Université Paris Saclay – Magistère – LDD – ENS

Analytical Mechanics
Tutorials
2020 – 2021

TD 1 : Interaction potential and system of particles

Exercice 1.1. One-dimensional oscillator

The goal of this exercise is to show that any one-dimensional conservative system is formally integrable. Here we will consider the calculation of the period of an oscillator.

We consider a particle of mass m moving along a straight axis Ox and undergoing a force deriving from a potential energy $V(x)$ having a minimum taken as the origin of the frame and which we will assume even $V(-x) = V(x)$. The mass oscillates around this position of stable equilibrium.

1. Write Newton's 2nd law of motion and the expression of the mechanical energy E of the system. Show that E is a constant of motion.

2. From the expression of the energy, show that for a general form of potential, it is possible to express the period of oscillation of the mass as the integral

$$T = 2\sqrt{2m}\int_0^{x_0} \frac{\mathrm{d}x'}{\sqrt{E - V(x')}}$$

where $x_0 \geqslant 0$ is defined by $E = V(x_0)$.

3. In the case of a potential of the form $V(x) = m\alpha|x|^n$ where $n > 0$ and $\alpha > 0$, express the period of oscillation T_n as a function of the energy E, the parameter α and the integral $I_n = \int_0^1 \frac{\mathrm{d}y}{\sqrt{1-y^n}}$ that you will not try to calculate. [1]

4. In the case of a harmonic potential $n = 2$, calculate the period of oscillation T_2 and show that the harmonic oscillator is isochronous. We denote $\alpha = \omega^2/2$ and we also give the value of the integral $I_2 = \pi/2$.

Exercice 1.2. Two-dimensional oscillator(s)

The objective of this exercise is to show that there is no differences between the problem of two one-dimensional particles and the problem of a two-dimensional particle.

We consider a particle of mass m moving in the plane Oxy. It is subjected to a force deriving from the potential energy $V(x, y)$ having a minimum taken as the origin of the frame.

1. Write Newton's 2nd law of motion and the expression of the mechanical energy E of the system. Show that E is a constant of motion.

2. In the case of a harmonic potential of the form $V(x, y) = \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2\right)$ solve Newton's equations and give the general solution. Describe the trajectory in the plane Oxy in the general case and in the case where $\omega_x = \omega_y$.

We now consider two particles of masses m moving along the axis Ox. We will denote by x_1 and x_2 the positions of the two particles along this axis. The two particles do not interact with each other but each undergo a force deriving from the same potential $V(x)$.

3. Write Newton's law of motion for the two particles and give the definition of the energy of the system. Check that it is a constant of motion.

---

[1] I_n can be expressed as a function of the Euler function $\Gamma(z)$ according to $I_n = \sqrt{\pi}\,\Gamma(1 + 1/n)/\Gamma(1/2 + 1/n)$

4. In the case of a harmonic potential $V(x) = \frac{1}{2}m\omega^2 x^2$, give the solution of the problem and the trajectory in the plane Ox_1x_2. Compare to the result of question 2.

We now consider that the two particles are subject to an interaction force which derives from the potential $V(x_2 - x_1)$ which depends only on the interdistance $r = x_2 - x_1$ between the particles.

5. Give the definition of the forces applying on each particles. Show that the law of action and reaction applies.

6. Give the definition of energy and show that it is conserved.

We will now consider the most general case of two particles of masses m_1 and m_2 moving along the axis Ox. The particles are subjected to forces deriving from the general potential $V(x_1, x_2)$.

7. What general form must the potential energy take for the particles to be considered in interaction?

8. Using the change of variables $y_i = \sqrt{m_i}x_i$, $i = 1, 2$, show that this problem is identical to the problem of a particle of unit mass in a two-dimensional space subject to the same potential. Generalize to the case of N particles in a three-dimensional space.

Exercice 1.3. N-body problem – Pair interaction potential

The goal of this exercice is to manipulate the concept of pair potential We consider a set of N particles of mass m and positions $\mathbf{r}_i$, $i = 1, \ldots, N$ which interact with each other through a pair interaction potential, [2] each pair of particles interacts with each other through a potential $\Phi(r)$ which depends only on the distance between the two particles. For example in the case of 3 particles the potential is written

$$V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \Phi(|\mathbf{r}_2 - \mathbf{r}_1|) + \Phi(|\mathbf{r}_3 - \mathbf{r}_1|) + \Phi(|\mathbf{r}_3 - \mathbf{r}_2|).$$

We can generalize this equation to the case of N particles by summing over the pairs, the total potential energy is therefore written

$$V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \sum_{i<j} \Phi(|\mathbf{r}_j - \mathbf{r}_i|)$$

where $\sum_{i<j} = \sum_{i=1}^{N} \sum_{j=i+1}^{N}$ is a sum over the set of $N(N-1)/2$ pairs of particles.

1. Give the expression of the energy of the system

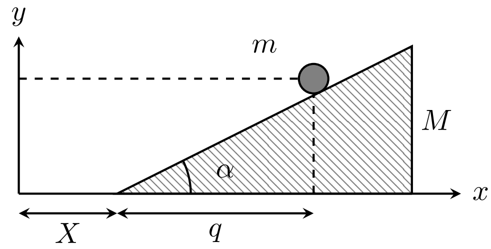
2. Check that the forces between the particles are additive, i.e. the force applying on the particle k is written as $\mathbf{F}_k = \sum_{i \neq k} \mathbf{F}_{i/k}$. Give the expression of the force $\mathbf{F}_{i/k}$ as a function of $\mathrm{d}\Phi/\mathrm{d}r$, $\mathbf{r}_i$ and $\mathbf{r}_k$.

[2] The electrostatic and gravitational interactions are examples of pair potentials.

TD 2 : Lagrange

Exercice 2.1. The moving inclined plane

A mass m moving in the xOy plane slides freely on an inclined plane. The angle of the inclined plane is given by α and its mass M. The inclined plane slides without friction along the horizontal axis Ox. The position of the inclined plane is identified by the coordinate X of its apex. The mass m is identified by its coordinates x and y. We consider the generalized coordinate $q = x - X$.



