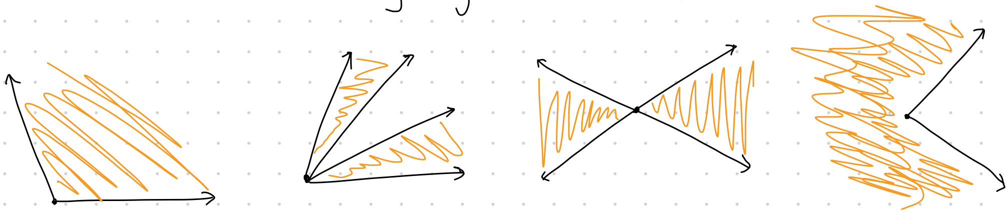
Generalized Inequalities

A cone is a **proper cone** if it is convex, closed, has an interior, and contains no lines (only rays, ie. it is pointed).



proper

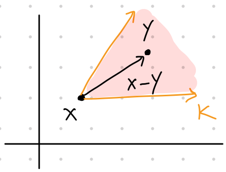
not proper

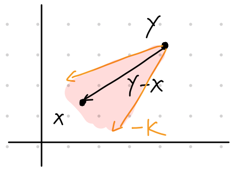
No rays condition: if $x, -x \in K$, $x=0$.

A proper cone K can define a partial ordering **generalized inequality**:

$x \preceq_K y$ if $y-x \in K$

Alternate: $x-y \in -K$





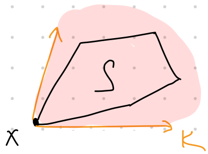
The inequality is **strict** if: $x \prec_K y$ if $y-x \in \text{int } K$.

Examples:

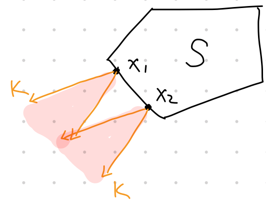
- Regular element-wise ordering on $\mathbb{R}^n$ is defined by cone $\mathbb{R}^n_+$.
- Cone $P \in S^n_+$ defines matrix inequality, i.e. $X \preceq Y$ if $Y-X \in S^n_+$

Since $\preceq_K$ only defines a partial ordering, not all elements are comparable and there may not be a "minimum" element.

$x \in S$ is **minimum** if $\forall y \in S$, $x \preceq_K y$, i.e. $S \subseteq x + K$.



$x \in S$ is **minimal** if $\forall y \in S$ $y \not\preceq_K x$, i.e. $(x - K) \cap S = \{x\}$.



not unique: x_1, x_2 both minimal.