Matrices and Vectors

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Introduction and Rationale

Entering the realm of ‘Linear Algebra’

- Linear can refer to lines (2D), planes (3D), anything ‘flat’
- Scaling (multiplying) something (vectors, matrices) by a number
- Adding these things (vectors, matrices) together

What for?

- Position/size of a window on the screen (2D point/vector)
- Velocity of a character in a game (vector)
- Image to identify an object (matrix/tensor)
- Motion of a robot

Contents

- Matrices and their types
- Matrix algebra (addition and product)
- Determinants
- Vectors definition and characterisation
- Vector algebra
- Vector spaces, basis and coordinates
- Dot product and angle between vectors
- Linear subspaces (line, plane, hyperplane)

Matrices

A matrix 'A' is an rectangular array of numbers of size $n \times m$, typically written as $A = (a_{i,j}) = (a_{ij})$ where $i = 1, \cdots, n$ and $j = 1, \cdots, m$ are the indexes.

$$A = (a_{ij}) = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m-1,1} & a_{m-1,2} & \cdots & a_{m-1,n-1} & a_{m-1,n} \\ a_{m,1} & a_{m,2} & \cdots & a_{,n-1} & a_{m,n} \end{bmatrix}$$

- The comma is often dropped but included here for clarity
- Matrix 'A' has 'n' rows and 'm' columns (**size** $n \times m$)
- The numbers a_{ij} in the matrix are called **entries** or **elements**

Matrix Operations: Addition

Given two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of equal size $n \times m$, the sum of A and B is a matrix C of size $n \times m$ with entries $C = (c_{ij}) = (a_{ij} + b_{ij})$.

- Add all the element of the same row and column
- Subtraction is similar $C = (c_{ij}) = (a_{ij} - b_{ij})$

Example 2×3 matrices:

$$\begin{bmatrix} 1 & 3 & -7 \\ 0 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 8 & -2 & 5 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 1 & -2 \\ 4 & 5 & 3 \end{bmatrix}$$

Types of Matrices

- A matrix A is called **square** when it has the same number of columns and rows, i.e. size is $n \times n$

$$A = \begin{bmatrix} 14 & -2 & 2 & 0 \\ -20 & 0 & -5 & 1 \\ 10 & -2 & 1 & 1 \\ 7 & -4 & 0 & 3 \end{bmatrix}$$

Types of Matrices

- A matrix A is called **square** when it has the same number of columns and rows, i.e. size is $n \times n$
- A square matrix $A = (a_{ij})$ is called **diagonal** when $a_{ij} = 0$ for all $i \neq j$ (a_{ii} is called the diagonal of the matrix)

$$A = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Types of Matrices

- A matrix A is called **square** when it has the same number of columns and rows, i.e. size is $n \times n$
- A square matrix $A = (a_{ij})$ is called **diagonal** when $a_{ij} = 0$ for all $i \neq j$ (a_{ii} is called the diagonal of the matrix)
- A matrix $A = (a_{ij})$ is called **(upper/lower) triangular** when all the elements below/above the diagonal are zeros

$$U = \begin{bmatrix} 7 & 4 & -1 & 3 \\ 0 & -2 & 7 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 7 & 6 & 0 \\ 8 & 1 & 0 & 9 \end{bmatrix}$$

Note: They do not need to be square

Matrix Operations: Product

Given two matrices $A = (a_{ij})$ of size $n \times r$ and $B = (b_{ij})$ of size $r \times m$, the product of 'A' and 'B' is a matrix $C = AB$ of size $n \times m$ with entries $C = (c_{ij})$:

$$c_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ir} b_{rj}$$

- A, B and C have different sizes!! (unless 'A' and 'B' square)
- Product works if number of columns of A same as number of rows of B
- $AB \neq BA$ (BA cannot be done!! unless 'A' and 'B' square)

