Gaussian Elimination

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Contents

General Systems of Linear Equations

Given 'm' linear equations on 'n' variables we want to find the solution(s) $x_1, x_2, \cdots, x_n$ fulfilling these equations.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

a_{ij} and b_i are just numbers.

This problem appears for instance:

- Solving electrical circuits
- Calculating position in GPS systems
- Machine learning (Linear regression)

Looking at Things in the Most Convenient Way

We saw vectors and matrices as two different entities, but:

- A vector $\mathbf{v} \in \mathbb{R}^n$ can be seen as a $n \times 1$ matrix

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- A matrix $A = (a_{ij})$ of size $n \times m$ can be seen as 'n' row vectors of dimension 'm' or 'm' column vectors of dimension 'n' packed together
- Considering vectors as matrices we can multiply them with other matrices

Matrix form of the General Systems of Linear Equations

Matrices and vectors (single column matrices) can be used to write the equations in a more compact form:

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \mathbf{b}$$

There are different ways of solving this type of problems:

- Cramer's Theorem
- Gaussian elimination + back substitution
- Inverse calculation (if $A\mathbf{x} = \mathbf{b}$ then $\mathbf{x} = A^{-1}\mathbf{b}$)

but... Do they have solution at all? Is there only one solution?

Cramer's Rule

For a system of 'n' linear equations in 'n' variables $A\mathbf{x} = \mathbf{b}$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad \ddots \qquad &\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

if the matrix A has non-zero determinant $\det(A) \neq 0$, the solution for $x_1, x_2, \cdots, x_n$ are given by:

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i is the matrix resulting from replacing column i in A by the vector $\mathbf{b} = [b_1, b_2, \cdots, b_n]$.

Example of Cramer's Rule

Find the solution of:

$$\begin{cases} 3x_1 + 3x_2 & = 1 \\ \quad\quad 4x_2 + 4x_3 & = 3 \\ \quad\quad 2x_2 + \ \ x_3 & = 0 \end{cases}$$

In matrix form $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 4 & 4 \\ 0 & 2 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

- $\det(A) = -12$
- x_1

Example of Cramer's Rule

Find the solution of:

$$\begin{cases} 3x_1 + 3x_2 \phantom{{}+4x_3} = 1 \\ \phantom{3x_1 + {}} 4x_2 + 4x_3 = 3 \\ \phantom{3x_1 + {}} 2x_2 + x_3 = 0 \end{cases}$$

In matrix form $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 4 & 4 \\ 0 & 2 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

- $\det(A) = -12$
- x_1

$$x_1 = \frac{1}{-12} \begin{vmatrix} \color{red}{1} & 3 & 0 \\ \color{red}{3} & 4 & 4 \\ \color{red}{0} & 2 & 1 \end{vmatrix} = \frac{-13}{-12} = \frac{13}{12}$$

Example of Cramer's Rule

Find the solution of:

$$\begin{cases} 3x_1 + 3x_2 = 1 \\ 4x_2 + 4x_3 = 3 \\ 2x_2 + x_3 = 0 \end{cases}$$

In matrix form $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 4 & 4 \\ 0 & 2 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

- $\det(A) = -12$
- $x_1 = \frac{13}{12}$, x_2

$$x_2 = \frac{1}{-12} \begin{vmatrix} 3 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{vmatrix} = \frac{9}{-12} = -\frac{3}{4}$$

Example of Cramer's Rule

Find the solution of:

$$\begin{cases} 3x_1 + 3x_2 \phantom{{}+4x_3} = 1 \\ \phantom{3x_1 + {}} 4x_2 + 4x_3 = 3 \\ \phantom{3x_1 + {}} 2x_2 + x_3 = 0 \end{cases}$$

In matrix form $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 4 & 4 \\ 0 & 2 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

- $\det(A) = -12$
- $x_1 = \frac{13}{12}$, $x_2 = -\frac{3}{4}$, x_3

$$x_3 = \frac{1}{-12} \begin{vmatrix} 3 & 3 & 1 \\ 0 & 4 & 3 \\ 0 & 2 & 0 \end{vmatrix} = \frac{-18}{-12} = \frac{3}{2}$$

