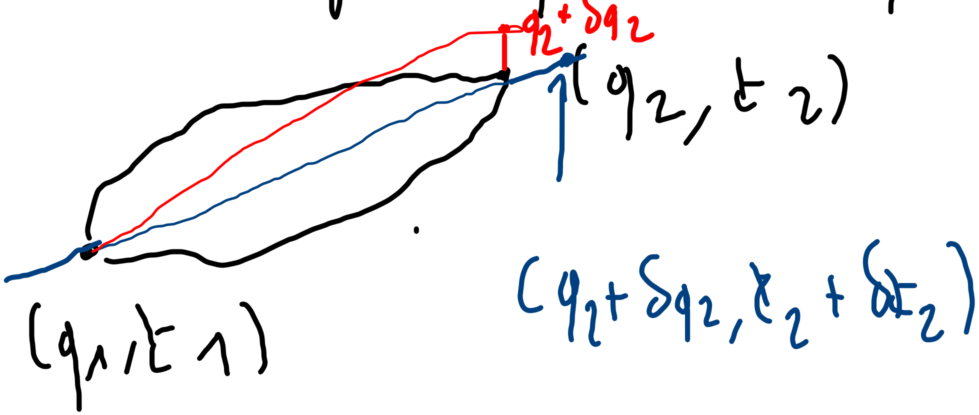
Exc 7.2 · H-J eq is PDE for

<u>Hamilton's principal f^n</u>

· Action is f^n<u>al</u> of the path connecting two pts q_1 and q_2, at time t_1 and t_2



$$S[q(t)] = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)\, dt$$

Depends on a path, on a f^n

- Motion in between (q_1, t_1) and (q_2, t_2) by a system whose Lagrangian is L is such that $\delta S\big|_{q=\bar{q}} = 0 \Leftrightarrow$ E-L eqs

$\Rightarrow$ Gives you $\bar{q}(t)$ extremal path (unique) with starting pt and end pt fixed

- Evaluate the f^{nal} S on $\bar{q}(t)$:

$$S[\bar{q}(t)] = \underbrace{\tilde{S}(q_1, t_1, q_2, t_2)}_{\text{Hamilton's pp } f^n}$$

- H-J formalism, starting pt is implicit and

$S(\underbrace{q, t}_{\text{End point}}) \rightarrow$ Implicitly depend on initial condition that we keep fixed

- For (q,t) endpoint, the momentum at this pt when following the E.O.M is given by

$$\boxed{p = \frac{\partial S}{\partial q}}$$

Ex 7.2

1) $$\boxed{\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0}$$ for any (q, t)

H-J eq

What you should not do:

2) $$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}\omega^2 m q^2 \implies \frac{\partial H}{\partial t} = 0$$

Reminder:

- For (q, p) satisfying Hamilton's eq (E.O.M)

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases} \implies \frac{dH}{dt} = \underbrace{\frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}} + \frac{\partial H}{\partial t}$$

$$= 0 + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \text{ so if } \frac{\partial H}{\partial t} = 0 \iff \underline{H(q(t), p(t)) = E = ct}$$