1. Establish the expression of the kinetic energy T as a function of the Cartesian coordinates of the two masses and then as a function of the generalized velocities $\dot{q}$ and $\dot{X}$.

2. Establish the expression of the potential energy U as a function of the Cartesian coordinates and then as a function of q.

3. Give the expression of the Lagrangian $\mathscr{L}(q, \dot{q}, \dot{X})$.

4. Write the Euler-Lagrange equations

5. Solve the Euler-Lagrange equations and describe the motion.

Exercice 2.2. Motion on a surface of revolution

A point mass m is forced to move on a surface of revolution around the axis Oz and passing through the origin of the frame. The mass undergoes the uniform vertical gravitationnal field $\mathbf{g} = -g\mathbf{e}_z$. We use cylindrical coordinates (r, θ, z). The surface is characterized by a function $z = h(r)$, strictly increasing, continuous and derivable over the interval $]0, \infty[$ whose exact shape will not be specified.

1. What is the number of degrees of freedom? Write the Lagrangian using the coordinates (r, θ).

2. Write the Euler-Lagrange equations and show that $\ell = r^2\dot{\theta}$ is a constant of motion. What is the physical meaning of this quantity ?

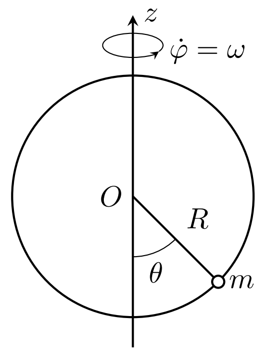
3. Remove the variable θ from the equation for r.

We will consider the case of a cone defined by the equation $z = \tan\alpha\ r$.

4. Write the equation of motion for r as $\ddot{r} = -\frac{\mathrm{d}V}{\mathrm{d}r}$. and give the expression of $V(r)$ as a function of r, g, α and ℓ.

5. Make a graphical analysis of the motion using the effective potential $V(r)$.

Exercice 2.3. Rotating ring



We consider a ring of radius R centered in O in uniform rotation at the angular velocity ω around the axis Oz. A point mass m slides without friction along this ring. The position of the mass is characterized by the angle θ defined on the diagram above. The mass m is subjected to the gravitational acceleration $\mathbf{g} = -g\mathbf{e}_z$.

1. Establish the expression of the Lagrangian for the mass m as a function of the variables θ and $\dot{\theta}$.

2. Write the Euler-Lagrange equation.

3. Identify the equilibrium positions of the mass m.

4. Study the stability of these equilibrium positions. Draw the evolution of the stable equilibrium position as a function to the parameter ω/ω_0.

Exercice 2.4. Inhomogeneous magnetic field

We will consider in this problem the motion of a charge q of mass m in an inhomogeneous and static magnetic field of fixed direction. We introduce an orthonormal frame $Oxyz$ and the unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. We will assume that the configuration of the magnetic field is such that $\mathbf{B}(x) = B(x)\mathbf{e}_y$. The motion of a charged particle in the direction of the magnetic field is a uniform translation and we will be interested here only in the motion in the plane (xOz).

The magnetic field $\mathbf{B}(x)$ is expressed as a function of the vector potential $\mathbf{A}(x)$ according to the relation $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. We also recall that the Lagrangian of a charged particle in a time-independent magnetic field is given by

$$\mathscr{L}(\mathbf{r}, \mathbf{v}) = \frac{1}{2}m\mathbf{v}^2 + q\mathbf{v} \cdot \mathbf{A}(\mathbf{r}) \tag{0.1}$$

1. Show that the vector potential can be written $\mathbf{A}(x) = A(x)\mathbf{e}_z$ and give the relation between $B(x)$ and $A(x)$.

2. Write the expression of the Lagrangian $\mathscr{L}(x, \dot{x}, \dot{z})$ as a function of $A(x)$ and the Cartesian coordinates and velocities of the particle in the plane (xOz).

3. Define and give the expression of the conjugated moments p_x and p_z as a function of the Cartesian positions and velocities.

4. Write the Euler-Lagrange equations and show that p_z is a constant. What type of coordinate is the coordinate z ?

5. Deduce that the motion of the particle along the axis Ox can be described by the following equation:

$$m\ddot{x} = -\frac{\mathrm{d}}{\mathrm{d}x}V_{\text{eff}}(x), \tag{0.2}$$

and give the expression of $V_{\text{eff}}(x)$ as a function of $A(x)$ and p_z, assuming that the origin of the effective potential is defined by $V_{\text{eff}}(x_0) = 0$ with x_0 defined by $qA(x_0) = p_z$.

6. Write the expression to the energy E and show that the energy is a constant of motion.

We will consider here the case of a charge coming from an area without magnetic field and moving towards an area of strong magnetic field. We will assume that the magnetic field intensity increases exponentially with the distance x, thus forming a “wall”. The field can therefore be written

$$B(x) = Be^{-ax}, \tag{0.3}$$

where $a > 0$ is a parameter that controls the growth of the magnetic field as a function of the distance x. The charge can come from $x = +\infty$ but cannot cross the “wall” and reach $x \to -\infty$.

7. Give the expression of $A(x)$ for this magnetic field configuration. We can choose $A(x) = 0$ for $x \to +\infty$.

8. We will assume that $p_z/qB > 0$. Show that the effective potential can be written as

$$V_{\text{eff}}(x) = D\left(1 - e^{-a(x-x_0)}\right)^2, \tag{0.4}$$

Give the expression of D and x_0 as a function of q, B, a and p_z.

9. Draw the function $V_{\text{eff}}(x)$ and show that there are bound orbits for $0 \leqslant E < D$ and free orbits for $E \geqslant D$.

10. Draw the shape of the bounded orbits $0 \leqslant E < D$ and free orbits $E \geqslant D$ in the plane (xOz).

11. We will consider the bounded orbits such as $0 \leqslant E < D$. We introduce the parameter θ defined by $\cos^2\theta = E/D$. Using the variables $u = e^{a(x-x_0)}$ and $\tau = \sqrt{2Da^2/m}t$ find a relation between $u' = \mathrm{d}u/\mathrm{d}\tau$, u and θ.