Example of Cramer's Rule

Find the solution of:

$$\begin{cases} 3x_1 + 3x_2 & = 1 \\ \quad\quad 4x_2 + 4x_3 & = 3 \\ \quad\quad 2x_2 + x_3 & = 0 \end{cases}$$

In matrix form $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 4 & 4 \\ 0 & 2 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

- $\det(A) = -12$
- $x_1 = \frac{13}{12}$, $x_2 = -\frac{3}{4}$, $x_3 = \frac{3}{2}$

Only if:

- Same number of equations and unknowns
- Matrix A is not singular ($\det(A) \neq 0$) has unique solution

Back Substitution

If the linear system of equations looks like:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\ddots \qquad \vdots \quad &\quad \vdots \\
a_{n-1n-1}x_{n-1} + a_{n-1n}x_n &= b_{n-1} \\
a_{nn}x_n &= b_n
\end{aligned}
$$

The solution is easy to find using back substitution

$$
\begin{aligned}
x_n &= \frac{b_n}{a_{nn}} \\
x_{n-1} &= \frac{b_{n-1} - a_{n-1n}x_n}{a_{n-1n-1}} \\
&\vdots
\end{aligned}
$$

Example of Back Substitution

Find the solution of:

$$\begin{cases} 2x_1 + 3x_2 & = 1 \\ \quad\quad x_2 + 2x_3 & = 3 \\ \quad\quad\quad\quad 2x_3 & = 4 \end{cases}$$

From the last equation:

$$x_3 = \frac{4}{2} = 2$$

Example of Back Substitution

Find the solution of:

$$\begin{cases} 2x_1 + 3x_2 = 1 \\ \phantom{2x_1 + {}} x_2 + 2x_3 = 3 \\ \phantom{2x_1 + x_2 + {}} 2x_3 = 4 \end{cases}$$

From the last equation:

$$x_3 = \frac{4}{2} = 2$$

From the second equation:

$$x_2 = 3 - 2x_3 = 3 - 2 \cdot 2 = -1$$

Example of Back Substitution

Find the solution of:

$$\begin{cases} 2x_1 + 3x_2 = 1 \\ x_2 + 2x_3 = 3 \\ 2x_3 = 4 \end{cases}$$

From the last equation:

$$x_3 = \frac{4}{2} = 2$$

From the second equation:

$$x_2 = 3 - 2x_3 = 3 - 2 \cdot 2 = -1$$

From the first equation:

$$x_1 = \frac{1}{2}(1 - 3x_2) = \frac{1}{2}(1 - 3 \cdot (-1)) = \frac{4}{2} = 2$$

If only it were always that easy!

Introduction to Gaussian Elimination

For a system of 'm' linear equations in 'n' variables $A\mathbf{x} = \mathbf{b}$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We can perform the following operations without changing the solution to the system:

- Change the order of the equations
- Multiply any equation by a number α. Example:

$$\alpha a_{11}x_1 + \alpha a_{12}x_2 + \cdots + \alpha a_{1n}x_n = \alpha b_1$$

- Substitute any equation by that equation plus/minus another one. Example:

$$(a_{11} - a_{21})x_1 + (a_{12} - a_{22})x_2 + \cdots + (a_{1n} - a_{2n})x_n = b_1 - b_2$$

Augmented Matrix for a System of Linear Equations

Since the system of linear equations can be written in matrix form as $A\mathbf{x} = \mathbf{b}$, we can define the **augmented matrix** $\tilde{A}$

$$\tilde{A} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

which also 'represents' the system of linear equations.

- Each row corresponds to an equation (ignoring the unknowns $x_1, x_2, \cdots, x_n$)
- Coefficient matrix (first columns) plus right hand side of the equation (last column)
- Equations can be recovered from the matrix

Gaussian Elimination

Is an algorithm to convert a matrix to upper triangular form (row echelon form) by using the following row operations

- Swapping two rows (change order of equations)
- Multiplying a row by a number different than zero (multiply an equation by a number)
- Adding a multiple of one row to another row (substitute one equation by it plus/minus another)

Gaussian elimination can be used to:

- Solve systems of linear equations
- Obtain the inverse of a matrix
- Test if a set of vectors is linearly independent
- Calculate the determinant of a matrix