Matrix Operations: Product Example

Product of 3×2 matrix (A) by 2×4 matrix (B) gives a 3×4 matrix ($C = (c_{ij})$)

$$\begin{bmatrix} 1 & 3 \\ 0 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & -1 & 3 \\ 4 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Matrix Operations: Product Example

c_{11}: take the first row of A and the first column of B multiply elements and sum

$$\begin{bmatrix} 1 & 3 \\ 0 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & -1 & 3 \\ 4 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Matrix Operations: Product Example

c_{12}: take the first row of A and the second column of B multiply elements and sum

$$\begin{bmatrix} \color{red}{1} & \color{red}{3} \\ 0 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & \color{red}{-2} & -1 & 3 \\ 4 & \color{red}{0} & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & \color{red}{-2} & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Matrix Operations: Product Example

c_{13}: take the first row of A and the third column of B multiply elements and sum

$$\begin{bmatrix} 1 & 3 \\ 0 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & -1 & 3 \\ 4 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & -2 & 2 & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

c_{14}: take the first row of A and the fourth column of B multiply elements and sum

$$\begin{bmatrix} 1 & 3 \\ 0 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & -1 & 3 \\ 4 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & -2 & 2 & 0 \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Matrix Operations: Product Example

c_{21}: take the second row of A and the first column of B multiply elements and sum

$$\begin{bmatrix} 1 & 3 \\ \color{red}{0} & \color{red}{-5} \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \color{red}{2} & -2 & -1 & 3 \\ \color{red}{4} & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & -2 & 2 & 0 \\ \color{red}{-20} & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Matrix Operations: Product Example

Keep going until you get the whole C matrix

$$\begin{bmatrix} 1 & 3 \\ 0 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & -1 & 3 \\ 4 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & -2 & 2 & 0 \\ -20 & 0 & -5 & 1 \\ 10 & -2 & 1 & 1 \end{bmatrix}$$

Sizes of matrices $(n \times r) \cdot (r \times m) = n \times m$

Matrix Transpose

The **transpose** of a matrix $A = (a_{ij})$ of size $n \times m$, written A^T, is a matrix of size $m \times n$ defined as $A^T = (a_{ji})$, i.e. the rows are exchanged by columns and the columns by rows.

Example:

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -5 \\ 1 & 2 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 0 & 1 \\ 3 & -5 & 2 \end{bmatrix}$$

Properties of Matrix Operations

Given three matrices 'A', 'B' and 'C' of appropriate sizes

- Commutative addition: $A + B = B + A$
- Associative addition: $(A + B) + C = A + (B + C)$
- Associative product: $A(BC) = (AB)C$
- Distributive left: $(A + B)C = AC + BC$
- Distributive right: $C(A + B) = CA + CB$
- Transpose: $(A^T)^T = A$
- Transpose sum: $(A + B)^T = A^T + B^T$
- Transpose product: $(AB)^T = B^T A^T$

Note: Even if 'A' and 'B' are square in general $AB \neq BA$ (hence two distributive laws)

Special Matrices

With the appropriate sizes:

- **Zero Matrix**: A matrix with all zero entries 0
  Properties (adding/product by zero):
  - If 'A' is a matrix: $A + 0 = 0 + A = A$
  - If 'A' is a matrix: $0A = 0$ (matching sizes)
  - If 'A' is a matrix: $A0 = 0$ (matching sizes)
- **Identity Matrix**: A square diagonal matrix with all ones in the diagonal I
  - If 'A' is a matrix: $AI = IA = A$

Notes:

- The zero matrix and the identity matrix play the role of '0' and '1' in the numbers
- The product with the identity matrix is commutative (exception to $AB \neq BA$)

Determinant of a Matrix

The **determinant** of a square matrix $A = (a_{ij})$, denoted $|A|$ or $\det(A)$, is a number:

- For 2×2 matrices

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- For 3×3 matrices

Positive sign

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Negative sign

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} - a_{32}a_{23}a_{11}$$

Matrix Cofactors

To calculate (recursively) determinants of large matrices we need to use **cofactors**

