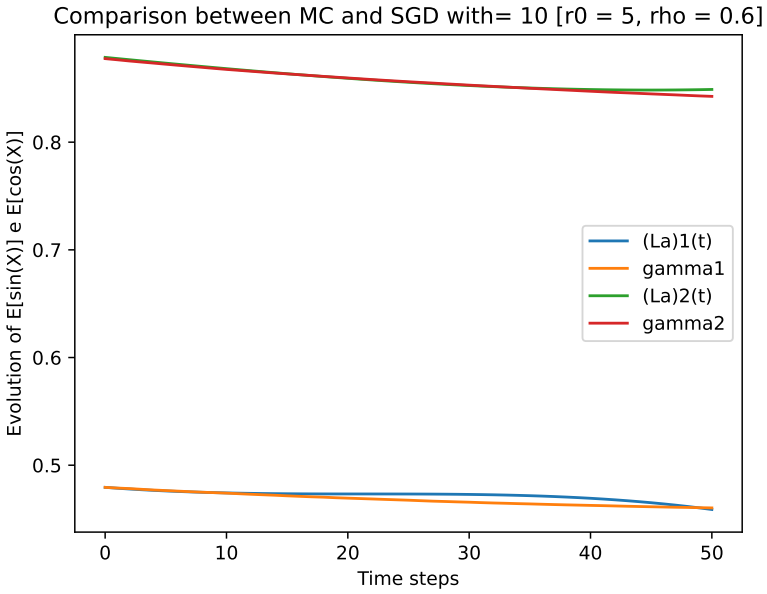
	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	3.48	0.90	2.69
$\rho = 0.7$	4.23	0.90	2.23
$\rho = 0.8$	5.14	0.90	1.79
$\rho = 0.9$	7.21	0.94	1.79

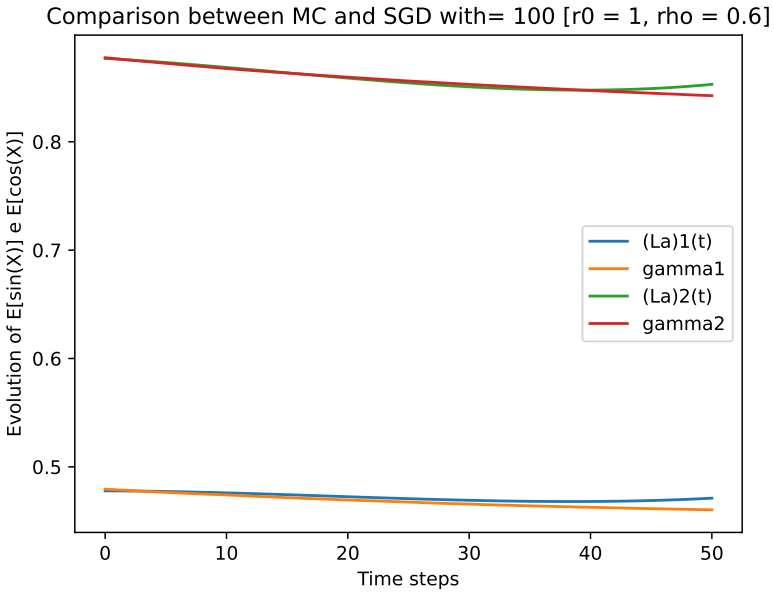
Tabella 9: Average execution times (in seconds s) with $M = 10000$

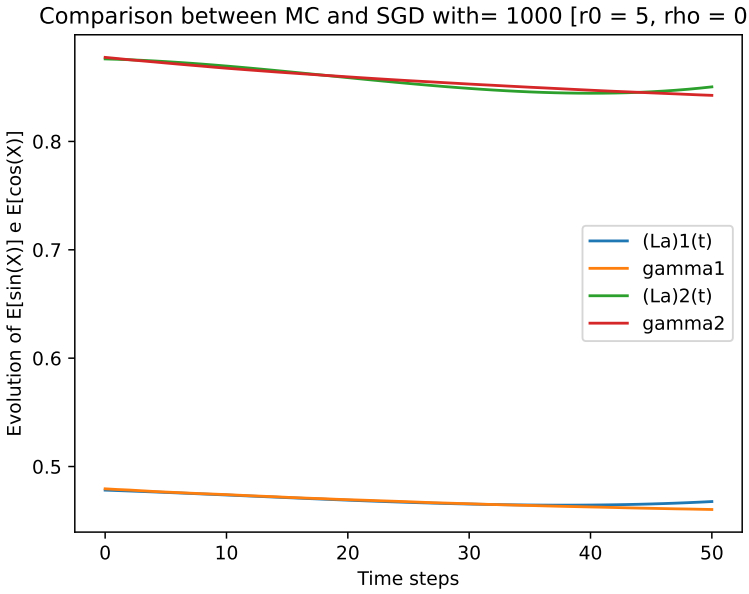
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	7	8	7.7	2	2	2	6	6	6
$\rho = 0.7$	9	11	9.4	2	2	2	5	5	5
$\rho = 0.8$	11	13	11.5	2	2	2	4	4	4
$\rho = 0.9$	15	18	16.1	2	3	2.1	4	4	4

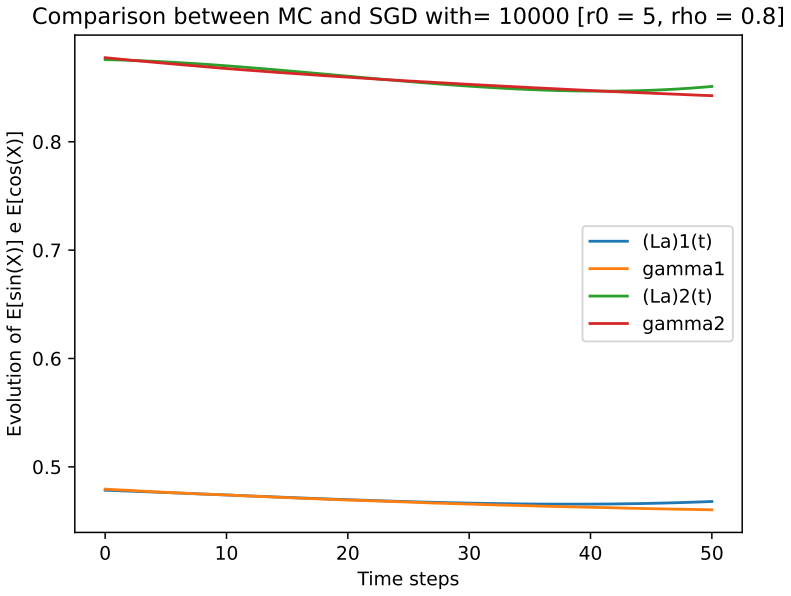
Tabella 10: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 5, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
10
20
30
40
50
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 100 [r0 = 1, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
10
20
30
40
50
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 1000 [r0 = 5, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
10
20
30
40
50
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 10000 [r0 = 5, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
10
20
30
40
50
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2

Case n = 4

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	1.72	2.90	4.55
$\rho = 0.7$	1.45	1.78	2.92
$\rho = 0.8$	15.90	2.54	2.16
$\rho = 0.9$	56.10	1.97	1.70

Tabella 11: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	60	2080	484	270	1460	819	950	1680	1288
$\rho = 0.7$	120	1350	412	150	850	505	350	1620	828
$\rho = 0.8$	120	22300	4501	250	1480	720	230	1510	609
$\rho = 0.9$	190	49999	15893.9	60	1520	560	210	1180	481

Tabella 12: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.26	0.39	0.77
$\rho = 0.7$	0.57	0.33	0.57
$\rho = 0.8$	0.42	0.43	0.34
$\rho = 0.9$	6.07	0.22	0.31

Tabella 13: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	30	150	64	40	160	95	70	310	188
$\rho = 0.7$	30	450	139	20	150	80	40	220	140
$\rho = 0.8$	30	210	104	30	230	104	20	200	84
$\rho = 0.9$	20	6840	1486	10	120	54	30	180	76

Tabella 14: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.16	0.12	0.23
$\rho = 0.7$	0.21	0.13	0.26
$\rho = 0.8$	0.32	0.12	0.15
$\rho = 0.9$	0.55	0.13	0.17

Tabella 15: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	20	40	24	10	30	18	10	80	34
$\rho = 0.7$	20	90	31	10	50	19	20	100	38
$\rho = 0.8$	20	100	48	10	30	17	10	30	21
$\rho = 0.9$	40	180	82	10	30	20	10	40	25

Tabella 16: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.56	0.13	0.2
$\rho = 0.7$	0.66	0.13	0.20
$\rho = 0.8$	0.99	0.14	0.16
$\rho = 0.9$	1.53	0.15	0.14

Tabella 17: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	10	19	13.1	2	7	2.9	4	6	4.6
$\rho = 0.7$	13	20	15.4	2	5	2.9	4	6	4.6
$\rho = 0.8$	19	37	23.3	2	11	3.2	3	4	3.8
$\rho = 0.9$	30	44	35.9	2	16	3.6	3	4	3.3

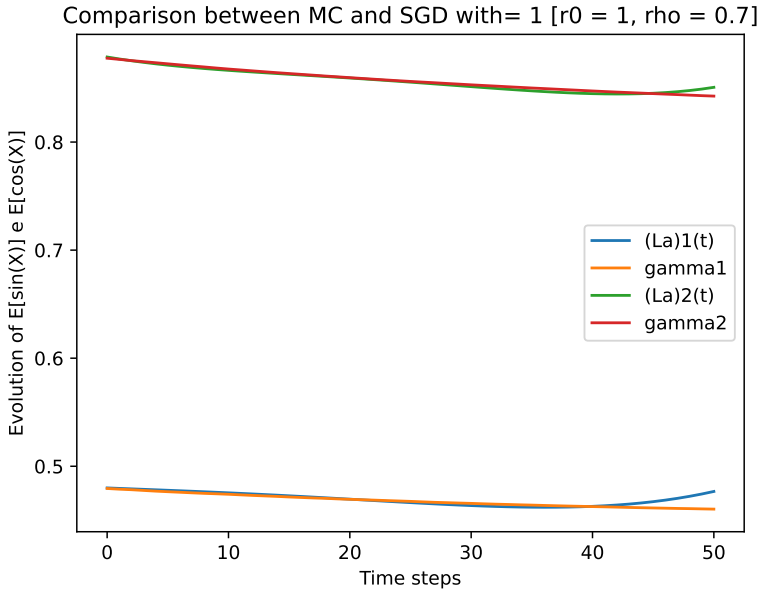
Tabella 18: Number of iterations m to achieve convergence with $M = 1000$

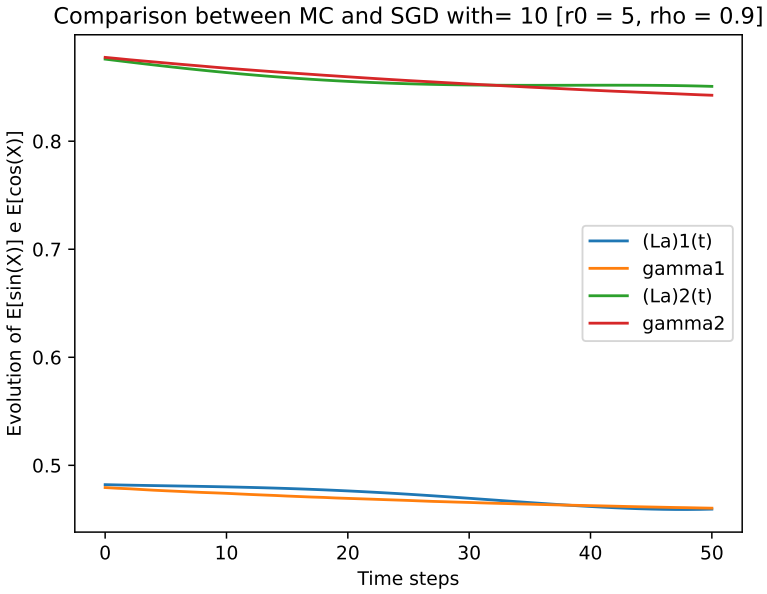
	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	6.26	1.07	2.12
$\rho = 0.7$	7.68	1.07	2.14
$\rho = 0.8$	10.63	1.06	1.71
$\rho = 0.9$	17.56	1.06	1.66

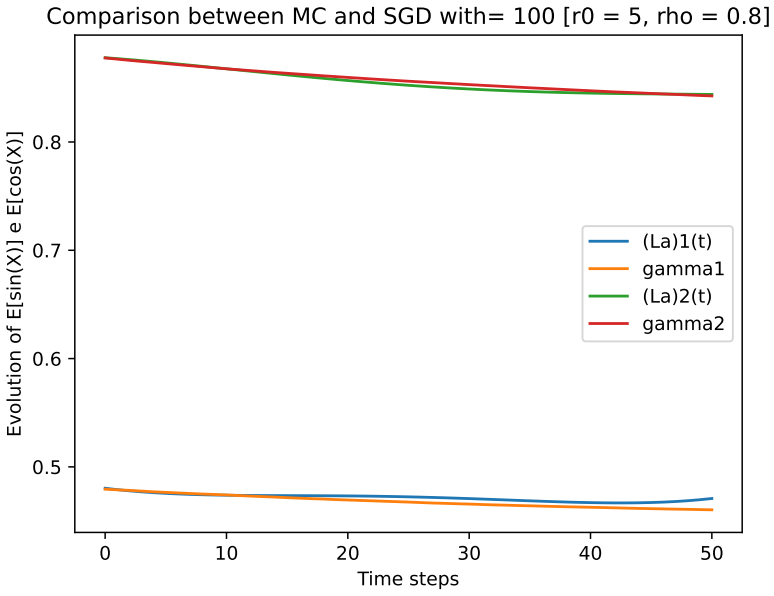
Tabella 19: Average execution times (in seconds s) with $M = 10000$

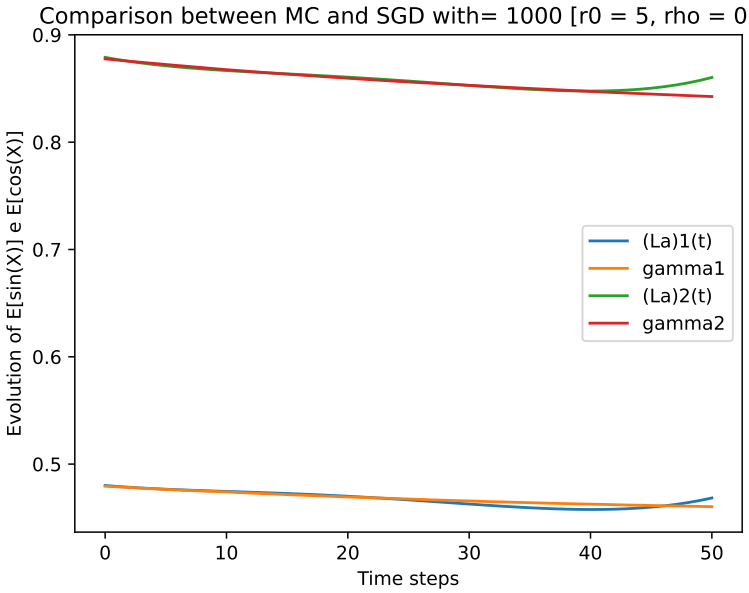
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	11	12	11.6	2	2	2	4	4	4
$\rho = 0.7$	14	15	14.3	2	2	2	4	4	4
$\rho = 0.8$	18	21	19.7	2	2	2	3	4	3.2
$\rho = 0.9$	29	35	32.7	2	2	2	3	4	3.1

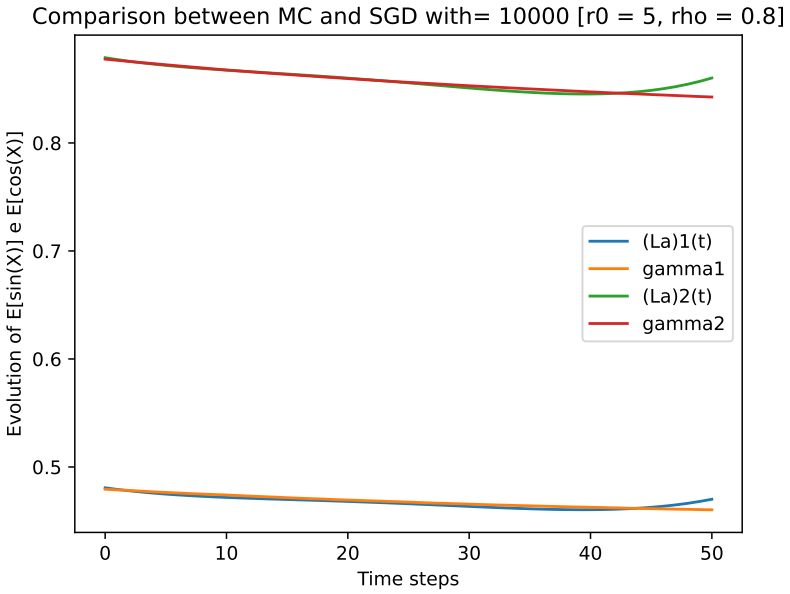
Tabella 20: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 5, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
0.5
0.6
0.7
0.8
0
10
20
30
40
50
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 100 [r0 = 5, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
0.5
0.6
0.7
0.8
0
10
20
30
40
50
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 1000 [r0 = 5, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
10
20
30
40
50


Comparison between MC and SGD with= 10000 [r0 = 5, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.8
0.7
0.6
0.5
0
10
20
30
40
50

Case n = 5

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	1.36	4.12	8.9
$\rho = 0.7$	2.67	2.78	3.51
$\rho = 0.8$	4.66	2.35	2.92
$\rho = 0.9$	90.29	3.76	2.65

Tabella 21: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	130	890	371	370	1760	866	610	2870	1714
$\rho = 0.7$	110	2470	729	240	1070	589	420	1050	753
$\rho = 0.8$	90	4260	1271	130	1190	496	240	1100	613
$\rho = 0.9$	440	49999	22338.7	200	1330	661	160	1380	565

Tabella 22: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.85	0.64	1.25
$\rho = 0.7$	0.88	0.53	0.73
$\rho = 0.8$	2.96	0.61	0.37
$\rho = 0.9$	2.35	0.58	0.53

Tabella 23: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	20	400	148	30	290	110	80	340	217
$\rho = 0.7$	20	630	157	20	190	89	40	250	132
$\rho = 0.8$	60	2110	513	20	280	104	30	100	68
$\rho = 0.9$	110	1210	411	30	250	103	20	210	93

Tabella 24: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.29	0.28	0.25
$\rho = 0.7$	0.48	0.25	0.40
$\rho = 0.8$	0.98	0.18	0.38
$\rho = 0.9$	1.31	0.29	0.19

Tabella 25: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	20	40	27	10	60	21	10	40	22
$\rho = 0.7$	30	80	49	10	60	19	10	60	33
$\rho = 0.8$	40	180	85	10	30	16	10	50	30
$\rho = 0.9$	70	180	114	10	60	24	10	40	16

Tabella 26: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	1.41	0.22	0.30
$\rho = 0.7$	1.73	0.17	0.26
$\rho = 0.8$	2.85	0.15	0.23
$\rho = 0.9$	5.33	0.19	0.28

Tabella 27: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	15	25	18.6	1	8	2.6	3	6	3.8
$\rho = 0.7$	20	24	22.5	1	7	2.2	3	4	3.3
$\rho = 0.8$	32	45	37.5	1	4	1.9	3	4	3.2
$\rho = 0.9$	53	104	70.3	1	6	2.6	3	6	3.5

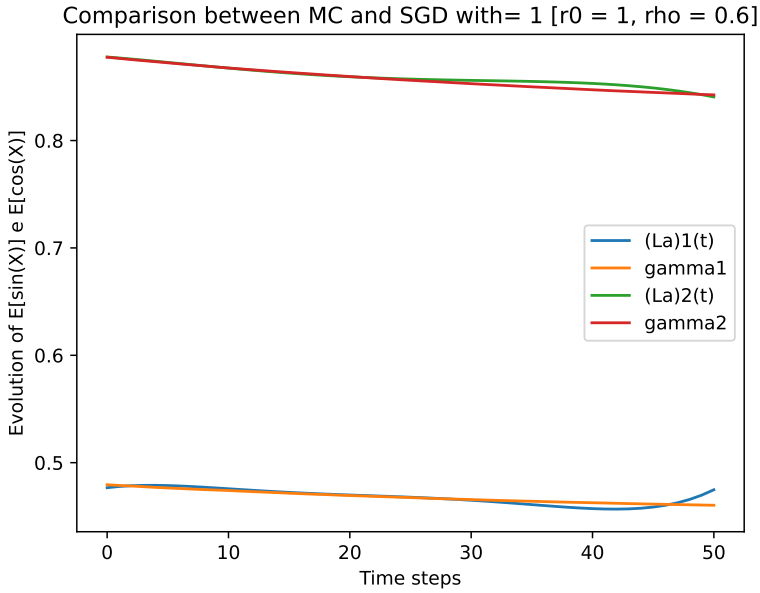
Tabella 28: Number of iterations m to achieve convergence with $M = 1000$

