Ex 8.2

$$H = \underbrace{\left(\frac{p_1^2}{2} + \frac{p_2^2}{2}\right) + \omega_0^2 \frac{\left(q_1^2 + q_2^2\right)}{2}}_{\text{Free-Hamiltonian}} + \underbrace{2\gamma q_1^2 q_2^2 \omega_0^2}_{\text{Coupling term}}$$

$$F_1(q_1, q_2, \theta_1, \theta_2) = \frac{1}{2}\omega_0 q_1^2 \cot(\theta_1) + \frac{1}{2}\omega_0 q_2^2 \cot(\theta_2)$$

$\to$ Sum of 2 generating f^n, 1 for 1^{st} system ...

$$\begin{cases} p_1 = \dfrac{\partial F}{\partial q_1} \\ I_1 = -\dfrac{\partial F}{\partial \theta_1} \end{cases} \Leftrightarrow \begin{cases} p_1 = \omega_0 q_1 \cot(\theta_1) \\ I_1 = -\dfrac{1}{2}\omega_0 q_1^2 \overbrace{\dfrac{\partial}{\partial \theta_1}(\cot(\theta_1))}^{= -\frac{1}{\sin^2(\theta_1)}} \end{cases}$$

$$I_1 = \frac{1}{2} \omega_0 q_1^2 \times \frac{1}{\sin^2(\theta_1)} \Leftrightarrow \boxed{q_1 = \pm\sqrt{\frac{2I_1}{\omega_0}} \sin(\theta_1)}$$

Choosing $\pm$ amounts only to a change of θ_1 by π

$$\boxed{q_1 = \sqrt{\frac{2I_1}{\omega_0}} \sin(\theta_1)}$$

$$p_1 = \omega_0 \cot(\theta_1) q_1 \Leftrightarrow \boxed{p_1 = \sqrt{2I_1\omega_0} \cos(\theta_1)}$$

- Similarly
$$\boxed{\begin{cases} q_2 = \sqrt{\frac{2I_2}{\omega_0}} \sin(\theta_2) \\ p_2 = \sqrt{2I_2\omega_0} \cos(\theta_2) \end{cases}}$$

2) $H = \underbrace{\omega_0 I_1 + \omega_0 I_2} + 2\gamma \cancel{\omega_0^2} \times \frac{4 I_1 I_2}{\cancel{\omega_0^2}} \sin^2(\theta_1) \sin^2(\theta_2)$

Free-H in action-angle coordinates

$$\boxed{H = \omega_0 (I_1 + I_2) + 8\gamma I_1 I_2 \sin^2(\theta_1) \sin^2(\theta_2)}$$

3) $$\sin^2(x)\sin^2(y) = \frac{1}{8}\left[2 - 2\cos 2x - 2\cos 2y + \cos 2(x-y) + \cos 2(x+y)\right]$$

$$H = \omega_0 (I_1 + I_2) + 2\gamma I_1 I_2 + \gamma I_1 I_2 \left[\cos(2(\theta_1 - \theta_2)) + \cos(2(\theta_1 + \theta_2)) - 2\cos(2\theta_1) - 2\cos(2\theta_2)\right]$$

$\hookrightarrow$ $\boxed{H_0 = \underbrace{\omega_0 (I_1 + I_2)}_{\text{Free-H}} + 2\gamma I_1 I_2}$

Net contribution of coupling over a period of θ_1 or θ_2

4) $H_0 = \omega_0(I_1+I_2) + 2\gamma I_1 I_2$

$$\begin{cases} \dot{I}_1 = -\dfrac{\partial H_0}{\partial \theta_1} = 0 \\ \dot{I}_2 = -\dfrac{\partial H_0}{\partial \theta_2} = 0 \end{cases}$$

<u>I_1 and I_2 cst</u> $\Longleftarrow$

$$\begin{cases} \dot{\theta}_1 = \dfrac{\partial H_0}{\partial I_1} = \omega_0 + \underline{2\gamma} I_2 \\ \dot{\theta}_2 = \omega_0 + 2\underline{\gamma I_1} \end{cases}$$