on the motion

Correct but useless

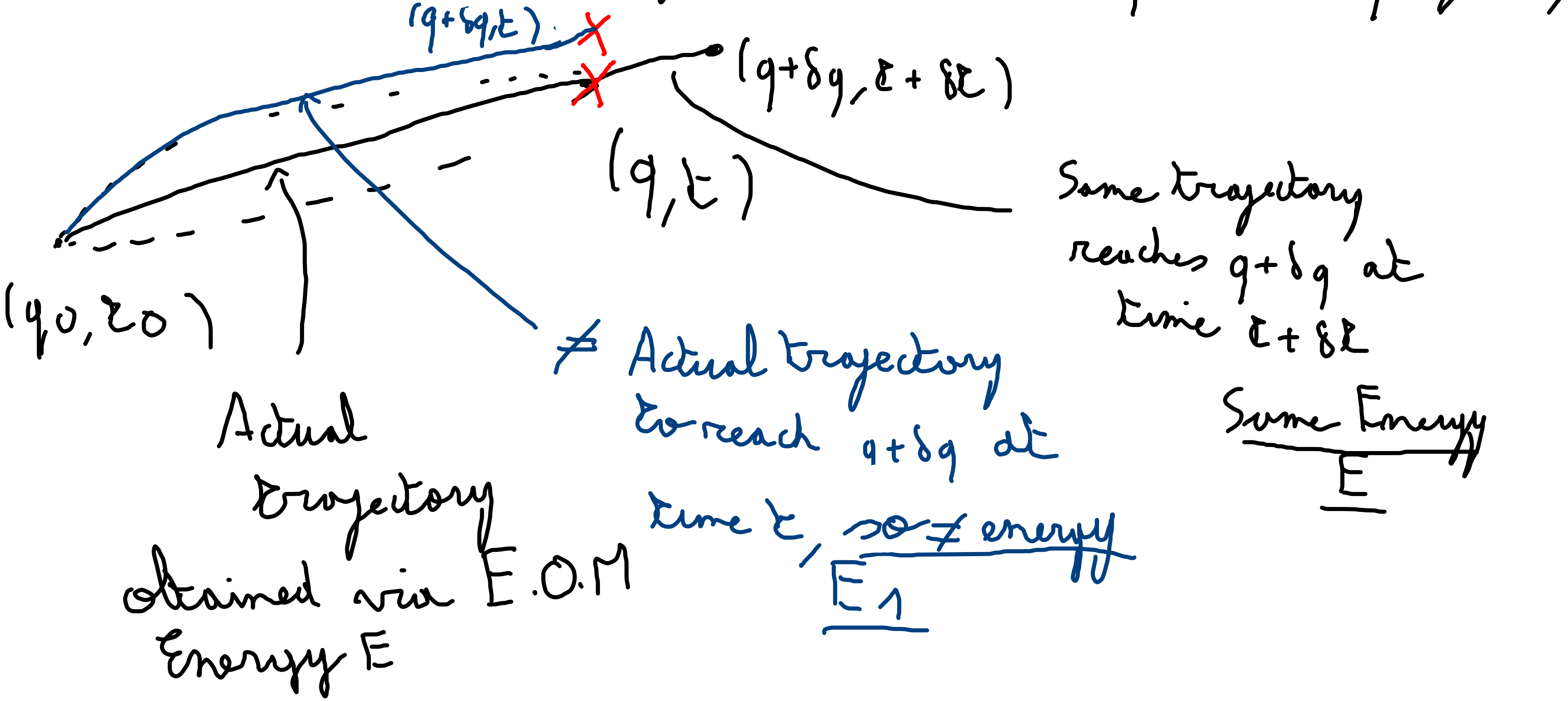
- $p = \frac{\partial S}{\partial q}(q,t)$ is value of momentum canonically conjugated q when on an optimal trajectory passing through q at time t (The initial conditions are implicit.)

⚠ It does not mean that $H\left(q, \frac{\partial S}{\partial q}(q,t)\right)$ is a ct!

- Indeed a priori $H\left(q, \frac{\partial S}{\partial q}(q,t)\right)$ depends on the endpoints q and t
- If q is changed without changing the time at which q is reached $\Rightarrow$ Different trajectory $\Rightarrow$ Different energy

$\Rightarrow \underline{\frac{\partial H}{\partial q} \neq 0}$ and similarly $\frac{\partial H}{\partial t} \neq 0$.

- However if q and t are varied so that the trajectory passing through (q,t) and $(q+\delta q, t+\delta t)$ is the same $\Longrightarrow$ Energy is the same at (q,t) and $(q+\delta q, t+\delta t)$



· How to choose δq and δt to stay on the same trajectory?

We want $H\left(q, \frac{\partial S}{\partial q}(q,t)\right) = H\left(q+\delta q, \frac{\partial S}{\partial q}(q+\delta q, t+\delta t)\right)$

$$\Leftrightarrow \delta q \frac{\partial H}{\partial q} + \frac{\partial H}{\partial p}\left(\frac{\partial^2 S}{\partial q^2}\delta q + \frac{\partial^2 S}{\partial q \partial t}\delta t\right) = 0$$

$$\Leftrightarrow \delta q\left[\frac{\partial H}{\partial q} + \frac{\partial H}{\partial p}\frac{\partial^2 S}{\partial q^2}\right] + \delta t \frac{\partial^2 S}{\partial q \partial t}\frac{\partial H}{\partial p} = 0$$

