Ex 7.2

1) $\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, \cancel{t}) = 0 \quad (*)$

2) ~~$\forall q, \; H(q, \frac{\partial S}{\partial q}) = E$~~ Not Correct

$S(q, t; \underline{\alpha}) \to$ 1 cst of integration

- Take S from the text and proves that satisfies $(*)$
- H not t-dependent $\Longrightarrow$ Ansatz: $S(q,t) = S_1(q) + S_2(t)$

$$\frac{\partial S}{\partial t} = \frac{\partial S_2}{\partial t}(t) \quad / \quad \frac{\partial S}{\partial q} = \frac{\partial S_1}{\partial q}(q)$$

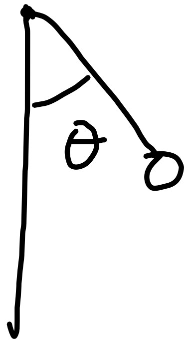
(*) then becomes: $\underbrace{\frac{\partial S_2}{\partial t}(t)}_{f^n \text{ of } t} + \underbrace{H\left(q, \frac{\partial S_1}{\partial q}(q)\right)}_{f^n \text{ of } q} = 0$

$$\frac{\partial S_2}{\partial t} = -E = \text{st}$$

$\hookrightarrow \boxed{S_2 = -Et}$

$$\boxed{H\left(q, \frac{\partial S_1}{\partial q}(q)\right) = E}$$

Ex 8.1] $H = \dfrac{1}{2ml^2}p^2 + mgl(1-\cos(\theta))$



$$\begin{cases} \omega = \sqrt{\dfrac{g}{l}} \\ ml^2 = 1 \end{cases} \Rightarrow H = \frac{p^2}{2} + \omega^2(1-\cos(\theta))$$

· Small amplitudes movements

$$\cos(\theta) = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + o(\theta^6) \qquad \theta \ll 1$$

$$H = \underbrace{\frac{p^2}{2} + \frac{\omega^2}{2}\theta^2}_{\text{H.O.}} \underbrace{- \frac{\omega^2}{8\times 3}\theta^4}_{\text{First non-linear system}} + o(\theta^6)$$

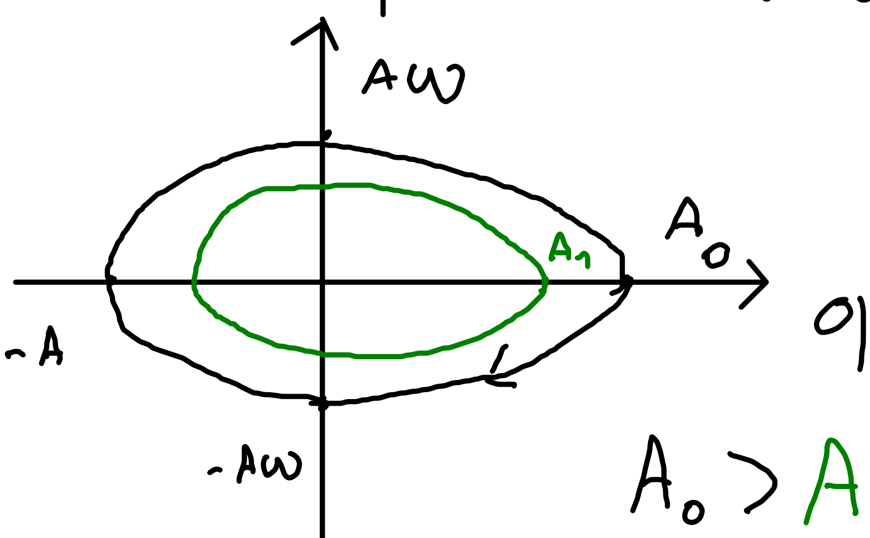
0) $\begin{cases} \dot{p} = -\omega^2 q \\ \dot{q} = p \end{cases} \iff \begin{cases} q(t) = A\cos(\omega t + \varphi) \\ p(t) = -A\omega \sin(\omega t + \varphi) \end{cases}$