Modif of frequencies by coupling

$\hookrightarrow$ I_i cst

$$\boxed{\begin{cases} \theta_2(t) = \tilde{\omega}_2(I_1)t + \theta_{2,0} \\ \theta_1(t) = \tilde{\omega}_1(I_2)t + \theta_{1,0} \end{cases}}$$

$\hookrightarrow$ E.O.M at 1st order in the perturbat°

5) $\boxed{\gamma I_i \ll \omega_0}$ → Small enough actions

→ Small enough amplitudes

to be in perturbative regime

- $\left| \begin{array}{l} \theta_2 \text{ ——— } \omega_0 + 2\gamma I_1 \\ \theta_1 \text{ oscillates at } \omega_0 + 2\gamma I_2 \end{array} \right.$

- $V(\theta_1, \theta_2, I_1, I_2) = \gamma I_1 I_2 [\cos(2(\theta_1 - \theta_2)) + \cos(2(\theta_1 + \theta_2))$
$- 2\cos(2\theta_1) - 2\cos(2\theta_2)]$

$2\omega_0 + \underbrace{2\gamma(I_1 + I_2)}_{\ll 2\omega_0} \sim 2\omega_0$

A. Vanishing

$\sim 2\omega_0$

↳ A. Vanishing

Oscillates at

$2\omega_0 + \underbrace{4\gamma I_1}_{\ll 2\omega_0}$ Almost

$\sim 2\omega_0$ ⇒ Vanishing over a period

$\boxed{\begin{array}{l} \omega_0 + 2\gamma I_2 - (\omega_0 + 2\gamma I_1) = 2\gamma(I_1 - I_2) \ll \omega_0 \\ \Rightarrow \text{A. Cst over a period} \Rightarrow \text{Net contribution} \end{array}}$

· Because of this A. cst term we talk of a resonant term. It is the fact that both free Harmonic oscillators have the same frequency ω_0 that made it possible

· $$\boxed{V_R(\theta_1-\theta_2, I_1, I_2) = \gamma I_1 I_2 \cos(2(\theta_1-\theta_2))}$$

$$\cancel{\left[V_{NR}(\theta_1, \theta_2, I_1, I_2) = \gamma I_1 I_2 \left[\cos(2(\theta_1+\theta_2)) - 2\cos(2\theta_1) - 2\cos(2\theta_2)\right]\right]}$$

6) (Exact resonance $\rightarrow V_R$ cst at 1st order of perturbat°

$\rightarrow \theta_1-\theta_2 = \text{cst} \rightarrow \tilde{\omega}_1 = \tilde{\omega}_2 \iff 2\gamma I_2 = 2\gamma I_1 \iff \boxed{I_1 = I_2}$

7)

$$F_2(\theta_1,\theta_2,J,P)=\theta_1\frac{J+P}{2}+\theta_2\frac{J-P}{2}$$

$(\theta_1,\theta_2,I_1,I_2)$

$\rightarrow (\varphi, Q, J, P)$

$$\begin{cases} I_1 = \dfrac{\partial F_2}{\partial \theta_1} \\ I_2 = \dfrac{\partial F_2}{\partial \theta_2} \\ \varphi = \dfrac{\partial F_2}{\partial J} \\ Q = \dfrac{\partial F_2}{\partial P} \end{cases} \Longleftrightarrow \begin{cases} \boxed{\begin{array}{l} I_1 = \dfrac{J+P}{2} \\ I_2 = \dfrac{J-P}{2} \end{array}} \\ \varphi = \dfrac{\theta_1+\theta_2}{2} \\ Q = \dfrac{\theta_1-\theta_2}{2} \end{cases} \Rightarrow \boxed{\begin{cases} \theta_1 = \varphi + Q \\ \theta_2 = \varphi - Q \end{cases}}$$

8) $H_0 = \omega_0(I_1+I_2)+2\gamma I_1 I_2 = \omega_0 J + \frac{\gamma}{2}(J^2-P^2)$

