Hypergraph Tectonics

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April 28, 2025

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1 Introduction

1.1 Cumulants and their tensors

We will derive cumulants from moments, following the construction outlined by [5]

Definition 1.1. If a single random variable X has a probability measure μ_X defined over $\mathbb{R}$, then the *expectation* of a function $f(X)$ will be

$$E\{f(X)\} = \int_{\mathbb{R}} f(x)\ d\ \mu.$$

Whenever $f(X) = X^r$, then $E\{f(X)\}$ is the r^{th} *moment of* X.

This definition aligns for the multivariate case.

Definition 1.2. If $\{X_i\}_{i=1}^n$ is a finite set of random variables, each with a probability measure μ_{X_i}, then the *expectation* if a multivariate function $f(X_1, \ldots, X_n)$ is defined to be

$$E\{f(X_1, \ldots, X_n)\} = \int_{\mathbb{R}^n} f(X_1, \ldots, X_n)\ d\mu$$

Here, we can consider traditional moments $E\{X_i^r\}$, but also cross moments $E\{X_i^r X_j^s\}$, and so on.

Definition 1.3. Let $\overrightarrow{X}$ denote the vector whose components are the random variables $\{X_i\}_{i=1}^n$. The *multivariate moment generating function* of $\overrightarrow{X}$ is

$$M_{\overrightarrow{X}}(\xi) = 1 + \left(\sum_{i=1}^{n} \xi_i E(X_i)\right) + \frac{1}{2!}\left(\sum_{i,j} \xi_i \xi_j E(X_i X_j)\right) + \frac{1}{3!}\left(\sum_{i,j,k} \xi_i \xi_j \xi_k E(X_i X_j X_k)\right) + \cdots,$$

where $\xi \in \mathbb{R}^n$, and the series is assumed to converge for all $|\xi|$ sufficiently small.

With this definition of the moments, we can define the cumulants via their generating function:

Definition 1.4. The *multivariate cumulant generating function* of $\overrightarrow{X}$ is

$$K_{\overrightarrow{X}}(\xi) = \log\left(M_{\overrightarrow{X}}(\xi)\right)$$

For a set of random variables $\{X_i\}_{i=1}^n$, we can construct arrays K^{n^r} for all integers a, where each array hold the values for the r^{th} multivariate cumulants. This is done by assigning the cumulant of $\prod_{j=1}^{r} X_{i_j}$ to the index $(i_1, i_2, \ldots, i_r)$ within the array.

Example 1.5. The following array gives theoretical 2^{nd} joint cumulants for 5 random variables

$$\begin{bmatrix} -2 & 3 & 4 & -2 & 0 \\ 1 & 0 & 1 & 7 & 0 \\ 9 & 31.5 & -5 & 0 & 0 \\ 6.53 & 2 & 0 & -2 & -6.53 \\ 1 & 3 & 4 & -2 & -6 \end{bmatrix}$$

where the cumulant of X_iX_j is found in the i^{th} row and j^{th} column.

Lemma 1.6. The array K^{n^r} consisting of the r^{th} multivariate cumulants is indeed a tensor.

Proof. Let $Y_j := \sum_{i=1}^{n} \alpha_{i,j} X_i$ for some number of Y_j. Then the moments of the Y_j will be the linear combination of the moments of X_i, and consequently the cumulants of Y_j will be the linear combination of the cumulants of Y_j. $\square$

We can thus call these the cumulant tensors, as they behave in the way we expect of a tensor. An important aspect of cumulants is that they can inform us if random variables are colinear. If the variables $X_1, X_2, \ldots, X_r$ are not colinear, then their joint cumulant will be zero, in other words the index $(1, 2, \ldots, r)$ within the cumulant array K^{n^r} will be zero.