Given a matrix $A = (a_{ij})$, the **cofactor** M_{kl} is the determinant of the matrix resulting from eliminating from 'A' row 'k' and column 'l' (submatrix) multiplied by $(-1)^{k+l}$

$$M_{kl} = (-1)^{k+l} \begin{vmatrix} a_{1,1} & \cdots & a_{1,l-1} & a_{1,l+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1,1} & \cdots & a_{k-1,l-1} & a_{k-1,l+1} & \cdots & a_{k-1,n} \\ a_{k+1,1} & \cdots & a_{k+1,l-1} & a_{k+1,l+1} & \cdots & a_{k+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,l-1} & a_{n,l+1} & \cdots & a_{n,n} \end{vmatrix}$$

Example of Matrix Cofactors

Obtain $M_{1,1}$ and $M_{2,3}$ of the matrix:

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

Example of Matrix Cofactors

Obtain $M_{1,1}$ and $M_{2,3}$ of the matrix:

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

- Cofactor $M_{1,1}$:

$$M_{1,1} = (-1)^{(1+1)} \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = (-1)^2(1 \cdot 2 - 1 \cdot 4) = -2$$

- Cofactor $M_{2,3}$:

$$M_{1,1} = (-1)^{(2+3)} \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = (-1)^5(0 \cdot 1 - 1 \cdot 2) = 2$$

Laplace Formula

Given a matrix $A = (a_{ij})$ the determinant of A can be calculated as:

$$\det(A) = \sum_{k=0}^{n} a_{k,l} M_{kl} \qquad \text{for any } 1 \leq l \leq n$$

or

$$\det(A) = \sum_{l=0}^{n} a_{k,l} M_{kl} \qquad \text{for any } 1 \leq k \leq n$$

Example:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

- You can use **any** row or column
- You can use it **recursively** for any matrix size

Example of Laplace Formula

Calculate the determinant of the matrix:

$$A = \begin{bmatrix} 0 & 7 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & -3 & 9 & 5 \\ 2 & -1 & 6 & 1 \end{bmatrix}$$

Example of Laplace Formula

Calculate the determinant of the matrix:

$$A = \begin{bmatrix} 0 & 7 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & -3 & 9 & 5 \\ 2 & -1 & 6 & 1 \end{bmatrix}$$

Solution:

$$\det(A) = 2(-1)^{2+2} \begin{vmatrix} 0 & 4 & 0 \\ 1 & 9 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= 2 \cdot 4(-1)^{1+2} \begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix}$$

$$= -8 \cdot (1 \cdot 1 - 2 \cdot 5) = -8 \cdot (1 - 10) = 72$$

Inverse of a Matrix

Given a square matrix A with $\det(A) \neq 0$, the **inverse** of A, denoted A^{-1}, is a matrix such that $AA^{-1} = A^{-1}A = I$ ('I' is identity matrix).

- Similar to $\frac{1}{a}$ for a real number 'a'
- Matrix with $\det(A) \neq 0$ is called non singular
- Inverse only of square matrices
- Product by inverse is commutative
- Calculating inverse 'by hand' is possible but laborious

Properties of Determinants and Inverse

For square matrices 'A', 'B' and 'C'

- $\det(A) = \det(A^T)$
- $\det(AB) = \det(A)\det(B)$
- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
- $\det(A^{-1}) = \frac{1}{\det(A)}$
- If 'A' is a diagonal matrix $\det(A)$ is the product of the entries of the diagonal

Cartesian Product

Given two sets A and B, the Cartesian product $A \times B$ is the set of ordered pairs of A and B.