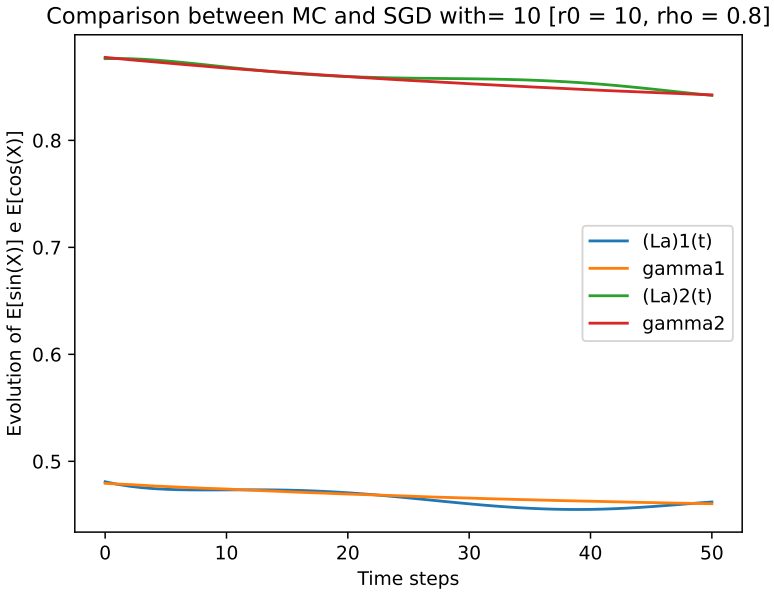
	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	16.95	1.42	3.38
$\rho = 0.7$	23.12	1.48	3.20
$\rho = 0.8$	36.69	1.3	3.22
$\rho = 0.9$	68.4	1.13	3.13

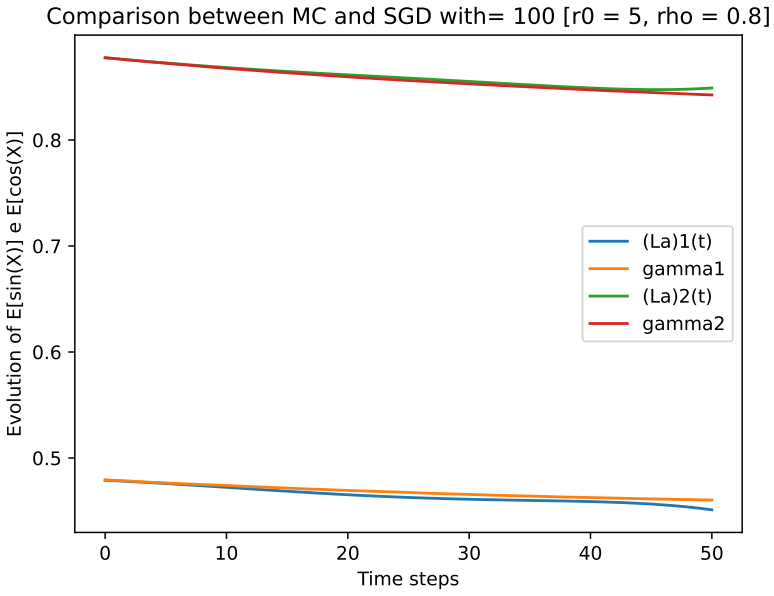
Tabella 29: Average execution times (in seconds s) with $M = 10000$

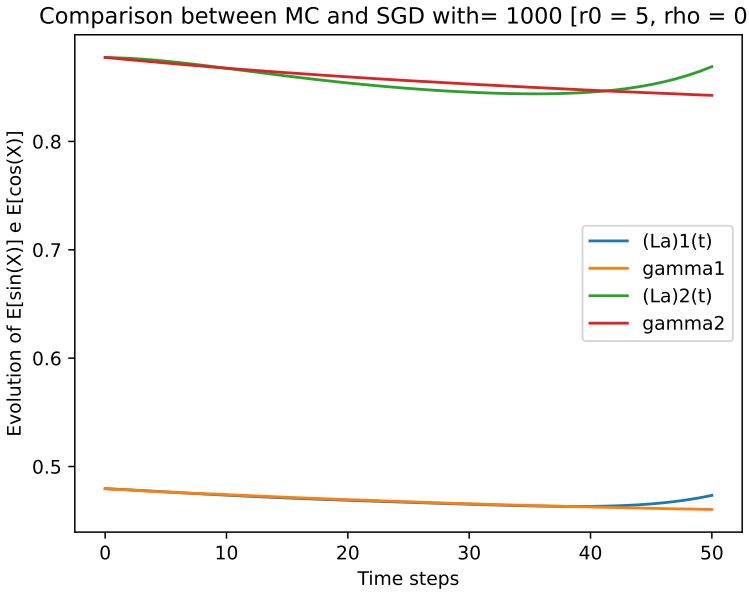
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	15	16	15.7	1	2	1.3	3	4	3.1
$\rho = 0.7$	21	23	21.5	1	2	1.4	3	3	3
$\rho = 0.8$	32	36	34.4	1	2	1.2	3	3	3
$\rho = 0.9$	59	66	63.4	1	1	1	2	3	2.9

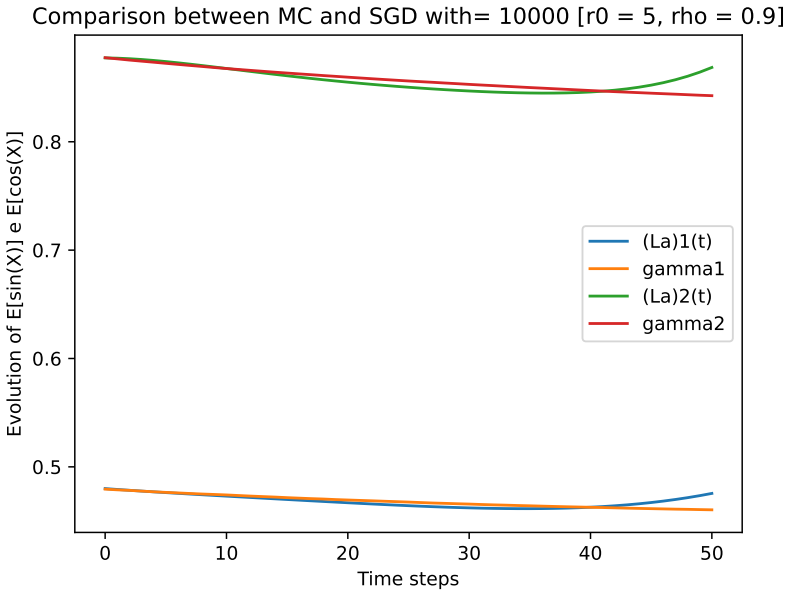
Tabella 30: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 10, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
10
20
30
40
50
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 100 [r0 = 5, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
10
20
30
40
50
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 1000 [r0 = 5, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
10
20
30
40
50
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 10000 [r0 = 5, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
10
20
30
40
50
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2

Case n = 6

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	5.18	4.60	8.74
$\rho = 0.7$	4.49	4.45	5.47
$\rho = 0.8$	19.83	5.0	4.59
$\rho = 0.9$	147.74	9.4	4.57

Tabella 31: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	130	2650	864	260	1430	791	410	2140	1250
$\rho = 0.7$	80	2080	731	210	1060	782	260	1370	766
$\rho = 0.8$	100	8230	3184	200	1550	809	260	1720	663
$\rho = 0.9$	180	49999	25371.6	220	6350	1360	190	2440	765

Tabella 32: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.68	0.33	0.74
$\rho = 0.7$	0.56	0.44	0.49
$\rho = 0.8$	2.31	0.53	0.51
$\rho = 0.9$	24.46	0.48	0.39

Tabella 33: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	30	240	89	20	180	75	30	380	167
$\rho = 0.7$	40	160	77	40	210	99	40	160	110
$\rho = 0.8$	50	1080	329	40	300	119	40	290	116
$\rho = 0.9$	120	21270	4210	10	240	108	30	220	89

Tabella 34: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.32	0.19	0.19
$\rho = 0.7$	0.64	0.13	0.25
$\rho = 0.8$	0.64	0.14	0.18
$\rho = 0.9$	1.54	0.16	0.17

Tabella 35: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	30	130	41	10	40	24	10	50	24
$\rho = 0.7$	40	290	81	10	30	16	10	60	31
$\rho = 0.8$	40	120	81	10	60	18	10	40	23
$\rho = 0.9$	110	300	196	10	60	20	10	40	22

Tabella 36: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	1.28	0.13	0.20
$\rho = 0.7$	1.87	0.12	0.19
$\rho = 0.8$	2.90	0.16	0.13
$\rho = 0.9$	6.89	0.11	0.16

Tabella 37: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	22	29	23.7	1	4	2.5	3	8	3.8
$\rho = 0.7$	31	43	34.6	1	4	2.1	2	7	3.5
$\rho = 0.8$	50	56	53.4	1	7	2.9	2	4	2.5
$\rho = 0.9$	105	174	127.3	1	6	2	2	4	2.9

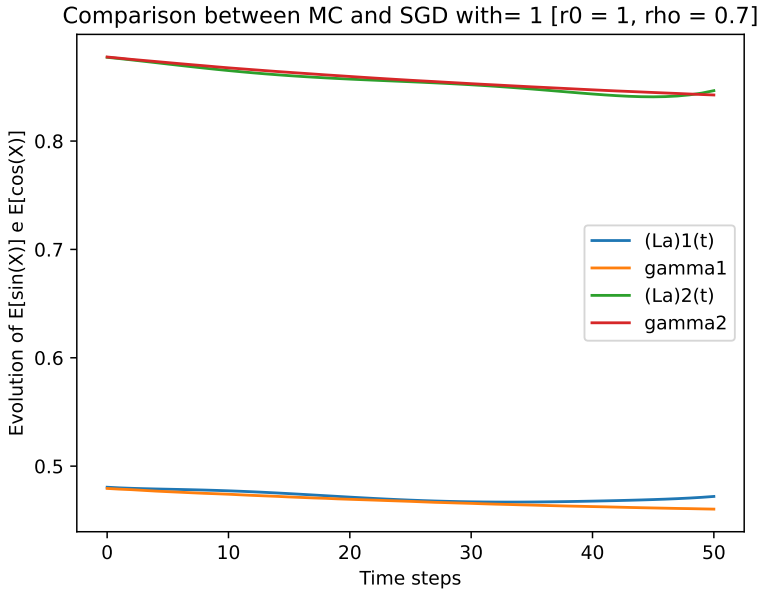
Tabella 38: Number of iterations m to achieve convergence with $M = 1000$

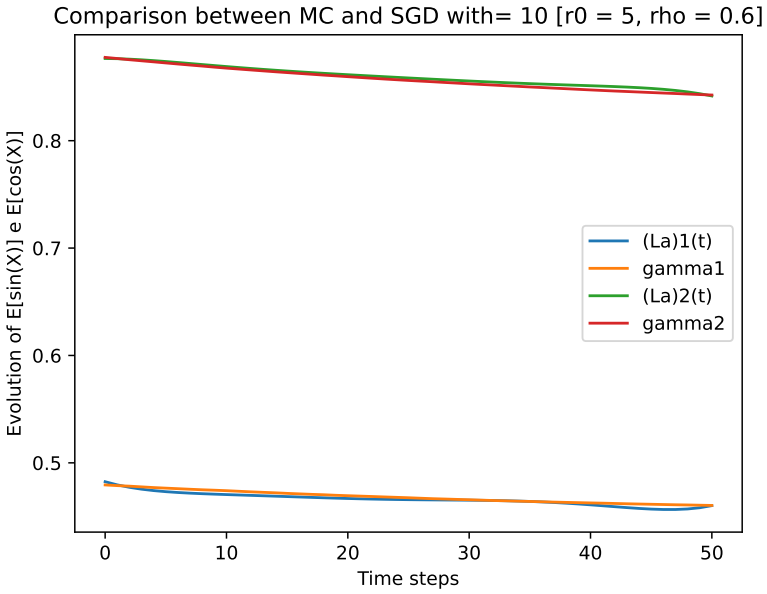
	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	16.77	0.77	1.92
$\rho = 0.7$	24.63	1.00	1.61
$\rho = 0.8$	41.76	0.77	1.61
$\rho = 0.9$	93.11	0.85	1.54

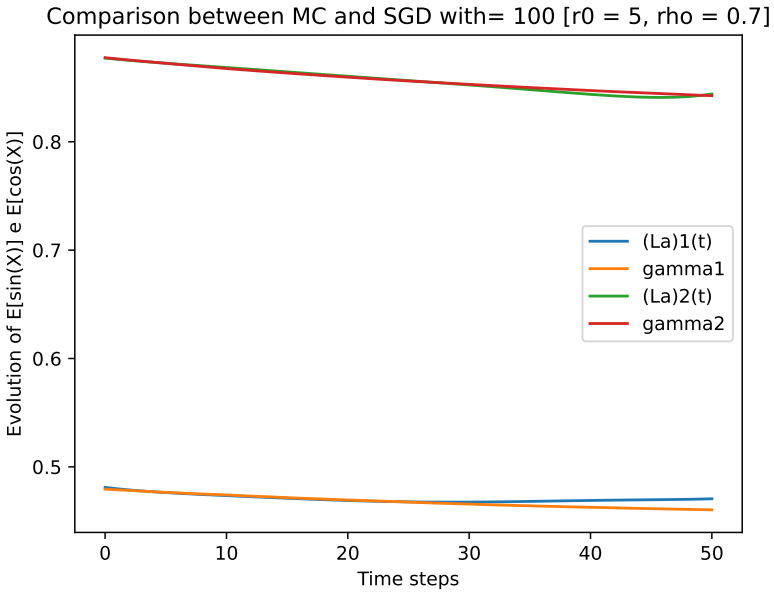
Tabella 39: Average execution times (in seconds s) with $M = 10000$

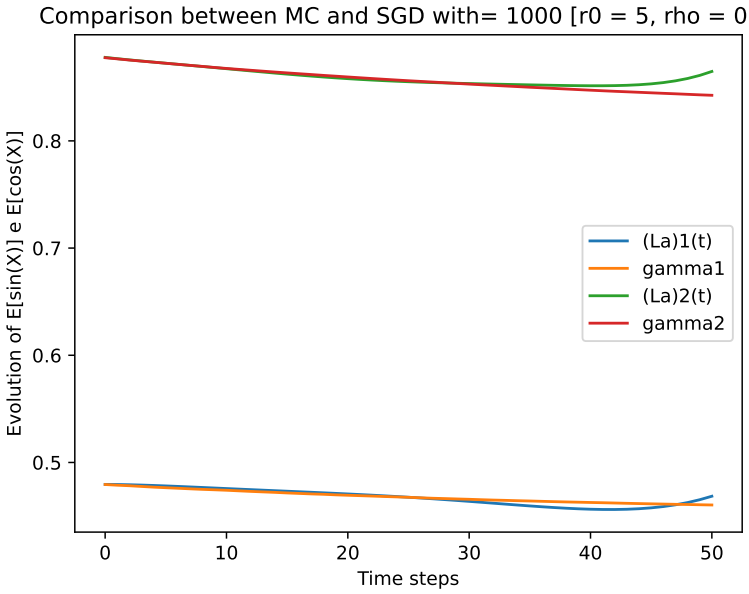
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	21	23	21.8	1	1	1	2	3	2.5
$\rho = 0.7$	31	34	31.9	1	2	1.3	2	3	2.1
$\rho = 0.8$	52	59	54.4	1	1	1	2	3	2.1
$\rho = 0.9$	114	128	121.3	1	2	1.1	2	2	2

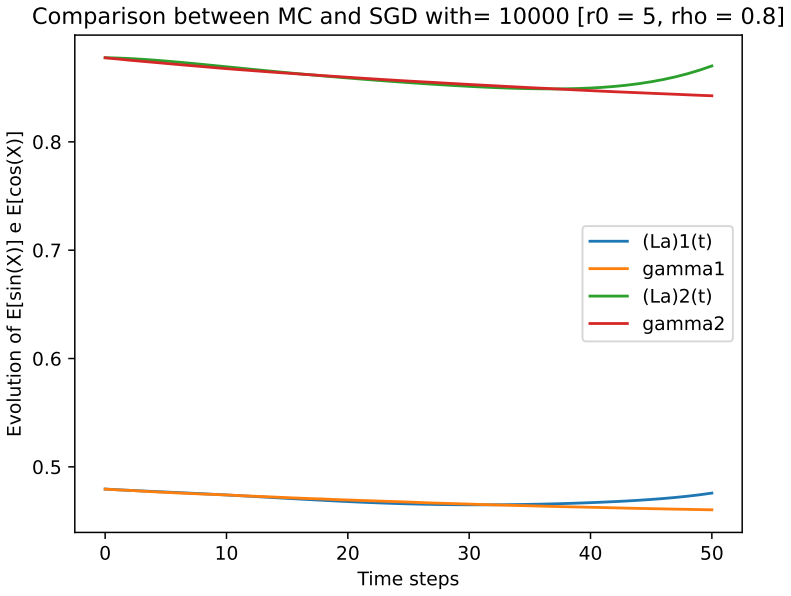
Tabella 40: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 5, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
10
20
30
40
50
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 100 [r0 = 5, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
10
20
30
40
50
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 1000 [r0 = 5, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
10
20
30
40
50
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 10000 [r0 = 5, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
10
20
30
40
50
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2

0.4 T = 1

Case n = 3

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	4.60	14.19	30.59
$\rho = 0.7$	6.06	8.29	13.83
$\rho = 0.8$	17.74	6.40	9.32
$\rho = 0.9$	29.5	5.47	5.58

Tabella 41: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	220	1470	683	1090	3170	2090	2110	6360	4544
$\rho = 0.7$	170	2540	921	280	2290	1199	890	4200	2029
$\rho = 0.8$	160	13840	2754	390	1590	947	580	2140	1350
$\rho = 0.9$	690	10410	4305	330	1740	812	230	1860	836

Tabella 42: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	1.22	1.81	3.94
$\rho = 0.7$	1.23	1.34	2.58
$\rho = 0.8$	2.29	1.33	1.53
$\rho = 0.9$	6.92	0.93	1.95

Tabella 43: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	50	330	159	90	440	234	200	1060	496
$\rho = 0.7$	10	500	160	70	500	173	140	500	323
$\rho = 0.8$	30	1210	298	50	480	172	110	410	194
$\rho = 0.9$	50	2210	904	50	290	120	20	550	253

Tabella 44: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.27	0.68	0.88
$\rho = 0.7$	0.39	0.47	0.92
$\rho = 0.8$	0.32	0.45	0.47
$\rho = 0.9$	0.37	0.59	0.59

Tabella 45: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	10	40	21	10	100	54	40	130	69
$\rho = 0.7$	10	50	30	10	70	38	20	110	74
$\rho = 0.8$	10	40	25	20	70	36	20	90	38
$\rho = 0.9$	10	50	29	20	150	46	30	90	47

Tabella 46: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.49	0.62	1.82
$\rho = 0.7$	0.75	0.54	1.21
$\rho = 0.8$	0.66	0.44	1.03
$\rho = 0.9$	0.82	0.46	0.75

Tabella 47: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	5	7	5.6	6	12	6.9	19	24	20.8
$\rho = 0.7$	5	27	8.9	5	9	6.3	13	16	14.4
$\rho = 0.8$	6	10	7.9	4	10	5.1	10	14	11.8
$\rho = 0.9$	7	18	9.7	4	13	5.4	8	13	8.9

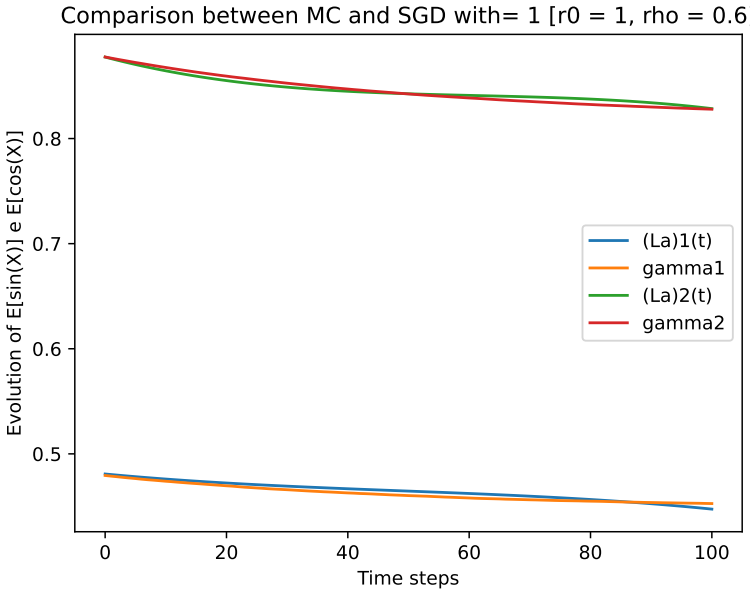
Tabella 48: Number of iterations m to achieve convergence with $M = 1000$