12. Deduce the values $u_\pm$ of the turning points $u' = 0$ of the trajectory.

13. Find an linear transformation of the coordinates $v = \alpha(u - \beta)$ such that we have the relation

$$v'^2 - \sin^2\theta\left(1 - v^2\right) = 0. \tag{0.5}$$

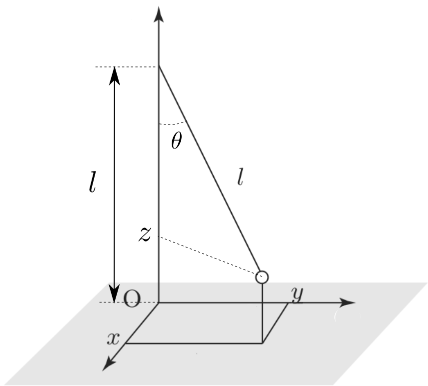
Give the expression of α and β as a function of θ.

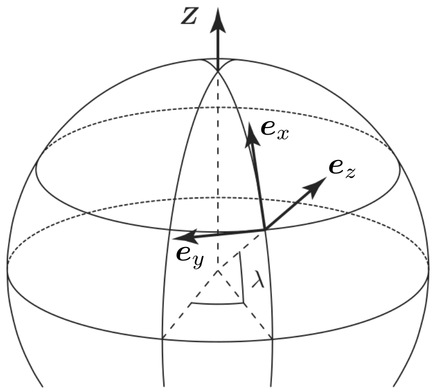
14. Using the equation $\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin(x)$, deduce the expression of $v(\tau)$ as a function of τ and θ. We can choose for example the origin of the time such that $v(\tau = 0) = 0$.

15. Give the expression of the position $x(\tau)$ as a function of τ and θ.

Exercice 2.5. Foucault's pendulum

Léon Foucault's pendulum is an experimental device designed to give a direct evidence of the rotation of the Earth with respect to a Galilean reference frame. The first public demonstration of this experiment dates back to 1851, the pendulum was suspended from the dome of the Pantheon in Paris where it is still possible to see it today. The objective of this exercise is to calculate the precession velocity of the pendulum as a function of the rotational velocity of the Earth and the latitude.





We note $\mathcal{R}_0$ the Galilean frame of reference in uniform translation with respect to the Sun, *i.e.* we neglect the curvature of the orbit of the Earth around the Sun during one day. The terrestrial frame of reference noted $\mathcal{R}_T$ and attached to the Earth rotates with respect to $\mathcal{R}_0$ at the angular velocity Ω around the Earth's axis of rotation of unit vector $\mathbf{Z}$.
In the terrestrial reference frame $\mathcal{R}_T$, the position of the pendulum of mass m and length ℓ is identified by the vector $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ where the vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ are tangent to the surface of the Earth and point respectively towards the North and West, while $\mathbf{e}_z$ points in the vertical direction through the center of the Earth. The vector $\mathbf{e}_z$ makes an angle λ with the equatorial plane. The point O of coordinate $(0,0,0)$ in the orthonormal base $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is at a distance R from the center of the Earth. The acceleration of gravity is $\mathbf{g} = -g\mathbf{e}_z$.
Using the constraint between the Cartesian coordinates of the pendulum, we will in the following questions express the variable z as a function of x and y so as to consider Foucault's pendulum as a two-dimensional problem:

1. Let θ be the angle that the pendulum makes with respect to the vector $-\mathbf{e}_z$. Express the coordinate z of the pendulum as a function of ℓ and θ. In the approximation of small angles $\theta \ll 1$, expand the expression of z up to the second order in θ.

2. Find a relation between the angle θ, the length ℓ and the coordinates x and y of the pendulum in the plane $(\mathbf{e}_x, \mathbf{e}_y)$. Simplify this relation in the case $\theta \ll 1$. Show that in the approximation of small angles we have $z = \alpha(x^2 + y^2)$ and give the expression of α as a function of ℓ.

3. Give the expression of the potential energy $U(x, y)$ of the pendulum using the small angle approximation as a function of the parameters of the problem.

In the following, we will express the kinetic energy of the pendulum as a function only of the coordinates (x, y) and the associated velocities $(\dot{x}, \dot{y})$:

4. Express the unit vector $\mathbf{Z}$ in the base $\mathbf{e}_x$ and $\mathbf{e}_z$.

5. Give the expression for the velocity $\mathbf{v} = \dot{\mathbf{r}}$ of the pendulum in the reference frame $\mathcal{R}_T$ as a function of x, $\dot{x}$, y and $\dot{y}$. Give a condition that allows us to neglect the projection of the velocity on the axis z. In the rest of the problem, we will neglect v_z in the reference frame $\mathcal{R}_T$.

6. Using the relation between velocities in two frames

$$\mathbf{v}_0 = \mathbf{v} + \Omega\mathbf{Z} \times (R\mathbf{e}_z + \mathbf{r}),$$

give the components of the velocity $\mathbf{v}_0$ of the pendulum in the Galilean frame of reference $\mathcal{R}_0$ in the base $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ as a function of the variables $(x, y, z, \dot{x}, \dot{y})$ and the parameters of the problem.

7. Expand $\mathbf{v}_o^2$ *as much as possible*, then simplify the expression obtained by neglecting
– the terms in Ω^2, *except* if they are multiplied by the radius of the Earth R,
– terms in z, *except* if they are multiplied by the radius of the Earth R.

8. By calculating the angular velocity Ω in radian per second and its square, justify the first approximation made in the previous question.

9. Within the framework of the approximations used in the previous questions, write the expression of the Lagrangian function $\mathscr{L} = \mathscr{L}(x, y, \dot{x}, \dot{y})$ of the pendulum in the form

$$\begin{aligned}\mathscr{L} = \frac{m}{2}\Big[\dot{x}^2 + \dot{y}^2 + 2\Omega\sin(\lambda)(\dot{y}x - y\dot{x}) - (2R\Omega\cos\lambda)\dot{y} - (R\Omega^2\sin 2\lambda)x\Big] \\ - mg'\alpha(x^2 + y^2) + C\end{aligned}$$

where you will specify the expressions of the constants g' and C as a function of m, g, R and λ.