Examples:

- $A = \{1, 2\}$ and $B = \{a, b, c\}$:
  $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$
- $A = \{1, 2\}$: $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ (a.k.a. A^2)
- For real numbers $\mathbb{R}$:
  $$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x, y) | x, y \in \mathbb{R}\}$$

Note: This extends to any number of sets

$$\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) | x_1, x_2, \cdots x_n \in \mathbb{R}\}$$

Vectors in R^2/R^n

Given the Cartesian product $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ a vector $\mathbf{v} \in \mathbb{R}^2$ is the ordered pair:

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Given the Cartesian product of $\mathbb{R}$ n-times, $\mathbb{R}^n$ a vector $\mathbf{v} \in \mathbb{R}^n$ is

the ordered pair:

$$\mathbf{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

- $x, y / x_1, x_2, \cdots$ are called **components**/**coordinates** of the vectors
- Vectors and points are (subtly) different
- Vectors have a length (norm) and a direction

Norm and Direction of a Vector

Given a vector $\mathbf{v} \in \mathbb{R}^n$ $\mathbf{v} = [x_1, x_2, \cdots, x_n]$:

- The **norm** of $\mathbf{v}$ (a.k.a length) is:

$$|\mathbf{v}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

  - In 2D/3D: $\sqrt{x^2 + y^2}$ / $\sqrt{x^2 + y^2 + z^2}$
  - Similar to the Euclidean distance
- A **unit vector** ($\hat{\mathbf{v}}$ hat) is a vector with norm $|\mathbf{v}| = 1$
- From any vector $\mathbf{v} \neq \mathbf{0}$ we can obtain a unit vector in the same direction as $\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$
- The **direction** of a vector $\mathbf{v}$ is (typically) given by its corresponding unit vector $\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$ (divide all components)

Example Norm and Direction of Vectors

Find the norm and 'direction' of the vectors:

- $\mathbf{v} = [1, 1] \in \mathbb{R}^2$:
- $\mathbf{v} = [1, 0, -2] \in \mathbb{R}^3$:
- $\mathbf{v} = [1, 1, 0, -1] \in \mathbb{R}^4$:

Example Norm and Direction of Vectors

Find the norm and 'direction' of the vectors:

- $\mathbf{v} = [1, 1] \in \mathbb{R}^2$:
  - Norm: $|\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$
  - Unit vector:

$$\mathbf{v} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

- $\mathbf{v} = [1, 0, -2] \in \mathbb{R}^3$:
  - Norm: $|\mathbf{v}| = \sqrt{1^2 + 0^2 + (-2)^2} = \sqrt{5}$
  - Unit vector:

$$\mathbf{v} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

- $\mathbf{v} = [1, 1, 0, -1] \in \mathbb{R}^4$:

Example Norm and Direction of Vectors

Find the norm and 'direction' of the vectors:

- $\mathbf{v} = [1, 1] \in \mathbb{R}^2$:
- $\mathbf{v} = [1, 0, -2] \in \mathbb{R}^3$:
- $\mathbf{v} = [1, 1, 0, -1] \in \mathbb{R}^4$:
  - Norm: $|\mathbf{v}| = \sqrt{1^2 + 1^2 + 0^2 + (-1)^2} = \sqrt{3}$
  - Unit vector:

$$\mathbf{v} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$$

Vector Algebra

We can define two operation for vectors $\mathbf{v} \in \mathbb{R}^n$ **addition** (subtraction) and **product by a number** (scalar).

- Given a vector $\mathbf{v} \in \mathbb{R}^n$ and a number (scalar) $a \in \mathbb{R}$ the product $a\mathbf{v}$ is a vector ($\mathbb{R}^n$) with all the components multipied by 'a'.
- Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ the sum of the vectors $\mathbf{u} + \mathbf{v} \in \mathbb{R}^n$ with components the sum of the components of $\mathbf{u}$ and $\mathbf{v}$.

Notes:

- Closure property: results are also vectors
- Fulfil commutative and associative properties $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ and $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{v} + (\mathbf{u} + \mathbf{w})$
- Subtraction $\mathbf{u} - \mathbf{v}$ can be defined by adding $\mathbf{u} + (-\mathbf{v})$ (change sign of all components of $\mathbf{v}$)

Example of Vector Algebra

- Product by a number: Given $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $a = 3$ find $a\mathbf{v}$
- Vector addition: Given $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ find $\mathbf{u} + \mathbf{v}$