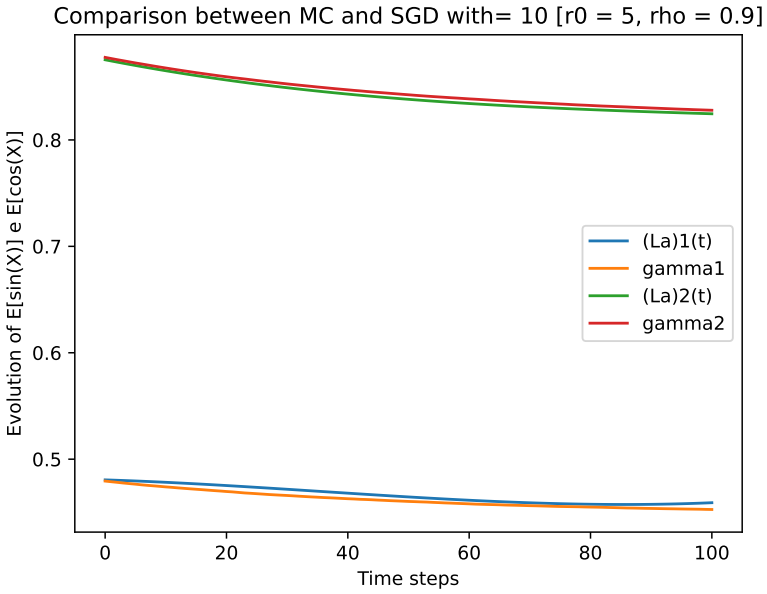
	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	4.71	5.83	18.25
$\rho = 0.7$	5.65	4.93	12.90
$\rho = 0.8$	6.19	3.93	9.96
$\rho = 0.9$	7.45	3.98	7.93

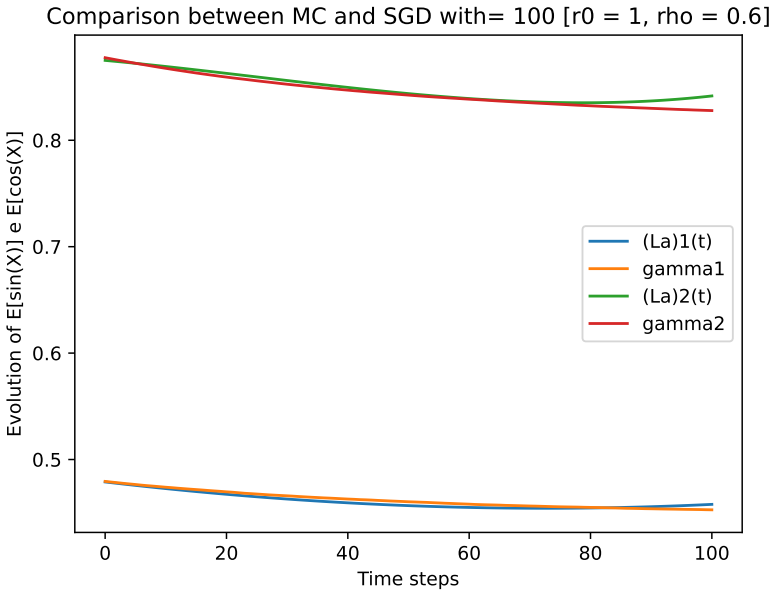
Tabella 49: Average execution times (in seconds s) with $M = 10000$

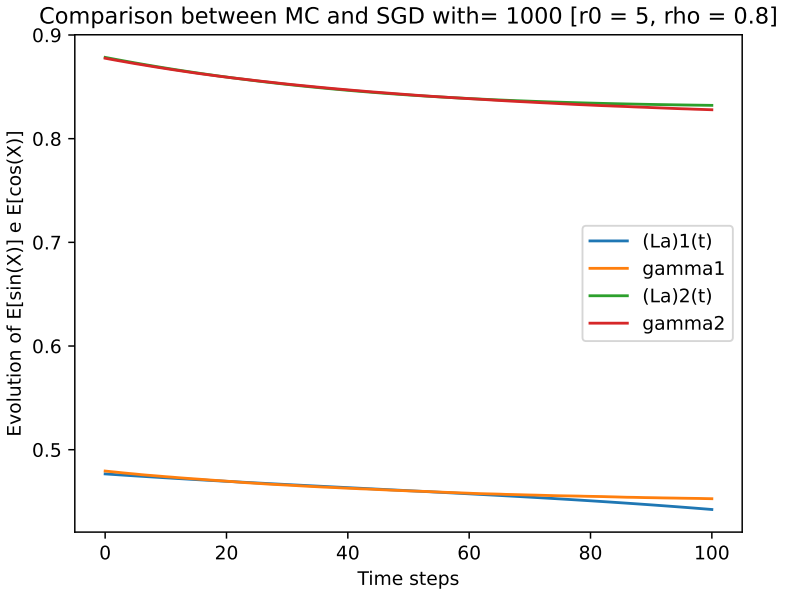
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	4	5	4.8	6	6	6	17	20	17.8
$\rho = 0.7$	5	7	5.5	5	5	5	13	13	13
$\rho = 0.8$	6	7	6.1	4	4	4	10	10	10
$\rho = 0.9$	7	11	7.7	4	4	4	8	8	8

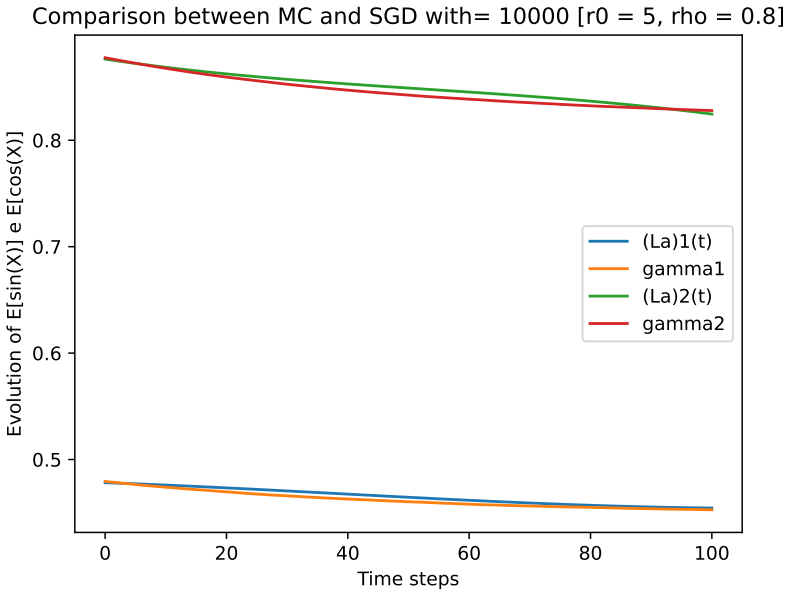
Tabella 50: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 5, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
20
40
60
80
100
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 100 [r0 = 1, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
20
40
60
80
100
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 1000 [r0 = 5, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
20
40
60
80
100


Comparison between MC and SGD with= 10000 [r0 = 5, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.8
0.7
0.6
0.5
0
20
40
60
80
100

Case n = 4

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	3.72	15.20	37.20
$\rho = 0.7$	6.40	6.31	15.76
$\rho = 0.8$	10.64	5.83	9.13
$\rho = 0.9$	97.45	5.70	7.09

Tabella 51: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	120	1280	517	1080	3820	2261	1830	8590	5523
$\rho = 0.7$	290	3140	949	400	1500	942	540	4030	2334
$\rho = 0.8$	170	3520	1504	300	1270	869	440	2330	1346
$\rho = 0.9$	720	49999	14405.8	280	1960	848	380	1850	1047

Tabella 52: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.92	2.69	6.23
$\rho = 0.7$	0.96	1.72	7.68
$\rho = 0.8$	2.55	1.45	5.01
$\rho = 0.9$	17.99	1.39	3.59

Tabella 53: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	50	180	117	110	590	340	260	890	548
$\rho = 0.7$	10	300	123	100	660	219	190	530	369
$\rho = 0.8$	40	910	325	70	290	183	100	450	256
$\rho = 0.9$	40	11880	2280	20	520	177	60	310	180

Tabella 54: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	1.36	2.10	3.85
$\rho = 0.7$	1.81	1.36	2.15
$\rho = 0.8$	1.61	1.74	1.55
$\rho = 0.9$	1.33	1.11	1.45

Tabella 55: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	10	130	40	20	150	62	60	140	106
$\rho = 0.7$	20	260	55	10	100	39	20	150	61
$\rho = 0.8$	20	80	49	20	80	48	20	90	44
$\rho = 0.9$	10	120	41	10	60	31	20	60	41

Tabella 56: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	2.17	1.38	3.64
$\rho = 0.7$	2.24	1.33	2.73
$\rho = 0.8$	3.97	1.31	2.31
$\rho = 0.9$	5.90	1.08	1.85

Tabella 57: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	7	12	9.1	4	10	5.9	13	21	15.3
$\rho = 0.7$	8	16	9.5	4	13	5.6	10	15	11.5
$\rho = 0.8$	10	40	16.2	3	11	5.6	8	15	9.8
$\rho = 0.9$	12	58	24.7	3	11	4.6	7	11	7.9

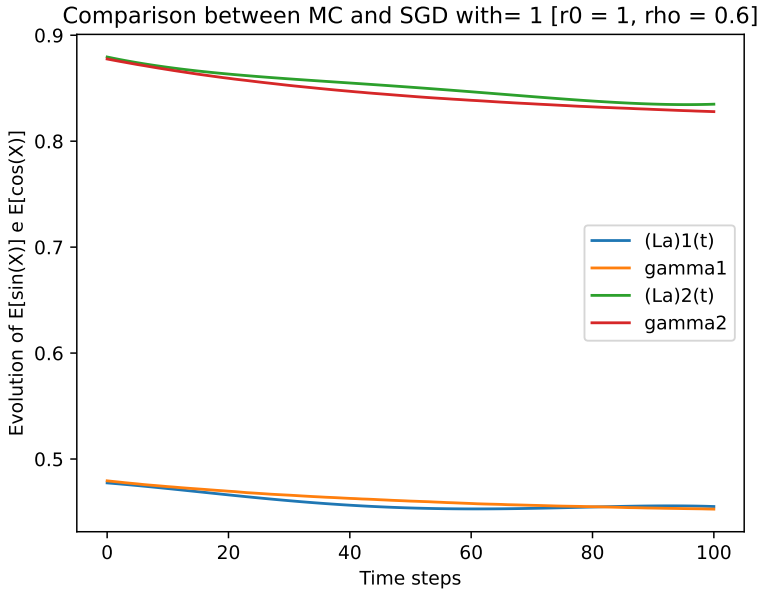
Tabella 58: Number of iterations m to achieve convergence with $M = 1000$

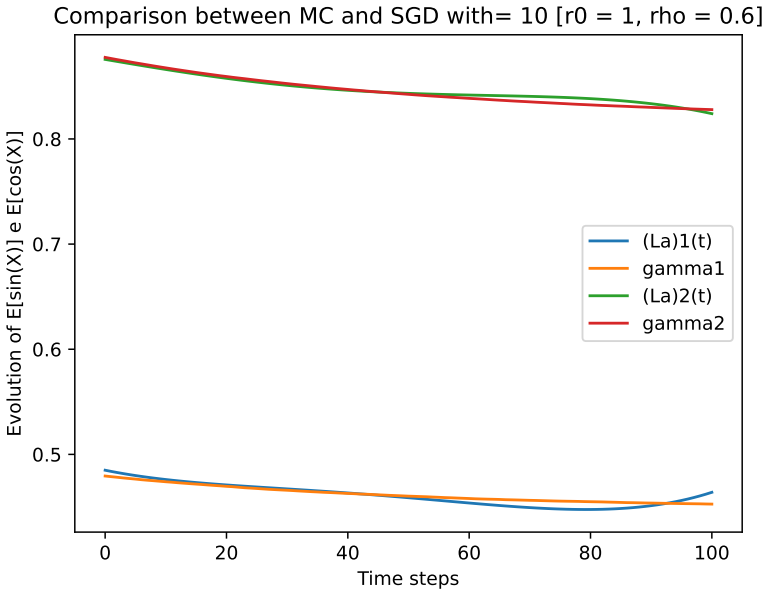
	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	17.09	10.90	33.12
$\rho = 0.7$	20.29	9.95	24.45
$\rho = 0.8$	25.10	9.10	16.95
$\rho = 0.9$	32.85	8.00	7.84

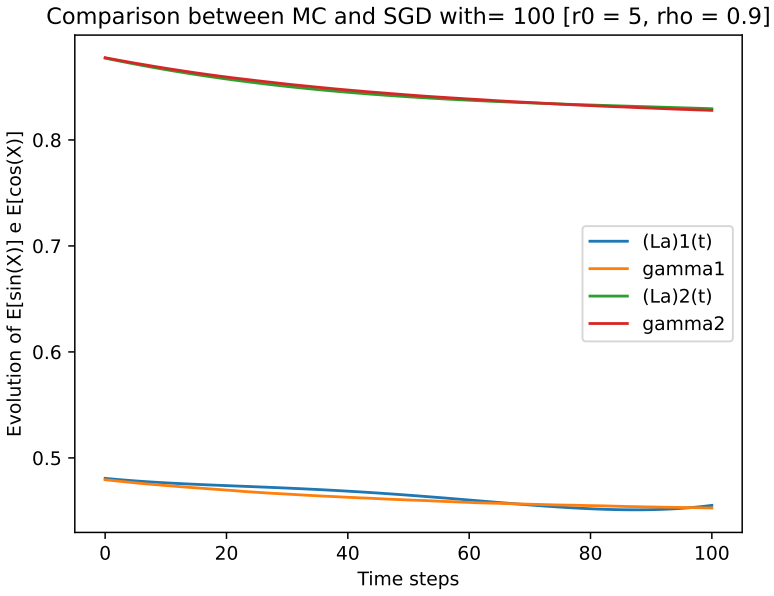
Tabella 59: Average execution times (in seconds s) with $M = 10000$

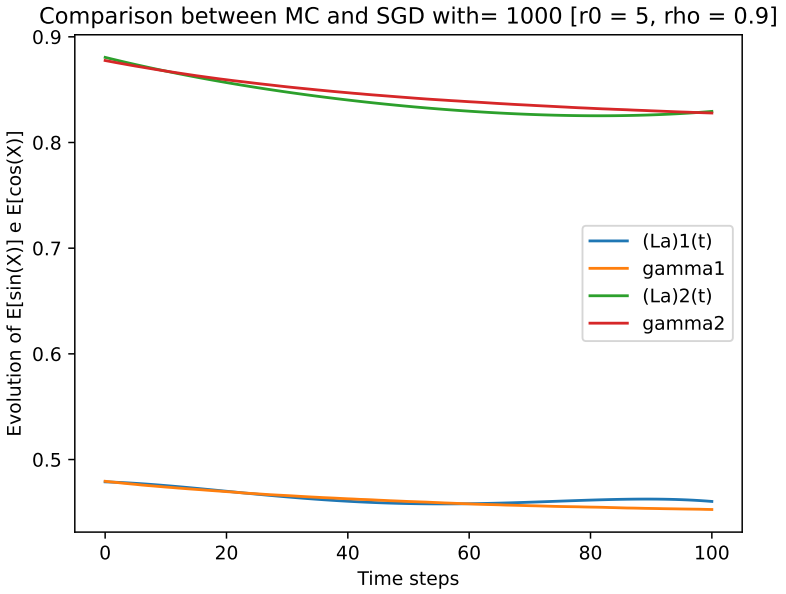
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	7	7	7	4	5	4.4	13	14	13.3
$\rho = 0.7$	7	9	8.3	4	4	4	10	10	10
$\rho = 0.8$	9	13	10.2	3	4	3.7	8	8	8
$\rho = 0.9$	12	15	13.3	3	3	3	7	7	7

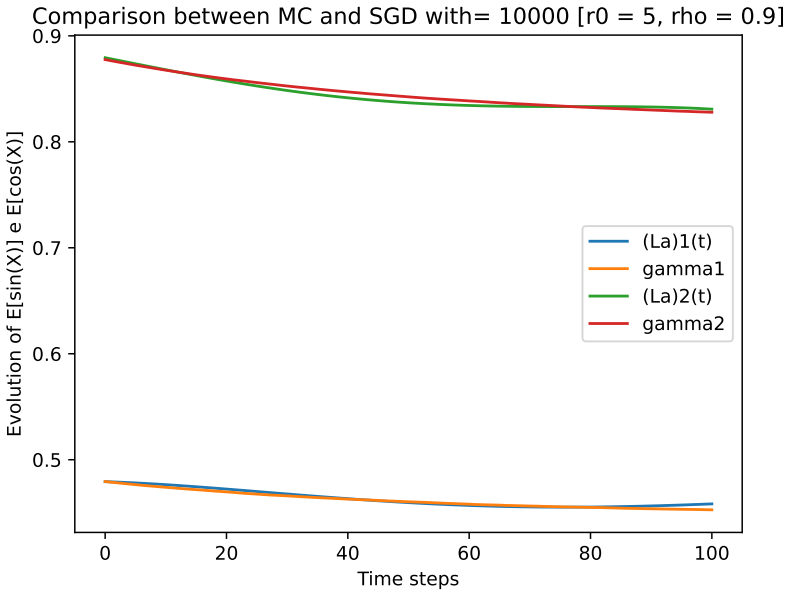
Tabella 60: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 1, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
20
40
60
80
100
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 100 [r0 = 5, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
20
40
60
80
100
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 1000 [r0 = 5, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
20
40
60
80
100


Comparison between MC and SGD with= 10000 [r0 = 5, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
20
40
60
80
100

Case n = 5

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	4.81	14.76	31.03
$\rho = 0.7$	9.11	7.29	17.28
$\rho = 0.8$	31.30	5.37	10.54
$\rho = 0.9$	90.16	6.65	5.83

Tabella 61: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	340	1340	709	1670	3170	2166	1690	8820	4546
$\rho = 0.7$	410	2260	1323	160	2300	1071	770	4350	2533
$\rho = 0.8$	240	24910	4579	90	1650	789	820	2910	1543
$\rho = 0.9$	570	49999	13249.9	470	1860	978	380	1350	857

Tabella 62: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	1.42	1.65	4.35
$\rho = 0.7$	1.71	1.87	2.07
$\rho = 0.8$	3.27	1.42	1.88
$\rho = 0.9$	1.47	1.26	1.81

Tabella 63: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	50	580	179	80	420	206	120	870	540
$\rho = 0.7$	60	1180	216	80	430	235	120	570	257
$\rho = 0.8$	50	1460	411	140	330	178	70	390	235
$\rho = 0.9$	40	750	185	50	370	157	90	490	228

Tabella 64: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.66	0.47	1.23
$\rho = 0.7$	0.63	0.69	0.87
$\rho = 0.8$	0.73	0.39	0.95
$\rho = 0.9$	1.93	0.49	0.43

Tabella 65: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	20	100	46	10	60	33	30	150	86
$\rho = 0.7$	20	150	44	10	130	48	20	90	61
$\rho = 0.8$	20	180	51	10	40	27	30	110	66
$\rho = 0.9$	40	370	135	10	90	34	10	60	30

Tabella 66: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	1.35	0.67	1.37
$\rho = 0.7$	1.48	0.52	0.92
$\rho = 0.8$	2.33	0.37	0.87
$\rho = 0.9$	3.61	0.53	0.74

Tabella 67: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	10	18	12.3	3	14	6.1	10	15	12.5
$\rho = 0.7$	11	16	13.4	3	8	4.7	8	10	8.4
$\rho = 0.8$	17	27	21.1	3	4	3.3	6	12	7.9
$\rho = 0.9$	23	66	32.9	3	11	4.8	6	8	6.7

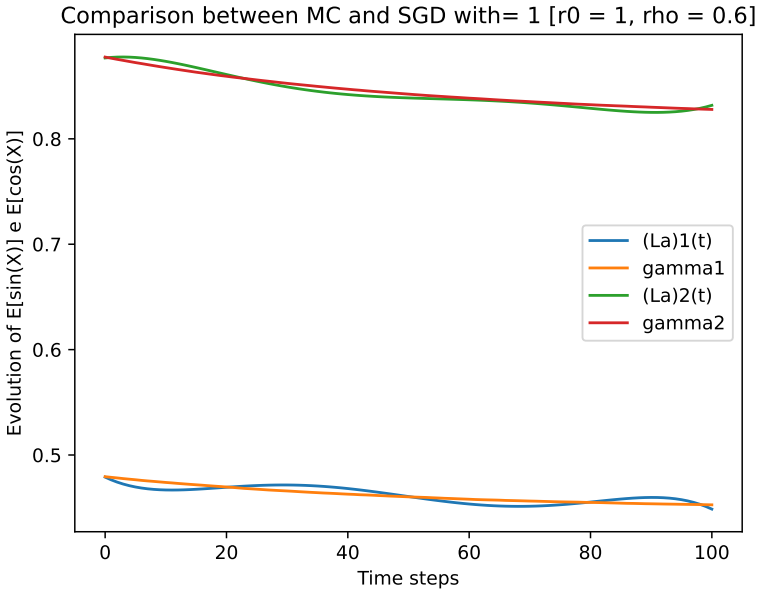
Tabella 68: Number of iterations m to achieve convergence with $M = 1000$