10. Write Euler-Lagrange equations

11. The equation for the coordinate x contains a constant term. Show that it can be removed by the change of variable $x = x' + x_e$. What is the expression of x_e as a function of the parameters of the problem? Which force present in the non-inertial reference frame is responsible for the displacement x_e?

12. Using the complex variable $u(t) = x'(t) + \mathrm{i}y(t)$, show that the Euler-Lagrange equations are equivalent to the equation

$$\ddot{u} - (2\mathrm{i}r)\,\dot{u} + g'u/\ell = 0,$$

specify the expression of r as a function of Ω and λ.

13. Using the change of variable $u(t) = U(t)e^{\mathrm{i}rt}$, write the equation of motion in the form $\ddot{U} + \omega^2 U = 0$. Give the expression of ω as a function of g', r and ℓ.

14. By noting $U(t) = X(t) + \mathrm{i}Y(t)$, describe the shape of the orbit corresponding to the parametric curve $(X(t), Y(t))$ in the plane (X, Y).

15. Using the geometric interpretation of the multiplication of a complex number by another complex number of unit modulus, describe the motion of the pendulum in the (x, y) plane. What is the angular velocity of the precession of the pendulum ?

16. The picture below shows Foucault's Pendulum installed in the Pantheon. The graduations (from 0 to 24) used to measure the precession of the pendulum appears to cover three quarters of the circle, comment.

TD 3 : Optimization and variational principle

Exercice 3.1. Optimization

1. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be the function $F(x, y) = x^2 + y^2 + xy$. Find the minimum of this function
2. Find the extrema of the function $F(x, y)$ by imposing the constraint $x^2 + y^2 = 1$.

Exercice 3.2. Catenoid

A film of soap between two circular rings of radius R centered around the Oz axis and distant by the length a forms the catenoid surface which is the minimum surface. In a cylindrical coordinate system, the surface is defined by a curve $r = r(z)$. The surface element is written $\mathrm{d}S = 2\pi r \mathrm{d}\ell = 2\pi r\sqrt{1 + r'^2}\mathrm{d}z$ where $r' = \mathrm{d}r/\mathrm{d}z$. The total area is thus written

$$S[r(z)] = 2\pi \int_{-a/2}^{a/2} F(r, r')\mathrm{d}z$$

where $F(r, r') = r(z)\sqrt{1 + r'(z)^2}$. The boundary conditions of the curve $r(z)$ are $r(\pm a/2) = R$.

1. Write the Euler equation for this functional without trying to solve it.
2. It is simpler here to consider Beltrami identity. Let $C = F - r'\dfrac{\partial F}{\partial r'}$, show that C is independent of z.
3. Knowing that the curve must be symmetrical with respect to the plane xOy, find the expression of the catenoid curve $r(z)$ as a function of C and determine the relation between C, a and R.

Exercice 3.3. Optical mirage

Optical mirage results from the deflection of light rays when they propagate in a medium having a strong gradient of the refractive index. This type of situation appears in particular in desert conditions where the very hot ground creates a temperature gradient near the surface.

Fermat's principle states that "The path taken by a ray between two given points is the path that can be traversed in the least time". The trajectory of the light ray which passes through the points A and B therefore minimizes the optical path length

$$L_{AB} = \int_A^B n\mathrm{d}s$$

We will assume that the light rays propagate in the plane (xOy) where Oy is the vertical axis. We try to determine the trajectory $y(x)$ of a light ray. We can write the differential element of the curvilinear coordinate as $\mathrm{d}s = \sqrt{1 + y'^2}\mathrm{d}x$, where $y' = \mathrm{d}y/\mathrm{d}x$. It is assumed that the refractive index $n(y)$ depends on the vertical position. The optical path length is therefore written

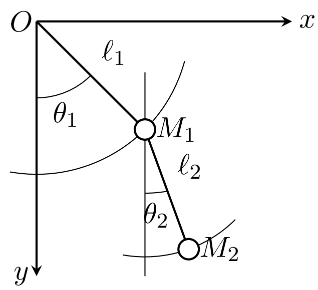
$$L_{AB} = \int_A^B n(y)\sqrt{1 + y'^2}\mathrm{d}x$$

1. Show that C defined by $C = \dfrac{n(y(x))}{\sqrt{1 + y'(x)^2}}$ is independent of x.
2. It is assumed that the refractive index varies according to $n(y) = C^2 + \alpha(y - y_0)$, where $\alpha > 0$ and y_0 are two constants. Derive the expression of the trajectory $y(x)$ and discuss its shape.

TD 4 : Oscillations

Exercice 4.1. Double plendulum

We consider a double pendulum consisting of two masses m_1 and m_2. The mass m_1 is connected to the fixed point O by a rigid rod of length ℓ_1, the mass m_2 is connected to the mass m_1 by a rigid rod of length ℓ_2. The positions of the masses are characterized by the angles θ_1 and θ_2 between the rods OM_1 and M_1M_2 with the vertical. Both masses are subjected to in the gravity field $\mathbf{g} = g\mathbf{e}_y$.



1. Establish the expression of the kinetic energy $T(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$.

2. Establish the expression of the potential energy $U(\theta_1, \theta_2)$.

3. Give the expression of the Lagrangian $\mathscr{L}(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$ as a function of the parameters

$$\alpha = \frac{m_2}{m_1 + m_2}, \quad \beta = \frac{\ell_2}{\ell_1}, \quad \omega_0 = \sqrt{\frac{g}{\ell_1}}, \quad I = (m_1 + m_2)\ell_1^2$$

4. Write the Euler-Lagrange equations

5. Write the Euler-Lagrange equations under the hypothesis of small oscillations $\theta \ll 1$.

6. Perform a harmonic expansion of the Lagrangian with respect to the coordinates and derive the corresponding Euler-Lagrange equations. Verify that you get the same set of equations obtained in the previous question.

7. Search for a solution of the form $\theta_i(t) = \mathrm{Re}(a_i e^{\mathrm{i}\omega t})$. What are the values of the eigenfrequencies ω?

