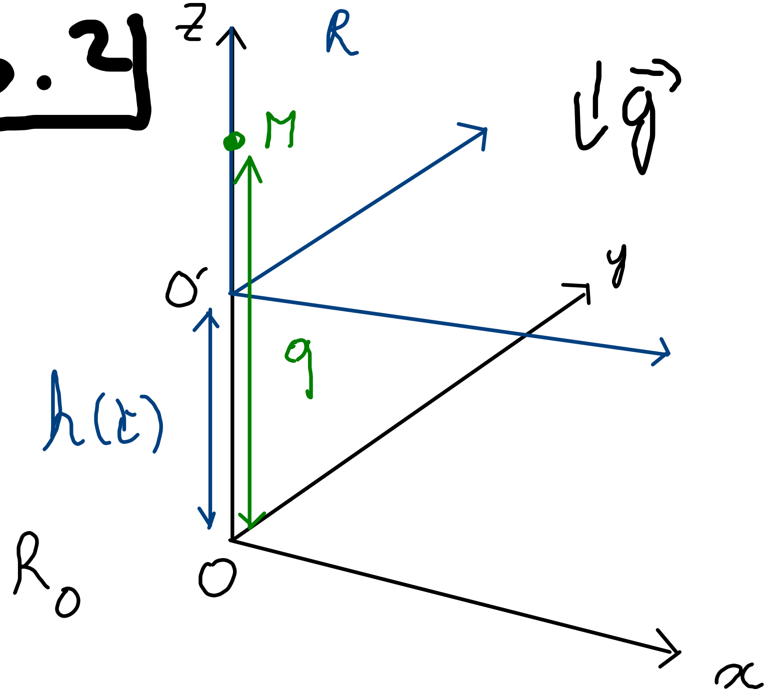
Ex 6.2



R non-galilean frame
$\Leftrightarrow \ddot{h}(t) \neq 0$

else R is uniform transl^{tio} with respect to R_0
$\Longrightarrow$ Galilean

1) Free falling mass in R_0 galilean :

$$\boxed{H(q,p) = \underbrace{\frac{p^2}{2m}}_{\text{Standard kinetic term}} + \underbrace{mg\,q}_{\text{Gravitational Potential}}}$$

2) ⓪ $(q,p) \to (Q,P)$

① $F_2(q,P) \to \frac{\partial F_2}{\partial q}$ or $\frac{\partial F_2}{\partial P}$

② Transformat° rules involve either (q,p) or (Q,P)

$$p = \pm \frac{\partial F_2}{\partial q} \quad / \quad Q = \pm \frac{\partial F_2}{\partial P}$$

③ Just learn sign for type 1, $F_1(q,Q)$:

$$p = + \frac{\partial F_1}{\partial q} \quad / \quad P = - \frac{\partial F_1}{\partial Q} \qquad \text{Type 1}$$

$$\boxed{p = + \frac{\partial F_2}{\partial q} \quad / \quad Q = + \frac{\partial F_2}{\partial P}} \qquad \text{Type 2}$$

$$\begin{cases} p = \dfrac{\partial F_2}{\partial q} \\[2ex] Q = \dfrac{\partial F_2}{\partial P} \end{cases}$$

Find $F_2(q, P)$

for $\begin{cases} Q = q - h(t) \\ P = p - m\dot{h}(t) \end{cases}$

$$\Longleftrightarrow \begin{cases} P + m\dot{h}(t) = \dfrac{\partial F_2}{\partial q} & (1) \\[2ex] q - h(t) = \dfrac{\partial F_2}{\partial P} & (2) \end{cases}$$

$$\boxed{F_2 = q(P + m\dot{h}) - P\,h(t)}$$

· $F_2(q, P)$ generates the transformat° $(q,p) \to (Q,P)$
so this transformat° is canonical

3) $\left(K(Q, P, t) = H(q,p) + \frac{\partial F_2}{\partial t}\right.$
↳ ∀ the type of generating fn

$$\boxed{F_2 = q(P + m\dot{h}) - P h(t)}$$

$$\frac{\partial F_2}{\partial t} = m q \ddot{h} - P \dot{h}$$

$$K = \frac{p^2}{2m} + mgq + mq\ddot{h} - \underline{P}\,\dot{h} \qquad (Q, \underline{P})$$

$$= \frac{(\underline{P} + m\dot{h})^2}{2m} + mg(Q + h(t)) + m(Q + h(t))\ddot{h} - \underline{P}\,\dot{h}$$

$$= \underbrace{\frac{\underline{P}^2}{2m} + mgQ}_{\text{Galilean H of free fall}} + \cancel{\underline{P}\dot{h}} + \frac{m(\dot{h})^2}{2} + mgh(t) + m(Q+h)\ddot{h} - \cancel{\underline{P}\dot{h}}$$