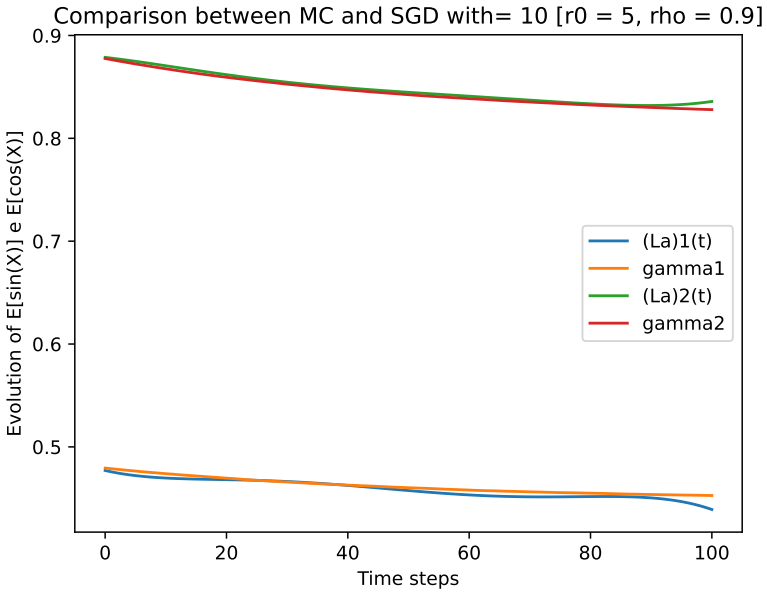
	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	13.12	4.45	13.18
$\rho = 0.7$	15.57	3.89	10.47
$\rho = 0.8$	20.77	3.93	8.27
$\rho = 0.9$	31.73	3.90	7.20

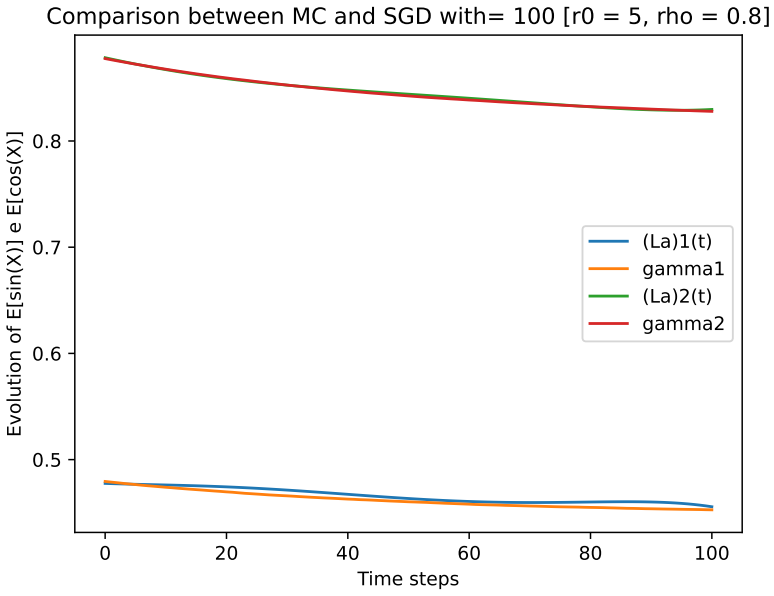
Tabella 69: Average execution times (in seconds s) with $M = 10000$

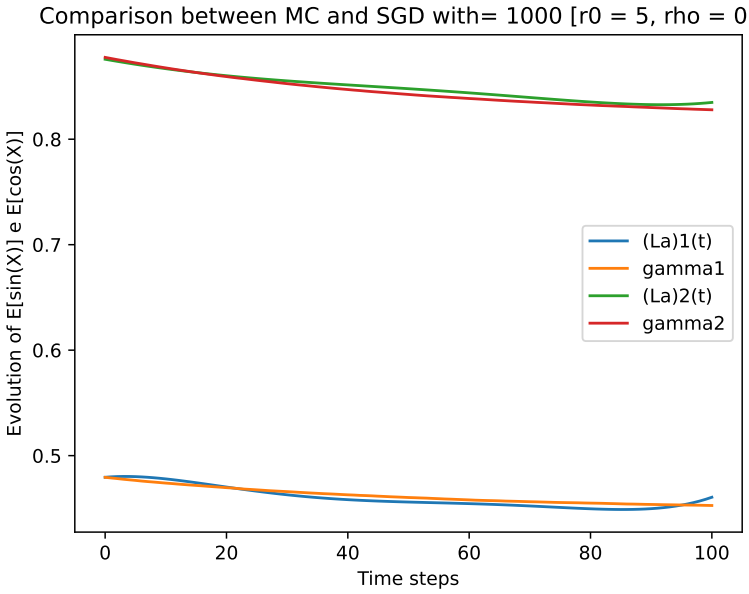
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	9	11	10	3	4	3.4	10	11	10.1
$\rho = 0.7$	11	13	11.9	3	3	3	8	8	8
$\rho = 0.8$	15	16	15.9	3	3	3	6	7	6.3
$\rho = 0.9$	22	31	24.3	3	3	3	5	6	5.5

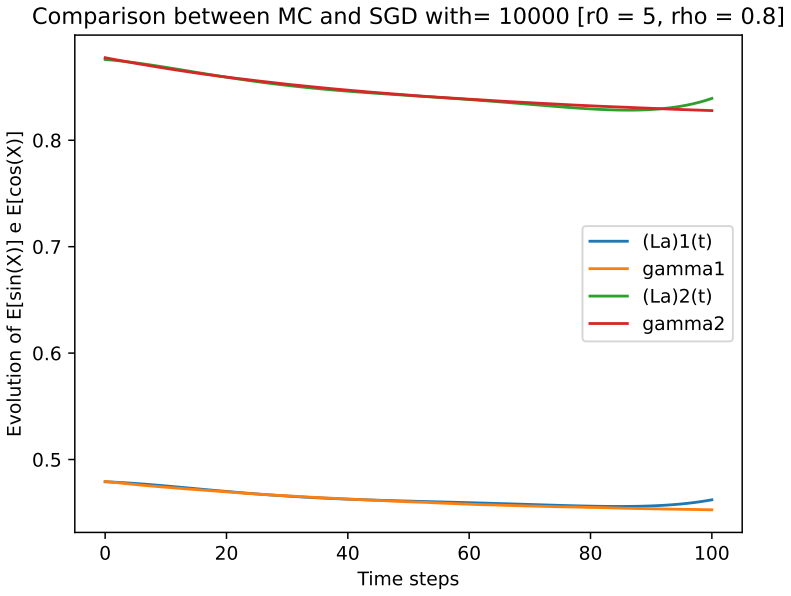
Tabella 70: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 5, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
20
40
60
80
100


Comparison between MC and SGD with= 100 [r0 = 5, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.8
0.7
0.6
0.5
0
20
40
60
80
100


Comparison between MC and SGD with= 1000 [r0 = 5, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
20
40
60
80
100
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 10000 [r0 = 5, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
20
40
60
80
100
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2

Case n = 6

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	7.97	13.95	35.22
$\rho = 0.7$	11.89	9.05	18.96
$\rho = 0.8$	17.13	10.48	9.50
$\rho = 0.9$	157.41	10.32	6.82

Tabella 71: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	170	1920	1142	610	3310	1991	2510	9890	5037
$\rho = 0.7$	320	6730	1702	210	2550	1292	770	3790	2713
$\rho = 0.8$	390	10610	2453	500	3180	1501	540	2200	1359
$\rho = 0.9$	590	49999	22514.8	430	5640	1475	380	2510	976

Tabella 72: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	1.13	2.55	6.69
$\rho = 0.7$	2.00	2.10	4.14
$\rho = 0.8$	5.07	2.23	2.89
$\rho = 0.9$	19.58	1.94	2.00

Tabella 73: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	40	220	96	100	420	216	230	880	565
$\rho = 0.7$	30	430	171	70	330	178	130	580	351
$\rho = 0.8$	40	1560	431	70	550	187	70	360	246
$\rho = 0.9$	100	6200	1661	60	270	164	70	330	170

Tabella 74: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	0.62	0.72	1.11
$\rho = 0.7$	0.86	0.52	0.85
$\rho = 0.8$	1.57	0.34	0.73
$\rho = 0.9$	1.55	0.54	0.46

Tabella 75: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	20	110	40	10	110	47	40	110	72
$\rho = 0.7$	30	140	56	20	60	34	30	70	55
$\rho = 0.8$	40	300	102	10	40	22	10	70	47
$\rho = 0.9$	40	320	100	10	100	35	10	60	30

Tabella 76: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	2.18	0.55	1.05
$\rho = 0.7$	2.51	0.53	0.89
$\rho = 0.8$	3.59	0.52	1.00
$\rho = 0.9$	6.78	0.48	0.73

Tabella 77: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	14	23	17.7	3	7	4.5	8	11	8.6
$\rho = 0.7$	16	26	20.4	3	8	4.2	6	9	7.2
$\rho = 0.8$	24	45	29.2	2	8	4.2	5	17	7.9
$\rho = 0.9$	38	111	54.9	3	6	3.9	5	7	5.9

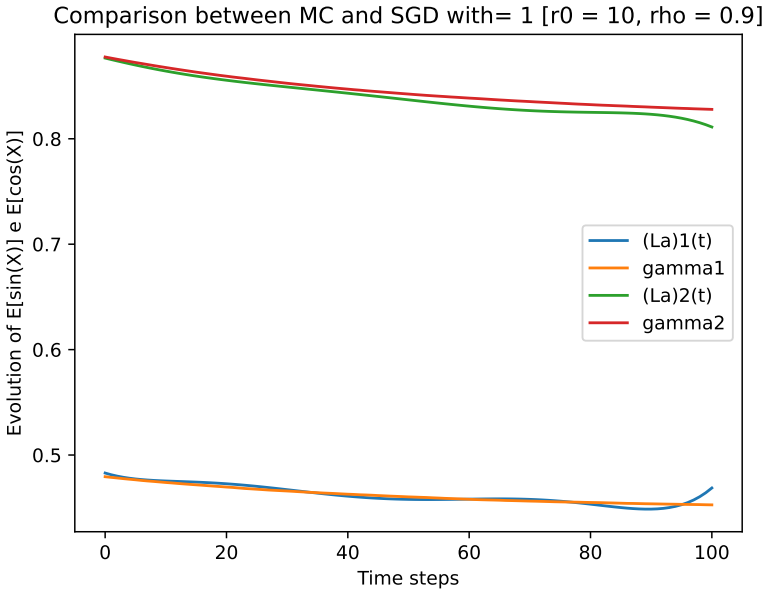
Tabella 78: Number of iterations m to achieve convergence with $M = 1000$

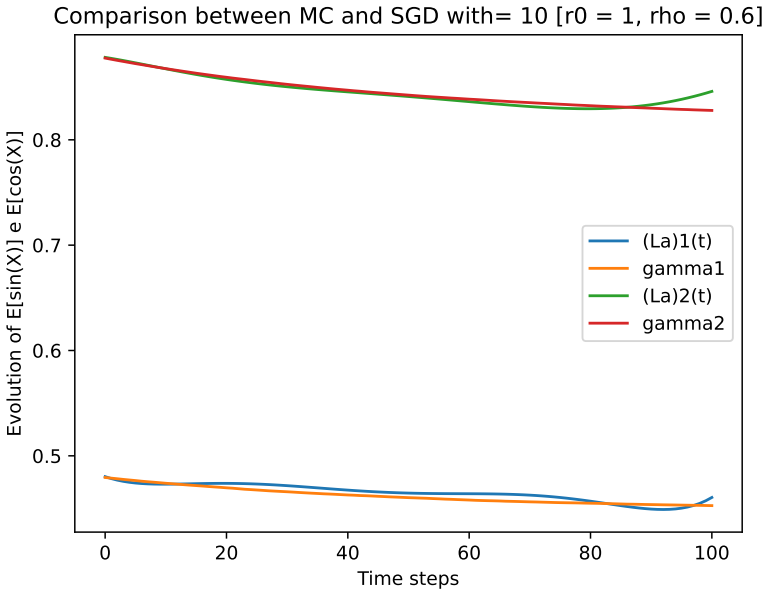
	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	20.21	4.72	12.51
$\rho = 0.7$	27.02	4.22	9.48
$\rho = 0.8$	39.52	3.61	7.80
$\rho = 0.9$	66.18	3.11	7.78

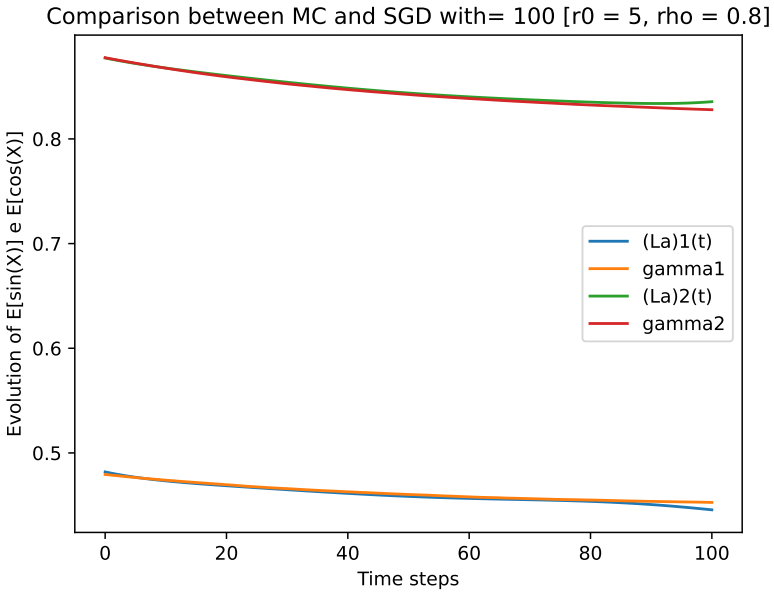
Tabella 79: Average execution times (in seconds s) with $M = 10000$

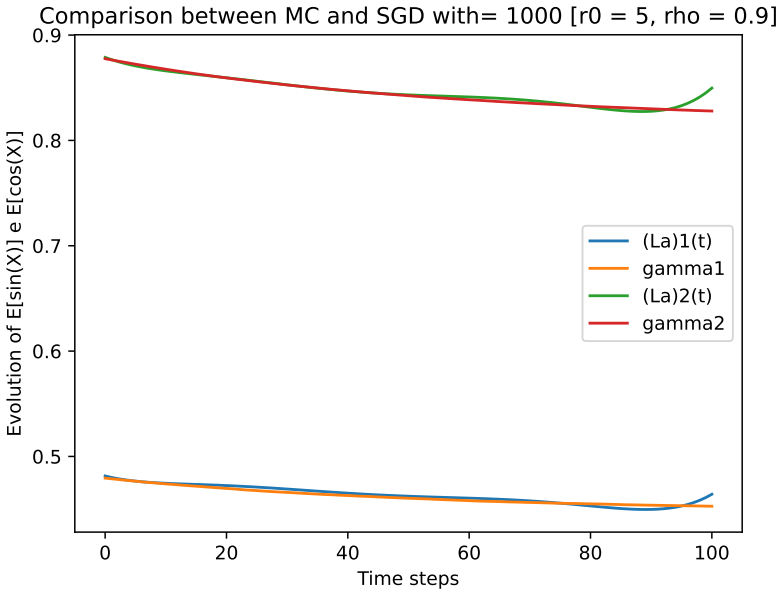
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	12	14	12.9	3	3	3	8	8	8
$\rho = 0.7$	16	19	17.2	2	3	2.7	6	7	6.1
$\rho = 0.8$	23	28	25.3	2	3	2.3	5	5	5
$\rho = 0.9$	41	45	42.3	2	2	2	5	5	5

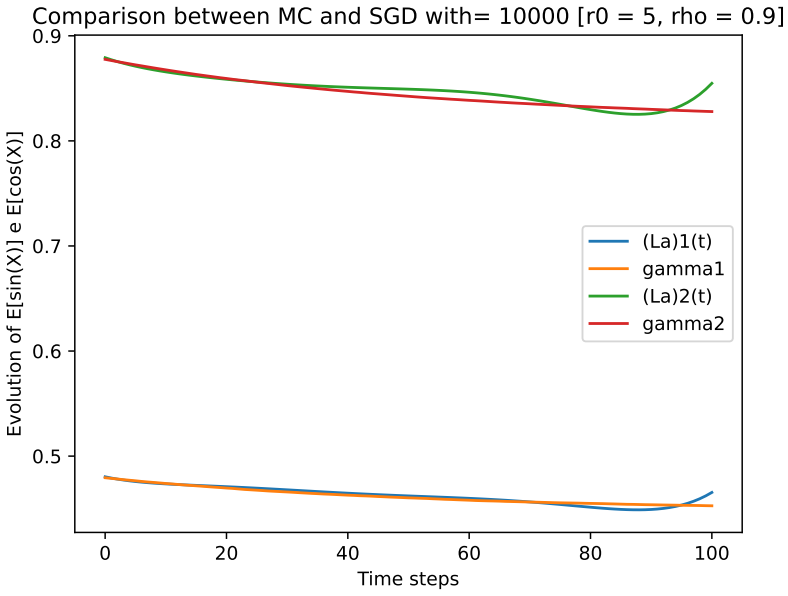
Tabella 80: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 1, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
20
40
60
80
100
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 100 [r0 = 5, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
20
40
60
80
100
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 1000 [r0 = 5, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
20
40
60
80
100


Comparison between MC and SGD with= 10000 [r0 = 5, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
20
40
60
80
100

0.5 T = 2

Case n = 3

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	22.35	80.57
$\rho = 0.7$	12.28	42.48
$\rho = 0.8$	34.07	19.80
$\rho = 0.9$	94.34	23.10

Tabella 81: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	1070	3030	1798	3760	10900	6485		overflow	
$\rho = 0.7$	200	1460	992	1310	4700	3419		overflow	
$\rho = 0.8$	360	8220	2753	950	2600	1597		overflow	
$\rho = 0.9$	560	24050	7602	880	2870	1863		overflow	

Tabella 82: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	3.91	11.72
$\rho = 0.7$	1.98	7.75
$\rho = 0.8$	7.45	5.10
$\rho = 0.9$	5.94	4.78

Tabella 83: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	50	500	275	260	1360	819		overflow	
$\rho = 0.7$	20	440	139	190	1110	544		overflow	
$\rho = 0.8$	80	2130	492	140	640	357		overflow	
$\rho = 0.9$	40	1300	416	120	830	335		overflow	

Tabella 84: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	1.2	2.03
$\rho = 0.7$	0.78	2.05
$\rho = 0.8$	1.26	1.67
$\rho = 0.9$	0.98	1.60

Tabella 85: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	10	140	50	20	260	85		overflow	
$\rho = 0.7$	10	50	33	30	150	86		overflow	
$\rho = 0.8$	10	160	53	30	130	70		overflow	
$\rho = 0.9$	10	90	41	40	110	67		overflow	

Tabella 86: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	1.02	5.35
$\rho = 0.7$	0.88	3.42
$\rho = 0.8$	0.88	2.31
$\rho = 0.9$	1.64	2.29

Tabella 87: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	2	11	5.3	21	36	27.5		overflow	
$\rho = 0.7$	3	14	4.6	12	29	17.7		overflow	
$\rho = 0.8$	3	12	4.6	10	18	11.9		overflow	
$\rho = 0.9$	3	29	8.5	8	25	11.9		overflow	

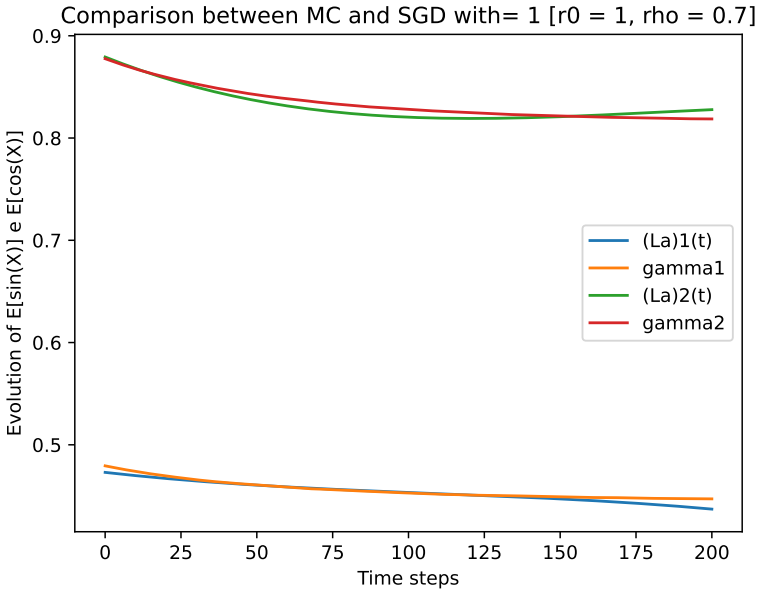
Tabella 88: Number of iterations m to achieve convergence with $M = 1000$