8. We consider the special case $\ell_1 = \ell_2$ and $m_1 = 3m_2$. Determine the eigenfrequencies and verify that the coordinates

$$q_+ = \theta_1 - \theta_2/2, \quad q_- = \theta_1 + \theta_2/2$$

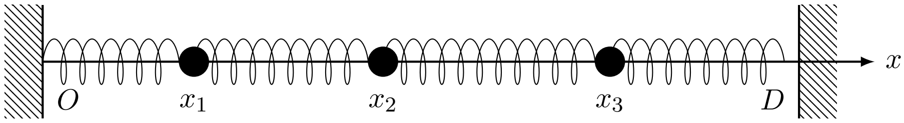
are normal coordinates

9. Give the expression of the approximated Lagrangian as a function of the normal coordinates q_+ and q_-.

Exercice 4.2. Three masses chains

We consider three point masses m movingalong the horizontal axis Ox. The positions of the three points are characterized by the coordinates $0 < x_1 < x_2 < x_3 < D$ and are connected to each other by identical springs of stiffness constant k and length at rest ℓ_0. The points x_1 and x_3 are both connected to two fixed points by the same type of spring at the coordinates $x = 0$ and $x = D$ respectively.

1. Write the expression of the kinetic energy $T(\dot x_1, \dot x_2, \dot x_3)$.



2. Write the expression of the potential energy $V(x_1, x_2, x_3)$ as a function of D, ℓ_0 and k.

3. Determine the equilibrium positions x_i^{eq} as a function of D.

4. Using the coordinates $y_i = \sqrt{m}\,(x_i - x_i^{\text{eq}})$, show that up to a constant, the Lagrangian is written as

$$\mathscr{L}(\mathbf{y}, \dot{\mathbf{y}}) = \frac{1}{2}\left(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2\right) - \omega_0^2\left(y_1^2 + y_2^2 + y_3^2 - y_1 y_2 - y_2 y_3\right),$$

where $\omega_0^2 = k/m$.

5. Write the Euler-Lagrange equations

6. By using the ansatz $y_i = a_i e^{\mathrm{i}\omega t}$, show that the problem is equivalent to the eigenvalue problem $A|a\rangle = \lambda|a\rangle$. Give the definition of $|a\rangle$, A and λ.

7. Find the eigenfrequencies and normal coordinates of the system.

TD 5 : Hamiltonian and phase space

Exercice 5.1. Hamiltonians

The objective of this exercise is to become familiar with the Legendre transformation and the construction of a Hamiltonian from a given Lagrangian.

1. We consider a classical charged particle that interacts with the vector potential $\mathbf{A}(\mathbf{r}, t)$ and the scalar potential $\phi(\mathbf{r}, t)$. In the classical framework the Lagrangian of this particle is written:

$$\mathscr{L}(\mathbf{r}, \mathbf{v}, t) = \frac{1}{2} m \mathbf{v}^2 + q\mathbf{A}(\mathbf{r}, t) \cdot \mathbf{v} - q\phi(\mathbf{r}, t).$$

Give the expression of the momentum $\mathbf{p}$ of the particle, then the Hamiltonian $\mathscr{H}(\mathbf{r}, \mathbf{p})$.

2. We consider a relativistic charged particle that interacts with the vector potential $\mathbf{A}(\mathbf{r}, t)$ and the scalar potential $\phi(\mathbf{r}, t)$. We will admit that in a relativistic framework the Lagrangian is written

$$\mathscr{L}(\mathbf{r}, \mathbf{v}, t) = -\frac{mc^2}{\gamma(\mathbf{v})} + q\mathbf{A}(\mathbf{r}, t) \cdot \mathbf{v} - q\phi(\mathbf{r}, t),$$

where $\gamma(\mathbf{v}) = \left(1 - \mathbf{v}^2/c^2\right)^{-1/2}$. Give the expression of the momentum $\mathbf{p}$ of the particle, then the Hamiltonian $\mathscr{H}(\mathbf{r}, \mathbf{p})$.

3. The Lagrangian of the double pendulum is written

$$\mathscr{L}(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)/I = \frac{1}{2}\dot\theta_1^2 + \frac{1}{2}\alpha\beta^2\dot\theta_2^2 + \alpha\beta\dot\theta_1\dot\theta_2\cos(\theta_1 - \theta_2) + \omega_0^2\cos\theta_1 + \alpha\beta\omega_0^2\cos\theta_2,$$

où $\alpha = m_2/(m_1 + m_2)$, $\beta = \ell_2/\ell_1$, $\omega_0 = \sqrt{g/l}$ et $I = (m_1 + m_2)\ell_1^2$. What is the expression of the Hamiltonian $\mathscr{H}(\theta_1, \theta_2, p_{\theta_1}, p_{\theta_2})$?

Exercice 5.2. Time-dependent Hamitonian

We consider the following Lagrangian

$$\mathscr{L}(x, \dot x, t) = \frac{1}{2} m \left(\dot x^2 - \omega^2 x^2\right) e^{\gamma t}$$

where the constants m, ω^2 and γ are positive.

1. Give the expression of the Hamiltonian for this system, is it time-independent ?

2. Write Hamiton's equations

3. Which physical problem does this Hamiltonian represent?

Exercice 5.3. Phase space

We consider a system with one degree of freedom of coordinate q and conjugated moment p. The Hamiltonian of the system is given by

$$H(q, p) = \frac{p^2}{2} + V(q)$$

where $V(q)$ is the potential energy. Give a physical interpretation of each of the following potentials that depend on a parameter $\alpha > 0$ and draw the corresponding phase portraits.

$$\begin{array}{ll} V_1(q,p) = \frac{\alpha}{2}q^2 & V_2(q,p) = -\frac{\alpha}{2}q^2 \\ V_3(q,p) = \alpha q & V_4(q,p) = \alpha e^q \\ V_5(q,p) = \alpha e^{-q^2/2} & V_6(q,p) = -\alpha e^{-q^2/2} \end{array}$$

TD 6 : Canonical transformation

Exercice 6.1. Free fall

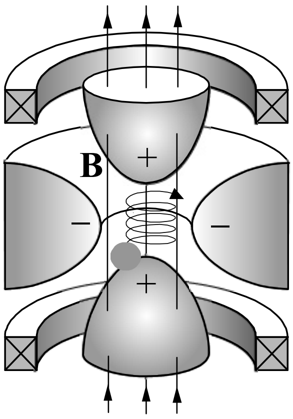
We consider the problem of free fall in a non-Galilean frame of reference (e.g. in an elevator). Let $\mathscr{R}_0$ be the terrestrial frame of reference considered as Galilean and $\mathscr{R}$ a non Galilean frame of reference in vertical rectilinear translation (non-uniform). We consider a mass m located in $\mathscr{R}_0$ by its vertical coordinate q and in $\mathscr{R}$ by $Q = q - h(t)$, where $h(t)$ characterise the position of origin of $\mathscr{R}$ with respect to the origin of $\mathscr{R}_0$. The conjugated momenta in $\mathscr{R}_0$ and $\mathscr{R}$ are denoted by p and $P = p - m\dot{h}(t)$ respectively.

