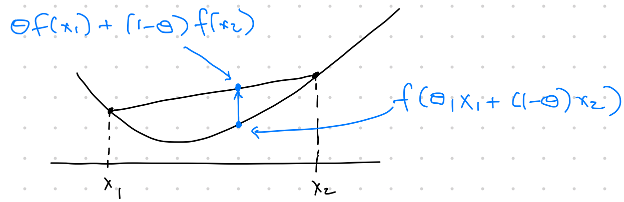
Convex Function Basics

Definition: A function $f: \mathbb{R}^n \to \mathbb{R}$ is **convex** if for $0 \le \theta \le 1$,

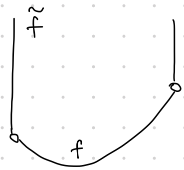
$$f(\theta x_1 + (1-\theta) x_2) \le \theta f(x_1) + (1-\theta) f(x_2)$$



The function is **strictly convex** if the inequality is strict $\forall\, x_1 \neq x_2$.

Extended Value Functions: It's convenient to have f be defined over all $\mathbb{R}^n$. We can extend any convex function f w/ $\operatorname{dom} f \subset \mathbb{R}^n$ to $\tilde{f}: \mathbb{R}^n \to \mathbb{R}$:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \operatorname{dom} f \\ \infty & \text{else} \end{cases}$$



allows us to just worry about $f: \mathbb{R}^n \to \mathbb{R}$, eg. in this thm.

w/ the convension $\infty \le \infty$, $\tilde{f}$ is still convex.

Thm: A function is convex iff it is convex along all lines emminating from any point in the domain, ie: "Convexity is defined over lines".

$$f: \mathbb{R}^n \to \mathbb{R} \text{ cvx iff } g(t) = f(x + tv) \text{ cvx } \forall\, x, v \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

pf: ($\Rightarrow$) $f: \mathbb{R}^n \to \mathbb{R}$ cvx.

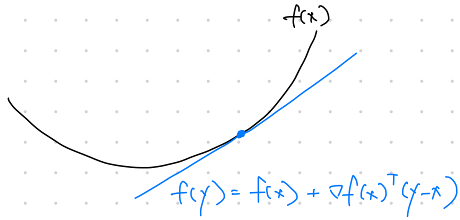
$$\begin{aligned} &g(\theta t_1 + (1-\theta) t_2) \\ &= f(x + (\theta t_1 + (1-\theta) t_2) v) \\ &= f(\theta(x + t_1 v) + (1-\theta)(x + t_2 v)) \\ &\le \theta f(x + t_1 v) + (1-\theta) f(x + t_2 v) \\ &= \theta g(t_1) + (1-\theta) g(t_2) \end{aligned}$$

($\Leftarrow$) $g(t) = f(x + tv)$ is cvx.

$$\begin{aligned} &g(\theta(1) + (1-\theta)(0)), \quad 0 \le \theta \le 1 \\ &= f(x_1 + \theta(x_2 - x_1)) \\ &\le \theta g(1) + (1-\theta) g(0) \\ &= \theta f(x_2) + (1-\theta) f(x_1) \end{aligned}$$

First-Order Condition: $f: \mathbb{R}^n \to \mathbb{R}$ is convex iff the 1-st order Taylor is a global underestimator $\forall x \in \mathbb{R}^n$.

$$f(y) \geq f(x) + \nabla f(x)^T (y-x) \quad \forall x, y$$



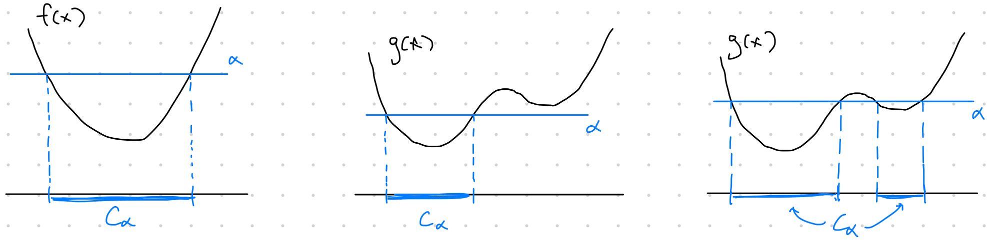
Second-Order Condition: $f: \mathbb{R}^n \to \mathbb{R}$ is convex iff f has positive curvature everywhere, ie. $\nabla^2 f(x) \succeq 0 \quad \forall x$.

cvx   not cvx

Geometrically, gaps between contour lines in <u>any dir</u> should be monotonically increasing (convex) or decreasing (concave).

Sublevel Sets: A sublevel set of $f: \mathbb{R}^n \to \mathbb{R}$ is defined as:

$$C_\alpha = \{ x \in \mathrm{dom}\, f \mid f(x) \le \alpha \}$$



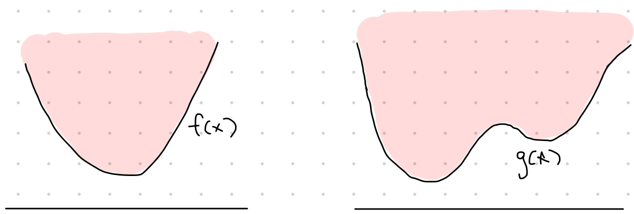
Convex

not Convex.

If f is convex, C_α is a convex set. The converse is not true.

Epigraph: The epigraph of $f: \mathbb{R}^n \to \mathbb{R}$ is defined as:

$$\mathrm{epi}\, f = \{ (x,t) \in \mathbb{R}^{n+1} \mid t \ge f(x) \}.$$



Convex

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$f: \mathbb{R}^n \to \mathbb{R}$ is convex iff $\mathrm{epi}\, f$ is a convex set.