Review of Arithmetic

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Table of Contents

Introduction Mathematical Notation & Concepts

- Considering mathematics as a language, you should be able to read an write it to some extent
- The language is pretty much standarised, but there are some different notations (standarised vs standarized)
- Maybe you are familiar with most of what we will see, but the language is more powerful than you can imagine (e.g. abstract algebra)
- Mathematical knowledge builds on top of more basic knowledge, sometimes you need to step back to move forward

Ask if there is something you don't understand!!

Set Theory

- Numbers are a type of set
- Set: A collection of elements: $S = \{\cdots\}$ (with curly brackets)
- Element in set: $a \in S$ (reads 'a' in 'S')
- Operations with sets:
  - Union/Intersection: $S \cup R / S \cap R$
  - Subtraction: $S \setminus R$ (elements of S not in R)
- Size of a set (a.k.a. cardinally) is the number of elements $|S|$
- Subset $R \subset S$ (all elements of R are in S)

Set Theory Example

Let $R = \{a, b, c, f, h\}$ and $S = \{b, f, \dagger, z\}$ be two sets

- Elements of R
- $R \cup S$
- $R \cap S$
- $R \setminus S$
- $S \setminus R$
- $|R|$
- $|R \setminus S|$

Set Theory Example

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- Elements of R a, b, c, f, h
- $R \cup S = \{a, b, c, f, h, \dagger, z\}$
- $R \cap S = \{b, f\}$
- $R \setminus S = \{a, c, h\}$
- $S \setminus R = \{\dagger, z\}$
- $|R| = 5$
- $|R \setminus S| = 3$

Types of Sets

There are different types of sets:

- Empty set: A set without elements $S = \{\emptyset\}$
- Universal set: Set with all the elements for a given context
- Finite sets: Sets with a finite cardinallity (size)
- Infinite sets: Sets with an infinite (∞) cardinallity (size)
- Ordered sets: Sets with an relation defined for pairs of elements (e.g. 'larger than', 'smaller than')

Sometimes sets are defined by some property, e.g.

$$S = \{a \quad | \quad a \text{ is a letter of the Greek alphabet}\}$$

Read: S is the set of elements 'a' such that ($|$) 'a' is a letter of the Greek alphabet.

Types of Sets Example

- Empty set:
- Universal set:
- Finite sets:
- Infinite sets:
- Ordered sets:

Types of Sets Example

- Empty set: People taller than 3m, elephants lighter than 1gr
- Universal set: Letters used on a book, digits to write a number, $\{0, 1\}$ in Boolean algebra
- Finite sets: Letters of the alphabet, students in this course, atoms in the universe
- Infinite sets: Natural numbers, real numbers, complex numbers
- Ordered sets: Letters of the alphabet, natural numbers, real numbers

Numbers and Their Computer Representation

There are different **sets** of numbers

- Natural numbers
- Integer numbers
- Rational numbers
- Real numbers

Note:

- They are infinite ordered sets ($a < b$, smaller than)
- One can define **operations** of elements in these sets addition ($+$) and multiplication ($\cdot$ or $\times$) leading to another number (in the set)
- Abstract algebra deals with sets and operations (e.g. vectors, matrices, Boolean algebra $\{0, 1\}$, functions)
- Subtraction and division are addition and product by an 'inverse' number (e.g. $4 - 3 = 4 + (-3)$ comp. hwr)

Natural Numbers ($\mathbb{N}$)

- Mathematically: $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$
- Some can be represented in computers as: `unsigned char` $(0 - 255)$, `unsigned short` $(0 - 65,535)$, `unsigned long` $(0 - 4,294,967,295)$, `unsigned long long` $(0 - 18,446,744,073,709,551,615)^{\dagger}$
- Addition $(+)$ and product $(\cdot)$ are well defined (not in computers)†
- Subtraction and division (inverse operation) sometimes not, e.g. '$4 - 10$' or '$1/3$' are not natural numbers†
- There are two special elements in $\mathbb{N}$: '0' and '1' (called identity elements) since '$a + 0 = a$' (sum) and '$a \cdot 1 = a$' (product)

†Careful when programming with natural numbers

Hilbert's Hotel

Consider a hotel with an infinite number of rooms $(1, 2, 3, \cdots)$, all of which are occupied. A new guest arrives and wants to get a room in the hotel. Can we accommodate the person?