By H-J $\frac{\partial S}{\partial t} = -H\left(q, \frac{\partial S}{\partial q}\right) \rightarrow \frac{\partial^2 S}{\partial q \partial t} = -\frac{\partial H}{\partial q} - \frac{\partial H}{\partial p}\frac{\partial^2 S}{\partial q^2}$

so $\frac{\delta q}{\delta t} = \frac{\frac{\partial H}{\partial q} + \frac{\partial H}{\partial p}\frac{\partial^2 S}{\partial q^2}}{\frac{\partial H}{\partial q} + \frac{\partial H}{\partial p}\frac{\partial^2 S}{\partial q^2}}\frac{\partial H}{\partial p} = \frac{\partial H}{\partial p}$ so $\boxed{\frac{\delta q}{\delta t} = \frac{\partial H}{\partial p}\left(q, \frac{\partial S}{\partial q}\right)}$

according to $\underline{\text{Hamilton's eq}}$

2) <u>What you should do:</u> $\boxed{\frac{\partial S}{\partial t}(q,t) + H\left(q, \frac{\partial S}{\partial q}(q,t)\right) = 0}$

① H is not explicitly time-dependent

We look for $S(q,t)$ in the form $S(q,t) = S_1(q) + S_2(t)$
<u>separable form</u>

Then $\frac{\partial S}{\partial q} = \frac{\partial S_1}{\partial q}(q)$ and $\frac{\partial S}{\partial t} = \frac{\partial S_2}{\partial t}(t)$ so that H-J

reads :
$$\underbrace{\frac{\partial S_2}{\partial t}(t)}_{\text{F}^n \text{ of } t \text{ only}} = - \underbrace{H\left(q, \frac{\partial S_1}{\partial q}(q)\right)}_{\text{F}^n \text{ of } q \text{ only}}$$

For the eq. to be satisfied both f^n have to be cst

- We name α this cst : $\left| \begin{array}{l} H\left(q, \dfrac{\partial S_1}{\partial q}(q)\right) = \alpha \\ \dfrac{\partial S_2}{\partial t}(t) = -\alpha \end{array} \right.$

$\hookrightarrow \boxed{S_2(t) = -\alpha t + \cancel{\alpha_0}}$

Irrelevant cst so we remove it

We separated the H-J eq into two eqs

α is called <u>the cst of separation</u>

so $S(q,t) = \overbrace{S_1(q)}^{=W(q)} - \alpha t$ and

$H\left(q, \dfrac{\partial S_1}{\partial q}(q)\right) = \alpha$ $\longrightarrow$ We see that α is actually like the energy E

· H - J $\qquad E = H\left(q, \frac{\partial W}{\partial q}\right)$

For H.O $\hookrightarrow \quad E = \frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2} m w^2 q^2$

$$\Longleftrightarrow \left(\frac{\partial W}{\partial q}\right)^2 = 2mE - m^2 \omega^2 q^2.$$

$$\Longrightarrow \frac{\partial W}{\partial q} = + \sqrt{2mE - m^2 \omega^2 q^2}$$

$$\frac{\partial W}{\partial q} = \frac{dW}{dq} \Longrightarrow \boxed{W(q) = \int_{q_0}^{q} \sqrt{2mE - m^2 w^2 q^2} + W(q_0)}$$

Finally $$\boxed{S(q,t;\alpha) = \int_{q_0}^{q} \sqrt{2m\alpha - m^2\omega^2\tilde{q}^2}\, d\tilde{q} - \alpha t}$$

② Another way to proceed is to check that the f^n given in the text satisfies the H-J equation

$$S(q,t) = -\alpha t + \int_{q_0}^{q} \sqrt{2m\alpha - m^2\tilde{q}^2\omega^2}\, d\tilde{q}$$

3) $\alpha = E$ = Energy of the system

4) $\beta \equiv \frac{\partial S}{\partial \alpha}(q,\alpha,t)$ where $S(q,\alpha,t) = -\alpha \times t + \int_{q_0}^{q} \sqrt{2m\alpha - m^2\tilde{q}^2\omega^2}\, d\tilde{q}$

<u>Assume</u>: q is a f^n of t and that it satisfies Hamilton's eqs

We have $\beta(\alpha, t) = \frac{\partial S}{\partial \alpha}(q(t), \alpha, t)$

<u>Goal</u>: $\frac{d\beta}{dt} = 0$

$$p \equiv \frac{\partial S}{\partial q}(q, \alpha, t) \longrightarrow p(t, \alpha) = \frac{\partial S}{\partial q}(q(t), \alpha, t)$$

$$\left\{ \begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} \\ \dot{q} &= +\frac{\partial H}{\partial p} \end{aligned} \right.$$

$$\frac{d\beta}{dt} = \frac{\partial \beta}{\partial q} \times \frac{dq}{dt} + \frac{\partial \beta}{\partial \alpha} \times \overbrace{\frac{d\alpha}{dt}}^{=0} + \frac{\partial \beta}{\partial t}$$