1.1.1 Random variables and their hypergraphs

Now that we have a clear way of identifying colinearity between random variables, we can construct a hypergraph. We let the set of vertices V be the set of random variables $\{X_1, \ldots, X_n\}$, and each hyperedge $e \in 2^V$ corresponds to a colinear relationship between random variables. For example, there will exist a hyperedge connecting X_2, X_4, X_7, and X_8 if there exists nonzero real numbers α_i such that

$$\alpha_2 X_2 + \alpha_4 X_4 + \alpha_7 X_7 + \alpha_8 X_8 = \alpha_0$$

1.2 What is a Hypergraph?

Definition 1.7. A *hypergraph* is a pair $(V, \mathcal{H})$, where $\mathcal{H} \subset 2^V$. A *uniform hypergraph* is a hypergraph where $\mathcal{H} \subset \binom{V}{k}$ for some integer k.

For our work, we will consider all hypergraphs to be uniform. If any hypergraph is not uniform, it can be made to be uniform.

Lemma 1.8. For any hypergraph $\mathcal{H}$, there is a functor which maps $\mathcal{H}$ to a uniform hypergraph $\overline{\mathcal{H}}$

Proof. Let k be the greatest number of vertices present in any hyperedge of $\mathcal{H}$. For any hyperedge e which contains less than $j < k$ vertices, adjoin to it $k - j$ vertices, so that the hyperedge becomes $\{v_1, v_2, \ldots, v_k, \infty_1, \infty_2, \ldots, \infty_{k-j}\}$. Fo each hyperedge we use distinct $\infty_i's$ to construct our new hypergraph. $\square$

With this lemma, we can regard all hypergraphs as uniform, so long as we retain the information of which vertices are our points at infinity.

1.3 Current Techniques

In the current literature, many techniques focus on defining eigenvalues for an adjacency tensor to a hypergraph. The work of [6] pioneered this path, as it gave a clear road map for defining eigenvalues and eigenvectors on a multilinear product. Following this, [2] utilizes a rescaled adjacency tensor to create an analog for spectral graph theory on hypergraphs. This is used to study bounds on average degrees and coloring of vertices. By considering the Laplacian tensor, constructed by subtracting from the diagonal multiway-array the rescaled adjacency tensor, [3] to study the connectivity of a hypergraph.

2 Main Theorem

This theory builds on work done by [4] and [1] in studying derivation algebras.

Definition 2.1. Let V be a finite set of size m and $\mathcal{H} \subset \binom{V}{n}$, where $\binom{V}{n}$ denotes subsets of size n. Label the entries of V by $\{1, 2, \ldots, m\}$. For a given field $\mathbb{F}$, define

$$\begin{aligned} \varphi : V &\to \mathbb{F}^m \\ i &\mapsto e_i \\ \tilde{\varphi} : V^n &\to (\mathbb{F}^m)^n \\ (i_1, i_2, \ldots, i_n) &\mapsto (\varphi(i_1), \varphi(i_2), \ldots, \varphi(i_n)) \end{aligned}$$

Definition 2.2. Let V be a finite set of size m and $\mathcal{H} \subset \binom{V}{n}$. We define the *adjacency tensor* corresponding to $\mathcal{H}$ as

$$\Gamma_{\mathcal{H}} : (\mathbb{F}^m)^n :=\to \mathbb{F} \tag{1}$$

$$\Gamma_{\mathcal{H}}(\tilde{\varphi}(X)) := \begin{cases} 1 & X \in \mathcal{H} \\ 0 & else \end{cases} \tag{2}$$

We will write $\Gamma := \Gamma_{\mathcal{H}}$ if the hypergraph is clear by context.

Definition 2.3. Let $\langle \Gamma \mid V_l \times \cdots V_1 \rightarrowtail V_0$ be a tensor. Its *derivation algebra* $\mathrm{Der}(\Gamma)$ is

$$Der(\Gamma) := \left\{ D \in End(V_0) \times \prod_{i=1}^{l} End(V_i) \mid \sum \langle \Gamma \mid D_\alpha v_\alpha, v_{\overline{\alpha}} \rangle = D_0 \langle \Gamma \mid v \rangle \right\}$$

Definition 2.4. For a set $A \in V$, define

$$\{x, y, A\} := \left\{ \{x, y, a\} \in \binom{V}{3} \mid a \in A \right\}.$$

If $A \subset \binom{V}{2}$, and our sets are unordered,. define

$$\{x, A\} := \left\{ \{x, a, b\} \in \binom{V}{3} \mid \mid \{a, b\} \in A \right\}$$

If an ordering is necessary, we will write such elements out more explicitly.