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We can move the guest currently in room 1 to room 2, the guest currently in room 2 to room 3, and so on, moving every guest from their current room 'n' to room 'n+1'. After this, room 1 is empty and the person gets room 1.

This works for any number of new guests.

Integer Numbers ($\mathbb{Z}$)

- Mathematically: $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, 3, \cdots\}$
- Some can be represented in computers as: `char` (-127 – 127), `short` (-32, 767 – 32, 767), `long` (-2, 147, 483, 647 – 2, 147, 483, 647,), `long long` (-9, 223, 372, 036, 854, 775, 807 – 9, 223, 372, 036, 854, 775, 807)†
- Addition ($+$) and product ($\cdot$) are well defined (not in computers)†
- Division (inverse operation) sometimes not, e.g. '10/5' is an integer but '5/10' is not†
- There are two special elements in $\mathbb{Z}$: '0' and '1' (called identity elements) since '$a + 0 = a$' (sum) and '$a \cdot 1 = a$' (product)

†Careful when programming with integer numbers

How Many Integer Numbers Are There?

A weird way of measuring sizes of sets:

Two sets have the same cardinallity if one can build a one-to-one 'map' from all elements of one set to all elements of the other set.

Examples:

- $S = \{a, b, c\}$ and $R = \{1, 2, 3\}$: $a \to 1$, $b \to 2$, $c \to 3$
- Natural numbers ($\mathbb{N}$) and even numbers $\{2, 4, 6, 8, \cdots\}$

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- Natural numbers ($\mathbb{N}$) and integers ($\mathbb{Z}$): $0 \to 0$, $1 \to 1$, $2 \to -1$, $3 \to 2$, $4 \to -2$, $5 \to 3$, $6 \to -3$,. . .

Rational Numbers ($\mathbb{Q}$)

- Mathematically: $\mathbb{Q} = \{\frac{a}{b} | a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\}\}$
- Do not have a basic data type
- Addition $(+)$ and product $(\cdot)$ are well defined
- Subtraction $(-)$ and division $(/)$ are well defined
- There are two special elements in $\mathbb{Q}$: '0' and '1' (called identity elements) since '$a + 0 = a$' (sum) and '$a \cdot 1 = a$' (product)

Note:

- Integers are included in the rationals, naturals are included in the integers ($\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$)
- The more 'complex' numbers have better properties for their operations

How Many Rational Numbers Are There?

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Yes, applying the process an infinite number of times.

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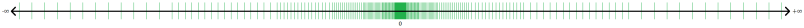
	1	2	3	4	5	6	7	8	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	...
2	$\frac{2}{1}$	$\cancel{\frac{2}{2}}$	$\frac{2}{3}$	$\cancel{\frac{2}{4}}$	$\frac{2}{5}$	$\cancel{\frac{2}{6}}$	$\frac{2}{7}$	$\frac{2}{8}$	...
3	$\frac{3}{1}$	$\frac{3}{2}$	$\cancel{\frac{3}{3}}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	...
4	$\frac{4}{1}$	$\cancel{\frac{4}{2}}$	$\frac{4}{3}$	$\cancel{\frac{4}{4}}$	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$	...
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	...
6	$\frac{6}{1}$	$\cancel{\frac{6}{2}}$	$\cancel{\frac{6}{3}}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	$\frac{6}{7}$	$\frac{6}{8}$	...
7	$\frac{7}{1}$	$\frac{7}{2}$	$\frac{7}{3}$	$\frac{7}{4}$	$\frac{7}{5}$	$\frac{7}{6}$	$\frac{7}{7}$	$\frac{7}{8}$	...
8	$\frac{8}{1}$	$\frac{8}{2}$	$\frac{8}{3}$	$\frac{8}{4}$	$\frac{8}{5}$	$\frac{8}{6}$	$\frac{8}{7}$	$\frac{8}{8}$	...
$\vdots$	$\vdots$								