$V_R = \gamma I_1 I_2 \cos(2(\theta_1-\theta_2)) = \frac{\gamma}{4}(J^2-P^2)\cos(4Q)$

9) $H = \omega_0 J + \frac{\gamma}{2} J^2 - \frac{\gamma}{2} P^2 + \frac{\gamma}{4}(J^2 - P^2)\cos(4Q)$

↳ φ independent

↳ $\dot{J} = -\frac{\partial H}{\partial \varphi} = 0 \implies \boxed{J = \text{cst}}$

10) Focus on the evolution of P and Q $\boxed{\gamma \neq 0}$

$$\begin{cases} \dot{P} = -\frac{\partial H}{\partial Q} = +\gamma (J^2 - P^2)\sin(4Q) \\ \dot{Q} = \frac{\partial H}{\partial P} = -\gamma P - \frac{\gamma P}{2}\cos(4Q) = -\frac{\gamma P}{2}\underbrace{[2 + \overbrace{\cos(4Q)}^{-1<}]}_{>0} \end{cases}$$

. When do they vanish? $\dot{Q} = 0 \iff \boxed{P = 0}$

$\dot{P} = 0 \iff \gamma J^2 \sin(4Q) = 0 \iff \boxed{\begin{array}{l} J = 0 \\ 4Q \equiv 0\ [\pi] \end{array}}$ Not considered after

$$\begin{cases} P = 0 \\ 4Q \equiv 0 \, [\pi] \end{cases} \iff \boxed{\begin{cases} P_0 = 0 \\ Q_0 = n \times \frac{\pi}{4}, \; n \in \mathbb{Z} \end{cases}}$$

11) $P = I_1 - I_2$ no resonance cond $I_1 = I_2 \iff \boxed{P = 0}$

$\Rightarrow$ It's one of the eq. position of the (P, Q) system

12) Motion in the phase portrait:

$\sin(4Q) > 0$

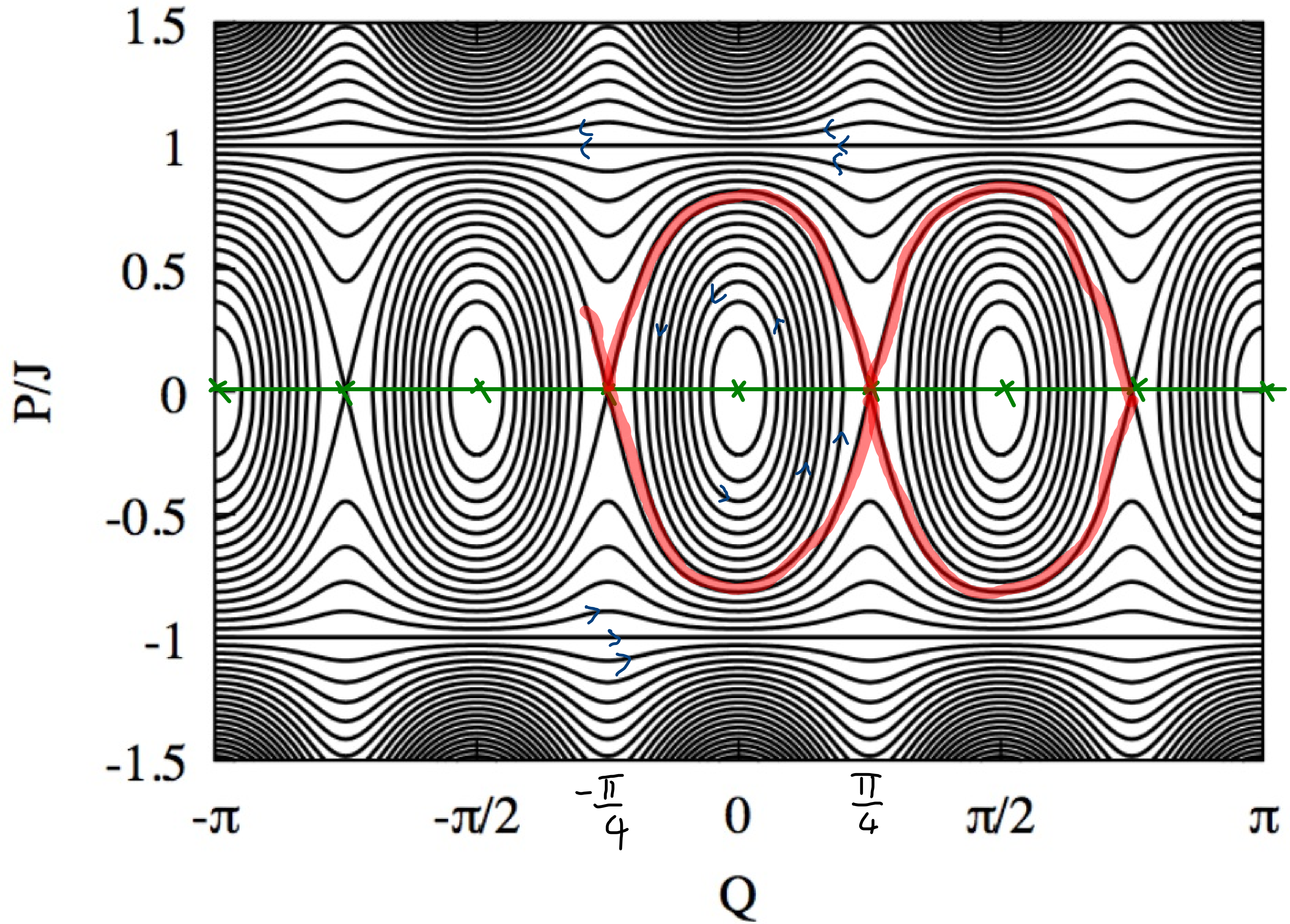
$\Uparrow$

· For P, $\dot{P} = (J^2 - P^2)\sin(4Q)$ If $\left| \frac{P}{J} \right| < 1$, $\left[0 < Q < \frac{\pi}{4} \right.$