Kinetic energy coming from relative velocity of frames

Potential energy due to the relative posit° of the frames

→ Term due to relative acceleration

· Galileon case: $\dot{h} = 0 \Leftrightarrow \dot{h} = \text{cst} = \alpha$

$\Leftrightarrow h(t) = \alpha t + \alpha_0$

$$K = \frac{p^2}{2m} + mQg + m\frac{\alpha^2}{2} + mg(\alpha t + \alpha_0)$$

$$\frac{\partial K}{\partial t} \underset{(1)}{=} mg\alpha \neq 0$$

· Gauge variance of $\mathcal{L}$: $\mathcal{L} \to \mathcal{L} + \frac{dF(Q,t)}{dt}$

gives 2 equivalent descript° of the system

For the same F: $H \longrightarrow H - \frac{\partial F(q,t)}{\partial t}$

Choosing a suitable F one can remove the explicit time dependence of K by $\tilde{K} \equiv K - \frac{\partial F}{\partial t}$, $\frac{\partial \tilde{K}}{\partial t} = 0$

(2) Using K, I can define E conserved with the dimens° of an energy

$E = K - mg\langle t \implies \frac{dE}{dt} = 0$

Ccl: • Several equivalent Hamiltonian for the same system

$\Longrightarrow$ No absolute definition of the energy of the system in this way

• When $\frac{\partial H}{\partial t} \neq 0$ but simple, we can find a conserved energy / find K s.t. $\frac{\partial K}{\partial t} = 0$

see https://physics.stackexchange.com/questions/287949/how-to-determine-the-lagrangians-true-explicit-dependence-on-time

$- + -$ = Do not depend on Q/P $\Rightarrow$ Do not affect the E.O.M

- Would also appear if R was Galilean $\Rightarrow$ Must not affect the E.O.M.

$-$ $= m(Q+h)\ddot{h}$

- Would not appear if R galilean ($\ddot{h}=0$)
- Contains $Q \Rightarrow$ Affects E.O.M $\Rightarrow$ Term of inertial force

- Hamilton's eq:

$$\begin{cases} \dot{P} = -\dfrac{\partial K}{\partial Q} = -mg - m\ddot{h} \\ \dot{Q} = +\dfrac{\partial K}{\partial P} = \dfrac{P}{m} \end{cases}$$

Ex 7.1 Classical 3D system described by $(\vec{q}(t), \vec{p}(t))$

$$Q \longrightarrow \underbrace{\Psi(\vec{x}, t)}_{\in \mathbb{C}}$$

- $z = \rho e^{i\theta} = x + iy$, $\boxed{\Psi(\vec{x}, t) = \overbrace{R(\vec{x}, t)}^{\in \mathbb{R}} e^{i \frac{\overbrace{S(\vec{x}, t)}^{\in \mathbb{R}}}{\hbar}}}$

- 1 complex Eq $\Leftrightarrow$ 2 coupled real Eqs.

- $\frac{\partial}{\partial t}(\Psi) = \dot{R} e^{i\frac{S}{\hbar}} + \frac{i}{\hbar} R \dot{S} e^{i\frac{S}{\hbar}} = (\dot{R} + \frac{i}{\hbar} R \dot{S}) e^{i\frac{S}{\hbar}}$

- $\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\cdot\ \frac{\partial}{\partial x}(\psi) = \frac{\partial}{\partial x}\left(R e^{i\frac{S}{\hbar}}\right) = \left(\frac{\partial R}{\partial x} + \frac{i}{\hbar} R \frac{\partial S}{\partial x}\right) e^{i\frac{S}{\hbar}}$$

$$\hookrightarrow \frac{\partial^2}{\partial x^2}(\psi) = \left(\frac{\partial^2 R}{\partial x^2} + \frac{i}{\hbar}\frac{\partial R}{\partial x}\frac{\partial S}{\partial x} + \frac{i}{\hbar} R \frac{\partial^2 S}{\partial x^2}\right) e^{i\frac{S}{\hbar}}$$

$$+ \left(\frac{\partial R}{\partial x}\cdot\frac{i}{\hbar}\frac{\partial S}{\partial x} - \frac{R}{\hbar^2}\left(\frac{\partial S}{\partial x}\right)^2\right) e^{i\frac{S}{\hbar}}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \left(\frac{\partial^2 R}{\partial x^2} - \frac{R}{\hbar^2}\left(\frac{\partial S}{\partial x}\right)^2 + i\left(\frac{2}{\hbar}\frac{\partial R}{\partial x}\frac{\partial S}{\partial x} + \frac{R}{\hbar}\frac{\partial^2 S}{\partial x^2}\right)\right) e^{i\frac{S}{\hbar}}$$

$$\boxed{\Delta(\psi) = \left(\Delta R - \frac{R}{\hbar^2}(\vec{\nabla} S)^2 + i\left(\frac{2}{\hbar}\vec{\nabla} S \cdot \vec{\nabla} R + \frac{R}{\hbar}\Delta(S)\right)\right) e^{i\frac{S}{\hbar}}}$$