Definition 2.5. The following notation will be used to define our tensors

- $E_{(i,j,k)}$ is the $n \times n \times n$ tensor (n to be determined explicitly or by context) with a 1 at position (i, j, k), and a zero elsewhere.
- $E_{\{i,j,k\}} := \sum_{\sigma \in Sym(i,j,k)} E_{(i^\sigma, j^\sigma, k^\sigma)}$ is the $n \times n \times n$ tensor with a 1 at position (a, b, c), where (a, b, c) is a permutation of (i, j, k), for all possible permutations of the three elements
- $E_{\{A,j,k\}} := \sum_{a \in A} \sum_{\sigma \in Sym(a,j,k)} E_{(a^\sigma, j^\sigma, k^\sigma)}$ is the sum of the tensors $E_{\{a,j,k\}}$, indexed by the elements $a \in A$. Similarly, $E_{(A,j,k)} := \sum_{a \in A} E_{(a,j,k)}$.

Definition 2.6. Let $\mathcal{H}$ be an arbitrary hypergraph with vertices V. Let $X = \{x_1, x_2, \ldots, x_a\} \subset V$. Define the *neighborhood* of X to be

$$N_{\mathcal{H}}(X) := \{\{y_1, y_2, \ldots, y_b\} \subset V \mid \{x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_b\} \in \mathcal{H}\}$$

Theorem 2.7. Consider a hypergraph $\mathcal{H}$ with n vertices (labeled 1-n, collectively in teh set V) and its associated tensor Γ. Let $\mathcal{D}$ denote the derivation algebra of Γ. Define D to be the matrix

$$D_0 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 0_{(n-2)\times(n-2)} \end{bmatrix},$$

where $0_{m \times m}$ is the $m \times m$ all zero matrix. Let (a, b, c) is any hyperedge in $\mathcal{H}$. Then after renaming vertices so that this hyperedge is labeled $(1, 2, c)$, (D_0, D_0, D_0) is a derivation in $\mathcal{D}$.

Proof. We shall utilize the notation of the Tucker product in our proof. As a 3-valent tensor, there are three modes in which to apply the product. The translation of the derivation conditions in terms of the Tucker product is as follows:

$$\Gamma(E_1 u_i, u_j, u_k) + \Gamma(u_i, E_2 u_j, u_k) + \Gamma(u_i, u_j, E_3 u_k) = 0, \quad \forall (u_i, u_j, u_k) \tag{3}$$

$$(\Gamma \times_1 E_1) + (\Gamma \times_2 E_2) + (\Gamma \times_3 E_3) = \sum_{m=1}^{3} \Gamma \times_m E_m = 0 \tag{4}$$

Furthermore, we will rewrite our tensor Γ more explicitly in terms of the previously defined $E_{\{i,j,k\}}$. To visualize how to read our derivation, we can imagine each vertex with label i get assigned a weight given by $(D_0)_{ii}$. So the vertex 1 gets weight 1, the vertex 2 gets weight -1, and all other vertices get weight 0.

Define the sets $A = N_{\mathcal{H}}(\{1,2\})$, and $X := \{\{i,j,k\} \in \mathcal{H} \mid \{1,2\} \not\in \{i,j,k\}\}$. Clearly, $\mathcal{H} = X \bigcup_{a \in A} \{a, 1, 2\}$. We can use these two sets to rewrite Γ as

$$\Gamma = E_{\{A,1,2\}} + \sum_{\{i,j,k\} \in X} E_{\{i,j,k\}}.$$

This allows us to write our Tucker product as

$$\begin{aligned}\sum_{m=1}^{3} \Gamma \times_m D_0 &= \sum_{m=1}^{3} \left(E_{\{A,1,2\}} + \sum_{\{i,j,k\} \in X} E_{\{i,j,k\}} \right) \times_m (E_{11} - E_{22}) \\ &= \left(\sum_{m=1}^{3} E_{\{A,1,2\}} \times_m (E_{11} - E_{22}) \right) + \left(\sum_{m=1}^{3} \sum_{\{i,j,k\} \in X} E_{\{i,j,k\}} \times_m (E_{11} - E_{22}) \right)\end{aligned}$$

