# Optimization Problems

**Standard Form**: min. $f_0(x)$ w.r.t. $x \in \mathbb{R}$

s.t. $f_i(x) \le 0$ , $i = 1, \dots, m$

$h_j(x) = 0$ , $j = 1, \dots, k$.

Implicit: x in all domains (or use extended value functions).

**Globally optimal**: $p^* = \inf_x \{ f_0(x) \mid f_i(x) \le 0, \ h_j(x) = 0 \}$.

$p^* = \infty$ if infeasible, $p^* = -\infty$ if unbounded below.

**Locally optimal**: $f_0(x) \le f_0(y) \quad \forall y \in \|x - y\|_2 \le R$

**Feasibility Problem**: min. $0$ wrt $x$

s.t. $f_i(x) \le 0$, $h_i(x) = 0$.

$p^* = 0$ & any $x$ satisfying constraint is optimal, else $p^* = \infty$.

**Equivalent Problems**: Can go from solution of one to another.

**Change of Variables**: For invertible $\phi : \mathbb{R}^n \to \mathbb{R}^n$ $\phi(z) = x$,

min. $f_0(\phi(z))$ wrt. $z$

s.t. $f_i(\phi(z)) \le 0$, $h_i(\phi(z)) = 0$ is equivalent.

$x^* = \phi^{-1}(z^*)$ to recover optimal $x$.

**Function Composition**: min. $\psi_0(f_0(x))$ w.r.t. $x$

s.t. $\psi_i(f_i(x)) \le 0$ , $\psi_j(h_j(x)) = 0$

is equiv. if: $\psi_0$ is strictly monotone increasing,

$\psi_i(z) \le 0$ iff $z \le 0$

$\psi_j(z) = 0$ iff $z = 0$

$x^*$ is also optimal for old problem.

**Slack Variables:** $f_i(x) \le 0$ iff $\exists\, s_i \ge 0$ $f_i(x) + s_i = 0$

$$\min.\ f_0(x) \quad \text{wrt. } x, s_i$$
$$\text{s.t. } f_i(x) + s_i = 0, \quad h_i(x) = 0.$$

$x^*$ optimal for old problem.

unlike change of vars, this does not need to be invertible.

**Eliminate Equality Constraint:** If $\exists\, \phi : \mathbb{R}^k \to \mathbb{R}^n$ s.t. $x = \phi(z)$ parameterizes feasible $x$ that satisfy equality constraints,

$$\min.\ f_0(\phi(z)) \quad \text{wrt. } z$$
$$\text{s.t. } f_i(\phi(z)) \le 0$$

is equivalent, and $x^* = \phi(z^*)$.

eg: $\min.\ f_0(x) \quad \text{wrt. } x$

s.t. $f_i(x) \le 0$

$Ax = b$.

$\Rightarrow$ Find one $x_0$ s.t. $Ax_0 = b$. Let $R(F) = N(A)$.

$\Rightarrow$ $\min.\ f_0(Fz + x_0) \quad \text{wrt. } z$

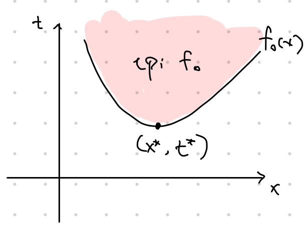
s.t. $f_i(Fz + x_0) \le 0$

is equivalent, $x^* = Fz^* + x_0$

**Optimize Out Variables:** $\inf_{x,y} f(x,y) = \inf_x \tilde{f}(x) = \inf_x \left[\inf_y f(x,y)\right]$

**Epigraph Form:** Do the optimization in "graph space"

$$\min.\ t \quad \text{wrt } x \in \mathbb{R}^n, t \in \mathbb{R}$$
$$\text{s.t. } f_0(x) \le t$$
$$f_i(x) \le 0, \quad h_i(x) = 0.$$




use $x^*$ in optimum $(x^*, t^*)$