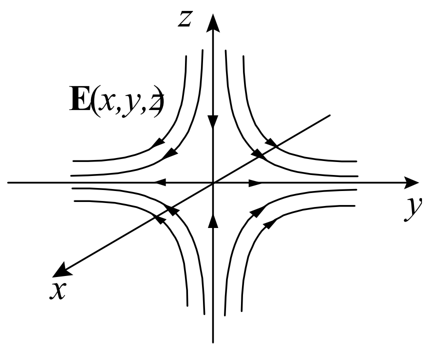
1. Give the expression of the Hamiltonian $H(q, p, t)$.

2. To show that the transformation $(q, p) \to (Q, P)$ is canonical, find a type 2 generating function $F_2(q, P)$ that generates it

3. Deduce the new Hamiltonian $K(Q, P, t)$ of the system. Discuss each term of K and identify the term responsible for the inertial force in the frame of reference $\mathscr{R}$.

Exercice 6.2. Piège de Penning





L'objectif de ce problème est d'étudier les pièges de Penning. Ces pièges constitués d'une combinaison de champs électrique et magnétique permettent de confiner des ions ou des électrons et sont utilisés pour des expériences de physique fondamentale et de métrologie.

Soit une particule chargée de charge q et de masse m. La position et le moment conjugué de cette particule sont repérés par les vecteurs $\mathbf{r}$ et $\mathbf{p}$ exprimés dans un repère cartésien $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. On rappelle que le Hamiltonien d'une particule chargée plongée dans un champ électromagnétique est donné par

$$H(\mathbf{r}, \mathbf{p}) = \frac{(\mathbf{p} - q\mathbf{A}(\mathbf{r}))^2}{2m} + q\Phi\left(\mathbf{r}\right)$$

où $\mathbf{A}(\mathbf{r})$ est le potentiel vecteur et $\Phi(\mathbf{r})$ et le potentiel scalaire. Au voisinage du centre d'un piège de Penning, on pourra considérer que le champ magnétique est uniforme et aligné le long de l'axe Oz, $\mathbf{B} = B\mathbf{e}_z$ et que le potentiel dont dérive le champ électrique est quadrupolaire. Les potentiels vecteur et scalaire s'écrivent donc

$$\mathbf{A}\left(\mathbf{r}\right) = \frac{m\omega}{2q}\left(x\mathbf{e}_y - y\mathbf{e}_x\right), \quad \Phi\left(\mathbf{r}\right) = \frac{m\Omega^2}{4q}\left(2z^2 - x^2 - y^2\right),$$

où $\omega = qB/m$ et Ω sont deux pulsations caractéristiques du piège.

1. Donner l'expression de $H(x, y, z, p_x, p_y, p_z)$ et montrer que dans le plan Oxy le piège de Penning ne fonctionne que si $\Omega_0^2 = \omega^2 - 2\Omega^2 > 0$.

Nous supposerons que la condition précédente est vérifiée. Pour simplifier les calculs, on utilisera un système d'unité tel que $m = 1$ et $\Omega_0 = 1$. On effectue une première transformation canonique $(x, y, p_x, p_y,) \to (q_+, q_-, p_+, p_-)$ générée par la fonction génératrice

$$S\left(p_x, y, q_+, q_-\right) = -p_x\left(q_+ + q_-\right) - \frac{y}{2}\left(q_+ - q_-\right).$$

Ce type de fonction génératrice suit les formules de génération

$$x = -\frac{\partial S}{\partial p_x}, \quad p_+ = -\frac{\partial S}{\partial q_+}, \quad p_y = \frac{\partial S}{\partial y}, \quad p_- = -\frac{\partial S}{\partial q_-}$$

2. Exprimer x, y, p_x et p_y en fonction de q_+ et q_-, p_+ et p_-.

3. On introduit les fréquences cyclotron et magnétron modifiées $\Omega_\pm = (\omega \pm 1)/2$. Exprimer H en fonction de Ω_+, Ω_-, q_+, q_-, p_+, p_-, p_z et z.

On effectue une deuxième transformation canonique $q_+, q_-, z, p_+, p_-, p_z \to \theta, \varphi, \phi, J, D, I$ où J, D et I sont des actions et où θ, φ et ϕ sont des angles. Cette transformation est générée par la fonction génératrice

$$F\left(q_+, q_-, z, \theta, \varphi, \phi\right) = \frac{q_+^2}{2}\cot\left(\theta\right) + \frac{q_-^2}{2}\cot\left(\varphi\right) + \frac{\Omega z^2}{2}\cot\left(\phi\right).$$

Cette transformation suit les règles de génération suivantes:

$$p_+ = \frac{\partial F_1}{\partial q_+}, \quad J = -\frac{\partial F_1}{\partial \theta}, \quad p_- = \frac{\partial F_1}{\partial q_-}$$
$$D = -\frac{\partial F_1}{\partial \varphi}, \quad p_z = \frac{\partial F_1}{\partial z}, \quad I = -\frac{\partial F_1}{\partial \phi}$$

4. Exprimer q_+, q_-, z, p_+, p_-, et p_z en fonction des nouveaux angles et actions.

5. Exprimer H en fonction des nouvelles actions.

TD 7 : Equation de Hamiltonien-Jacobi

Exercice 7.1. Limite classique de l'équation de Schrödinger

On considère un système quantique décrit par un ensemble de coordonnées $\mathbf{x}$ et une fonction d'onde $\psi(\mathbf{x}, t)$. Le système est en interaction avec potentiel $V(\mathbf{x}, t)$ qui dépend du temps. L'évolution de la fonction d'onde est donnée par l'équation de Schrödinger dépendante du temps.