Now, by construction,

$$\begin{aligned} E_{\{x,y,z\}} \times_1 E_{rr} &= \begin{cases} E_{(r,y,z)} + E_{(r,z,y)} & x = r \\ 0 & else \end{cases} \\ E_{\{x,y,z\}} \times_2 E_{rr} &= \begin{cases} E_{(y,r,z)} + E_{(z,r,y)} & x = r \\ 0 & else \end{cases} \\ E_{\{x,y,z\}} \times_3 E_{rr} &= \begin{cases} E_{(y,z,r)} + E_{(z,y,r)} & x = r \\ 0 & else \end{cases}, \end{aligned}$$

where the condition of $x = r$ is to say that $r \in \{x, y, z\}$. While the specific equality is unknown (as in is it that $x = r$, $y = r$, or $z = r$), $\{x, y, z\}$ is a generic hyperedge. $E_{\{x,y,z\}}$ has nonzero values only at coordinates corresponding to permutations of $\{x, y, z\}$, and $()\times_m E_{rr}$ annihilates anything in the m^{th}-axis which is not in the r^{th} coordinate. This allows us to simplify our previous sum

$$\sum_{m=1}^{3} \Gamma \times_m D_0 = \sum_{m=1}^{3} E_{\{A,1,2\}} \times_m (E_{11} - E_{22})$$

From here, we can evaluate the sums over m and σ explicitly.

$$\sum_{m=1}^{3} \Gamma \times_m D_0 = \sum_{m=1}^{3} E_{\{A,1,2\}} \times_m (E_{11} - E_{22})$$

$$\begin{aligned}&\big((E_{(1,A,2)}+E_{1,2,A})-(E_{(2,1,A)}+E_{(2,A,1)})\big)\\+&\big((E_{(A,1,2)}+E_{2,1,A})-(E_{(1,2,A)}+E_{(A,2,1)})\big)\\+&\big((E_{(A,2,1)}+E_{2,A,1})-(E_{(1,A,2)}+E_{(A,1,2)})\big)\end{aligned}$$

This sum is separated vertically into the three modes. We only need to utilize four permutations from $Sym(a,1,2)$ for each mode as the only permutations which contribute non trivially are the ones which have either 1 or 2 in the coordinate corresponding to the particular mode. Now, we can reorganize our sum (and distribute negatives)

$$\begin{aligned}&E_{(1,A,2)}+E_{1,2,A}-E_{(2,1,A)}-E_{(2,A,1)}\\+&E_{(A,1,2)}+E_{2,1,A}-E_{(1,2,A)}-E_{(A,2,1)}\\+&E_{(A,2,1)}+E_{2,A,1}-E_{(1,A,2)}-E_{(A,1,2)}\\=&(E_{(1,A,2)}-E_{(1,A,2)})+(E_{1,2,A}-E_{(1,2,A)})\\+&(E_{(A,1,2)}-E_{(A,1,2)})+(E_{2,1,A}-E_{(2,1,A)})\\+&(E_{(A,2,1)}-E_{(A,2,1)})+(E_{2,A,1}-E_{(2,A,1)})\\=&0\end{aligned}$$

As $\sum\limits_{m=1}^{3}\Gamma\times_m D=0$, (D_0,D_0,D_0) is indeed a derivation in $\mathcal{D}$. □

We can now create a family of derivations utilizing this theorem.

Definition 2.8. Let $\overrightarrow{x}=(x_1,x_2,\dots,x_{k+1})\in\mathbb{F}^{k+1}$ denote an arbitrary vector where for all i, $x_i\neq 0$. Define

$$\begin{aligned}E_k(\overrightarrow{x}):=&\sum_{i=1}^{k+1}x_iE_{2i-1}-x_iE_{2i},\quad E_{jj}\in M_n(\mathbb{Q}),\\E_k(\overrightarrow{x}):=&Diag(x_1,-x_1,x_2,-x_2,\dots,x_{k+1},-x_{k+1},0,0,\dots,0)\in M_n(\mathbb{Q})\\F_k:=&\sum_{i=1}^{k+2}(-1)^{i-1}E_{ii},\quad E_{ii}\in M_n(\mathbb{Q})\\F_k:=&Diag((-1)^0,(-1)^1,(-1)^2,\dots,(-1)^k,(-1)^{k+1},0,0,\dots,0)M_n(\mathbb{Q}),\end{aligned}$$

Note that $D_0=\frac{1}{x_1}E_0(\overrightarrow{x})=F_0$.