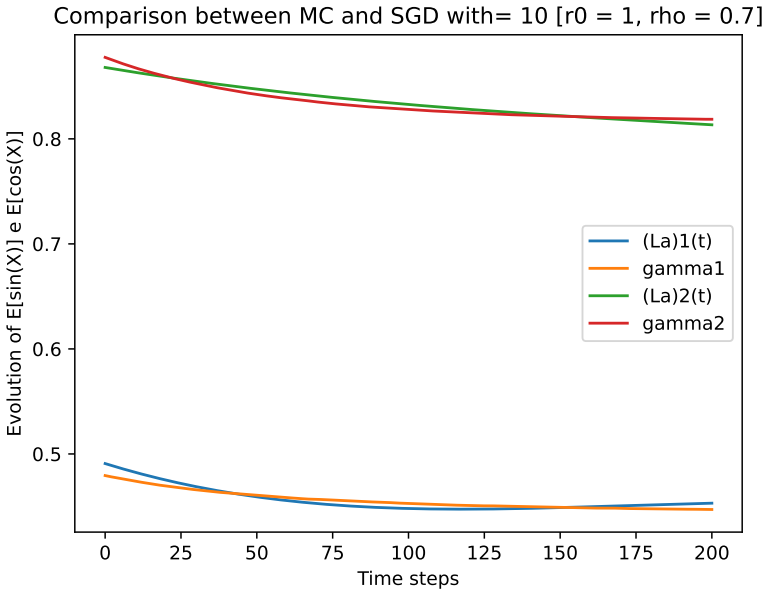
	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	4.67	38.12
$\rho = 0.7$	4.30	24.13
$\rho = 0.8$	5.42	19.99
$\rho = 0.9$	5.61	15.43

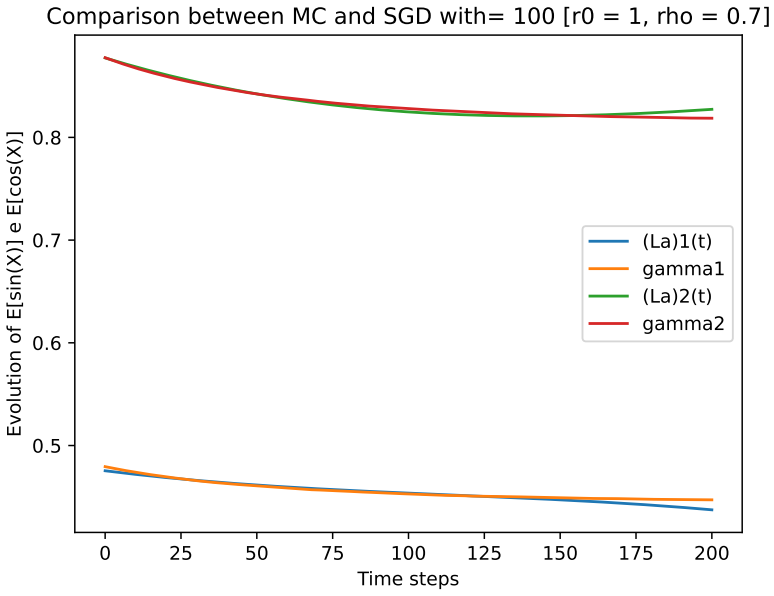
Tabella 89: Average execution times (in seconds s) with $M = 10000$

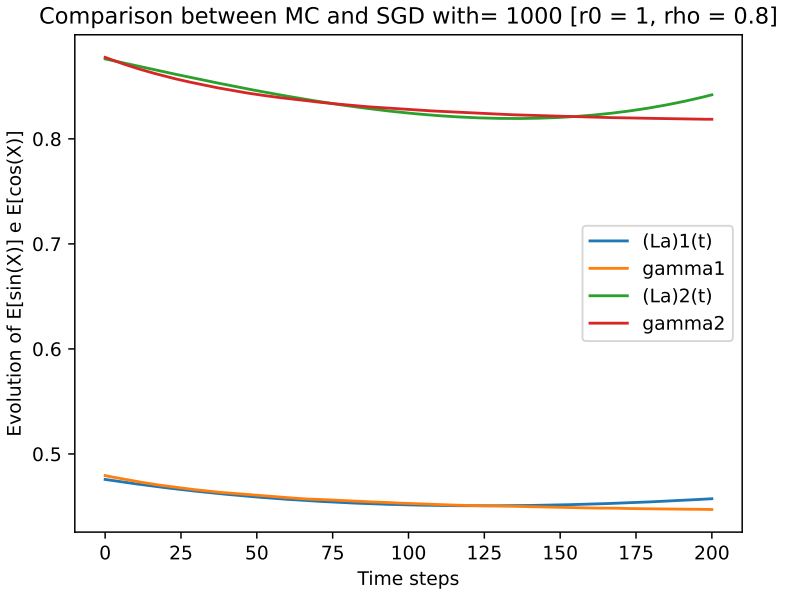
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	2	3	2.5	15	26	20		overflow	
$\rho = 0.7$	2	3	2.3	12	17	12.7		overflow	
$\rho = 0.8$	2	4	2.9	9	14	10.6		overflow	
$\rho = 0.9$	3	3	3	8	9	8.2		overflow	

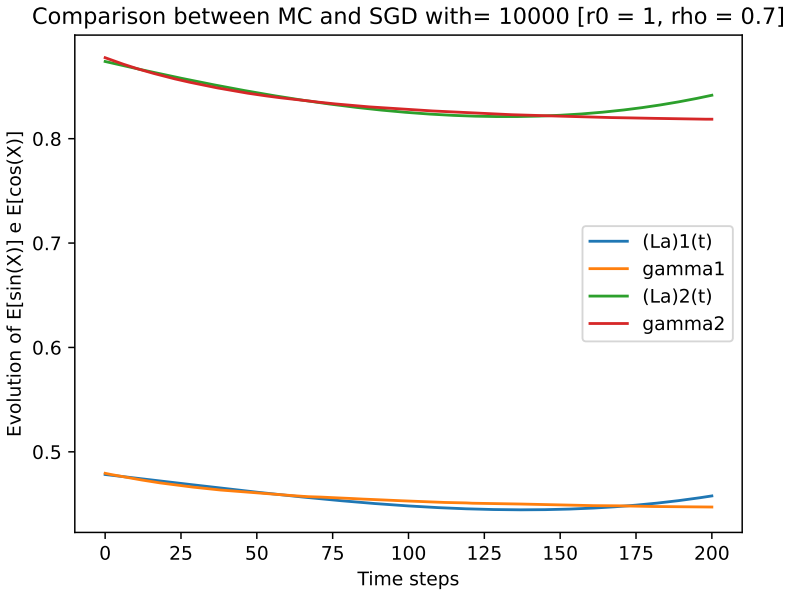
Tabella 90: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 1, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
25
50
75
100
125
150
175
200
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 100 [r0 = 1, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
25
50
75
100
125
150
175
200
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 1000 [r0 = 1, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
25
50
75
100
125
150
175
200
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 10000 [r0 = 1, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
25
50
75
100
125
150
175
200
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2

Case n = 4

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	29.67	104.08
$\rho = 0.7$	19.5	58.47
$\rho = 0.8$	44.65	23.28
$\rho = 0.9$	271.35	17.65

Tabella 91: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	620	4050	2331	2890	11970	7908		overflow	
$\rho = 0.7$	670	3850	1529	1570	6540	4510		overflow	
$\rho = 0.8$	490	13550	3428	280	4230	1806		overflow	
$\rho = 0.9$	1560	49999	20370.8	670	2480	1312		overflow	

Tabella 92: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	4.56	13.23
$\rho = 0.7$	4.85	7.61
$\rho = 0.8$	6.83	5.09
$\rho = 0.9$	29.93	4.01

Tabella 93: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	80	620	310	340	1380	886		overflow	
$\rho = 0.7$	20	810	322	290	1090	519		overflow	
$\rho = 0.8$	40	2420	438	80	680	348		overflow	
$\rho = 0.9$	60	6660	1909	110	690	273		overflow	

Tabella 94: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	0.84	2.54
$\rho = 0.7$	1.13	1.82
$\rho = 0.8$	1.41	1.38
$\rho = 0.9$	4.74	1.27

Tabella 95: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	10	90	31	30	190	90		overflow	
$\rho = 0.7$	10	110	40	30	100	64		overflow	
$\rho = 0.8$	10	150	42	30	70	49		overflow	
$\rho = 0.9$	20	950	169	10	80	45		overflow	

Tabella 96: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	1.43	3.93
$\rho = 0.7$	1.27	3.09
$\rho = 0.8$	1.34	2.35
$\rho = 0.9$	1.92	2.17

Tabella 97: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	4	11	6	13	20	16.4		overflow	
$\rho = 0.7$	3	11	5.3	10	21	12.8		overflow	
$\rho = 0.8$	4	11	5.6	8	13	9.7		overflow	
$\rho = 0.9$	4	23	8	7	14	9.1		overflow	

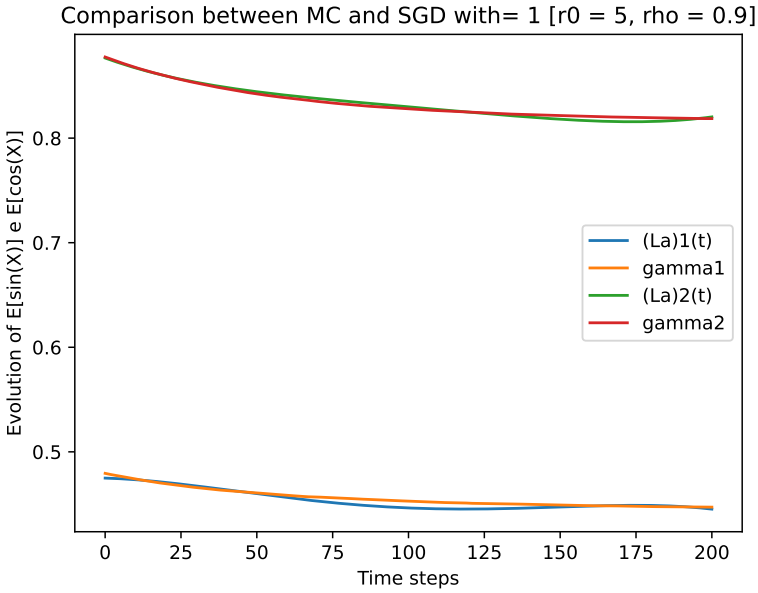
Tabella 98: Number of iterations m to achieve convergence with $M = 1000$

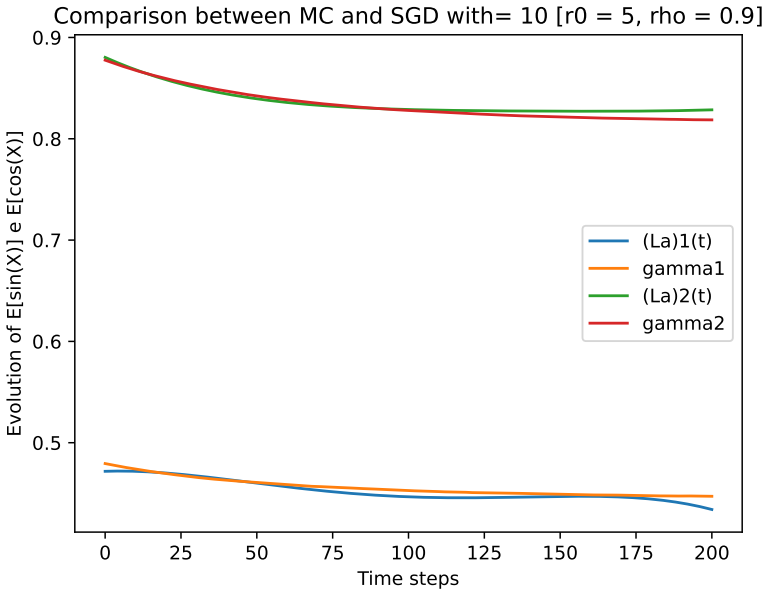
	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	7.24	30.59
$\rho = 0.7$	7.20	22.75
$\rho = 0.8$	7.77	18.19
$\rho = 0.9$	8.51	16.25

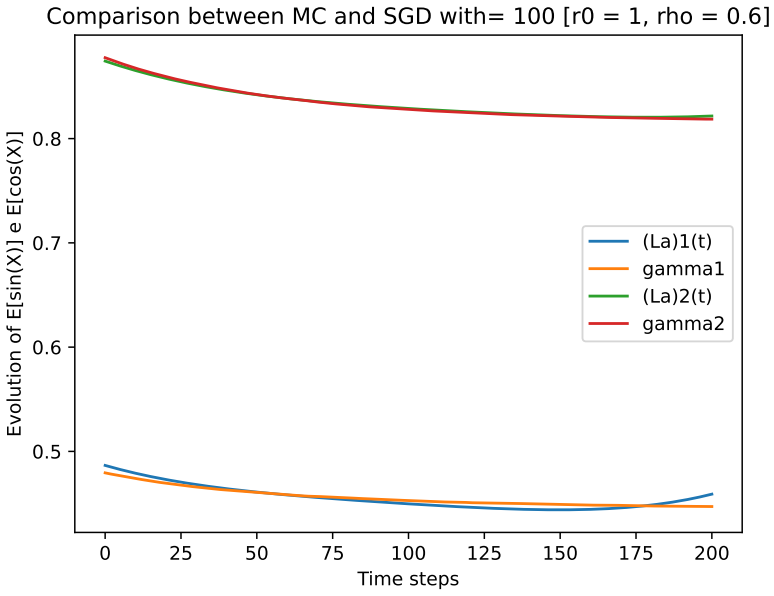
Tabella 99: Average execution times (in seconds s) with $M = 10000$

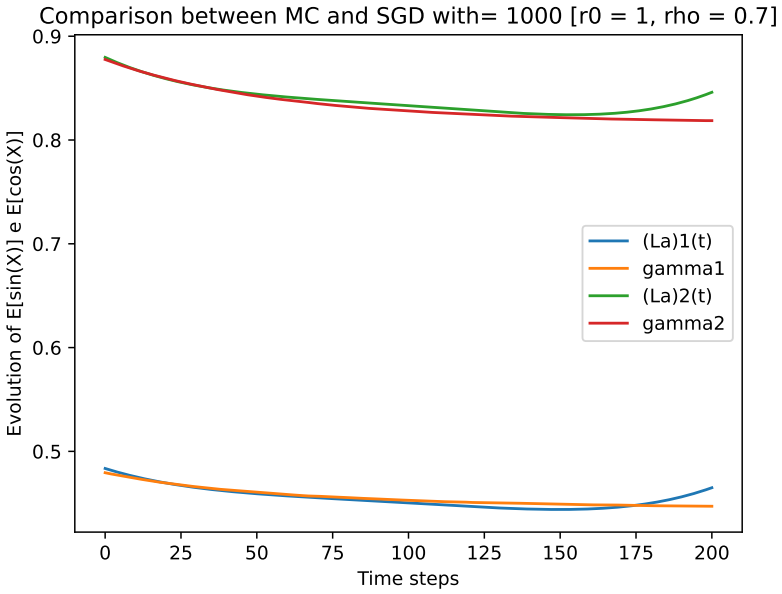
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	3	4	3.1	13	13	13		overflow	
$\rho = 0.7$	3	4	3.1	10	10	10		overflow	
$\rho = 0.8$	3	4	3.3	8	8	8		overflow	
$\rho = 0.9$	3	4	3.6	7	7	7		overflow	

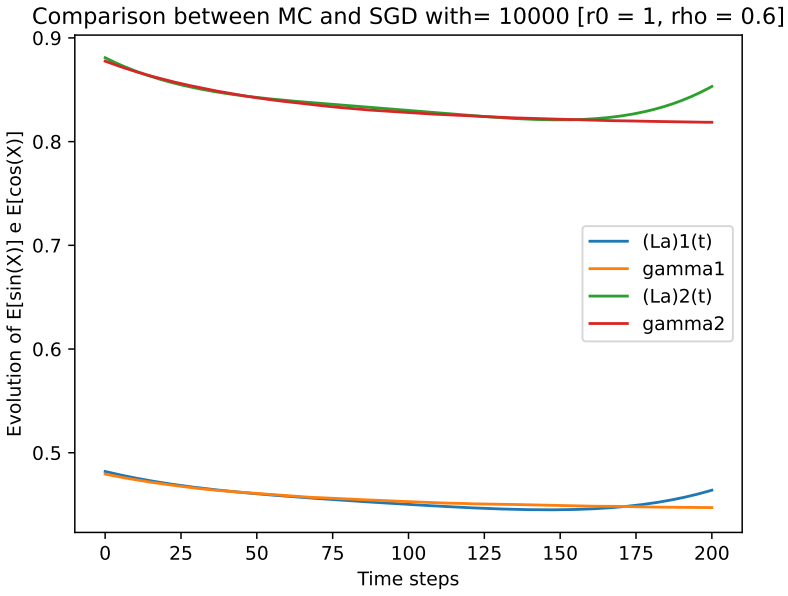
Tabella 100: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 5, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
25
50
75
100
125
150
175
200


Comparison between MC and SGD with= 100 [r0 = 1, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.8
0.7
0.6
0.5
0
25
50
75
100
125
150
175
200


Comparison between MC and SGD with= 1000 [r0 = 1, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
25
50
75
100
125
150
175
200


Comparison between MC and SGD with= 10000 [r0 = 1, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
25
50
75
100
125
150
175
200

Case n = 5

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	28.57	133.75
$\rho = 0.7$	33.20	54.61
$\rho = 0.8$	59.02	42.73
$\rho = 0.9$	306.42	20.43

Tabella 101: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	820	3420	2167	4540	17080	10248		overflow	
$\rho = 0.7$	490	6630	2459	2280	8430	4194		overflow	
$\rho = 0.8$	470	9120	4432	1410	7030	3278		overflow	
$\rho = 0.9$	1160	49999	23456.7	400	4680	1570		overflow	

Tabella 102: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	4.13	13.39
$\rho = 0.7$	5.30	7.85
$\rho = 0.8$	18.94	3.82
$\rho = 0.9$	53.15	5.07

Tabella 103: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	90	490	272	380	1610	883		overflow	
$\rho = 0.7$	100	570	333	180	950	517		overflow	
$\rho = 0.8$	70	6300	1230	90	510	252		overflow	
$\rho = 0.9$	80	24990	3503	150	1130	335		overflow	

Tabella 104: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	1.99	2.05
$\rho = 0.7$	1.05	2.14
$\rho = 0.8$	1.84	2.11
$\rho = 0.9$	3.72	1.92

Tabella 105: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	20	190	68	40	130	70		overflow	
$\rho = 0.7$	10	80	36	20	170	73		overflow	
$\rho = 0.8$	10	180	63	30	150	72		overflow	
$\rho = 0.9$	10	680	127	30	120	65		overflow	

Tabella 106: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	1.59	3.27
$\rho = 0.7$	1.98	2.5
$\rho = 0.8$	1.86	2.56
$\rho = 0.9$	2.85	1.93

Tabella 107: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	4	10	6.1	10	17	12.4		overflow	
$\rho = 0.7$	4	16	7.6	8	13	9.6		overflow	
$\rho = 0.8$	5	12	7.1	7	14	9.8		overflow	
$\rho = 0.9$	7	18	11	5	12	7.4		overflow	

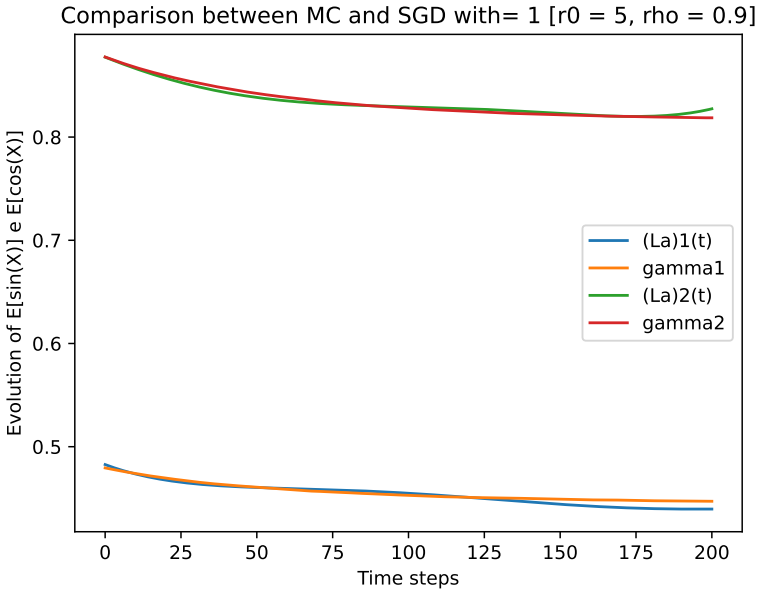
Tabella 108: Number of iterations m to achieve convergence with $M = 1000$

