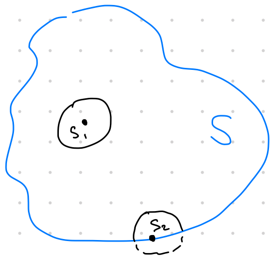
Sets

Interior In Euclidean space, a point $s \in S$ is an **interior point** if there is a Euclidean ball B centered at s s.t. $B \subseteq S$.

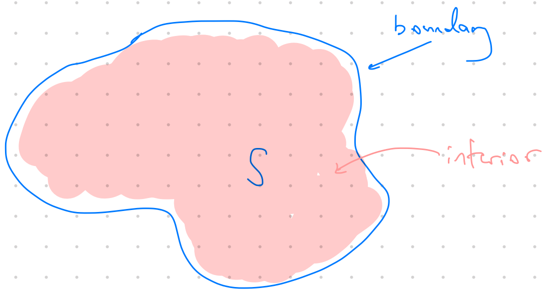


s_1 is an interior point, s_2 is not.

The **interior of a set** is the set of all interior points.

(Def works for any metric).

Similarly, **closure point** x means all balls contain a point in S.
The **boundary** are the points in the closure that are not in the interior.



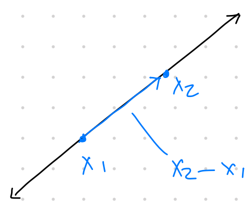
closure = boundary $\cup$ interior.

the closure of the open set is the closed set including its boundary.

Lines, rays, segments

A line through x_1, x_2: $x_1 + \theta(x_2 - x_1)$ $\theta \in \mathbb{R}$

$$\Rightarrow (1-\theta)x_1 + \theta x_2.$$



A ray: remove -ve direction from x_1,

$$\Rightarrow \theta \geq 0$$

A segment: don't go past x_2 in a ray, $0 \leq \theta \leq 1$.

A set S is convex if it also contains every line segment between two points in the set: $s_1, s_2 \in S \Rightarrow \theta s_1 + (1-\theta)s_2 \in S$, $0 \le \theta \le 1$.

Convex combination: generalize line segment to multiple points,

$$x = \theta_1 x_1 + \cdots + \theta_n x_n, \quad \theta_1 + \cdots + \theta_n = 1, \quad \theta_i \ge 0.$$

Thm: convex sets contain every convex combination of its pts.

pf: $k=2$ true by def. Induction. assume true for k points.

Let $\theta_1 + \cdots + \theta_{k+1} = 1$, consider:

$$\theta_1 x_1 + \cdots + \theta_k x_k + \theta_{k+1} x_{k+1}$$

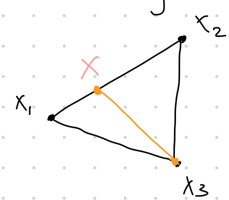
$$= \theta_{1:k}\left[\overbrace{\frac{\theta_1}{\theta_{1:k}} x_1 + \cdots + \frac{\theta_k}{\theta_{1:k}} x_k}^{x \in S \text{ from assumption}}\right] + \theta_{k+1} x_{k+1}$$

where $\theta_{1:k} = \theta_1 + \cdots + \theta_k$

$$= (1 - \theta_{k+1})\, x + \theta_{k+1} x_{k+1}$$

By def of convexity, must be in set. □

Geometrically:



The convex hull of a set S is the set of all convex combination of the elements of S: $\text{conv}(S) = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in S,\ \theta_i \ge 0,\ \sum \theta_i = 1\}$.

The convex hull is the smallest convex set that contains S.

pf: this is kind of a tautology, since we constructed the set from def of cvx. Remove any pt $x \in \text{conv}(S)$, new set is not cvx @ x.

The convex hull of a convex set is itself.

Affine Sets

An affine set is a set that also contains every line through two points in the set: $x_1, x_2 \in S$, $\theta x_1 + (1-\theta) x_2 \in S$.

It's always convex, since every line contains every line segment.

An affine combination is: $x = \theta_1 x_1 + \cdots + \theta_K x_K$, $\theta_1 + \cdots + \theta_K = 1$.

An affine set must contain all affine comb of its pts.

pf. same as convex combination

The affine hull of a set S is the set of all affine combinations of its points.

For any $x_0 \in S$, the set $V = \{x - x_0 \mid x \in S\}$ is a subspace.

pf: Take $v_1, v_2 \in V$.

Affine comb: $\alpha(v_1 + x_0) + \beta(v_2 + x_0) + (1-\alpha-\beta) x_0 \in C$

$\Rightarrow (\alpha v_1 + \beta v_2) + x_0 \in C$

$\Rightarrow \alpha v_1 + \beta v_2 \in V$.

Clearly, $0 \in V$ by taking $x_0 - x_0$. □

$\Rightarrow C = \{x_0 + v \mid v \in V\}$, i.e. an affine set is shifted subspace and the dimension of the affine space is $\dim V$.