Example of Vector Algebra

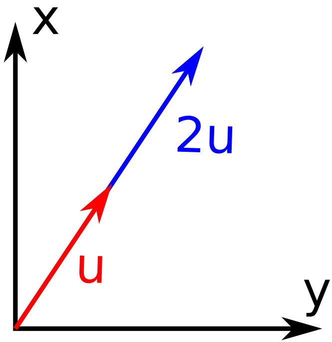
- Product by a number: Given $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $a = 3$ find $a\mathbf{v}$

$$a\mathbf{v} = 3 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

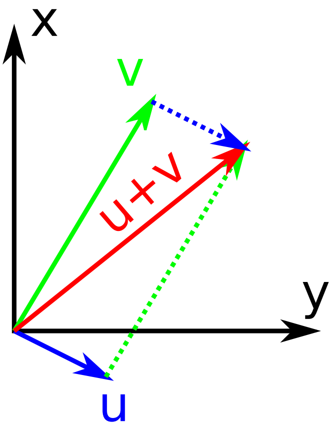
- Vector addition: Given $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ find $\mathbf{u} + \mathbf{v}$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Scalar Multiplication



Addition of vectors



Vector Spaces

The set of vectors (e.g. $\mathbb{R}^n$) together with the addition and product by a number (scalar) is called a **vector space**

Properties:

- Closure, commutative, associative (we saw)
- Identity elements:
  - Vector with all zeros: $\mathbf{v} + \mathbf{0} = \mathbf{v}$
  - Number (scalar) '1': $1\mathbf{v} = \mathbf{v}$
- Inverse element: For $\mathbf{v} \in \mathbb{R}^n$ there is $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
- Distributive:
  - $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$
  - $a(\mathbf{v} + \mathbf{u}) = a\mathbf{v} + a\mathbf{u}$

Note: Similar properties as real numbers

Linear Combination

Let $a_1, a_2, \cdots, a_n \in \mathbb{R}$ be 'n' numbers (scalars) and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be 'n' vectors, the **linear combination** of the vectors is a vector $\mathbf{v}$:

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$$

- The sum of the vectors multiplied by the scalars
- Linear combination is an important concept in linear algebra

Example: Scalars $2, \pi, 3 \in \mathbb{R}$ and vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$$2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \pi\begin{bmatrix} 5 \\ 3 \end{bmatrix} + 3\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 + 5\pi \\ -1 + 3\pi \end{bmatrix}$$

Linear Independence

A set of non-zero vectors $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ are said to be **linearly independent** when the only scalars $a_1, a_2, \cdots, a_n \in \mathbb{R}$ that fulfil:

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n = \mathbf{0}$$

are $a_1 = a_2 = \cdots = a_n = 0$.

If the vectors are not linearly independent, then they are called **linearly dependent**, i.e. some a_i are not zero.

If the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ are linearly dependent (at least) one vector can be stated as a linear combination of the rest (solve for one vector, e.g. $a_1 \neq 0$):

$$\mathbf{u}_1 = -\frac{a_2}{a_1}\mathbf{u}_2 - \frac{a_3}{a_1}\mathbf{u}_2 \cdots - \frac{a_n}{a_1}\mathbf{u}_n$$

Examples of Linear (In)dependence in $\mathbb{R}^2$

- The vectors of R^2 $\mathbf{u}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ are linearly dependent because $1\mathbf{u}_1 + 2\mathbf{u}_2 = 0$ $(\frac{4}{-2} = \frac{2}{-1})$.
- The vectors of R^2 $\mathbf{u}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ are linearly independent because $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 = 0$ happens only for $a_1 = a_2 = 0$
- The vectors of R^2 $\mathbf{u}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ are linearly dependent because $2\mathbf{u}_1 + 3\mathbf{u}_2 + \mathbf{u}_3 = 0$

Note: We do not know of a systematic way of testing for independence (yet)

Basis of a Vector Space

A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is said to be a basis when:

- They are linearly independent, i.e.