Lemma 2.9. After renaming the vertices of $\mathcal{H}$, choose $X:=\{\{2i-1,2i,A_i\}\}_{i=1}^{k+1}\subset\mathcal{H}$ such that

$$\begin{aligned}A_i:=&N_{\mathcal{H}}(\{2i-1,2i\}),\\\forall j\neq i,\,2j,\ 2j-1\notin A_i\\\forall\{x,y,z\}\in\mathcal{H},\forall i,\,2i\in\{x,y,z\}&\iff 2i-1\in\{x,y,z\}\end{aligned}$$

With a different renaming of the vertices, choose $Y := \{\{i, i+1, A_i\}\}_{i=1}^{k+1} \subset \mathcal{H}$ such that

$$\begin{aligned} A_i :=& N_{\mathcal{H}}(\{i, i+1\}), \\ \forall \{x,y,z\} \in \mathcal{H},\ i \in \{x,y,z\} \implies& (i+1 \in \{x,y,z\}) \text{ XOR } (i-1 \in \{x,y,z\}) \\ \forall i, j,\ \{i, i+1, j\}, \{i, i+1, j+1\} \notin \mathcal{H}, \end{aligned}$$

Let Γ_X (Γ_Y) be the associated tensor of $\mathcal{H}$ with respect to the labeling corresponding to the set X (the set Y), and $\mathcal{D}_X$ ($\mathcal{D}_Y$) the derivation algebra of Γ_X (Γ_Y). Then $(E_k(\overrightarrow{x}), E_k(\overrightarrow{x}), E_k(\overrightarrow{x})) \in \mathcal{D}_X$, and $(F_k, F_k, F_k) \in \mathcal{D}_Y$.

Proof. First, choices of X and Y are always possible by setting $k = 0$ and choosing a single hyperedge in $\mathcal{H}$.

We will first show that $(F_k, F_k, F_k) \in \mathcal{D}_Y$. This requires strong induction with the two previous cases, and so a direct computation of the second base case of $k = 1$ is necessary. For our second base case, our derivation is (F_1, F_1, F_1), with

$$F_1 := = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 0_{(n-3)\times(n-3)} \end{bmatrix}$$

We factor our tensor as

$$\Gamma_Y := \left(E_{\{1,2,A_1\}}\right) + \left(E_{\{2,3,A_2\}}\right) + \left(\sum_{\substack{\{x,y,z\}\in\mathcal{H} \\ \{1,2\}\not\in\{x,y,z\} \\ \{2,3\}\not\in\{x,y,z\}}} E_{\{x,y,z\}} \right)$$

By the conditions used to rename our vertices, these three sets partition all hyperedges in our tensor. Our Tucker product now evaluates to

$$\begin{aligned} &\sum_{m=1}^{3} \Gamma_Y \times_m F_k = \\ &\sum_{m=1}^{3} \left(E_{\{1,2,A_1\}} + E_{\{2,3,A_2\}} + \left(\sum_{\substack{\{x,y,z\}\in\mathcal{H} \\ \{1,2\}\not\in\{x,y,z\} \\ \{2,3\}\not\in\{x,y,z\}}} E_{\{x,y,z\}} \right)\right) \times_m F_k \\ &= \sum_{m=1}^{3} \left(E_{\{1,2,A_1\}} \times_m (E_{11} - E_{22} + E_{33})\right) + \left(E_{\{2,3,A_2\}} \times_m (E_{11} - E_{22} + E_{33})\right) \end{aligned}$$

$$+\left(\sum_{\substack{\{x,y,z\}\in\mathcal{H}\\ \{1,2\}\not\subseteq\{x,y,z\}\\ \{2,3\}\not\subseteq\{x,y,z\}}} E_{\{x,y,z\}}\right)\times_m (E_{11}-E_{22}+E_{33})$$