The solution to a linear eq'n $C = \{x \mid a^T x = b\}$ is an affine set.

pf: $V = \{x \mid a^T x = 0\}$, pick $a^T x_0 = b \Rightarrow C = \{x_0 + v \mid a^T v = 0\}$.

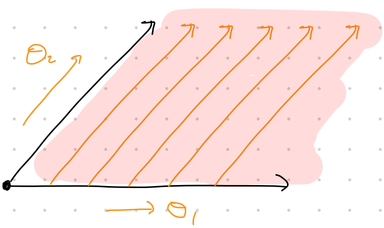
Cones

A set is a **cone** if it contains all non-negative scaling of a point.

$\Rightarrow x \in K \Rightarrow \theta x \in K, \ \theta \geq 0.$

Ray is a cone — Convex cone — non-convex cone

Thm: A cone is convex iff it contains every conic combination of its points.



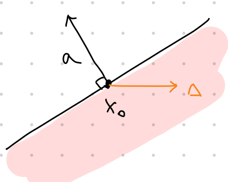
A **conic combination** of a set S is $\{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in S, \ \theta_i \geq 0\}$,

and a **conic hull** is the conic combination of all points in S.

Common Convex Sets

Basics: Vector spaces, affine spaces, empty set, singleton sets,

Hyperplanes & Halfspaces:



$H = \{x \mid a^T x = b\}$ is affine.

$H = \{x \mid a^T x \leq b\}$ is convex, but not affine.

$a^T x_0 = b.$

$a^T(x_0 + \Delta) = a^T x_0 + a^T \Delta.$

$= b + a^T \Delta.$

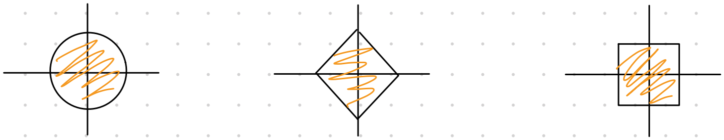
$$a^T \Delta = \begin{cases} \geq 0 & \text{if } \Delta \text{ acute to } a \\ \leq 0 & \text{if } \Delta \text{ obtuse to } a \\ 0 & \text{if } \Delta \perp \text{ to } a \end{cases}$$

$\leftarrow$ halfspace (acute / obtuse cases)

$\leftarrow$ hyperplane ($\perp$ case)

Balls, Ellipsoids, Norm-Cones

Any norm-ball centered at x_c is convex: $B = \{x \mid \|x - x_c\| \le r\}$.



L_2-norm L_1-norm L_∞-norm

pf: Let $x_1, x_2 \in B$, $0 \le \theta \le 1$. By Δ-ineq:

$$\|\theta x_1 + (1-\theta)x_2 - x_c\| = \|\theta x_1 + (1-\theta)x_2 - \theta x_c - (1-\theta)x_c\|$$
$$= \|\theta(x_1 - x_c) + (1-\theta)(x_2 - x_c)\|$$
$$\le \theta\|x_1 - x_c\| + (1-\theta)\|x_2 - x_c) \le r.$$

Any matrix $P \in S^n_{++}$ also induces an inner product norm that forms an ellipse. This is a norm-ball under a specific norm, so convex.

Let: $P \in S^n_{++}$. Since P is symmetric, $P = Q \Lambda Q^T$, where Q is orthogonal w/ evecs as columns, Λ is diagonal of evals, $\lambda_i > 0$.

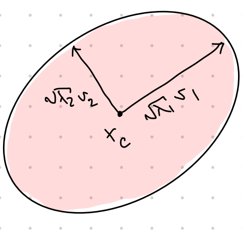
Then: $x^T P x = x^T Q \Lambda Q^T x = z^T \Lambda z$ where $z = Q^T x$.

So $\|\cdot\|_{\langle\cdot,\cdot\rangle_P}$ is $\|\cdot\|_2$ w/ i-th element stretched by λ_i.

So to "normalize" back to L_2, we should scale i-th elem by $\frac{1}{\sqrt{\lambda_i}}$

To make this nicer, we can consider P^{-1} & scale by $\sqrt{\lambda_i}$

(shortest semi-axis of P-ellipse is longest of P^{-1}-ellipse).



So Ellipse defined as

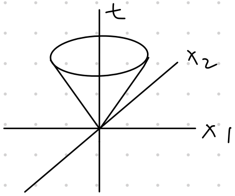
$$E = \{x \mid x^T P^{-1} x \le 1\}$$

scale doesn't matter, can push into P^{-1}.

Or centered at x_c,

$$E = \{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\}$$

A norm cone is a convex cone of successively larger norm balls,



$$K = \{ (x,t) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n, \ \|x\| \le t \}$$

pf convexity: Suppose $(x_1, t_1), (x_2, t_2) \in K$, $x_1, x_2 \in \mathbb{R}^n$.

$$\|\theta x_1 + (1-\theta) x_2\| \le \theta \|x_1\| + (1-\theta)\|x_2\| \le \theta t_1 + (1-\theta) t_2 \quad \square.$$

Polyhedra