Real Numbers ($\mathbb{R}$)

- Irrational numbers: Number that cannot be written as a/b (e.g. π, $\sqrt{2}$)
- Between two different rational numbers there is always an irrational number
- Real numbers are the set of rational and irrational numbers
- There are more real numbers (uncountably infinite) than natural numbers (countably infinite)
- Main type of number for this course
- Some can be represented in computers as: `float`, `double` and `long double` (Floating Point IEEE 754)



Real Number Arithmetic

Refers to the operations (sum and product) that can be done with numbers in the sets and their rules.

Properties:

- Closure: If $a, b \in \mathbb{R}$ then $a + b \in \mathbb{R}$ and $a \cdot b \in \mathbb{R}$
- Commutative: If $a, b \in \mathbb{R}$ then $a + b = b + a$ and $a \cdot b = b \cdot a$
- Associative: $\forall a, b, c \in \mathbb{R} \quad a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- Identity element (0/1): If $a \in \mathbb{R}$ then $a + 0 = a$ and $a \cdot 1 = a$
- Inverse: If $a \in \mathbb{R} \setminus \{0\}$ then $b, c \in \mathbb{R}$ exist such that $a + b = 0$ and $a \cdot c = 1$
- Distributive w.r.t. addition: If $a, b, c \in \mathbb{R}$ then $a \cdot (b + c) = a \cdot b + a \cdot c$

Some Examples of Real Number Arithmetic

- Closure: $2 \in \mathbb{N}$ and $\sqrt{2} \in \mathbb{R}$ but $\sqrt{2} + 2 \in \mathbb{R}$ since $\mathbb{N} \subset \mathbb{R}$ (type casting)
- Associative: $(10^{30} + (-10^{30})) + 1 = 10^{30} + ((-10^{30}) + 1)$ (not in a computer)
- Inverse:
  - $\frac{2}{5} \in \mathbb{R}$ and taking $\frac{5}{2} \in \mathbb{R}$ then $\frac{2}{5} \cdot \frac{5}{2} = \frac{2 \cdot 5}{5 \cdot 2} = 1$
  - $\frac{\pi}{6} \in \mathbb{R}$ and taking $-\frac{\pi}{6} \in \mathbb{R}$ then $\frac{\pi}{6} + \left(-\frac{\pi}{6}\right) = 0$
- Distributive w.r.t. addition: $2 \cdot (3 + 5) = 2 \cdot 3 + 2 \cdot 5 = 16$

This is trivial, but some of these properties can be true or not for other mathematical entities with huge implications (e.g. for matrices A and B the product $A \cdot B \neq B \cdot A$)

Rational Number Arithmetic

All properties of real numbers also apply to rational numbers, but:

- For $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ the sum is $\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot b}{b \cdot d} \in \mathbb{Q}$
  - Note: for $\frac{a}{b} \in \mathbb{Q}$ and $c \in \mathbb{N}$ we have $\frac{a}{b} + c = \frac{a}{b} + \frac{c}{1} = \frac{a + c \cdot b}{b}$
- For $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ the product is $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \in \mathbb{Q}$
- For $\frac{a}{b}$ the inverse is $\frac{1}{a/b} = \frac{b}{a}$

Examples:

- $\frac{1}{2} + \frac{2}{3} = \frac{3+4}{6} = \frac{7}{6}$
- $\frac{4}{5} + 1 = \frac{4+5}{5} = \frac{9}{5}$
- $\frac{x^2}{y} + \frac{2y}{z} = \frac{x^2 z + 2y^2}{yz}$

Exponential

- If we add 'n' times $a \in \mathbb{R}$ we get:
  $a + a + \cdots + a = \sum_{i=0}^{n} a = n \cdot a$ (sum of 'a' from $i = 0$ to 'n')
- If we multiply 'n' times $a \in \mathbb{R}$ we get:
  $a \cdot a \cdot a \cdots a = \prod_{i=0}^{n} a = a^n$ (product of 'a' from $i = 0$ to 'n')