Operation 1: Swapping Two Rows

- Each row 'i' of the matrix $\tilde{A}$ corresponds to an equation
- The order of the equations does not change the system

$$\tilde{A} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

- We can do this for any pairs of rows
- The new matrix has the same solution (we just wrote the equations in a different order)

Operation 2: Row Product by a Number

- Each row 'i' of the matrix $\tilde{A}$ corresponds to an equation
- The equation does not change when multiplying by a number (e.g. multiply first row by $\frac{1}{a_{11}}$)

$$\tilde{A} = \left[\begin{array}{cccc|c} \frac{a_{11}}{a_{11}} & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} & \frac{b_1}{a_{11}} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} & \frac{b_1}{a_{11}} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

- We can do this for any entry a_{ij} (given $a_{ij} \neq 0$)
- We get a 1 in some entry of the matrix

Operation 3: Row Linear Combination

- Replacing one row a_{ij} by $a_{ij} + \alpha a_{kj}$ (for $j = 1, 2, \cdots, n$) does not change the solution of the system
- This is a **linear combination** of rows

$$\tilde{A} \rightarrow \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} + \alpha a_{11} & a_{22} + \alpha a_{12} & \cdots & a_{2n} + \alpha a_{1n} & b_2 + \alpha b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

- We can do this for any pairs of lines 'i' and 'k'
- If we take $\alpha = -\frac{a_{21}}{a_{11}}$ we make a zero entry in the matrix

Operation 3: Row Linear Combination

- Replacing one row a_{ij} by $a_{ij} + \alpha a_{kj}$ (for $j = 1, 2, \cdots, n$) does not change the solution of the system
- This is a **linear combination** of rows

$$A = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \color{red}{0} & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & \cdots & a_{2n} - \frac{a_{21}}{a_{11}} a_{1n} & b_2 - \alpha b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

- We can do this for any pairs of lines 'i' and 'k'
- If we take $\alpha = -\frac{a_{21}}{a_{11}}$ we make a zero entry in the matrix
- Alternatively we can set a_{11} to '1' and take $\alpha = -a_{21}$

Types of Solutions of Systems of Linear Equations

So far we assumed systems of linear equations have a solution (Cramer's rule and back substitution), but...

- If there are more unknowns than equations ($n > m$)
  - Infinite solutions
  - No solution
- If there are same unknowns as equations ($n = m$)
  - One solution
  - Infinite solutions
  - No solution
- If there are fewer unknowns than equations ($n < m$)
  - No solution
  - One solution
  - Infinite solutions

Gaussian elimination also helps identifying the number of solutions (if any)

Example of Gaussian Elimination (One Solution)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix

$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 2 & 4 & 0 \\ 2 & 4 & 8 & 6 \\ 2 & 7 & 12 & 4 \end{array}\right]$$

We can do one of these things:

- Swapping two rows
- Multiplying a row by a number different than zero
- Adding a multiple of one row to another row

Example of Gaussian Elimination (One Solution)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix

$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 2 & 4 & 0 \\ 2 & 4 & 8 & 6 \\ 2 & 7 & 12 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 4 & 8 & 6 \\ 0 & 2 & 4 & 0 \\ 2 & 7 & 12 & 4 \end{array}\right]$$

We can do one of these things:

- Swapping two rows: Row 1 and Row 2
- Multiplying a row by a number different than zero
- Adding a multiple of one row to another row

Example of Gaussian Elimination (One Solution)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix

$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 2 & 4 & 0 \\ 2 & 4 & 8 & 6 \\ 2 & 7 & 12 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 4 & 8 & 6 \\ 0 & 2 & 4 & 0 \\ 2 & 7 & 12 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 2 & 4 & 0 \\ 2 & 7 & 12 & 4 \end{array}\right]$$

We can do one of these things:

- Swapping two rows
- Multiplying a row by a number different than zero: Row 1 $\times \frac{1}{2}$
- Adding a multiple of one row to another row

Example of Gaussian Elimination (One Solution)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix

$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 2 & 4 & 0 \\ 2 & 4 & 8 & 6 \\ 2 & 7 & 12 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 2 & 4 & 0 \\ 2 & 7 & 12 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 2 & 4 & 0 \\ 0 & 3 & 4 & -2 \end{array}\right]$$

We can do one of these things:

- Swapping two rows
- Multiplying a row by a number different than zero
- Adding a multiple of one row to another row: Row 3 - 2$\times$ Row 1

Example of Gaussian Elimination (One Solution)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix

$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 2 & 4 & 0 \\ 2 & 4 & 8 & 6 \\ 2 & 7 & 12 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 2 & 4 & 0 \\ 0 & 3 & 4 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 4 & -2 \end{array}\right]$$

We can do one of these things:

- Swapping two rows
- Multiplying a row by a number different than zero: Row 2 $\times \frac{1}{2}$
- Adding a multiple of one row to another row

Example of Gaussian Elimination (One Solution)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix

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We can do one of these things:

- Swapping two rows
- Multiplying a row by a number different than zero
- Adding a multiple of one row to another row: Row 3 - 3$\times$ Row 2

Example of Gaussian Elimination (One Solution)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix

$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 2 & 4 & 0 \\ 2 & 4 & 8 & 6 \\ 2 & 7 & 12 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

We can do one of these things:

- Swapping two rows
- Multiplying a row by a number different than zero
- Adding a multiple of one row to another row

The augmented matrix $\tilde{A}$ has been converted to upper triangular and we can 'recover' the system of linear equations

Example of Gaussian Elimination (One Solution)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix

$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 2 & 4 & 0 \\ 2 & 4 & 8 & 6 \\ 2 & 7 & 12 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

- The original three equations ($m = 3$) in three variables ($n = 3$) now become:

$$\begin{aligned} x + 2y + 4z &= 3 \\ y + 2z &= 0 \\ -2z &= -2 \end{aligned}$$

- Solving through back substitution $z = 1$, $y = -2z = -2$, $x = 3 - 2y - 4z = 3$
- The linear system has **one solutions** ($n = m$ **after Gaussian elimination**)

Example of Gaussian Elimination (Infinite Solutions)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix

$$\tilde{A} = \left[\begin{array}{ccc|c} 3 & 18 & 21 & 6 \\ 2 & 12 & 14 & 4 \\ 1 & 6 & 7 & 2 \end{array}\right]$$

We can do one of these things:

- Swapping two rows
- Multiplying a row by a number different than zero
- Adding a multiple of one row to another row

Example of Gaussian Elimination (Infinite Solutions)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix

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We can do one of these things:

- Swapping two rows: Row 1 and Row 3
- Multiplying a row by a number different than zero
- Adding a multiple of one row to another row

Example of Gaussian Elimination (Infinite Solutions)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix

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We can do one of these things:

- Swapping two rows
- Multiplying a row by a number different than zero
- Adding a multiple of one row to another row: Row 2 - 2× Row 1

Example of Gaussian Elimination (Infinite Solutions)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix

$$\tilde{A} = \left[\begin{array}{ccc|c} 3 & 18 & 21 & 6 \\ 2 & 12 & 14 & 4 \\ 1 & 6 & 7 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 6 & 7 & 2 \\ 0 & 0 & 0 & 0 \\ 3 & 18 & 21 & 6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 6 & 7 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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The augmented matrix $\tilde{A}$ has been converted to upper triangular and we can 'recover' the system of linear equations

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Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix

$$\tilde{A} = \left[\begin{array}{ccc|c} 3 & 18 & 21 & 6 \\ 2 & 12 & 14 & 4 \\ 1 & 6 & 7 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 6 & 7 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- From the three equations ($m = 3$) in three variables ($n = 3$) only one is left:

$$x + 6y + 7z = 2$$
$$x = 2 - 6y - 7z$$

- If we input values for 'y' and 'z' we get the value of 'x' which fulfils the remaining equation
- The linear system has **infinite solutions** ($n > m$ **after Gaussian elimination**)

Example of Gaussian Elimination (No solution)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix:

$$\tilde{A} = \left[\begin{array}{ccc|c} 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 5 \\ 5 & 10 & 1 & 4 \end{array}\right]$$

We can do one of these things:

- Swapping two rows
- Multiplying a row by a number different than zero
- Adding a multiple of one row to another row

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- Swapping two rows
- Multiplying a row by a number different than zero
- Adding a multiple of one row to another row: Row 2 - 3$\times$ Row 1

Example of Gaussian Elimination (No solution)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix:

$$\tilde{A} = \left[\begin{array}{ccc|c} 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 5 \\ 5 & 10 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 5 & 10 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