<u>For H_0</u>

$$H_0 = \frac{p^2}{2} + \frac{\omega^2}{2} q^2 \longrightarrow \boxed{E = \frac{A^2 \omega^2}{2} = cst}$$

$\partial \omega$

2)



$A_0 > A_1$

$$E = \frac{p^2}{2} + \frac{\omega^2}{2} q^2$$

$\underline{\varphi = 0} \rightarrow t = 0, \quad q_0 = A$

$p_0 = 0$

1). $\frac{1}{4}\varepsilon\omega^2 q^4 \ll \frac{1}{2}\omega^2 q^2$ Assumpt°

$\Longleftrightarrow \frac{\varepsilon q^2}{2} \ll 1 \rightarrow$ Small enough amplitude

$\Longrightarrow$ Small enough energies

· $F_1(q,\theta) = \frac{1}{2}\omega q^2 \cot(\theta)$

$$\begin{cases} p = \frac{\partial F_1}{\partial q} \\ J = -\frac{\partial F_1}{\partial \theta} \end{cases} \Longleftrightarrow \begin{cases} \boxed{p = \omega q \cot(\theta)} \\ J = -\frac{1}{2}\omega q^2 \frac{\partial}{\partial \theta}(\cot(\theta)) \end{cases}$$

$$\frac{\partial}{\partial\theta}(\cot(\theta)) = \frac{\partial}{\partial\theta}\left(\frac{\cos(\theta)}{\sin(\theta)}\right) = \frac{-\sin^2(\theta) - \cos^2(\theta)}{\sin^2(\theta)}$$

$$= -\frac{1}{\sin^2(\theta)}$$

$$\boxed{J = +\frac{1}{2}\omega q^2 \times \frac{1}{\sin^2(\theta)}}$$

$$\hookrightarrow \quad q^2 = \frac{2J}{\omega}\sin^2(\theta)$$

$$\Longleftrightarrow \quad q = \pm\sqrt{\frac{2J}{\omega}}\sin(\theta)$$

Amounts to a shift of θ by π

$$\Longrightarrow \quad \boxed{q = +\sqrt{\frac{2J}{\omega}}\sin(\theta)} \quad \boxed{p = \sqrt{2J\omega}\cos(\theta)}$$

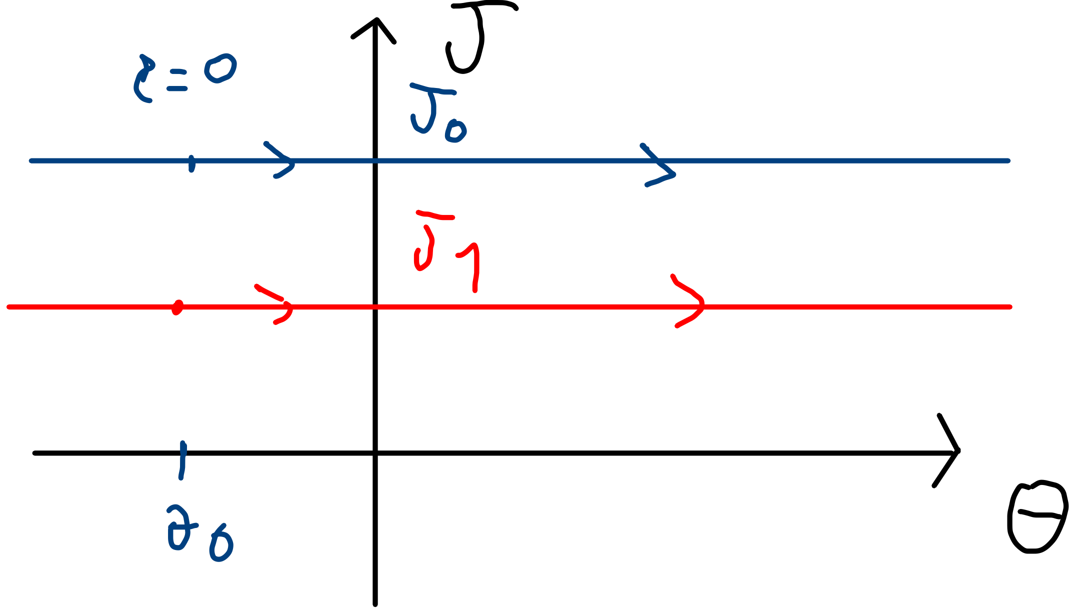
$$H_0 = \frac{p^2}{2} + \frac{\omega^2}{2} q^2 = \frac{\omega}{\cancel{2}} \times \cancel{2} J \times \cos^2(\theta) + \frac{\omega^{\cancel{2}}}{\cancel{2}} \times \frac{\cancel{2} J}{\cancel{\omega}} \times \sin^2(\theta)$$