Assume S is C^2

$$= \frac{\partial^2 S}{\partial q \partial \alpha} \dot{q} + 0 + \frac{\partial^2 S}{\partial t \partial \alpha}$$

Schwarz Thm

$$= \frac{\partial}{\partial \alpha}\left(\frac{\partial S}{\partial q} \right) \dot{q} + \frac{\partial}{\partial \alpha}\left(\frac{\partial S}{\partial t} \right)$$

$$\frac{dB}{dt} = \frac{\partial}{\partial \alpha}\left(\frac{\partial S}{\partial q}\right)\dot{q} + \frac{\partial}{\partial \alpha}\left(\frac{\partial S}{\partial t}\right)$$

$$H(q(t), p(t,\alpha))$$

$$= \frac{\partial}{\partial \alpha}\Big(\underbrace{\frac{\partial S}{\partial q}}_{=p}\Big)\frac{\partial H}{\partial p} + \frac{\partial}{\partial \alpha}\left(\frac{\partial S}{\partial t}\right)$$

$$\frac{\partial H}{\partial \alpha} = \frac{\partial H}{\partial p} \times \frac{\partial p}{\partial \alpha}$$

$$= \underbrace{\frac{\partial p}{\partial \alpha} \times \frac{\partial H}{\partial p}}_{=\frac{\partial H}{\partial \alpha}} + \frac{\partial}{\partial \alpha}\left(\frac{\partial S}{\partial t}\right) = \frac{\partial}{\partial \alpha}\left[H(q,p) + \frac{\partial S}{\partial t}\right]$$

$$= \frac{\partial}{\partial \alpha}\Big[\underbrace{H\Big(q, \frac{\partial S}{\partial q}\Big) + \frac{\partial S}{\partial t}}_{\substack{=0 \\ \text{by } \underline{H.J}}}\Big]$$

$$\Rightarrow \boxed{\frac{dB}{dt} = 0}$$

5) $(q,p) \longrightarrow (\beta,\alpha)$ using

$S(q,\alpha,t)$ as a type 2 generating f^n

$$\begin{cases} p = +\dfrac{\partial S}{\partial q} \\ \beta = +\dfrac{\partial S}{\partial \alpha} \end{cases}$$

eqs. of canonical transformat°

$\longrightarrow$ Same eq as before

· Hamiltonian ? $K = H + \dfrac{\partial S}{\partial t} = 0$ by H-J eq.

· Hamilton's eqs ?

$$\begin{cases} \dot{\beta} = \dfrac{\partial K}{\partial \alpha} = 0 \\ \dot{\alpha} = -\dfrac{\partial K}{\partial \beta} = 0 \end{cases}$$

$\Longleftrightarrow$ α and β are cst of the motion

6) $\beta(q,t,\alpha) = \dfrac{\partial S}{\partial \alpha} \Rightarrow \beta(q,t,\alpha) = \dfrac{\partial}{\partial \alpha}\left(-\alpha t + \displaystyle\int_{q_0}^{q} \sqrt{2m\alpha - m^2\tilde{q}^2\omega^2}\, d\tilde{q}\right)$

$$= -t + \int_{q_0}^{q} \frac{m\, d\tilde{q}}{\sqrt{2m\alpha - m^2\tilde{q}^2\omega^2}}$$

Now:

$q \dashrightarrow q(t)$

$\beta \dashrightarrow \beta(q(t), \alpha, t)$ But $\dot{\beta} = 0$

$$\beta + t = \int_{q_0}^{q(t)} \frac{m\, d\tilde{q}}{\sqrt{2m\alpha - m^2\tilde{q}^2\omega^2}}$$

$$\int_{q_0}^{q(t)} \frac{m\, d\tilde{q}}{\sqrt{2m\alpha - m^2\tilde{q}^2 w^2}} = \sqrt{\frac{m}{2\alpha}} \int_{q_0}^{q(t)} \frac{d\tilde{q}}{\sqrt{1 - \frac{m w^2}{2\alpha}\tilde{q}^2}}$$

$$v = \sqrt{\frac{m}{2\alpha}}\, w\, \tilde{q} \qquad = \frac{1}{w} \int_{\sqrt{\frac{m}{2\alpha}} w q_0}^{\sqrt{\frac{m}{2\alpha}} w q(t)} \frac{dv}{\sqrt{1 - v^2}}$$

$$= \frac{1}{w}\left[\operatorname{Arcsin}\left(w\sqrt{\frac{m}{2\alpha}}\, q(t)\right) - \text{Cste} \right]$$

- $\operatorname{Arcsin}\left(w\sqrt{\frac{m}{2\alpha}}\, q(t)\right) = w(\beta + t) + \text{cste}$

$$\boxed{q(t) = \frac{1}{w}\sqrt{\frac{2\alpha}{m}} \sin\left(w(\beta+t) + \text{cste}\right)}$$

- Meaning of α and β ?