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\Delta(\psi) + V(\vec{x},t)\psi$$

$$\Leftrightarrow i\hbar\left(\dot{R} + \frac{i\dot{S}}{\hbar}R\right)\cancel{e^{i\frac{S}{\hbar}}} = \left(-\frac{\hbar^2}{2m}\left(\Delta R - \frac{R}{\hbar^2}(\vec{\nabla}S)^2\right) - i\frac{\hbar^2}{2m}\left(\frac{2}{\hbar}\vec{\nabla}S\cdot\vec{\nabla}R + \frac{R}{\hbar}\Delta(S)\right) + V(\vec{x},t)R\right)\cancel{e^{i\frac{S}{\hbar}}}$$

$$\Leftrightarrow \begin{cases} -\dot{S}R = -\frac{\hbar^2}{2m}\Delta R + \frac{R}{2m}(\vec{\nabla}S)^2 + VR \\ \hbar\dot{R} = -\frac{\hbar}{m}\vec{\nabla}S\cdot\vec{\nabla}R - \frac{\hbar}{2m}R\Delta(S) \end{cases}$$

$$\Leftrightarrow \begin{cases} \dot{S} + \frac{1}{2m}(\vec{\nabla}S)^2 + V = \frac{\hbar^2}{2m}\frac{\Delta(R)}{R} \\ \end{cases}$$

- $R = \sqrt{\rho} \longrightarrow \begin{cases} \dot{R} = \dfrac{\dot{\rho}}{2\sqrt{\rho}} \\ \nabla(R) = \dfrac{\nabla(\rho)}{2\sqrt{\rho}} \end{cases}$

- $$\frac{\hbar \dot{\rho}}{2\sqrt{\rho}} = -\frac{\hbar}{m} \times \frac{\vec{\nabla} S \cdot \vec{\nabla}(\rho)}{2\sqrt{\rho}} - \frac{\hbar}{2m}\sqrt{\rho}\,\Delta(S)$$

$$\Longleftrightarrow \dot{\rho} + \frac{\vec{\nabla}(S)}{m} \cdot \vec{\nabla}(\rho) + \frac{\rho\,\Delta(S)}{m} = 0$$

(Continuity Eq.)

$$\Longleftrightarrow \boxed{\dot{\rho} + \vec{\nabla} \cdot \Big(\rho \underbrace{\frac{\vec{\nabla}(S)}{m}}_{=\vec{v}}\Big) = 0}$$

Conservat° Eq for the Probability

Very small

$$\underbrace{\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\vec{\nabla}(S)\right)^2 + V(\vec{x},t)}_{= H(\vec{x},\, \vec{\nabla}(S),\, t)} = \underbrace{\frac{\hbar^2}{2m} \frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}}}$$

Extra term of Quantum origin

Classical Hamilton-Jacobi eq for a particle in the potential V

- Classical 3D system $(\vec{q}, \vec{p})$, Dynamics $H \Rightarrow \frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}, t) = 0$ HJ

Q — $\Psi(\vec{x}, t)$, Dynamics $\hat{H}$, Schrödinger $\Leftrightarrow$ | Conservat° of proba

| Modified HJ

2 real variables

- 1D classical system (q,p)

Dynamics given by $H(q,p)$ $\dashrightarrow$ $\begin{cases} \dot{p} = -\dfrac{\partial H}{\partial q} \\ \dot{q} = +\dfrac{\partial H}{\partial p} \end{cases}$

$\Leftrightarrow \begin{cases} \dot{p} = \{p, H\} \\ \dot{q} = \{q, H\} \end{cases} \Leftrightarrow \boxed{\dot{f}(q,p) = \{f, H\}}$

$\{\,,\,\}$ Poisson bracket

- 1D quantum system Canonical Quantization

$(q,p) \longrightarrow (\hat{q}, \hat{p})$ 2 operators on an Hilbert space = Space of states of the system

- 1D Q system described by $|\psi\rangle$ state, vector of an Hilbert space $\mathcal{H}$

Dynamics $H(q,p) \longrightarrow H(\hat{q},\hat{p})$ operator acting on $\mathcal{H}$

Heisenberg eq: $\boxed{\dot{O}(\hat{q},\hat{p}) = \frac{1}{i\hbar}[O(\hat{q},\hat{p}),\hat{H}]}$

$\{\,,\,\} \longrightarrow \frac{1}{i\hbar}[\,,\,]$

$\boxed{\{q,p\} = 1}$

$\boxed{[\hat{q},\hat{p}] = i\hbar\hat{\mathbb{1}}}$

- How do you like Class and Q trajectory?
- Heisenberg eq. $\Longleftrightarrow$ Schrödinger eq. on ψ

  cf Wikipedia page
- We just derived the Schrödinger eq for a particle in a potential
- Other Q: Feynmann Path Integral (equivalent)

- Ref: Cohen-Tannoudji, QM

- dB-B theory: Q system,

in standard Q mech only a Ψ wavefn
$\Rightarrow$ No trajectory, no localized particle

- dBB I have a particle + a wavefn
and the velocity of it is given by $\vec{v} \propto \frac{\vec{\nabla}(S)}{m}$

- Then particle evolves with the slightly modified H-J!