Given a number $a \in \mathbb{R}$ and two natural numbers $n, m \in \mathbb{N}$:

- Any number to the power of 0 is 1: $a^0 = 1$
- Sum of exponents: $a^{n+m} = a^n \cdot a^m$
- Negative exponent: $a^{-n} = \frac{1}{a^n}$
- Product of exponents: $(a^n)^m = a^{n \cdot m}$
- Root: $a^{\frac{1}{m}} = \sqrt[m]{a}$ (m-th root means $c^m = a$)
- For $b \in \mathbb{R}$: $a^n \cdot b^n = (a \cdot b)^n$

Examples of Exponential

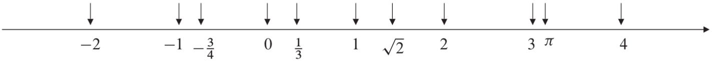
- 2^{-3}
- $3^2 \cdot 27$
- $(3^2)^3$
- $4^3 \cdot 16^2$
- $\frac{32}{4}$
- $\left(\frac{12}{2^3}\right)^2$
- $(12)^5$

Examples of Exponential

- $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
- $3^2 \cdot 27 = 3^2 \cdot 3^3 = 3^{2+3} = 3^5$
- $(3^2)^3 = 3^{2\cdot 3} = 3^6$
- $4^3 \cdot 16^2 = (2^2)^3 \cdot (2^4)^2 = 2^{2\cdot 3} \cdot 2^{4\cdot 2} = 2^6 \cdot 2^8 = 2^{6+8} = 2^{14}$
- $\frac{32}{4} = \frac{2^5}{2^2} = 2^5 \cdot 2^{-2} = 2^{5-2} = 2^3$
- $\left(\frac{12}{2^3}\right)^2 = \left(\frac{3\cdot 2^2}{2^3}\right)^2 = \left(3 \cdot 2^2 \cdot 2^{-3}\right)^2 = \left(3 \cdot 2^{2-3}\right)^2 = \left(3 \cdot 2^{-1}\right)^2$
  $= 3^2 \cdot 2^{-1\cdot 2} = 3^2 \cdot 2^{-2} = \frac{3}{2^2}$
- $(12)^5 = (3 \cdot 2^2)^5 = 3^5 \cdot (2^2)^5 = 3^5 \cdot 2^{2\cdot 5} = 3^5 \cdot 2^{10}$

The Real Line & Inequalities

Since the real numbers $\mathbb{R}$ is an ordered ordered, we can represent the elements of $\mathbb{R}$ as a line: **the real line**



Order is defined by the binary relation $<$ (or $>$, $\leq$, $\geq$) used to define inequalities.

Rules for Inequalities: Given the numbers $a, b, c \in \mathbb{R}$

- Adding a number: If $a < b$ then $a \pm c < b \pm c$
- Positive multiplication: If $a < b$ and $c > 0$ then $a \cdot c < b \cdot c$
- Negative multiplication: If $a < b$ and $c < 0$ then $a \cdot c > b \cdot c$
- Inverse: If $a > 0$ then $1/a > 0$
- Order of the inverse: If $0 < a < b$ then $1/b < 1/a$

Intervals on $\mathbb{R}$

Intervals are subsets of $\mathbb{R}$ defined by (groups of) inequalities.

There are different types:

- Open: $(a, b) = \{x \in \mathbb{R} \mid a < x < b$ (all numbers between a and b)
- Closed: $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b$ (all numbers between a and b, included)
- Half-open:
  - $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b$ (all numbers between a and b, b included)
  - $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b$ (all numbers between a and b, a included)
  - $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x$ (all numbers larger than a)

They can be combined using set union $(a, b) \bigcup [c, \infty)$ (for $a < b < c$)

Examples of Intervals

Assuming $x \in \mathbb{R}$:

- $x - 4 \geq 0$:
- $-3x < 9$:
- $x^2 < 4$:
- $x^2 \geq 9$:

Examples of Intervals

Assuming $x \in \mathbb{R}$:

