Convexity-Preserving Operations

Non-negative Scaling: If f cvx, $\alpha f(x)$ cvx for $\alpha \geq 0$.

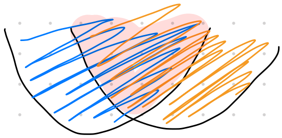
Addition: If f_1, f_2 cvx, $f_1 + f_2$ cvx.

pf.
$$\begin{aligned}(f_1+f_2)(\theta x+(1-\theta)y) &= f_1(\theta x+(1-\theta)y) + f_2(\theta x+(1-\theta)y) \\ &\leq \theta(f_1(x)+f_2(x)) + (1-\theta)(f_1(y)+f_2(y)) \\ &= \theta(f_1+f_2)(x) + (1-\theta)(f_1+f_2)(y).\end{aligned}$$

Composition w/ Linear: If f cvx, $g(x) = f(Ax+b)$ cvx.

pf:
$$\begin{aligned}g(\theta x+(1-\theta)y) &= f(A(\theta x+(1-\theta)y)+b) \\ &= f(\theta(Ax+b) + (1-\theta)(Ay+b)) \\ &\leq \theta f(Ax+b) + (1-\theta) f(Ay+b) \\ &= \theta g(x) + (1-\theta) g(y).\end{aligned}$$

Max: If $f_1, \dots, f_n$ are cvx, $g(x) = \max\{f_1(x), \dots, f_n(x)\}$ is cvx.



$\text{epi } g = \bigcap_i \text{epi } f_i$, which is a cvx set.

Supremum: If $f(x,y)$ is cvx in x, then $g(x) = \sup_y f(x)$ is cvx.

pf:
$$\begin{aligned}g(\theta x_1+(1-\theta)x_2) &= \sup_y f(\theta x_1+(1-\theta)x_2, y) \\ &\leq \sup_y \theta f(x_1,y) + (1-\theta) f(x_2,y) \\ &\leq \theta\left[\sup_y f(x_1,y)\right] + (1-\theta)\left[\sup_y f(x_2,y)\right] \\ &= \theta g(x_1) + (1-\theta) g(x_2)\end{aligned}$$

$$\text{epi } g(x) = \underbrace{\bigcap_y \text{epi } f(x,y)}_{\text{Intersection of infinite cvx sets} \Rightarrow \text{cvx.}}$$

For each y, a cvx set

Composition: For scalar functions, $g(x) = h(f(x))$ cvx if

① h is cvx & non-decreasing, f cvx

② h is cvx & non-increasing, f $-$cvx

pf: $g'(x) = h'(f(x))f'(x)$

$$g''(x) = h''(f(x))f'(x)^2 + h'(f(x))f''(x)$$

① : $g''(x) = [\geq 0][\geq 0] + [\geq 0][\geq 0] \geq 0$

② : $g''(x) = [\geq 0][\geq 0] + [\leq 0][\leq 0] \geq 0$

Generalize to vector functions: $f: \mathbb{R}^n \to \mathbb{R}^k$, $h: \mathbb{R}^k \to \mathbb{R}$.

$g(x) = h(f_1(x), \ldots, f_k(x))$, each $f_i: \mathbb{R}^n \to \mathbb{R}$

Then, composition rules apply to each arg, ie.

- g_i cvx & $\nearrow$, f_i cvx.
- g_i cvx & $\searrow$, f_i $-$cvx.

Minimum: If $f(x,y)$ is cvx in (x,y), and if C is a cvx set, then

$g(x) = \inf_{y \in A} f(x,y)$ is cvx.

pf: Let: $y_1 = \text{argmin}_y\, f(x_1,y)$, $y_2 = \text{argmin}_y\, f(x_2,y)$

("argmin" may not be attainable, general gist is fine)

$\Rightarrow g(\theta x_1 + (1-\theta)x_2) = \inf_y f(\theta x_1 + (1-\theta)x_2, y)$

$\leq f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2)$

by def of inf, if is $\leq$ any particular choice of y.

$\leq \theta f(x_1,y_1) + (1-\theta)f(x_2,y_2)$

$= \theta g(x_1) + (1-\theta)g(x_2)$

Perspective: If $f: \mathbb{R}^n \to \mathbb{R}$ cvx, then its perspective

$g: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ $g(x,t) = t f(x/t)$ is also cvx.

pf: $(x, t, r) \in \text{epi } g \Rightarrow r \geq t f(x/t)$

$\Rightarrow r/t \geq f(x/t)$

$\Rightarrow (x/t, r/t) \in \text{epi } f.$

$\Rightarrow \text{per}(\text{epi } g) = \{(x/t, r/t)\} = \text{epi } f$

$\Rightarrow \text{epi } g = \text{per}^{-1}(\text{epi } f)$, epi f cvx $\Rightarrow$ epi g cvx $\Rightarrow g$ cvx.