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n = \mathbf{0}$$

  means $a_1 = a_2 = \cdots = a_n = 0$
- Any vector $\mathbf{v}$ can be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ (i.e. the set of vectors $\{\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is linearly dependent)

- The **dimension** (d) of a vector space is the cardinality (size) of a basis
- **Any** linearly independent set of 'd' vectors is a basis
  - Robotics, 2D/3D games, rotating screens

Standard Basis of a Vector Space

- The **standard basis** on $\mathbb{R}^2$ is $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (dimension 2)
  - $a\mathbf{e}_1 + b\mathbf{e}_2 = \mathbf{0}$ only when $a = b = 0$
  - If $\mathbf{v} = [x, y]$ then $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2$
- The **standard basis** on $\mathbb{R}^3$ is $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (dimension 3).

Note: For $\mathbf{v} = [x, y]$ the coordinates are 'x' and 'y'

Dot Product

Given two vectors $\mathbf{v} = [v_1, v_2, \cdots, v_n]$ and $\mathbf{w} = [w_1, w_2, \cdots, w_n]$ their **dot product** or **scalar product** $(\mathbf{v} \cdot \mathbf{w})$ is the number:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{i=1}^{n} v_i w_i$$

Takes two vectors and returns a number (scalar)

Properties:

- Commutative: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
- Distributive: $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- Scalar multiplication: $(a\mathbf{v}) \cdot (b\mathbf{w}) = (ab)\mathbf{v} \cdot \mathbf{w}$
- Not associative: $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w}) \neq (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$

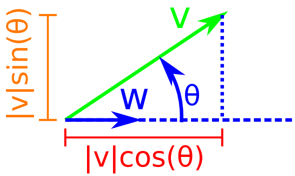
Geometrical Interpretation of the Dot Product

Given two vectors $\mathbf{v}$ and $\mathbf{w}$ the dot product is (also):

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta$$

where θ is the angle between the two vectors $\mathbf{v}$ and $\mathbf{w}$.

If we assume, e.g. $\mathbf{w}$ is a unit vector, i.e. $|\mathbf{w}| = 1$ we get $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| \cos\theta$



Perpendicular projection of $\mathbf{v}$ along $\mathbf{w}$ (length)

Geometrical Interpretation of the Dot Product (ii)

- The angle between one vector and itself is $\theta = 0$, i.e. $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}||\mathbf{v}|\cos(0) = |\mathbf{v}|^2 = a_1^2 + a_2^2 + \cdots + a_n^2$
- Dot product is used to define/calculate angles between vectors
- Two non zero vectors $\mathbf{v}$ and $\mathbf{w}$ are perpendicular (90°) if their dot product is zero $\mathbf{v} \cdot \mathbf{w} = 0$
- The dot product is negative if the vectors point in 'opposite' directions, i.e. $\cos(\theta) < 0$ or $|\theta| > \frac{\pi}{2}$

Examples of Dot Product

Calculate the dot products and check if the vectors are perpendicular

- $\mathbf{u} = [1, 3]$ and $\mathbf{v} = [-6, 2]$
- $\mathbf{u} = [-3, 2, 1, 0]$ and $\mathbf{v} = [0, 2, -4, 5]$
- $\mathbf{u} = [2, 1, 0]$ and $\mathbf{v} = [1, 2, 2]$

Examples of Dot Product

Calculate the dot products and check if the vectors are perpendicular

- $\mathbf{u} = [1, 3]$ and $\mathbf{v} = [-6, 2]$
$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot (-6) + 2 \cdot 3 = 0$$
Perpendicular.
- $\mathbf{u} = [-3, 2, 1, 0]$ and $\mathbf{v} = [0, 2, -4, 5]$
$$\mathbf{u} \cdot \mathbf{v} = (-3) \cdot 0 + 2 \cdot 2 + 1 \cdot (-4) + 0 \cdot 5 = 0$$
Perpendicular.
- $\mathbf{u} = [2, 1, 0]$ and $\mathbf{v} = [1, 2, 2]$
$$\mathbf{u} \cdot \mathbf{v} = 2 \cdot 1 + 1 \cdot 2 + 0 \cdot 2 = 4$$
- **Bonus**: angle between $\mathbf{u} = [1, 1]$ and $\mathbf{v} = [0, 3]$
$$\cos(\theta) = \frac{1 \cdot 0 + 1 \cdot 3}{\sqrt{1^2 + 1^2}\sqrt{0^2 + 3^2}} = \frac{3}{\sqrt{2}\sqrt{3^2}} = \frac{1}{\sqrt{2}}$$
Angle is 45°