$$=\sum_{m=1}^{3}\left(E_{\{1,2,A_1\}}\times_m (E_{11}-E_{22}+E_{33})\right)+\left(E_{\{2,3,A_2\}}\times_m (E_{11}-E_{22}+E_{33})\right)$$

$$=0+0=0$$

Now that both base cases have been verified, we may proceed with the induction. Let $A := N_{\mathcal{H}}$, i.e. the pairs of vertices which lie in a hyperedge with $k+2$, and $B := \binom{V-\{k+2\}}{3}\cap\mathcal{H}$, i.e. the hyperedges which are not adjacent to $k+2$. This allows us to partition our tensor via adjacency with $k+2$. We factor our tensor as

$$\Gamma_Y=\left(\sum_{\{r,s\}\in A} E_{\{r,s,k+3\}}\right)+\left(\sum_{\{x,y,z\}\in\binom{V-\{k+3\}}{3}\cap\mathcal{H}} E_{\{x,y,z\}}\right)$$

We will now continue on with evaluating the Tucker product for the derivation.

$$\sum_{m=1}^{3}\Gamma_Y\times_m F_k$$

$$=\sum_{m=1}^{3}\left(\sum_{\{r,s\}\in A} E_{\{r,s,k+2\}}+\sum_{\{x,y,z\}\in B} E_{\{x,y,z\}}\right)\times_m (-1)^k(E_{k+1,k+1}-E_{k+2,k+2})$$

$$+\sum_{m=1}^{3}\Gamma\times_m F_{k-2,k-2}$$

$$=(-1)^k\sum_{m=1}^{3}\left(\sum_{\{r,s\}\in A} E_{\{r,s,k+2\}}\right)\times_m (E_{k+1,k+1}-E_{k+2,k+2})$$

Where $\sum_{m=1}^{3}\left(\sum_{\{x,y,z\}\in B} E_{\{x,y,z\}}\right)\times_m (E_{k+1,k+1}-E_{k+2,k+2})=0$ because by construction neither $k+1$ nor $k+2$ are in any $\{x,y,z\}\in B$, and so the tucker product will always evaluate to zero. We proceed with our calculations. For the following we factor out $(-1)^k$ for brevity, as our end result is to show that the sum evaluates to zero.

$$\sum_{m=1}^{3}\left(\sum_{\{r,s\}\in A} E_{\{r,s,k+2\}}\right)\times_m (E_{k+1,k+1}-E_{k+2,k+2})$$

$$
\begin{aligned}
&= \sum_{m=1}^{3} \sum_{\substack{\{r,s\}\in A\\ k+1=s}} \big(E_{\{r,k+1,k+2\}} \times_m (E_{k+1,k+1} - E_{k+2,k+2})\big)\\
&+ \sum_{m=1}^{3} \sum_{\substack{\{r,s\}\in A\\ k+1\notin\{r,s\}}} \big(E_{\{r,s,k+2\}} \times_m (E_{k+1,k+1} - E_{k+2,k+2})\big)
\end{aligned}
$$

By the construction of Y we have that the second summand evaluates to zero as there is no pair of vertices adjacent to $k+2$ which does not include $k+1$. We define $A' := N_{\mathcal{H}}(\{k+1,k+2\})$. Our sum thus further simplifies as

$$
\begin{aligned}
&\sum_{m=1}^{3} \sum_{a\in A'} \big(E_{\{a,k+1,k+2\}} \times_m (E_{k+1,k+1} - E_{k+2,k+2})\big)\\
&= \big((E_{(k+1,A',k+2)} + E_{(k+1,k+2,A')} - E_{(k+2,A',k+1)} - E_{(k+2,k+1,A')})\\
&\quad + (E_{(A',k+1,k+2)} + E_{(k+2,k+1,A')} - E_{(A',k+2,k+1)} - E_{(k+1,k+2,A')})\\
&\quad +(E_{(A',k+2,k+1)} + E_{(k+2,A',k+1)} - E_{(A',k+1,k+2,)} - E_{(k+1,A',k+2)})\big)\\
&=0
\end{aligned}
$$