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = -\frac{\hbar^2}{2m} \boldsymbol{\nabla}^2 \psi(\mathbf{x}, t) + V(\mathbf{x}, t)\psi(\mathbf{x}, t)$$

On peut toujours exprimer la fonction d'onde sous la forme

$$\psi(\mathbf{x}, t) = R(\mathbf{x}, t) e^{\mathrm{i}S(\mathbf{x},t)/\hbar},$$

où $R(\mathbf{x}, t)$ et $S(\mathbf{x}, t)$ sont deux fonctions réelles. L'utilisation de ces variables est à la base de la formulation hydrodynamique de la mécanique quantique selon de Broglie (1892-1987) et Bohm(1917-1992).

1. A partir de l'équation de Schrödinger, en déduire le système d'équations aux dérivées partielles pour les fonctions $\rho(\mathbf{x}, t) = R^2(\mathbf{x}, t)$ et $S(\mathbf{x}, t)$.

2. Prendre la limite $\hbar \to 0$ et en déduire une interpretation classique de la fonction $S(\mathbf{x}, t)$.

Exercice 7.2. Oscillateur harmonique

Soit un oscillateur harmonique décrit par les variables (q, p). L'hamiltonien s'écrit

$$H(q, p) = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 q^2$$

1. Exprimer l'équation de Hamilton-Jacobi.

2. Montrer que l'action peut s'exprimer sous la forme

$$S(q, \alpha, t) = \sqrt{2m} \int^q \mathrm{d}u \sqrt{\alpha - \frac{1}{2} m\omega^2 u^2} - \alpha t,$$

où α est une constante d'intégration.

3. Que représente la constante α ?

4. On définit $\beta = \frac{\partial S}{\partial \alpha}$. Dans un cadre général, sachant que $S(q, \alpha, t)$ est une fonction de q, α et du temps t (en supposant q et α indépendants) et en utilisant les équations de Hamilton, montrer que β est une constante du mouvement.

5. Démontrer la même chose en utilisant le fait que $S(q, \alpha, t)$ peut-être vu comme une fonction génératrice de 2ème espèce de la transformation canonique $(q, p) \to (\beta, \alpha)$.

6. En déduire l'expression de $q(t)$.

TD 8 : Perturbations et résonances

Exercice 8.1. Oscillateur non-linéaire

L'objectif de ce problème est de calculer la période d'un oscillateur faiblement non linéaire en fonction de l'amplitude du mouvement

Soit un oscillateur non-linéaire de coordonnées q et de moment p. L'Hamiltonien du système s'écrit

$$H(q,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{1}{4}\varepsilon\omega^2 q^4$$

Les deux premiers termes correspondent à un oscillateur harmonique de pulsation ω. Le dernier terme correspond à une correction que l'on va considérer comme perturbative, c'est-à-dire $\varepsilon q^2 \ll 1$.

On considère la transformation angle-action $(q,p) \to (\theta, J)$ déterminée par la fonction génératrice F_1 de première espèce

$$F_1(q,\theta) = \frac{1}{2}\omega q^2 \cot\theta$$

et on rappelle les règles de génération des fonctions génératrices de première espèce

$$p = \frac{\partial F_1}{\partial q}, \quad J = -\frac{\partial F_1}{\partial \theta}$$

1. Exprimer q et p en fonction de θ et J

2. Dessiner l'allure du portrait de phase pour le Hamiltonien $H_0(q,p)$ (sans perturbation $\varepsilon = 0$).

3. Exprimer le Hamiltonien H_0 en fonction des variables angle-action (θ, J).

4. Dessiner l'allure du portrait de phase de $H_0(\theta, J)$

5. Exprimer le hamiltonien H en fonction des variables angle-action (θ, J)

6. A l'aide de la relation

$$\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta,$$

moyenner l'expression de V sur la variable d'angle θ, on exprimera $\langle H\rangle = H_0 + \int_0^{2\pi} V(\theta,J)\frac{d\theta}{2\pi}$ le hamiltonien moyen ainsi obtenu.

7. Etablir les équations de Hamilton relatives aux variables θ et J pour le hamiltonien $\langle H\rangle$.

8. Intégrer ces équations et établir la formule exprimant la pulsation $\Omega(J)$ des oscillations en fonction de l'action J.

9. A l'ordre le plus bas en ε donner la relation entre l'amplitude du mouvement A et la variable d'action J.

10. En déduire l'expression de la période $T(A)$ en fonction de l'amplitude du mouvement, au premier ordre en ε.

Exercice 8.2. Résonance non-linéaire

Nous allons étudier dans ce problème le cas d'une résonance non-linéaire entre deux vibrations. Ce problème intervient dans beaucoup de problèmes de physique allant des résonances dans les systèmes mécaniques jusqu'aux résonances à l'échelle atomique et moléculaire (dans une version quantique). Ce problème est inspiré de l'article Roberts and Jaffé, J. Chem. Phys. ***99****, 2495 (1993).*

Nous allons considérer ici le système modèle composé de deux oscillateurs harmoniques dégénérés couplés par un terme nonlinéaire quartique. L'Hamiltonien du système s'écrit

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{\omega_0^2}{2}\left(q_1^2 + q_2^2\right) + 2\gamma\omega_0^2 q_1^2 q_2^2$$

où ω_0 est la pulsation harmonique des oscillateurs, et où on a choisi un système d'unité tel que la masse des oscillateurs est l'unité $m = 1$. Le paramètre γ est un paramètre qui caractérise le couplage entre les oscillateurs. On va exprimer tout d'abord cet Hamiltonien en terme des coordonnées angle-action. Pour cela nous allons effectuer une première transformation canonique $(q_1, q_2, p_1, p_2) \to (\theta_1, \theta_2, I_1, I_2)$ définie par la fonction génératrice de 1ère espèce,

$$F_1(q_1, q_2, \theta_1, \theta_2) = \frac{1}{2}\omega_0 q_1^2 \cot\theta_1 + \frac{1}{2}\omega_0 q_2^2 \cot\theta_2,$$

avec les règles de transformations associées

$$p_i = \frac{\partial F_1}{\partial q_i}, \quad I_i = -\frac{\partial F_1}{\partial \theta_i}, \quad i = 1, 2.$$