Linear Subspaces

A **linear subspace** of $\mathbb{R}^n$ is a vector space of all the linear combinations of a set of linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m\}$ where 'm' ($m < n$) is the dimension of the subspace.

- Closure: Addition of vectors remains in the space
- Result of product of number and vector remains in the space
- The vector '$\mathbf{0}$' is in the subspace

A **hyperplane** of $\mathbb{R}^n$ is a linear subspace of dimension $n - 1$

This is a 'fancy' way of naming lines and planes (which extends to nD).

Lines in $\mathbb{R}^2$

Take one vector in $\mathbf{v} = [a, b] \in \mathbb{R}^2$, it defines a line with points:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \mathbf{v} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$$

i.e. $x = \lambda a$ and $y = \lambda b$ for any number $\lambda \in \mathbb{R}$. Solving for λ we get $\lambda = \frac{x}{a}$ and $\lambda = \frac{y}{b}$, therefore $\frac{x}{a} = \frac{y}{b}$:

$$\boxed{bx - ay = 0}$$

or $y = \frac{b}{a}x$ (equation of a line).

- General equation of a line $y = mx + y_0$
- Subspace goes through the point $[0, 0]$, i.e. $x = 0$ and $y = 0$
- A line in $\mathbb{R}^2$ is a **hyperplane**
- Observation, we can write the line equation as the dot product of two vectors:

$$\begin{bmatrix} b \\ -a \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Planes in $\mathbb{R}^3$

Take two vectors in $\mathbf{u} = [u_1, u_2, u_3], \mathbf{v} = [v_1, v_2, v_3] \in \mathbb{R}^3$, they define a plane with points:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda_u \mathbf{u} + \lambda_v \mathbf{v} = \lambda_u \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \lambda_v \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Eliminating λ_u and λ_v we get to the equation of a plane through the point $[0, 0, 0]$ (origin) in $\mathbb{R}^3$:

$$\boxed{ax + by + cz = 0}$$

- General equation of a plane $ax + by + cz = d$
- Numbers 'a', 'b' and 'c' depend on $\mathbf{u}$ and $\mathbf{v}$
- We can write the line equation as the dot product of two vectors:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Hyperplanes in $\mathbb{R}^n$

Take one vector $\mathbf{v} = [v_1, v_2, \cdots, v_n] \in \mathbb{R}^n$, the set of vectors $\mathbf{x} = [x_1, x_2, \cdots, x_n] \in \mathbb{R}^n$ perpendicular to $\mathbf{v}$, i.e. $\mathbf{v} \cdot \mathbf{x} = 0$, form a hyperplane (subspace of dimension $n-1$)

- Equation of a line (in R^2), plane (in R^3), 3D space (in R^4),... is: $\mathbf{v} \cdot \mathbf{x} = 0$ (with proper dimensions)
- The line, plane, 3D space,... is perpendicular to '$\mathbf{v}$'
- Intersections of hyperplanes reduces dimension
  - Intersection of two lines (1D) in 2D is a point (0D)
  - Intersection of two planes (2D) in 3D is a line (1D)
  - Intersection of three planes (2D) in 3D is a point (0D)
- This can be extended to affine hyperplanes $\mathbf{v} \cdot \mathbf{x} = a$ (not going through the origin)

Summary

Important stuff:

- Concept of Matrix and types (Gaussian elimination)
- Determinants (solution of systems of equations)
- Inverse matrix (Gaussian elimination)

Useful stuff (to understand what you'll bee doing & more):

- Vector algebra
- Vector spaces
- Dot product
- Subspaces