- $x - 4 \geq 0$: $x \geq 4$ as an interval $[4, \infty)$
- $-3x < 9$: $x > 3$ as an interval $(3, \infty)$
- $x^2 < 4$: $-2 < x < 2$ as an interval $(-2, 2)$
- $x^2 \geq 9$: $x \geq 3$ and $x \leq -3$ as an interval $(-\infty, 3] \cup [3, \infty)$

Solve for $x \in \mathbb{R}$:

- $-3x + 3 > 6$

$$\begin{aligned}
-3x + 3 > 6 \quad &\Rightarrow \quad -3x + 3 - 3 > 6 - 3 \\
&\Rightarrow \quad -3x > 3 \\
&\Rightarrow \quad -3x \cdot \frac{1}{3} > 3 \cdot \frac{1}{3} \\
&\Rightarrow \quad -x > 1 \\
&\Rightarrow \quad x < 1 \qquad (-\infty, 1)
\end{aligned}$$

Examples of Inequalities

Solve for $x \in \mathbb{R}$:

- $\frac{1}{2x} > 5$

$$
\begin{aligned}
\frac{1}{2x} > 5 \quad & \Rightarrow \quad 2\frac{1}{2x} > 2 \cdot 5 \\
& \Rightarrow \quad \frac{1}{x} > 10 \\
& \Rightarrow \quad \text{If } x > 0 \text{ then } x \cdot \frac{1}{x} > 10x \\
& \Rightarrow \quad \text{If } x > 0 \text{ then } 1 > 10x \\
& \Rightarrow \quad \text{If } x > 0 \text{ then } \frac{1}{10} > \frac{1}{10} \cdot 10x \\
& \Rightarrow \quad \text{If } x > 0 \text{ then } \frac{1}{10} > x \qquad \left(-\infty, \frac{1}{10}\right) \\
& \Rightarrow \quad x > 0 \text{ and } x < \frac{1}{10} \qquad \left(0, \frac{1}{10}\right)
\end{aligned}
$$

Examples of Inequalities

Solve for $x \in \mathbb{R}$:

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$$
\begin{aligned}
\frac{1}{2x} > 5 \quad & \Rightarrow \quad 2\frac{1}{2x} > 2 \cdot 5 \\
& \Rightarrow \quad \frac{1}{x} > 10 \\
& \Rightarrow \quad \text{If } x < 0 \text{ then } x\frac{1}{x} < 10x \\
& \Rightarrow \quad \text{If } x < 0 \text{ then } 1 < 10x \\
& \Rightarrow \quad \text{If } x < 0 \text{ then } \frac{1}{10} < \frac{1}{10} \cdot 10x \\
& \Rightarrow \quad \text{If } x < 0 \text{ then } \frac{1}{10} < x \qquad \left(\frac{1}{10}, \infty\right) \\
& \Rightarrow \quad x < 0 \text{ and } x > \frac{1}{10} \qquad \{\emptyset\}
\end{aligned}
$$

Solve for $x \in \mathbb{R}$:

- $\frac{1}{2x} > 5$

$$\frac{1}{2x} > 5 \quad \Rightarrow \quad 2\frac{1}{2x} > 2 \cdot 5$$
$$\Rightarrow \quad \frac{1}{x} > 10$$
$$\Rightarrow \quad x > 0 \text{ and } x < \frac{1}{10} \qquad \left(0, \frac{1}{10}\right)$$

Equalities

The solution to equalitites (if any) are points in the real line.

If $x \in \mathbb{R}$, e.g.

- $ax + b = 0$: sol. $x = \frac{-b}{a}$
- $ax^2 + bx + c = 0$: sol. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (for $b^2 - 4ac > 0$)
- $\frac{ax+b}{cx+d} = 1$: sol. $x = \frac{d-b}{a-c}$ (for $a \neq c$)

Notes:

- Some equalities might not have solution $(x^2 + 1 = 0)$
- Some equalities might not have a formula (numerical methods)

Summary

- Basic sets & operations
- Different types of numbers & their computer representation
- Representation of numbers is limited on a computer
- Arithmetic rules & powers
- The real line
- Inequalities & intervals