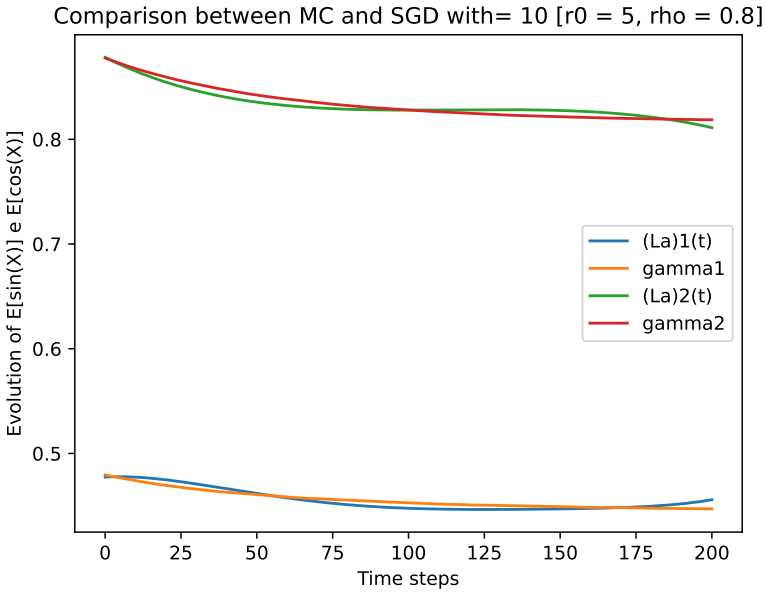
	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	10.72	26.45
$\rho = 0.7$	11.51	21.05
$\rho = 0.8$	15.16	15.97
$\rho = 0.9$	16.24	13.36

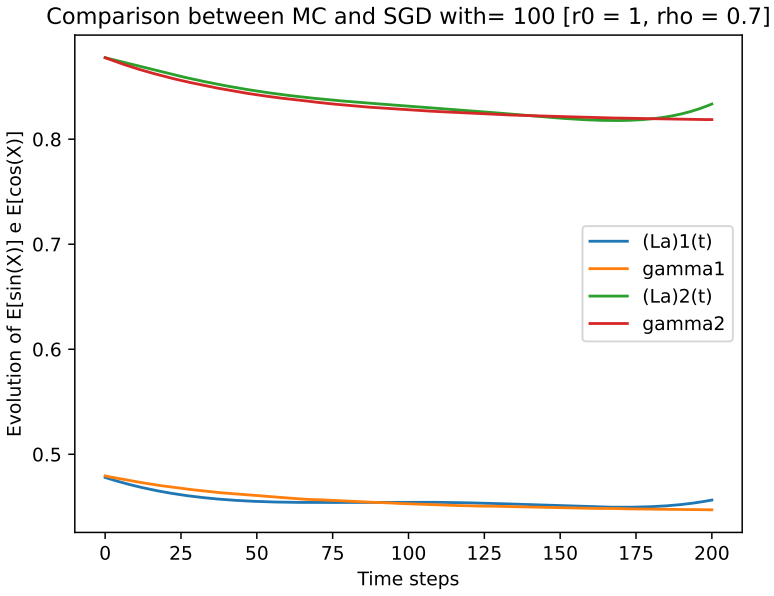
Tabella 109: Average execution times (in seconds s) with $M = 10000$

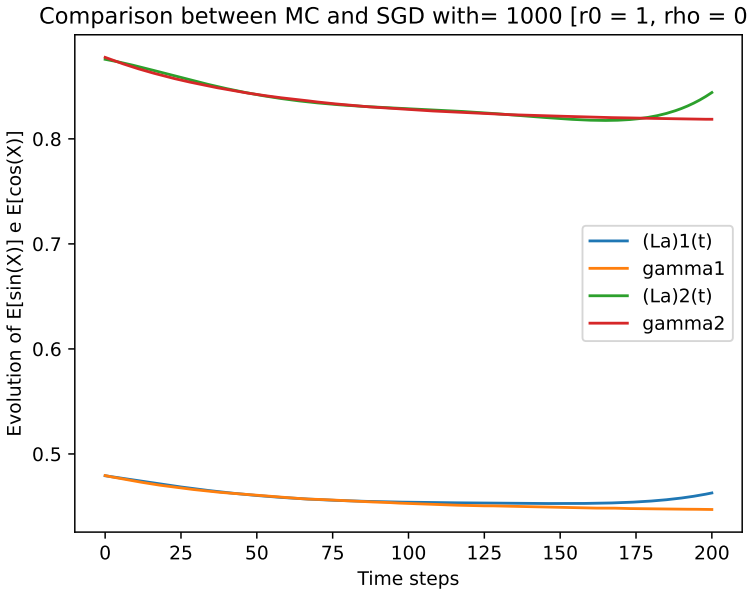
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	4	5	4.1	10	11	10.1		overflow	
$\rho = 0.7$	4	5	4.4	8	8	8		overflow	
$\rho = 0.8$	5	8	5.8	6	7	6.1		overflow	
$\rho = 0.9$	5	8	6.2	5	6	5.1		overflow	

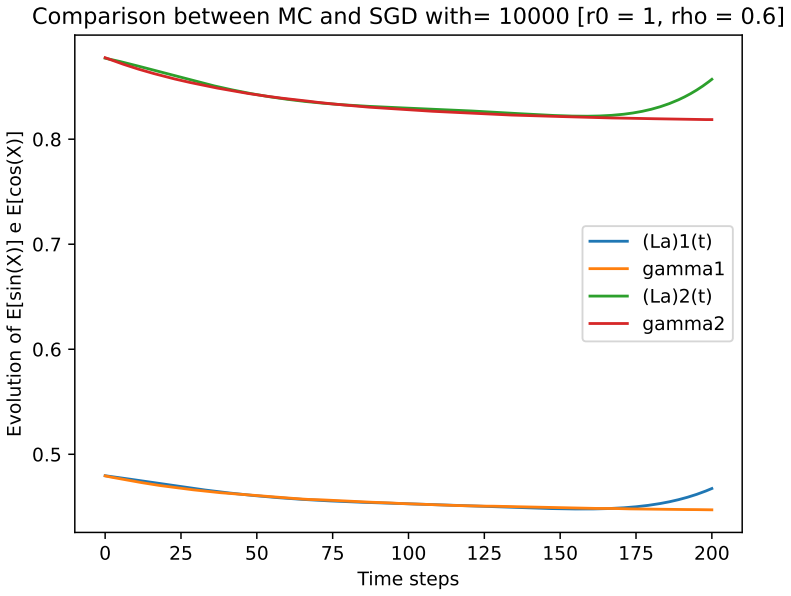
Tabella 110: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 5, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
(La)1(t)
gamma1
(La)2(t)
gamma2
0
25
50
75
100
125
150
175
200
Time steps


Comparison between MC and SGD with= 100 [r0 = 1, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
(La)1(t)
gamma1
(La)2(t)
gamma2
0
25
50
75
100
125
150
175
200
Time steps


Comparison between MC and SGD with= 1000 [r0 = 1, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
25
50
75
100
125
150
175
200
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 10000 [r0 = 1, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
25
50
75
100
125
150
175
200
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2

Case n = 6

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	32.08	143.46	
$\rho = 0.7$	17.72	59.82	
$\rho = 0.8$	45.81	38.29	56.83
$\rho = 0.9$	415.62	20.29	33.01

Tabella 111: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	530	5630	2424	4710	21280	10855		overflow	
$\rho = 0.7$	380	3370	1336	990	8130	4511		overflow	
$\rho = 0.8$	900	12100	3456	1050	4840	2893	2490	6850	4171
$\rho = 0.9$	7640	49999	31384.5	440	4390	1532	1620	4050	2405

Tabella 112: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	3.39	14.24	
$\rho = 0.7$	5.58	6.85	
$\rho = 0.8$	19.75	7.34	8.91
$\rho = 0.9$	15.37	5.23	5.81

Tabella 113: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	50	640	219	410	1280	917		overflow	
$\rho = 0.7$	60	710	359	130	900	442		overflow	
$\rho = 0.8$	90	7070	1271	90	870	472	290	990	569
$\rho = 0.9$	130	2720	988	70	670	337	100	750	369

Tabella 114: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	1.07	3.43	
$\rho = 0.7$	0.80	2.40	
$\rho = 0.8$	3.97	2.23	4.02
$\rho = 0.9$	9.14	1.61	2.75

Tabella 115: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	10	50	35	60	170	111		overflow	
$\rho = 0.7$	10	60	26	40	160	78		overflow	
$\rho = 0.8$	10	520	129	30	190	72	70	260	127
$\rho = 0.9$	20	1540	295	20	80	52	50	140	88

Tabella 116: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	2.03	3.15	
$\rho = 0.7$	3.47	2.27	
$\rho = 0.8$	3.84	2.44	7.27
$\rho = 0.9$	3.87	1.91	7.67

Tabella 117: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	5	10	6.8	8	24	10.7		overflow	
$\rho = 0.7$	6	32	11.8	6	12	7.7		overflow	
$\rho = 0.8$	7	29	13	5	15	8.3	20	31	23.4
$\rho = 0.9$	8	32	13.1	5	10	6.5	13	50	24.7

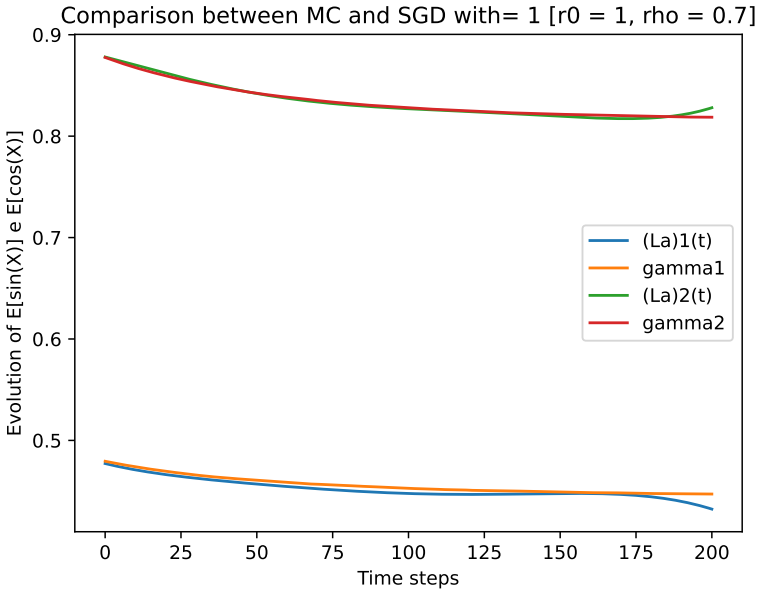
Tabella 118: Number of iterations m to achieve convergence with $M = 1000$

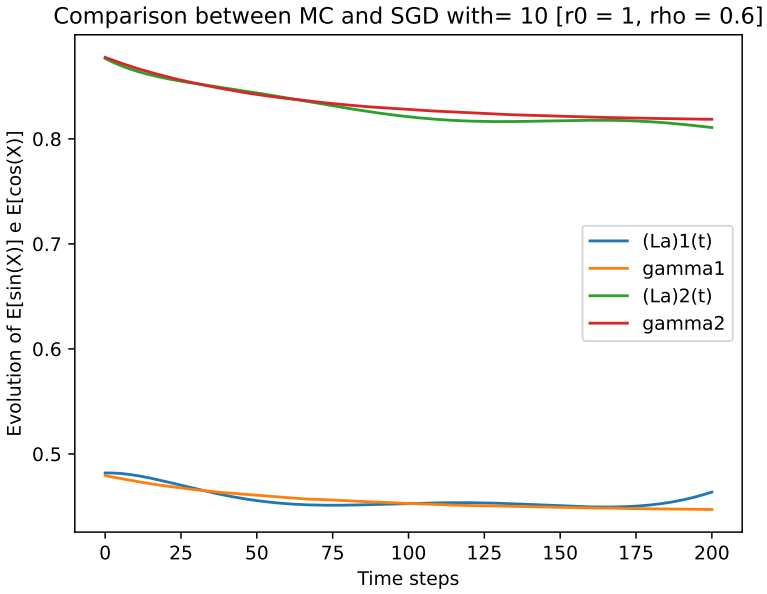
	$r_0 = 1$	$r_0 = 5$	$r_0 = 10$
$\rho = 0.6$	17.80	27.06	
$\rho = 0.7$	20.13	19.79	
$\rho = 0.8$	23.78	16.89	67.48
$\rho = 0.9$	27.38	16.50	45.50

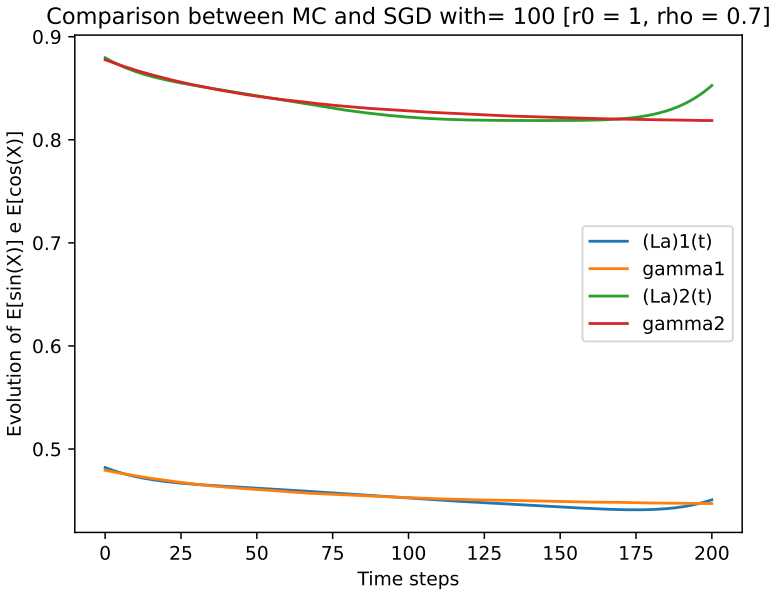
Tabella 119: Average execution times (in seconds s) with $M = 10000$

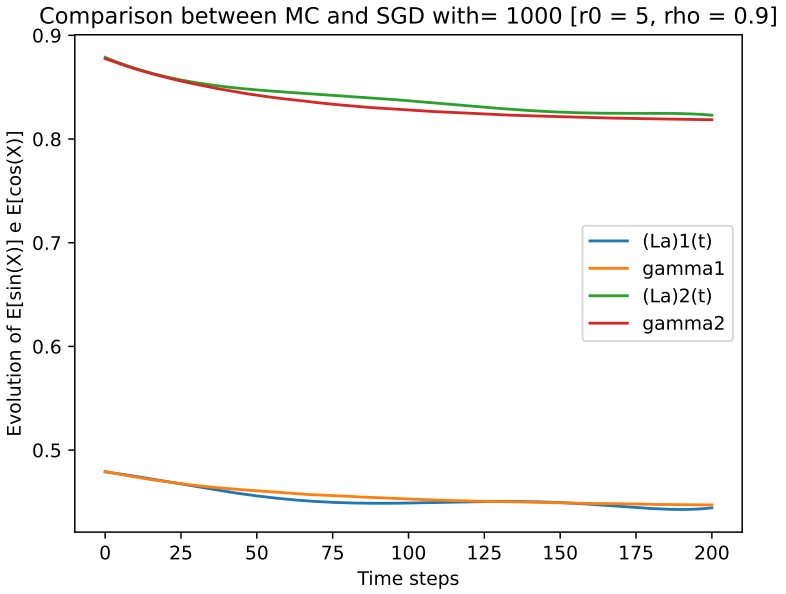
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	5	7	5.4	8	9	8.2		overflow	
$\rho = 0.7$	6	7	6.1	6	6	6		overflow	
$\rho = 0.8$	7	8	7.2	5	6	5.1	17	24	18.8
$\rho = 0.9$	7	9	8.3	5	5	5	12	17	13.6

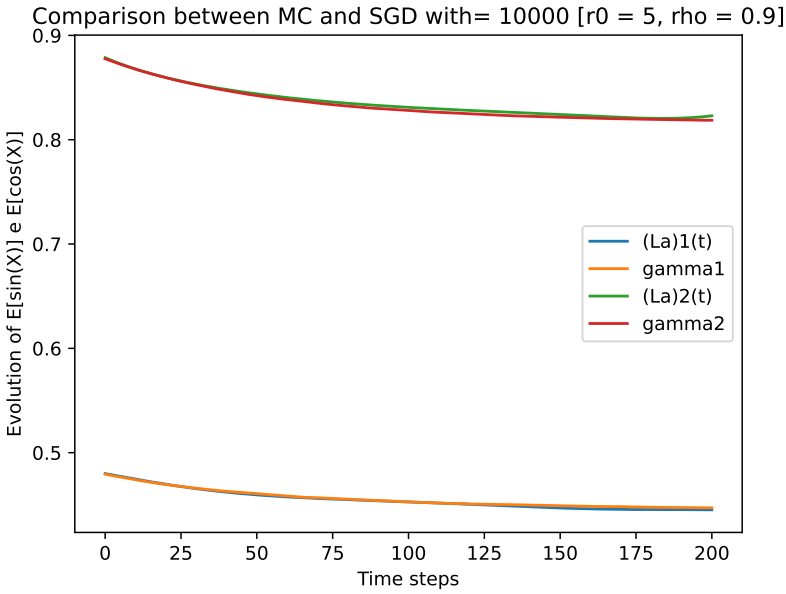
Tabella 120: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 1, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
(La)1(t)
gamma1
(La)2(t)
gamma2
0
25
50
75
100
125
150
175
200
Time steps


Comparison between MC and SGD with= 100 [r0 = 1, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
0.9
0.8
0.7
0.6
0.5
(La)1(t)
gamma1
(La)2(t)
gamma2
0
25
50
75
100
125
150
175
200
Time steps


Comparison between MC and SGD with= 1000 [r0 = 5, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
25
50
75
100
125
150
175
200


Comparison between MC and SGD with= 10000 [r0 = 5, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
25
50
75
100
125
150
175
200

0.6 T = 4

Case n = 3

	$r_0 = 1$
$\rho = 0.6$	148.41
$\rho = 0.7$	98.14
$\rho = 0.8$	99.71
$\rho = 0.9$	269.29

Tabella 121: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	3450	12910	6042		overflow			overflow	
$\rho = 0.7$	1150	7610	3999		overflow			overflow	
$\rho = 0.8$	1030	9770	4062		overflow			overflow	
$\rho = 0.9$	900	49999	10512.9		overflow			overflow	

Tabella 122: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$
$\rho = 0.6$	15.35
$\rho = 0.7$	10.90
$\rho = 0.8$	8.20
$\rho = 0.9$	72.03

Tabella 123: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	170	1180	521		overflow			overflow	
$\rho = 0.7$	110	740	370		overflow			overflow	
$\rho = 0.8$	110	690	280		overflow			overflow	
$\rho = 0.9$	50	12920	2475		overflow			overflow	

Tabella 124: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$
$\rho = 0.6$	3.48
$\rho = 0.7$	5.24
$\rho = 0.8$	5.58
$\rho = 0.9$	15.62

Tabella 125: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	30	140	68		overflow			overflow	
$\rho = 0.7$	20	310	101		overflow			overflow	
$\rho = 0.8$	30	330	108		overflow			overflow	
$\rho = 0.9$	20	1360	307		overflow			overflow	

Tabella 126: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$
$\rho = 0.6$	2.06
$\rho = 0.7$	3.32
$\rho = 0.8$	3.08
$\rho = 0.9$	2.61

Tabella 127: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	4	8	4.7		overflow			overflow	
$\rho = 0.7$	4	16	7.8		overflow			overflow	
$\rho = 0.8$	3	19	7.1		overflow			overflow	
$\rho = 0.9$	4	9	6		overflow			overflow	