We can do one of these things:

- Swapping two rows
- Multiplying a row by a number different than zero
- Adding a multiple of one row to another row: Row 3 - 5× Row 1

Example of Gaussian Elimination (No solution)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix:

$$\tilde{A} = \left[\begin{array}{ccc|c} 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 5 \\ 5 & 10 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

We can do one of these things:

- Swapping two rows: Row 2 and Row 3
- Multiplying a row by a number different than zero
- Adding a multiple of one row to another row

Example of Gaussian Elimination (No solution)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix:

$$\tilde{A} = \left[\begin{array}{ccc|c} 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 5 \\ 5 & 10 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

We can do one of these things:

- Swapping two rows
- Multiplying a row by a number different than zero
- Adding a multiple of one row to another row

The augmented matrix $\tilde{A}$ has been converted to upper triangular and we can 'recover' the system of linear equations

Example of Gaussian Elimination (No solution)

Find the solution of $A\mathbf{v} = \mathbf{b}$ for the system with the following augmented matrix:

$$\tilde{A} = \left[\begin{array}{ccc|c} 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 5 \\ 5 & 10 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

- The original three equations ($m = 3$) in three variables ($n = 3$) now become

$$\begin{aligned} x + 2y &= 1 \\ z &= -1 \\ 0 &= 2 \qquad ???? \end{aligned}$$

- The last 'reconstructed' equation does not make sense
- The linear system has **no solution** ('nonsense' equation **after Gaussian elimination**)

Rank of a Matrix

The **rank** of an $m \times n$ matrix A ($rank(A)$) is the maximum number of linearly independent rows or columns (seen as vectors) of A

- The rank of A is smaller or equal than $\min(n, m)$
- The maximum size of a square submatrix of A with determinant different than zero
- (In practice) The number of non-zero rows in the matrix after running the Gaussian Elimination algorithm

The ranks of 'A' and '$\tilde{A}$' tell whether a system of 'm' linear equations $A\mathbf{x} = \mathbf{b}$ in 'n' variables has solutions and how many

- If $rank(A) = rank(\tilde{A}) = n$ then there is **one solution**
- If $rank(A) = rank(\tilde{A}) < n$ then there are **infinite solution**
- If $rank(A) < rank(\tilde{A})$ then there is **no solution**

Examples of Rank of a Matrix

- Matrix of rank 1:

$$\tilde{A} = \left[\begin{array}{ccc|c} 3 & 18 & 21 & 6 \\ 2 & 12 & 14 & 4 \\ 1 & 6 & 7 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 6 & 7 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- Matrix of rank 2:

$$A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 6 & 0 \\ 5 & 10 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Matrix of rank 3:

$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 2 & 4 & 0 \\ 2 & 4 & 8 & 6 \\ 2 & 7 & 12 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

Examples of $rank(A)$ vs $rank(\tilde{A})$ vs 'n'

All the examples had three unknowns ($n = 3$)

- $rank(A) = rank(\tilde{A}) = 1 < 3$ (**infinite solutions**):

$$\tilde{A} = \left[\begin{array}{ccc|c} 3 & 18 & 21 & 6 \\ 2 & 12 & 14 & 4 \\ 1 & 6 & 7 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 6 & 7 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- $rank(A) = 2$ and $rank(\tilde{A}) = 3$ (**no solution**):

$$\tilde{A} = \left[\begin{array}{ccc|c} 2 & 4 & 0 & 2 \\ 3 & 6 & 0 & 5 \\ 5 & 10 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

- $rank(A) = rank(\tilde{A}) = 3$ (**one solution**):

$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 2 & 4 & 0 \\ 2 & 4 & 8 & 6 \\ 2 & 7 & 12 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

Interpretation of Systems of Linear Equations

There are different ways of understanding systems of linear equations:

- Linear combination. Looking at the matrix A as columns

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

  Can the vector **b** be stated as a **linear combination** of the columns of A? What are the scalars $x_1, x_2, \cdots, x_n$?