Now, we must show that $(E_k(\overrightarrow{x}), E_k(\overrightarrow{x}), E_k(\overrightarrow{x})) \in \mathcal{D}_X$. Unlike the previous family of derivations, we do not need to rely on strong induction. Define $A := N_{\mathcal{H}}(\{2k+1, 2k+3\})$. Our evaluation of the derivation is as follows:

$$
\begin{aligned}
&\sum_{m=1}^{3} \Gamma \times_m E_k(\overrightarrow{x})\\
&= \sum_{m=1}^{3} \left(\sum_{\substack{\{r,s,t\}\in\mathcal{H}\\ 2k+2\notin\{r,s,t\}}} E_{\{r,s,t\}} + E_{\{2k+1,2k+2,A\}} \right) \times_m \left(\sum_{i=1}^{k+1} x_i E_{2i-1} - x_i E_{2i} \right)\\
&= \sum_{m=1}^{3} \Gamma \times_m E_{k-1}((x_1, x_2, \ldots, x_k))\\
&+ \sum_{m=1}^{3} \left(\sum_{\substack{\{r,s,t\}\in\mathcal{H}\\ 2k+2\notin\{r,s,t\}}} E_{\{r,s,t\}} + E_{\{2k+1,2k+2,A\}} \right) \times_m (x_{k+1}E_{2k+1} - x_{k+1}E_{2k+2})\\
&\quad = \sum_{m=1}^{3} \big(E_{\{2k+1,2k+2,A\}} \times_m (x_{k+1}E_{2k+1})\big) - \big(E_{\{2k+1,2k+2,A\}} \times_m (x_{k+1}E_{2k+2})\big)\\
&\quad = 0
\end{aligned}
$$

□

3 Interpreting our Derivations

We will now aim to better understand what thee theorems tell us about our hypergraphs. The renaming coming from Y is used to view our hypergraph as it is in the following figures.

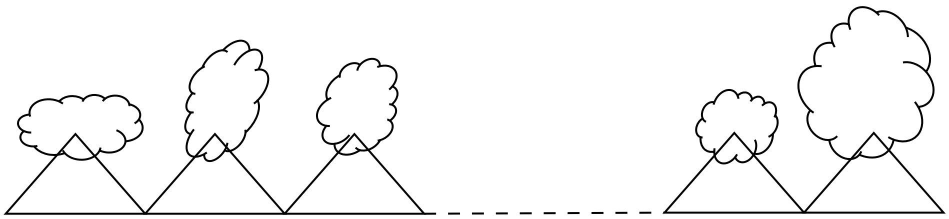

Figure 1: Mountain Ranges

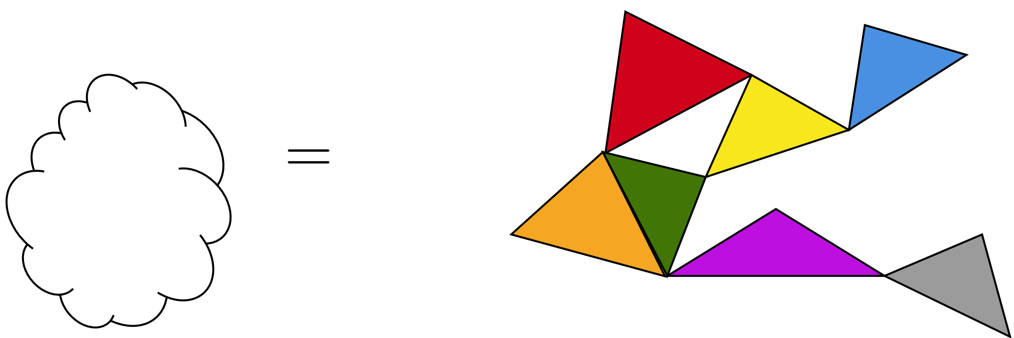

Figure 2: Continents

In Figure 1, we can imagine certain distinguished hyperedges, which we call *mountains*, forming a *mountain range*, where each mountain connects to an adjacent mountain at a vertex. At the top of the mountains are *continents*, which in the Figure 2 we can see represent an arbitrary conglomerate of hyperedges. Continents may intersect with each other, but the key is that the bases of the mountain range forms the given structure. What our theorem says is that if we assign weights to each vertex as in Figure 3, if we take any hyperedge in our hypergraph and add together the weights on each vertex, they sum to zero.