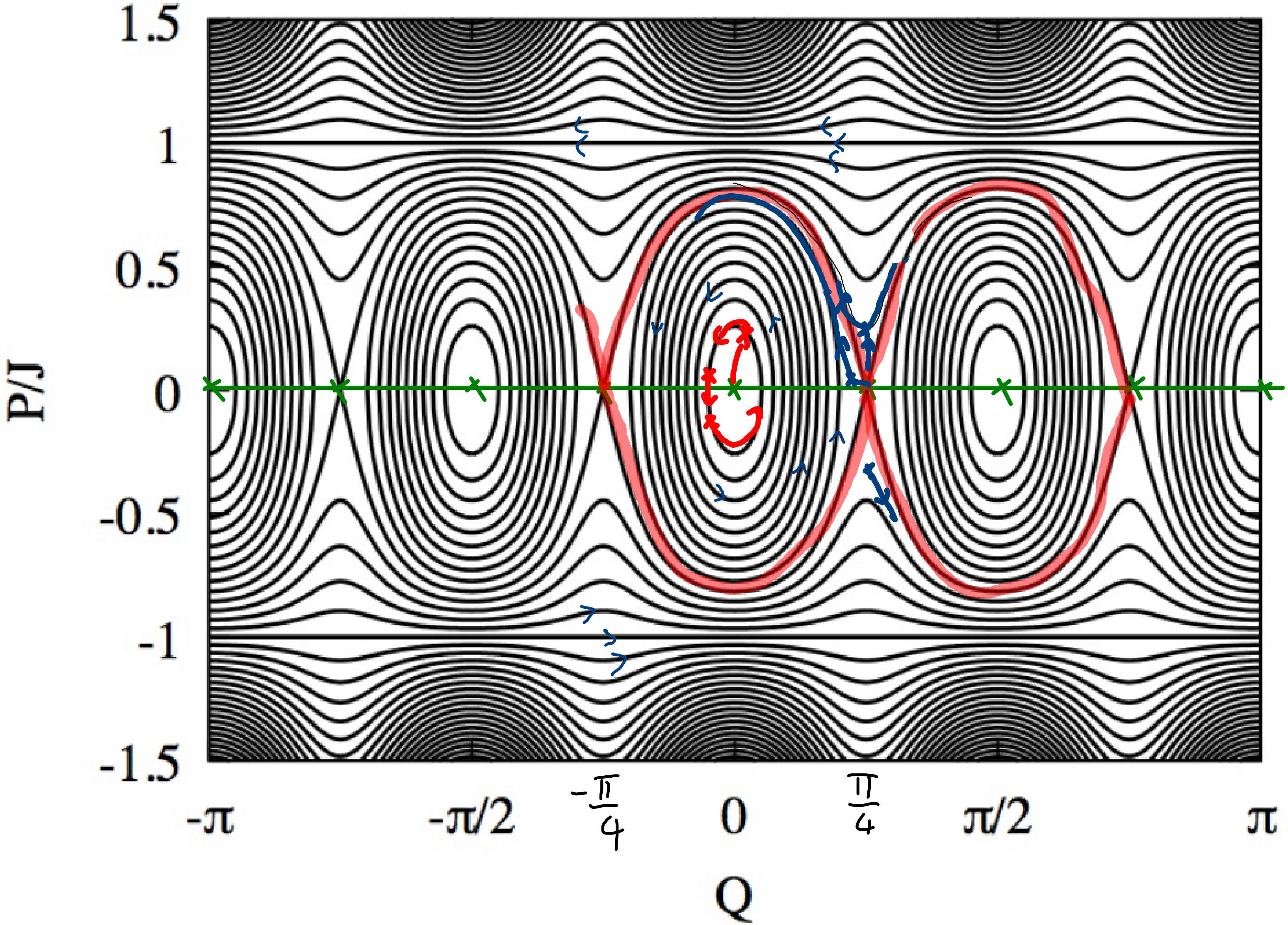
$\Rightarrow \dot{P} > 0$, $P \nearrow$

$\dot{Q} = -\frac{\gamma P}{2}[2 + \cos(4Q)]$ If $\left| \begin{array}{l} P > 0, \; Q \searrow \\ P < 0, \; Q \nearrow \end{array} \right.$; $\left[0 > Q > -\frac{\pi}{4} \right.$, $P \searrow$

12)



12)



- $Q = n\frac{\pi}{2}, \; n \in \mathbb{Z} \longrightarrow$ <u>Stable</u>

  $Q = (2n+1) \times \frac{\pi}{4}, \; n \in \mathbb{Z} \longrightarrow$ <u>Unstable</u>

13) Looks like a non-linear oscillator/simple pendulum

$$1 - \frac{q^2}{2} + \dots$$

$$\tilde{H} = \frac{p^2}{2} - W_0 \cos(q) + \text{cst}$$

$$H = \underbrace{W_0 J + \frac{\gamma}{2} J^2}_{\text{Only } J \text{ dependent}} - \frac{\gamma}{2} P^2 + \frac{\gamma}{4} (\underbrace{J^2 - P^2}_{\simeq J^2 \text{ since } P \ll J}) \cos(4Q)$$

$$H_a \simeq -\frac{\gamma}{2}P^2 + \frac{\gamma}{4}J^2\cos(4Q) \rightarrow \neq \text{ with } \tilde{H}$$

are the signs

$$\hookrightarrow \begin{cases} \dot{P} = +\gamma J^2 \sin(4Q) \\ \dot{Q} = -\gamma P \end{cases} \Longrightarrow \ddot{Q} = -\gamma^2 J^2 \sin(4Q)$$

$$\Longleftrightarrow \boxed{\ddot{Q} + \gamma^2 J^2 \sin(4Q) = 0}$$

↳ Equat° of simple pendulum

Good Luck Everyone!

Feel free to contact me:

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