$$\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -3 & +2 & 0 \\ +2 & -5 & 2 \\ 0 & 2 & -3 \end{pmatrix}}_{=M} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

- Eingenvalues of the matrix

$\Leftrightarrow$ $\omega$ s.t. $\underline{q_i = a_i e^{i\omega t}}$

- $\underline{\omega_1} \in Sp(M)$, $q_1 = a_1 e^{i\omega_1 t}$

Solve for $a_i$

$$H(\mathbf{r},\mathbf{p}) = \frac{(\mathbf{p} - q\mathbf{A}(\mathbf{r}))^2}{2m} + q\Phi(\mathbf{r})$$ $\longrightarrow$ Classical e.m. Hamiltonian

$$\mathbf{A}(\mathbf{r}) = \frac{m\omega}{2q}(x\mathbf{e}_y - y\mathbf{e}_x), \quad \Phi(\mathbf{r}) = \frac{m\Omega^2}{4q}\left(2z^2 - x^2 - y^2\right),$$

$$\vec{p} - q\vec{A} = \begin{vmatrix} p_x + q\frac{m\omega}{2q}y \\ p_y - q\frac{m\omega}{2q}x \\ p_z \end{vmatrix} \Rightarrow (\vec{p} - q\vec{A})^2 = \left(p_x + \frac{m\omega}{2}y\right)^2 + p_z^2 + \left(p_y - \frac{m\omega}{2}x\right)^2$$

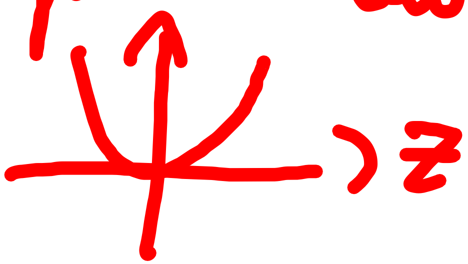
$$= \vec{p}^{\,2} + m\omega(yp_x - xp_y) + \frac{m^2\omega^2}{4}(x^2 + y^2)$$

- $$H = \frac{\vec{p}^{\,2}}{2m} + \frac{\omega}{2}(yp_x - xp_y) + \frac{m\omega^2}{8}(x^2+y^2) + \frac{m\Omega^2}{4}(2z^2 - x^2 - y^2)$$

$$H = \frac{\vec{p}^{\,2}}{2m} + \frac{\omega}{2}(yp_x - xp_y) + \frac{m\Omega^2}{2}z^2 + \frac{m}{8}(\omega^2 - 2\Omega^2)(x^2+y^2)$$

$$H = \underbrace{\frac{\vec{p}^{\,2}}{2m}}_{\text{Standard Kinetic Term}} + \underbrace{\frac{\omega}{2}(y p_x - x p_y)}_{\text{Mixing } \vec{p}/\vec{r}} + \underbrace{\frac{m\Omega^2}{2}}_{\text{Harmonic Oscillator Potential}} z^2 + \underbrace{\frac{m}{8}(\omega^2 - 2\Omega^2)(x^2+y^2)}$$

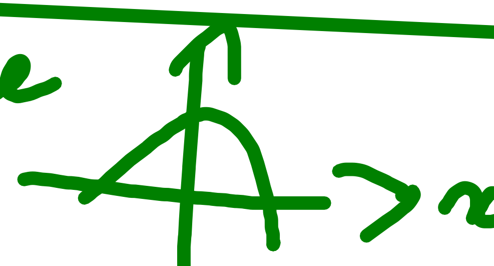
Confinement along z ok




Trapping only if

$$\boxed{\omega^2 - 2\Omega^2 > 0}$$

else




particles are ejected from the "trap"

a system of units such that $m = 1$ and $\Omega_0 = 1$. We perform a first canonical transformation $(x, y, p_x, p_y,) \to (q_+, q_-, p_+, p_-)$ generated by following generating function

$$S(p_x, y, q_+, q_-) = -p_x(q_+ + q_-) - \frac{y}{2}(q_+ - q_-).$$

This generating function gives the canonical transformation from the following relations

$$x = -\frac{\partial S}{\partial p_x}, \quad p_+ = -\frac{\partial S}{\partial q_+}, \quad p_y = \frac{\partial S}{\partial y}, \quad p_- = -\frac{\partial S}{\partial q_-}$$

$$\left\{\begin{array}{l} x = q_+ + q_- \\ p_+ = p_x + \dfrac{y}{2} \\ p_y = -\dfrac{1}{2}(q_+ - q_-) \\ p_- = p_x - \dfrac{y}{2} \end{array}\right. \iff \boxed{\left\{\begin{array}{ll} x = q_+ + q_- & y = p_+ - p_- \\ p_x = \dfrac{p_+ + p_-}{2} & p_y = -\dfrac{1}{2}(q_+ - q_-) \end{array}\right.}$$

$$H = \frac{\vec{p}^{\,2}}{2\underbrace{m}_{=1}} + \frac{\omega}{2}(yp_x - xp_y) + \frac{\overbrace{m}^{=1}\Omega^2}{2} z^2 + \frac{\overbrace{m}^{=1}}{8}\underbrace{(\omega^2 - 2\Omega^2)}_{=\Omega_0^2 = 1}(x^2+y^2)$$

$$\boxed{H_z = \frac{p_z^2}{2} + \frac{\Omega^2 z^2}{2}}$$

$$\begin{cases} x = q_+ + q_- \\ p_x = \dfrac{p_+ + p_-}{2} \end{cases} \qquad y = p_+ - p_- \qquad p_y = -\frac{1}{2}(q_+ - q_-)$$