$\alpha \longrightarrow$ Energy which is cst so $\boxed{\alpha = \frac{\dot{q}_0^2}{2m} + \frac{1}{2} m\omega^2 q_0^2}$

$\beta \to$? $t = -\beta \Rightarrow q(-\beta) = \sqrt{\frac{2\alpha}{m}} \sin(\text{cst} \times \omega)$

$\boxed{\beta = -t_0}$ $= q_0$

Fixing α and β means fixing the initial conditions

- Lanczos, The Variational Pp of Mechanics

For details on why Hamilton-Jacobi is powerful and meaning

<u>Chapter VIII</u>

Ex 7.3

$$H(q,p) = \frac{p^2}{2m} + mgq$$

- $\frac{\partial H}{\partial t} = 0 \longrightarrow S(q,E,t) = -E \times t + W(q)$

- $\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0 \longrightarrow \boxed{E = H\left(q, \frac{\partial W}{\partial q}\right)}$

- $E = \left(\frac{\partial W}{\partial q}\right)^2 \times \frac{1}{2m} + mgq \Leftrightarrow \frac{\partial W}{\partial q} = \sqrt{2mE - 2m^2 g q}$

- $\boxed{S(q,E,t) = -E \times t + \int^{q} \sqrt{2mE - 2m^2 g \tilde{q}}\, d\tilde{q}}$

q and not q^2

- $\beta \equiv \frac{\partial S}{\partial E}(q, E, t) = -t + \int^{q} \frac{m \, d\tilde{q}}{\sqrt{2mE - 2m^2 g \tilde{q}}}$

- $\frac{d\beta}{dt} = 0$ when $(q(t), p(t))$ satisfy E.O.M

- $\beta + t = \int_{q_0}^{q(t)} \frac{m \, d\tilde{q}}{\sqrt{2mE - 2m^2 g \tilde{q}}}$

$$\frac{1}{\sqrt{1-u}} = \frac{d}{du}\left(-2\sqrt{1-u}\right)$$

$$= \sqrt{\frac{m}{2E}} \int_{q_0}^{q(t)} \frac{d\tilde{q}}{\sqrt{1 - \frac{mg}{E}\tilde{q}}}$$

$$u = \frac{mg}{E}\tilde{q}$$

$$= \frac{1}{g}\sqrt{\frac{E}{2m}} \int_{q_0 \frac{mg}{E}}^{\frac{mg}{E} q(t)} \frac{du}{\sqrt{1-u}} = -2\frac{1}{g}\sqrt{\frac{E}{2m}} \left[\sqrt{1-u}\right]_{\#_0}^{\#}$$

$$\beta + t = -\frac{2}{g}\sqrt{\frac{E}{2m}}\left[\sqrt{1-\frac{mg}{E}q(t)} - \sqrt{1-\frac{mg}{E}q_0}\right]$$

$$\Leftrightarrow -\sqrt{\frac{m}{2E}} \times g\,(\beta+t) + \sqrt{1-\frac{mg}{E}q_0} = \sqrt{1-\frac{mg}{E}q(t)}$$

$$\Leftrightarrow \cancel{1} - \frac{mg}{E}q(t) = \frac{m}{2E}\times g^2(\beta+t)^2 - \sqrt{\frac{2m}{E}}\,g\sqrt{1-\frac{mg}{E}q_0}\,(\beta+t) + \cancel{1} - \frac{mg}{E}q_0$$

$$\Leftrightarrow \boxed{q(t) = -\frac{g}{2}(\beta+t)^2 + \sqrt{\frac{2E}{m}}\sqrt{1-\frac{mg}{E}q_0}\,(\beta+t) + q_0}$$

- Meaning of $E \longrightarrow$ Energy

$\beta \rightarrow ?$ $\quad t = -\beta \longrightarrow q(-\beta) = q_0$

$\boxed{\beta = -t_0}$

- Second term could be expressed with q_0

$$\sqrt{\frac{2E}{m}} \times \sqrt{1 - \frac{mgq_0}{E}}$$