$\boxed{H_0 = J\omega}$ $\rightarrow$ General form of action-angle Hamiltonian for 1 d.o.f system

Does not depend on θ

4) Hamilton's eq:

$$\begin{cases} \dot{J} = -\dfrac{\partial H_0}{\partial \theta} = 0 \\ \dot{\theta} = +\dfrac{\partial H_0}{\partial J} = \omega \end{cases} \Leftrightarrow \boxed{\begin{cases} J = cst \\ \theta(t) = \omega t + \theta_0 \end{cases}}$$



$\underline{\omega > 0}$

$J_0 > J_1$

5) $H = H_0 + \frac{1}{4}\varepsilon \omega^2 q^4 = J\omega + \frac{1}{4}\varepsilon \omega^2 \left(\frac{2J}{\omega}\right)^2 \sin^4(\theta)$

$$\boxed{H(J,\theta) = J\omega + \varepsilon J^2 \sin^4(\theta)}$$

$\longrightarrow$ It does depend on θ

$\Rightarrow$ J will be a f^n of time, but not <u>at first order</u>

6) We linearize the perturbat° term:

$$\sin^4(\theta) = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i} \right)^4$$

$$= \left(\frac{e^{2i\theta} - 2 + e^{-2i\theta}}{-4} \right)^2$$

$$= \dots$$

$$\sin^4(\theta) = \frac{3}{8} - \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$$

$$H(J,\theta) = J\omega + \varepsilon J^2 \left[\frac{3}{8} - \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta) \right]$$

- We start with an unperturbed periodic system

→ Expect the full system to be close to a periodic system

→ How much is the full system changed over a period of the old system i.e. $\theta \to \theta + 2\pi$?

(1) E.O.M then $\frac{1}{2\pi}\int_0^{2\pi} \times \, d\theta$ → J is not modified
θ <u>is</u>

(2) We average the Hamiltonian :

$$\langle H\rangle = \langle H_0\rangle + \left\langle \frac{3}{8}\varepsilon J^2 \right\rangle + \left\langle \left(-\frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)\right)\varepsilon J^2 \right\rangle$$

- $\langle H_0 \rangle = \int_0^{2\pi} \underbrace{J\omega}_{ct} \frac{d\theta}{2\pi} = J\omega$
- $\left\langle \frac{3}{8} \varepsilon J^2 \right\rangle = \varepsilon J^2 \times \frac{3}{8}$

- $\langle \cos(2\theta) \rangle = 0$ — π - periodic in θ
- $\langle \cos(4\theta) \rangle = 0$ — $\frac{\pi}{2}$ periodic in θ

- ⚠ $\left\langle \cos\left(\frac{\theta}{2}\right) \right\rangle \neq 0$ — 4π - periodic in θ