$$\cdot\; yp_x - xp_y = \frac{(p_+ - p_-)(p_+ + p_-)}{2} + \frac{(q_+ + q_-)}{2}(q_+ - q_-)$$

$$= \frac{p_+^2 - p_-^2 + q_+^2 - q_-^2}{2}$$

$$\boxed{yp_x - xp_y = \frac{1}{2}(p_+^2 + q_+^2) - \frac{1}{2}(q_-^2 + p_-^2)}$$

$$\cdot\, x^2 + y^2 = (q_+ + q_-)^2 + (p_+ - p_-)^2$$

$$= (q_+^2 + p_+^2) + (q_-^2 + p_-^2) - 2p_+p_- + 2q_+q_-$$

$$\cdot\, p_x^2 + p_y^2 = \frac{(p_+ + p_-)^2}{4} + \frac{(q_+ - q_-)^2}{4}$$

$$= \frac{1}{4}\left[q_+^2 + p_+^2 + q_-^2 + p_-^2 + 2p_+p_- - 2q_+q_-\right]$$

$$\cdot\, H_+ = \left(\frac{\omega}{4} + \frac{1}{8} + \frac{1}{8}\right)(q_+^2 + p_+^2) = \frac{\omega + 1}{4}(q_+^2 + p_+^2)$$

· $H_+ = \frac{\Omega_+}{2}\left(q_+^2 + p_+^2\right)$

· $H_- = \left(-\frac{\omega}{4} + \frac{1}{8} + \frac{1}{8}\right)\left(q_-^2 + p_-^2\right)$

$$\boxed{H_- = -\frac{\Omega_-}{2}\left(q_-^2 + p_-^2\right)}$$

· $\frac{1}{8}\left(-2p_+p_- + 2q_+q_-\right) + \frac{1}{8}\left(2p_+p_- - 2q_+q_-\right)$

$= 0$

$$\boxed{H = H_z + \frac{\Omega_+}{2}\left(q_+^2 + p_+^2\right) - \frac{\Omega_-}{2}\left(q_-^2 + p_-^2\right)}$$

$+$ and $-$ are decoupled

$\Rightarrow$ Sum of 3 Harmonic Oscillators

A second canonical transformation $q_+, q_-, z, p_+, p_-, p_z \rightarrow \theta, \varphi, \phi, J, D, I$ is performed where $J$, $D$ and $I$ are called actions and where $\theta$, $\varphi$ and $\phi$ are called angles. This transformation is generated by the generating function

$$F(q_+, q_-, z, \theta, \varphi, \phi) = \frac{q_+^2}{2}\cot(\theta) + \frac{q_-^2}{2}\cot(\varphi) + \frac{\Omega z^2}{2}\cot(\phi).$$

This transformation derive from the following relations

$$p_+ = \frac{\partial F_1}{\partial q_+}, \quad J = -\frac{\partial F_1}{\partial \theta}, \quad p_- = \frac{\partial F_1}{\partial q_-}$$
$$D = -\frac{\partial F_1}{\partial \varphi}, \quad p_z = \frac{\partial F_1}{\partial z}, \quad I = -\frac{\partial F_1}{\partial \phi}$$

$\cot(\theta) = \frac{1}{\tan(\theta)}$

$\frac{d}{d\theta}(\tan(\theta)) = \frac{1}{\cos^2(\theta)}$

$\cdot\, p_+ = q_+ \cot(\theta)$ / $\cdot\, J = -\frac{q_+^2}{2}\frac{d}{d\theta}(\cot(\theta)) = -\frac{q_+^2}{2} \times \frac{-\frac{1}{\cos^2(\theta)}}{\tan^2(\theta)}$

$= \frac{q_+^2}{2\sin^2(\theta)}$

$\Rightarrow \boxed{q_+ = \pm\sqrt{2J}\sin(\theta)}$

$\hookrightarrow \boxed{p_+ = \sqrt{2J}\cos(\theta)}$

We take +

$\boxed{q_+ = \sqrt{2J}\sin(\theta)}$

$$\begin{cases} q_+ = \sqrt{2J}\sin(\theta) \\ p_+ = \sqrt{2J}\cos(\theta) \end{cases} \Rightarrow \boxed{H_+ = \Omega_+ \times J}$$

$$H_+ = \frac{\Omega_+}{2}\left(q_+^2 + p_+^2\right)$$

- $$\boxed{H_- = -\Omega_- \times D}$$

- $$\begin{cases} p_z = \Omega z \cot(\phi) \\ I = \dfrac{\Omega z^2}{2\sin^2(\phi)} \end{cases} \Rightarrow \boxed{\begin{cases} z = \sqrt{\dfrac{2I}{\Omega}}\sin(\phi) \\ p_z = \sqrt{2I\Omega}\cos(\phi) \end{cases}}$$

$$\boxed{H = \Omega_+ J - \Omega_- D + \Omega I}$$

⚠ Only depends on action variable

$$\begin{cases} \dot{J} = -\dfrac{\partial H}{\partial \Theta} \\ \dot{\Theta} = +\dfrac{\partial H}{\partial J} \end{cases} \iff \begin{cases} \dot{J} = 0 \longrightarrow \text{Action is cst} \\ \dot{\Theta} = \Omega_+ \end{cases}$$

$\hookrightarrow \Theta(t) = \Omega_+ t + \Theta_0$

- Perturbation th : • Modification of pulsation of $\Theta$
  $\Omega_+ \pm \varepsilon$
- J remains cst

$w_1 = w_0 \rightarrow$ Lower

$w_2 = \sqrt{3}\, w_0 \rightarrow$ Upper Branch

$\simeq 1{,}7$