1. Exprimer q_1, q_2, p_1 et p_2 en fonction de θ_1, θ_2, I_1 et I_2.

2. Exprimer l'Hamiltonien $H(\theta_1, \theta_2, I_1, I_2)$ en fonction des nouvelles coordonnées et moments.

3. En utilisant l'égalité suivante

$$\sin^2(x)\sin^2(y) = \frac{1}{8}\left[2 - 2\cos 2x - 2\cos 2y + \cos 2(x-y) + \cos 2(x+y)\right]$$

exprimer l'Hamiltonien sous la forme

$$H(\theta_1, \theta_2, I_1, I_2) = H_0(I_1, I_2) + V(\theta_1, \theta_2, I_1, I_2)$$

où $H_0 = \langle H \rangle_{\theta_1, \theta_2}$ est la moyenne de H sur les angles θ_1 et θ_2 (il contient ainsi l'ensemble des termes indépendant de θ_1 et θ_2) et V contient le reste des termes. Donner les expressions de H_0 et V.

4. Nous allons tout d'abord étudier H_0 qui correspond à l'Hamiltonien non-résonant. Exprimer les équations de Hamilton pour $I_1, I_2, \theta_1, \theta_2$ à partir de H_0. En déduire les fréquences du système $\tilde{\omega}_1(I_1, I_2)$ et $\tilde{\omega}_2(I_1, I_2)$ associés aux variations temporelles des angles θ_1 et θ_2.

Par la suite, on supposera que les actions I_1, I_2 restent faibles $\gamma I_i \ll \omega_0$ avec $i = 1, 2$.

5. Exprimer le terme de couplage $V(\theta_1, \theta_2, I_1, I_2)$ sous la forme

$$V(\theta_1, \theta_2, I_1, I_2) = V_\mathrm{R}(\theta_1 - \theta_2, I_1, I_2) + V_\mathrm{NR}(\theta_1, \theta_2, I_1, I_2)$$

et justifier de manière succinte (sans faire de calcul) pourquoi V_R est un terme de couplage de résonance.

Par la suite nous allons négliger le terme V_NR et nous allons donc considérer l'Hamiltonien approché

$$H(\theta_1, \theta_2, I_1, I_2) \approx H_0(I_1, I_2) + V_\mathrm{R}(\theta_1 - \theta_2, I_1, I_2). \tag{0.6}$$

6. A partir de l'expression de V_R justifier pourquoi la condition de résonance est donnée par $\tilde{\omega}_1(I_1, I_2) = \tilde{\omega}_2(I_1, I_2)$ et en déduire la relation entre I_1 et I_2 définissant l'hypersurface de l'espace des phases correspondant à la résonance (on supposera $\gamma \neq 0$).

7. Nous allons étudier plus en détail la dynamique autour de la résonance. Pour cela on introduit la transformation canonique $(\theta_1, \theta_2, I_1, I_2) \to (\varphi, Q, J, P)$ définie à partir de la fonction génératrice de 2ème espèce

$$F_2(\theta_1, \theta_2, J, P) = \theta_1 \frac{J+P}{2} + \theta_2 \frac{J-P}{2},$$

avec les règles de transformation

$$I_1 = \frac{\partial F_2}{\partial \theta_1}, \quad I_2 = \frac{\partial F_2}{\partial \theta_2}, \quad \varphi = \frac{\partial F_2}{\partial J}, \quad Q = \frac{\partial F_2}{\partial P}$$

Exprimer θ_1, θ_2, I_1 et I_2 en fonction de φ, Q, J, et P.

8. Montrer que l'Hamiltonien H, donné par l'équation (0.6) écrit en termes des nouvelles variables (φ, Q, J, P) s'écrit

$$H(J, Q, P) = \omega_0 J + \frac{\gamma}{2} J^2 - \frac{\gamma}{2} P^2 + \frac{\gamma}{4} \left(J^2 - P^2\right) \cos(4Q) \tag{0.7}$$

9. En déduire que J est une quantité conservée.

10. Identifier les positions d'équilibre $Q = Q_0$ et $P = P_0$.

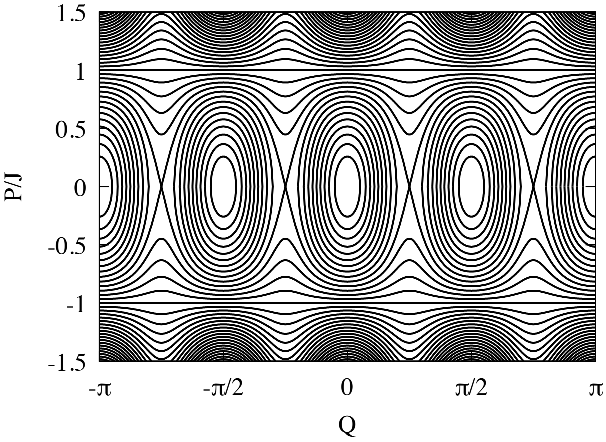
11. Reprendre la condition de résonance trouvée à la question 7) et montrer que la résonance correspond à la valeur $P = 0$.

Le portrait de phase de l'Hamiltonien H Eq. (0.7) est représenté sur la figure suivante dans le plan (Q, P).

12. Identifier sur la figure les positons d'équilibres stables, instables et la séparatrice.

13. Pour $P < J$, identifier un modèle physique bien connu dont le portrait de phase ressemble au présent portrait de phase et proposer une approximation sur H pour le montrer.

14. Décrire qualitativement la dynamique à partir de l'un des point d'équilibre stable en terme des coordonnées initiales (q_i, p_i).


1.5
1
0.5
0
-0.5
-1
-1.5
P/J
-π
-π/2
0
π/2
π
Q