Secular term

$$\boxed{\langle H\rangle = J\omega + \varepsilon J^2 \times \frac{3}{8}}$$

7) $$\begin{cases} \dot{J} = -\dfrac{\partial \langle H \rangle}{\partial \theta} \\[2ex] \dot{\theta} = +\dfrac{\partial \langle H \rangle}{\partial J} \end{cases} \iff \begin{cases} \dot{J} = 0 \\[2ex] \dot{\theta} = \omega + \dfrac{3}{4}\varepsilon J \end{cases}$$

$$\iff \begin{cases} J = \text{cst} \\[1ex] \theta(t) = \underbrace{\left(\omega + \dfrac{3}{4}\varepsilon J\right)}_{=\,\omega^{(1)}} t + \theta_0 \end{cases}$$

Action not changed, Frequency <u>is changed</u> at first order in ε

8) $$\Omega(J) = \omega + \frac{3}{4}\varepsilon J$$

Period
Frequency of the system depends on J, so on the energy and so on the amplitude of 1D non-linear oscillator

9) Look for $J = f^n(A)$ at lowest order in ε

$$H(q,p) = \frac{p^2}{2} + \frac{q^2 w^2}{2} + \frac{\varepsilon}{4} w^2 q^4$$

$\hookrightarrow$ When q is maximum, $q = A$

and $\dot{q} = p$ (Hamilton's eq) $\longrightarrow p = 0$

$$\Rightarrow \boxed{E = \frac{A^2 w^2}{2} + \frac{\varepsilon}{4} w^2 A^4}$$

• Also $H = \boxed{Jw + \varepsilon \times \frac{3}{8} J^2 + O(\varepsilon^2) = E + O(\varepsilon^2)}$

· We expand $A = A^{(0)} + \varepsilon A^{(1)} + o(\varepsilon^2)$

$$E = \frac{(A^{(0)} + \varepsilon A^{(1)})^2}{2}\omega^2 + \frac{\varepsilon}{4}\omega^2 (A^{(0)} + \varepsilon A^{(1)})^4 + o(\varepsilon^2)$$

$$E = \underbrace{J\omega}_{\text{0th order}} + \varepsilon \times \underbrace{\frac{3}{8}J^2}_{\text{1st order}} + o(\varepsilon^2)$$

To lowest order, 0th order

Solve for $A^{(1)}$

$$E = \underbrace{\frac{A^{(0)2}\omega^2}{2}}_{\text{0th order}} + \varepsilon \Big[\underbrace{A^{(0)} A^{(1)} \omega^2 + \frac{\omega^2 A^{(0)4}}{4}}_{\text{1st order}} \Big] + o(\varepsilon^2)$$

$$\bar{J}\omega = \frac{A^{(0)2}\omega^2}{2} \iff \boxed{\bar{J} = \frac{A^{(0)2}\omega}{2}}$$

10) $$T(A) = \frac{2\pi}{\Omega(\bar{J}(A))} = \frac{2\pi}{\omega + \frac{3}{4}\varepsilon \bar{J}(A)}$$

$$= \frac{2\pi}{\omega} \times \frac{1}{1 + \frac{3}{4}\frac{\varepsilon}{\omega}\bar{J}(A)}$$

Rk: To get 1^{st} order in ε of $T(A)$ you just need 0^{th} order of $\bar{J}(A)$ in ε

$$T(A) = \frac{2\pi}{w} \times \frac{1}{1 + \frac{3}{4} \times \frac{\varepsilon}{w} \times \left[\frac{A^2 w}{2} + o(\varepsilon)\right]}$$

$$= \frac{2\pi}{w}\left[1 - \frac{3}{4} \times \frac{A^2 w}{2w}\varepsilon + o(\varepsilon^2)\right]$$

$$= \frac{2\pi}{w}\left[1 - \frac{3}{8} \times A^2 \times \varepsilon + o(\varepsilon^2)\right]$$

To go beyond

- Chapter V, London-Lifshitz I