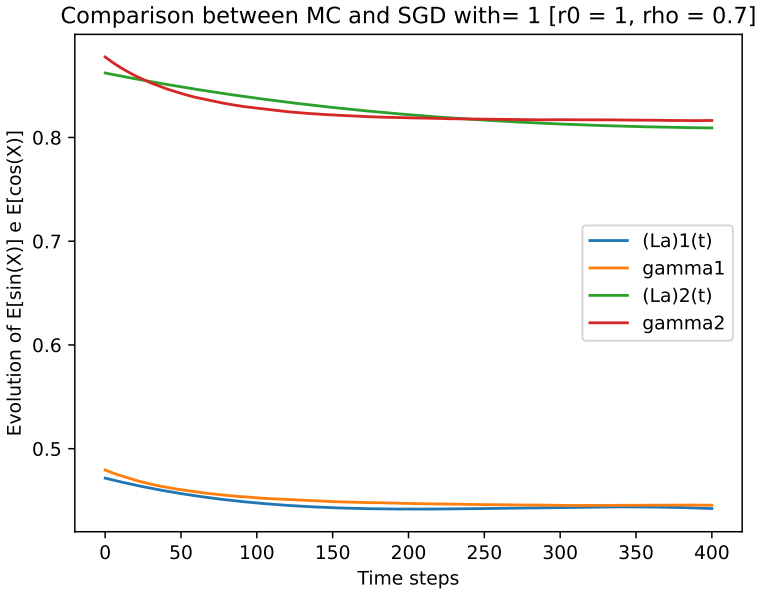
Tabella 128: Number of iterations m to achieve convergence with $M = 1000$

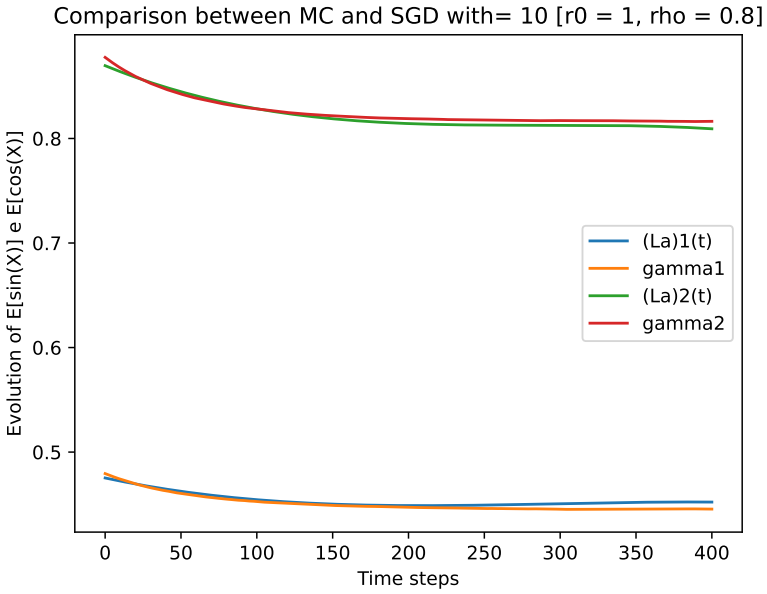
	$r_0 = 1$
$\rho = 0.6$	16.91
$\rho = 0.7$	16.04
$\rho = 0.8$	12.17
$\rho = 0.9$	12.15

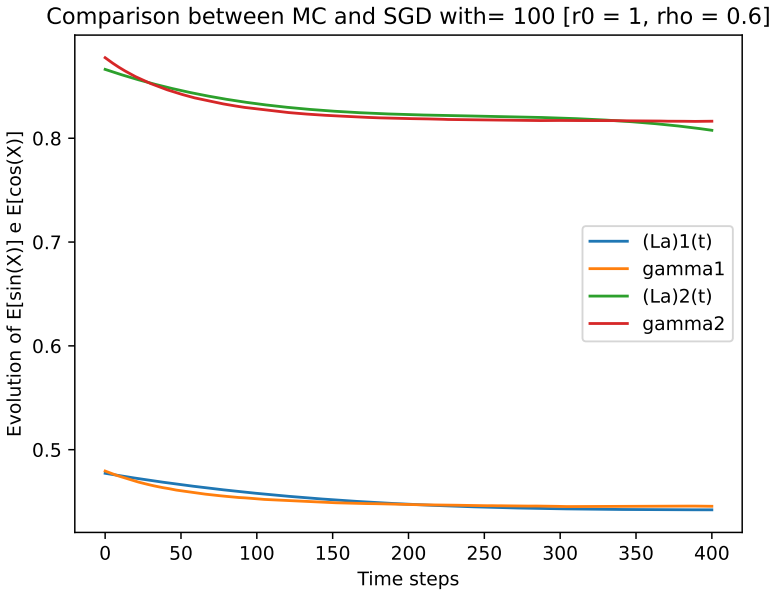
Tabella 129: Average execution times (in seconds s) with $M = 10000$

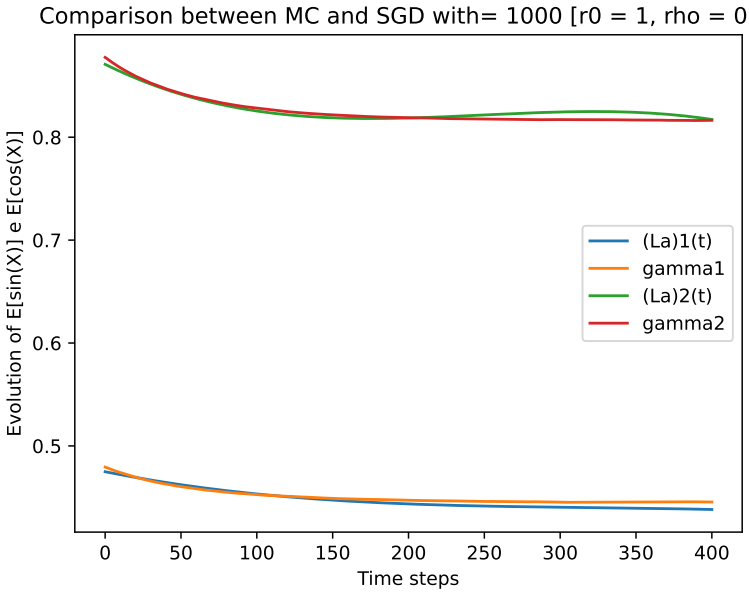
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	4	4	4		overflow			overflow	
$\rho = 0.7$	4	4	4		overflow			overflow	
$\rho = 0.8$	3	3	3		overflow			overflow	
$\rho = 0.9$	3	3	3		overflow			overflow	

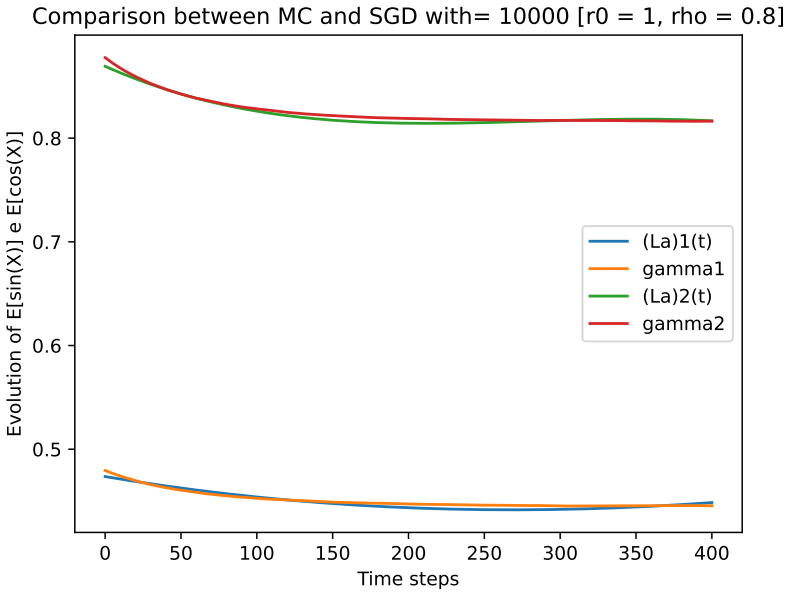
Tabella 130: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 1, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
50
100
150
200
250
300
350
400
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 100 [r0 = 1, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
50
100
150
200
250
300
350
400
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 1000 [r0 = 1, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
(La)1(t)
gamma1
(La)2(t)
gamma2
0
50
100
150
200
250
300
350
400
Time steps


Comparison between MC and SGD with= 10000 [r0 = 1, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
(La)1(t)
gamma1
(La)2(t)
gamma2
0
50
100
150
200
250
300
350
400
Time steps

Case n = 4

	$r_0 = 1$
$\rho = 0.6$	145.11
$\rho = 0.7$	102.23
$\rho = 0.8$	177.10
$\rho = 0.9$	890.75

Tabella 131: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	3320	11980	5645		overflow			overflow	
$\rho = 0.7$	1020	12700	3974		overflow			overflow	
$\rho = 0.8$	740	25910	6650		overflow			overflow	
$\rho = 0.9$	2470	49999	34868.5		overflow			overflow	

Tabella 132: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$
$\rho = 0.6$	13.09
$\rho = 0.7$	13.76
$\rho = 0.8$	22.88
$\rho = 0.9$	313.63

Tabella 133: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	120	820	446		overflow			overflow	
$\rho = 0.7$	250	820	473		overflow			overflow	
$\rho = 0.8$	70	4380	778		overflow			overflow	
$\rho = 0.9$	170	34500	10453		overflow			overflow	

Tabella 134: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$
$\rho = 0.6$	4.71
$\rho = 0.7$	2.61
$\rho = 0.8$	5.88
$\rho = 0.9$	37.38

Tabella 135: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	40	160	88		overflow			overflow	
$\rho = 0.7$	30	90	49		overflow			overflow	
$\rho = 0.8$	10	390	108		overflow			overflow	
$\rho = 0.9$	10	2810	703		overflow			overflow	

Tabella 136: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$
$\rho = 0.6$	4.57
$\rho = 0.7$	3.25
$\rho = 0.8$	3.55
$\rho = 0.9$	4.56

Tabella 137: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	4	18	9.5		overflow			overflow	
$\rho = 0.7$	3	27	6.7		overflow			overflow	
$\rho = 0.8$	3	18	7.4		overflow			overflow	
$\rho = 0.9$	3	41	9.5		overflow			overflow	

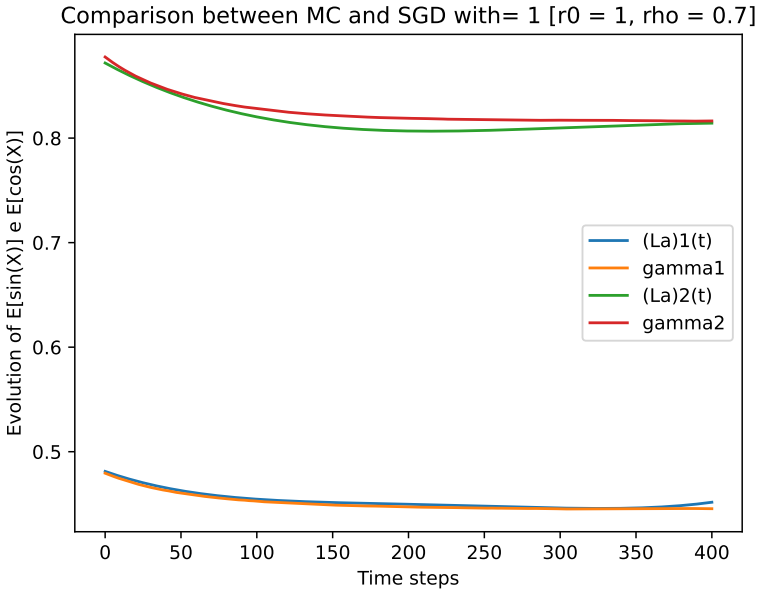
Tabella 138: Number of iterations m to achieve convergence with $M = 1000$

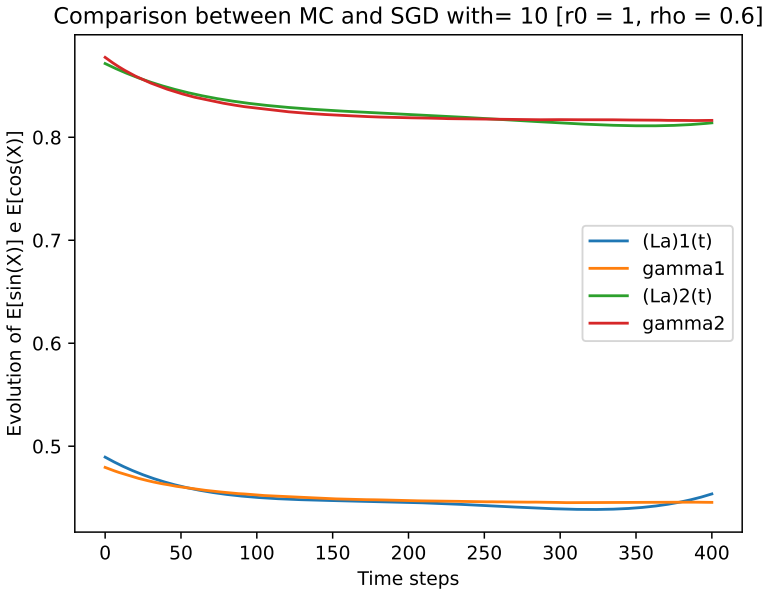
	$r_0 = 1$
$\rho = 0.6$	14.42
$\rho = 0.7$	14.35
$\rho = 0.8$	14.42
$\rho = 0.9$	14.42

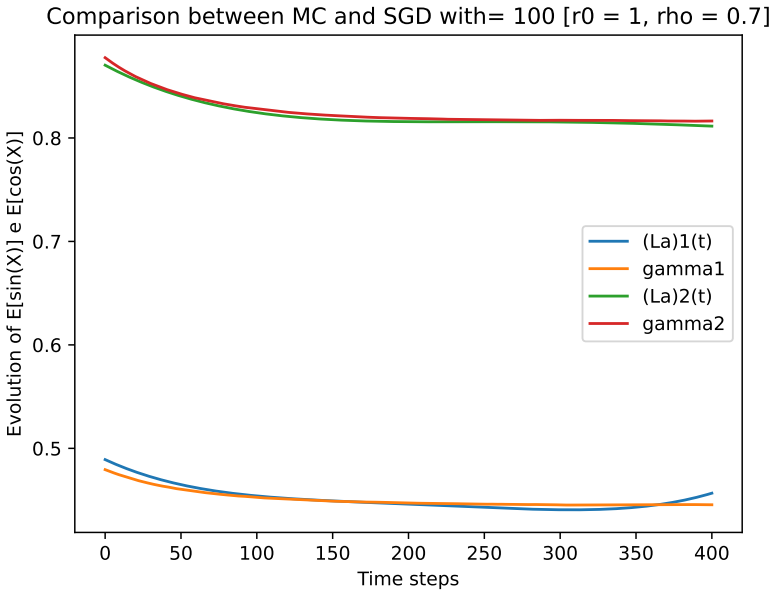
Tabella 139: Average execution times (in seconds s) with $M = 10000$

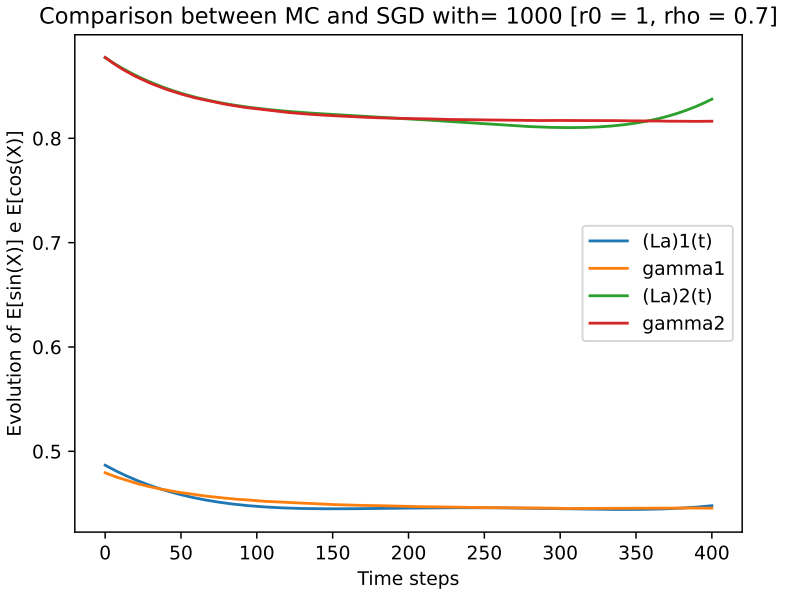
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	3	3	3		overflow			overflow	
$\rho = 0.7$	3	3	3		overflow			overflow	
$\rho = 0.8$	3	3	3		overflow			overflow	
$\rho = 0.9$	3	3	3		overflow			overflow	

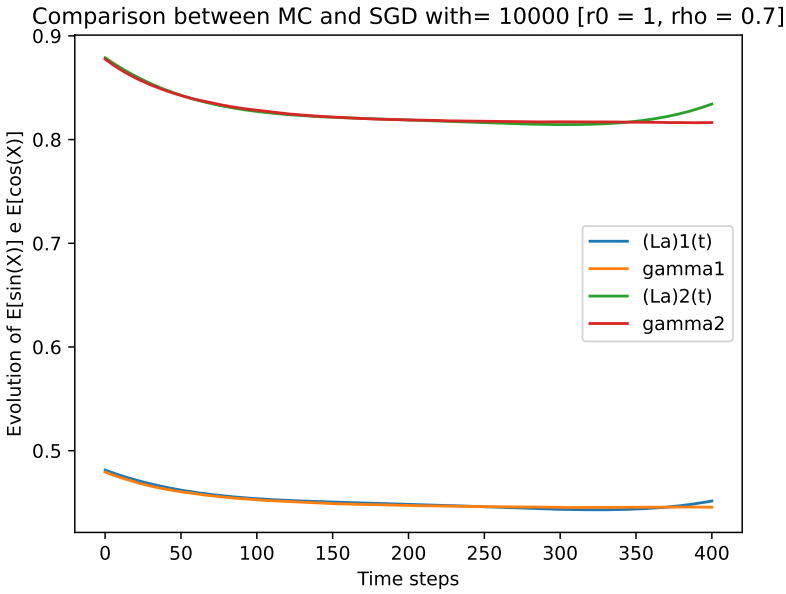
Tabella 140: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 1, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
50
100
150
200
250
300
350
400
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 100 [r0 = 1, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
0
50
100
150
200
250
300
350
400
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 1000 [r0 = 1, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
0.5
0.6
0.7
0.8
0
50
100
150
200
250
300
350
400
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2


Comparison between MC and SGD with= 10000 [r0 = 1, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
0.5
0.6
0.7
0.8
0.9
0
50
100
150
200
250
300
350
400
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2

Case n = 5

	$r_0 = 1$
$\rho = 0.6$	187.19
$\rho = 0.7$	100.55
$\rho = 0.8$	343.00
$\rho = 0.9$	1090.67

Tabella 141: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	2410	17200	7261		overflow			overflow	
$\rho = 0.7$	1490	6980	3871		overflow			overflow	
$\rho = 0.8$	640	42740	13311		overflow			overflow	
$\rho = 0.9$	2840	49999	41910.3		overflow			overflow	

Tabella 142: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$
$\rho = 0.6$	30.49
$\rho = 0.7$	13.62
$\rho = 0.8$	22.37
$\rho = 0.9$	423.87

Tabella 143: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	120	3880	1020		overflow			overflow	
$\rho = 0.7$	180	860	455		overflow			overflow	
$\rho = 0.8$	120	2540	743		overflow			overflow	
$\rho = 0.9$	190	38600	14211		overflow			overflow	

Tabella 144: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$
$\rho = 0.6$	3.98
$\rho = 0.7$	3.58
$\rho = 0.8$	13.71
$\rho = 0.9$	108.12

Tabella 145: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	30	150	68		overflow			overflow	
$\rho = 0.7$	20	100	61		overflow			overflow	
$\rho = 0.8$	40	910	234		overflow			overflow	
$\rho = 0.9$	10	12040	1843		overflow			overflow	

Tabella 146: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$
$\rho = 0.6$	4.24
$\rho = 0.7$	7.74
$\rho = 0.8$	2.61
$\rho = 0.9$	2.29

Tabella 147: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	3	14	7.8		overflow			overflow	
$\rho = 0.7$	3	58	14.2		overflow			overflow	
$\rho = 0.8$	3	7	4.8		overflow			overflow	
$\rho = 0.9$	2	6	4.2		overflow			overflow	

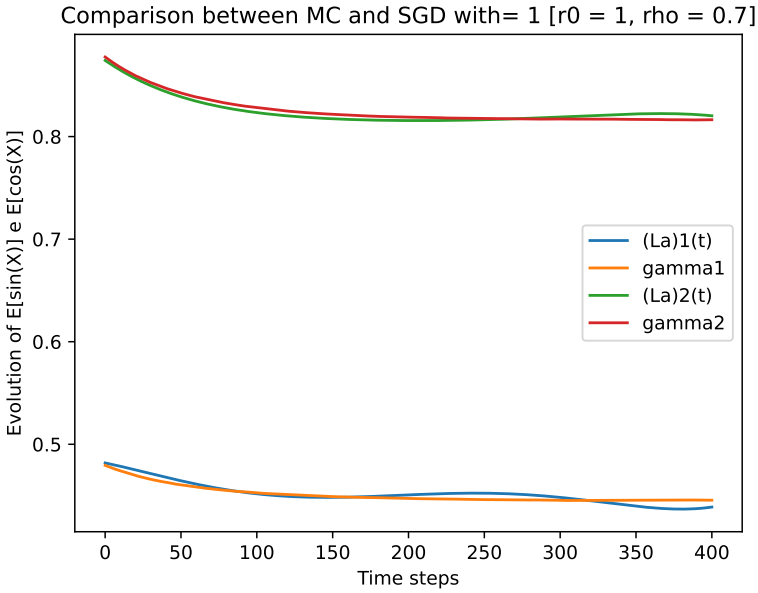
Tabella 148: Number of iterations m to achieve convergence with $M = 1000$