- Geometrical interpretation: Each equation:

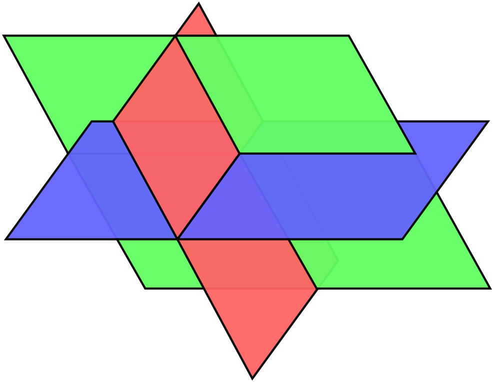
$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

  corresponds to a hyperplane in $\mathbb{R}^n$.
  Do these hyperplanes intersect? Where $x_1, x_2, \cdots, x_n$?

Geometrical Interpretation in $\mathbb{R}^3$

In $\mathbb{R}^3$ we can draw things, and hyperplanes are just planes.

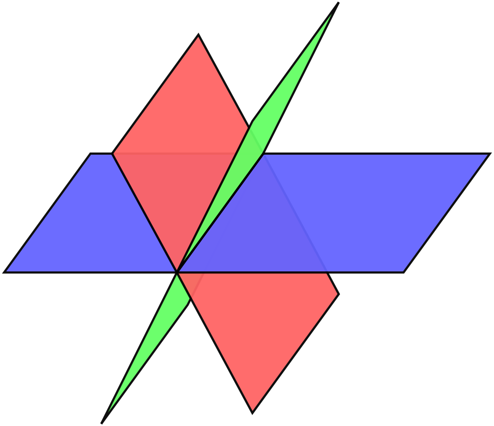
- The planes intersect at one point (**one solution**)



Geometrical Interpretation in $\mathbb{R}^3$

In $\mathbb{R}^3$ we can draw things, and hyperplanes are just planes.

- The planes intersect at one point (**one solution**)
- The planes intersect in one line (**infinite solutions**)



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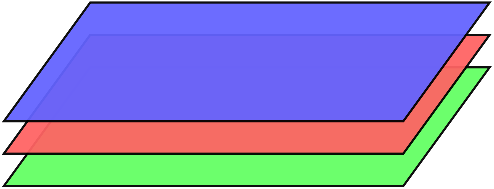
- The planes intersect in one line (**infinite solutions**)
- The planes are all the same (**infinite solutions**)
- The planes do not intersect
  - Because they are parallel (**no solution**)



Geometrical Interpretation in $\mathbb{R}^3$

In $\mathbb{R}^3$ we can draw things, and hyperplanes are just planes.

- The planes intersect at one point (**one solution**)
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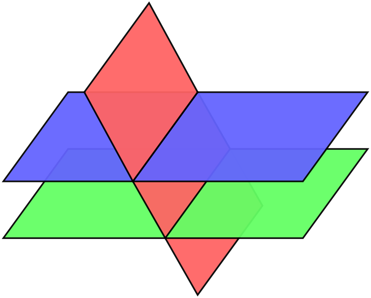
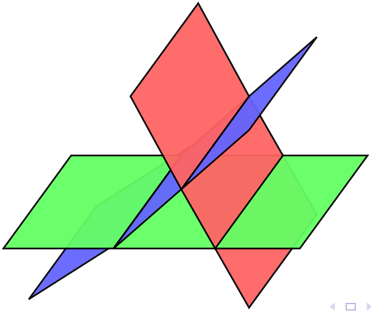
- The planes are all the same (**infinite solutions**)
- The planes do not intersect
  - Because they are parallel (**no solution**)
  - Because two are parallel (**no solution**)



Geometrical Interpretation in $\mathbb{R}^3$

In $\mathbb{R}^3$ we can draw things, and hyperplanes are just planes.

- The planes intersect at one point (**one solution**)
- The planes intersect in one line (**infinite solutions**)
- The planes are all the same (**infinite solutions**)
- The planes do not intersect
  - Because they are parallel (**no solution**)
  - Because two are parallel (**no solution**)
  - Just because (**no solution**)



Summary

- Solving systems of linear equations $A\mathbf{x} = \mathbf{b}$ in 'three ways'
  - Cramer's rule (if $\det(A) \neq 0$)
  - Back substitution
  - Gaussian elimination + back substitution
- Rank of a matrix
- Rank condition to find whether there is a solution
- Geometrical interpretation in $3D$