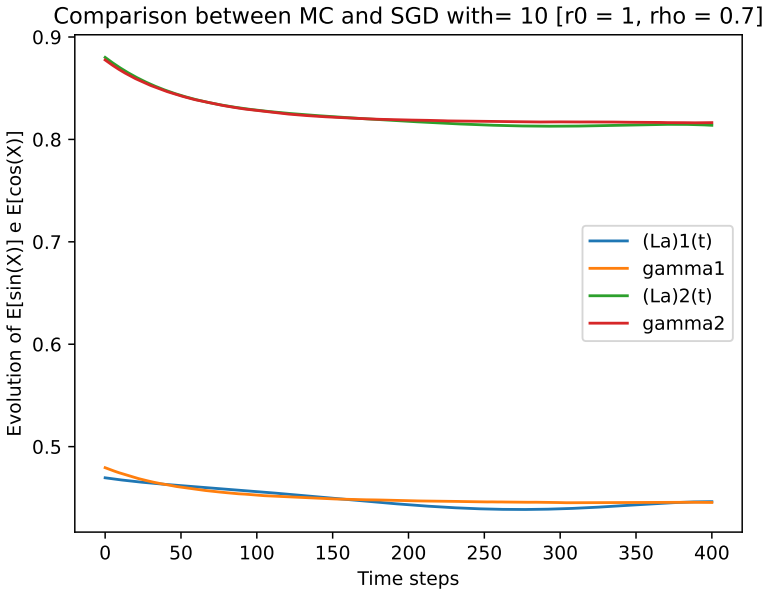
	$r_0 = 1$
$\rho = 0.6$	15.91
$\rho = 0.7$	12.07
$\rho = 0.8$	11.51
$\rho = 0.9$	13.16

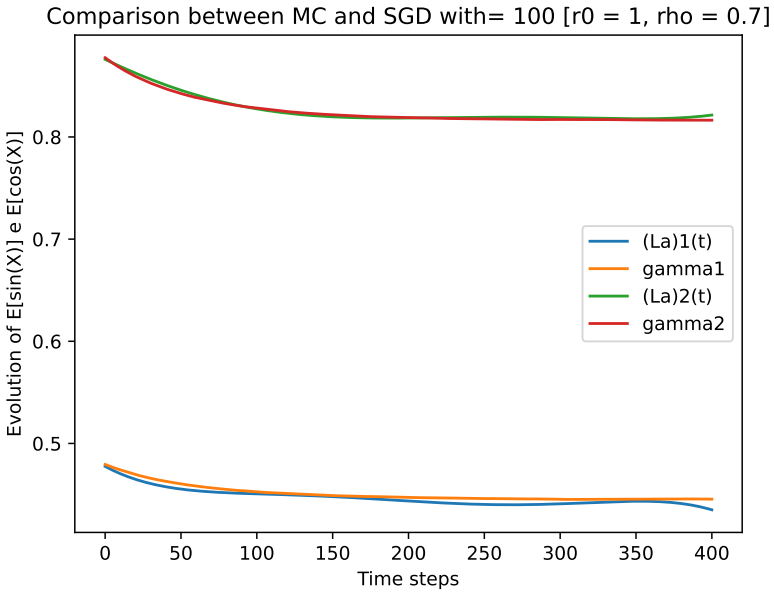
Tabella 149: Average execution times (in seconds s) with $M = 10000$

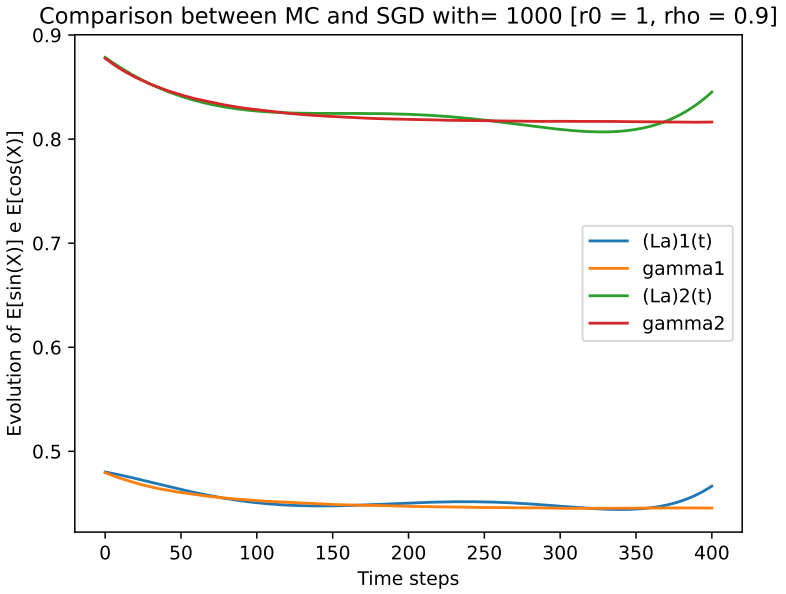
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	2	3	2.9		overflow			overflow	
$\rho = 0.7$	2	3	2.2		overflow			overflow	
$\rho = 0.8$	2	3	2.1		overflow			overflow	
$\rho = 0.9$	2	3	2.4		overflow			overflow	

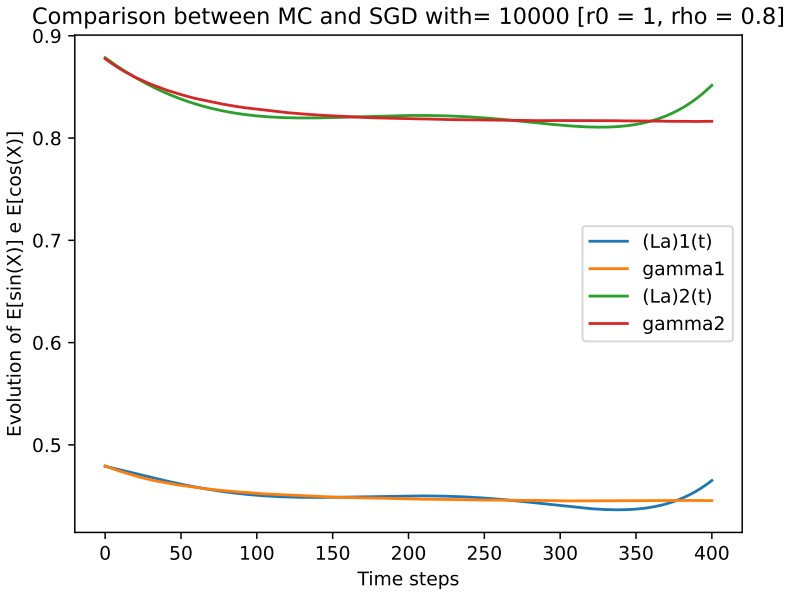
Tabella 150: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 1, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
50
100
150
200
250
300
350
400


Comparison between MC and SGD with= 100 [r0 = 1, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.8
0.7
0.6
0.5
0
50
100
150
200
250
300
350
400


Comparison between MC and SGD with= 1000 [r0 = 1, rho = 0.9]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
50
100
150
200
250
300
350
400


Comparison between MC and SGD with= 10000 [r0 = 1, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
50
100
150
200
250
300
350
400

Case n = 6

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	146.81	
$\rho = 0.7$	104.65	
$\rho = 0.8$	412.79	241.22
$\rho = 0.9$	813.41	195.51

Tabella 151: Average execution times (in seconds s) with $M = 1$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	2280	10450	5647		overflow			overflow	
$\rho = 0.7$	1460	9530	4026		overflow			overflow	
$\rho = 0.8$	660	49999	15788.9	3320	17940	9134		overflow	
$\rho = 0.9$	820	49999	31095.4	2200	26720	7477		overflow	

Tabella 152: Number of iterations m to achieve convergence with $M = 1$

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	26.03	
$\rho = 0.7$	9.16	
$\rho = 0.8$	67.37	32.54
$\rho = 0.9$	452.89	45.88

Tabella 153: Average execution times (in seconds s) with $M = 10$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	190	2900	848		overflow			overflow	
$\rho = 0.7$	120	620	296		overflow			overflow	
$\rho = 0.8$	170	7200	2200	310	2520	1056		overflow	
$\rho = 0.9$	50	49999	14881.8	260	4210	1503		overflow	

Tabella 154: Number of iterations m to achieve convergence with $M = 10$

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	6.87	
$\rho = 0.7$	5.74	
$\rho = 0.8$	9.53	18.47
$\rho = 0.9$	34.43	36.04

Tabella 155: Average execution times (in seconds s) with $M = 100$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	50	230	111		overflow			overflow	
$\rho = 0.7$	20	240	93		overflow			overflow	
$\rho = 0.8$	20	650	154	80	390	269		overflow	
$\rho = 0.9$	20	3830	557	70	1110	528		overflow	

Tabella 156: Number of iterations m to achieve convergence with $M = 100$

	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	4.71	
$\rho = 0.7$	3.50	
$\rho = 0.8$	3.03	46.11
$\rho = 0.9$	7.87	131.98

Tabella 157: Average execution times (in seconds s) with $M = 1000$

	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	4	26	7.6		overflow			overflow	
$\rho = 0.7$	3	12	5.6		overflow			overflow	
$\rho = 0.8$	2	9	4.9	27	120	73.5		overflow	
$\rho = 0.9$	3	79	12.7	27	503	212.4		overflow	

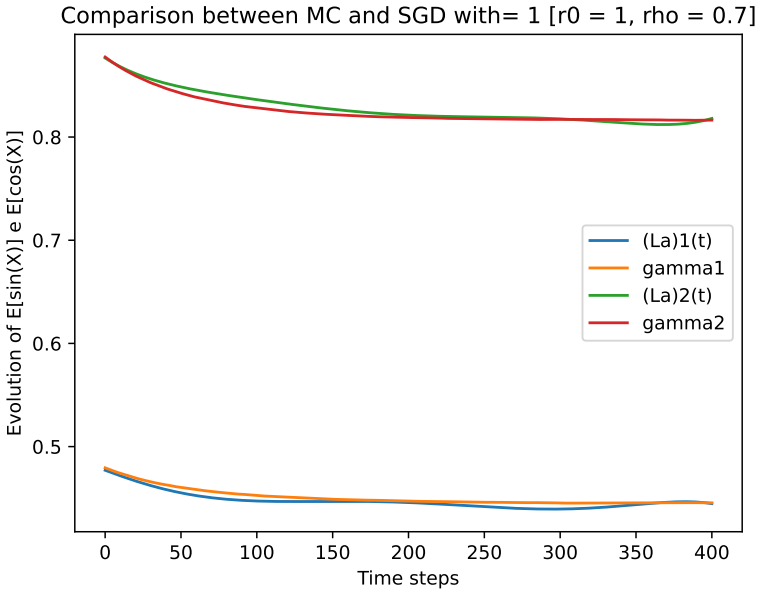
Tabella 158: Number of iterations m to achieve convergence with $M = 1000$

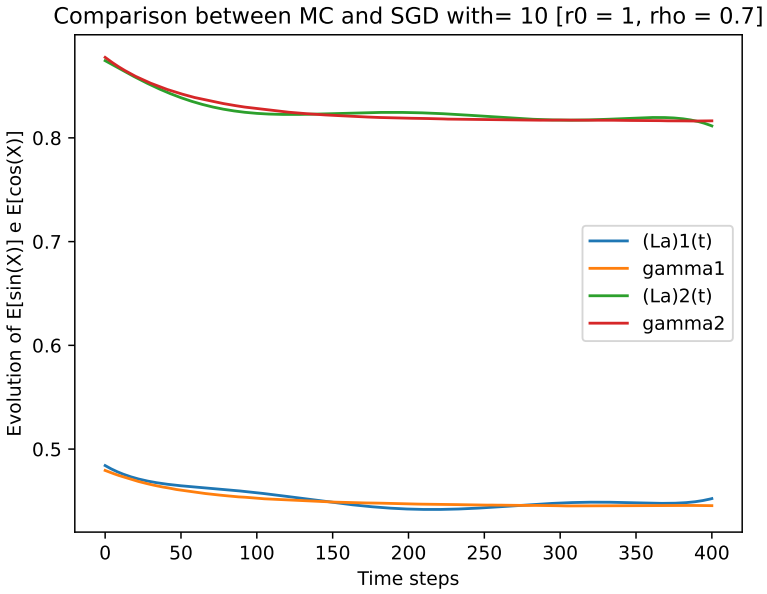
	$r_0 = 1$	$r_0 = 5$
$\rho = 0.6$	13.45	
$\rho = 0.7$	14.13	
$\rho = 0.8$	14.80	339.72
$\rho = 0.9$	20.18	402.01

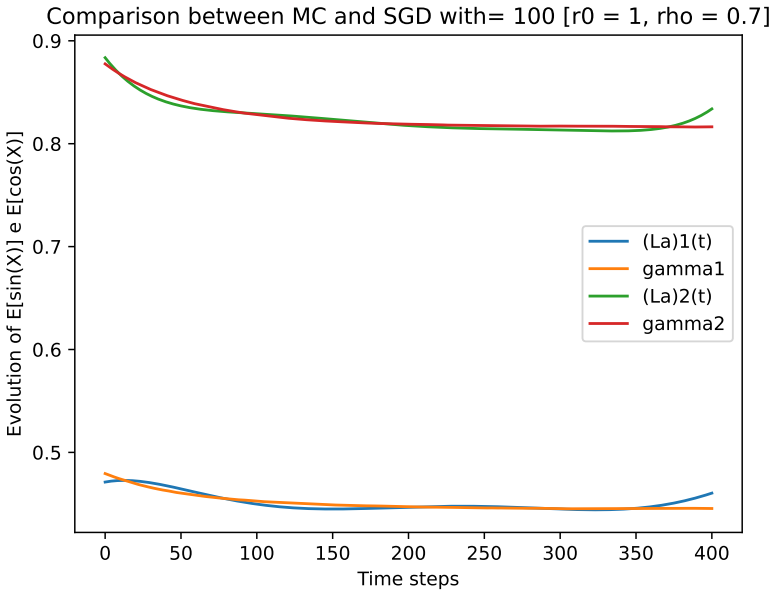
Tabella 159: Average execution times (in seconds s) with $M = 10000$

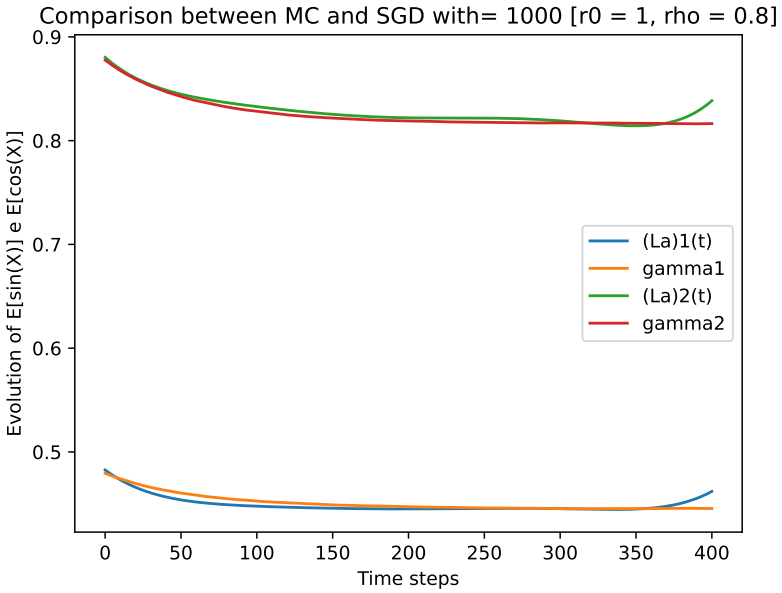
	$r_0 = 1$ min	$r_0 = 1$ max	$r_0 = 1$ average	$r_0 = 5$ min	$r_0 = 5$ max	$r_0 = 5$ average	$r_0 = 10$ min	$r_0 = 10$ max	$r_0 = 10$ average
$\rho = 0.6$	2	2	2		overflow			overflow	
$\rho = 0.7$	2	3	2.1		overflow			overflow	
$\rho = 0.8$	2	3	2.2	16	96	49.2		overflow	
$\rho = 0.9$	3	3	3	15	187	60.5		overflow	

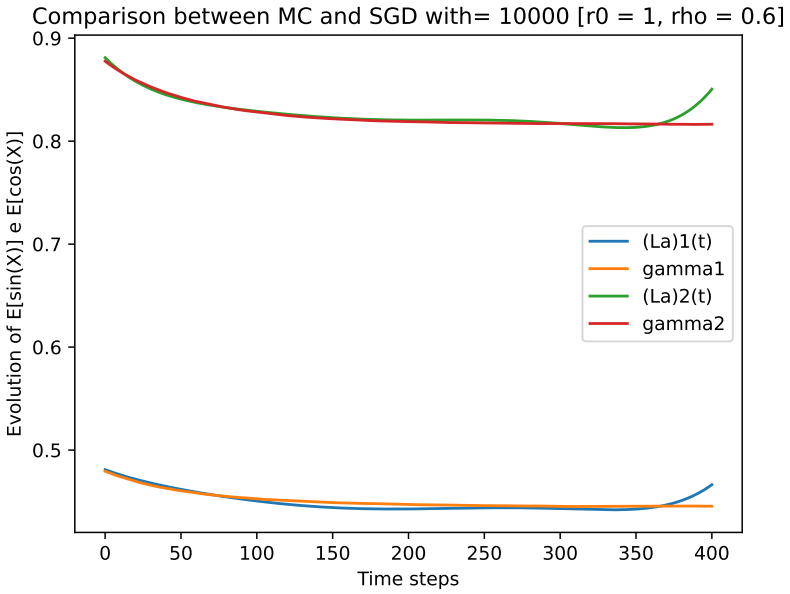
Tabella 160: Number of iterations m to achieve convergence with $M = 10000$




Comparison between MC and SGD with= 10 [r0 = 1, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
0.8
0.7
0.6
0.5
(La)1(t)
gamma1
(La)2(t)
gamma2
0
50
100
150
200
250
300
350
400
Time steps


Comparison between MC and SGD with= 100 [r0 = 1, rho = 0.7]
Evolution of E[sin(X)] e E[cos(X)]
0.9
0.8
0.7
0.6
0.5
(La)1(t)
gamma1
(La)2(t)
gamma2
0
50
100
150
200
250
300
350
400
Time steps


Comparison between MC and SGD with= 1000 [r0 = 1, rho = 0.8]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
50
100
150
200
250
300
350
400


Comparison between MC and SGD with= 10000 [r0 = 1, rho = 0.6]
Evolution of E[sin(X)] e E[cos(X)]
Time steps
(La)1(t)
gamma1
(La)2(t)
gamma2
0.9
0.8
0.7
0.6
0.5
0
50
100
150
200
250
300
350
400

0.7 Observations and Conclusions

The study showed that as the parameter M increases (leaving the others fixed), execution times and the number of iterations needed to achieve convergence are considerably reduced. However, in the case of $M = 10^4$ the times increase again compared to the previous oness. Therefore, it follows that $M = 10^3$ is the best case study for both time and number of iterations. This result is independent of the choice of T.

As regards the sections $T = 0.5$ and $T = 1$, we have that for small values of M the choices of $r_0 = 1$ and $\rho = 0.6, 0.7$ often generate the best case considering times and number of iterations. But when we increase M, the values that are most efficient are $r_0 = 5, 10$ and $\rho = 0.8, 0.9$, rather than $r_0 = 1$ which has relatively larger iteration numbers.

Observing the graphs, a recurring peak is found in the final instant of the approximate solution. However, this peak disappears when you look at the graph related to the same case history belonging to the section with the next final time. This is evidence that the cause of this peak can be given by the choice of the polynomial base and not by the approximation itself.

Considering the outcomes of the study, the convergence of the method slows down as the final instant increases T. There is also, always with the increase of T, a greater presence of non convergence of the method, mainly in cases with $M = 1$.

Finally, we note that in the cases $T = 2$ and $T = 4$ the phenomenon of overflow occurs. In the section $T = 2$ it only concerns the case $r_0 = 10$, with any ρ or n, except for the case $n = 6$ where it only applies for the values $\rho = 0.6$ and 0.7. As for the section $T = 4$ it occurs for all cases with $r_0 = 10$. It also occurs for all cases with $r_0 = 5$, except for the case $n = 6$ where it occurs only for the values $\rho = 0.6$ and 0.7.