While clouds may intersect, it is relatively easy to check if this occurs. If we were to create a new hypergraph by deleting the mountain range, we can check for connected components.

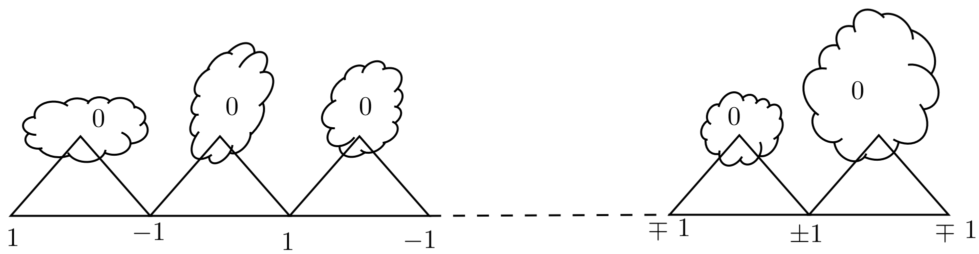

Figure 3: Weights on hyperedges in a mountain range

Definition 3.1. Given a hypergraph $\mathcal{H} = (V, E)$, the *incidence graph* of $\mathcal{H}$ is a bipartite graph $\mathcal{G} = (V_V \bigcup V_E, E_H)$ constructed as follows:

- V_V are the vertices of $\mathcal{H}$.
- V_E are vertices corresponding to the hyperedges e of $\mathcal{H}$.
- There is an edge from a vertex $v' \in V_V$ to a vertex in $e \in V_E$ if the vertex v' in $\mathcal{H}$ is contained in the hyperedge e in $\mathcal{H}$.

Lemma 3.2. A hypergraph $\mathcal{H}$ is connected if and only if its incidence graph $\mathcal{G}$ is connected.

Proof. If a hypergraph is connected, then between any two vertices there is a path connecting them where the path follows adjacency through hyperedges. In the incidence graph, this same path can be followed. If vertex v_1 and v_2 are connected via shared adjacency in hyperedge e_1, on the incidence matrix v_1 and v_2 are both adjacent to e_1, and so a path connects them. Therefore, any path in $\mathcal{H}$ corresponds to a path in $\mathcal{G}$, and vice versa. □

By this lemma, checking if a hypergraph is connected is equivalent to checking if a bipartite graph is connected.

For the labeling of X, we have the following figure

In Figure 4, we have a central continent which has coming from it various *peninsulas*. Our theorem says that if we assign weights to each vertex as in Figure 5, on any hyperedge the sum of the weights on the vertices equals to zero.

Corollary 3.3. With an appropriate relabeling to avoid unwanted overlaps, linear combinations of $(E_a(\overrightarrow{x_a}), E_a(\overrightarrow{x_a}), E_a(\overrightarrow{x_a}))'s$ and $(F_b, F_b, F_b)'s$ (with varying $a's$ and $b's$) form a derivation which is interpretable as a combination of continents, mountain ranges, and peninsulas.

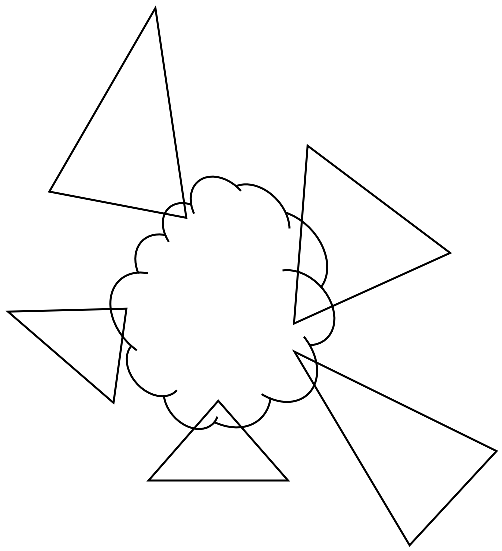

Figure 4: Peninsulas off the Mainland

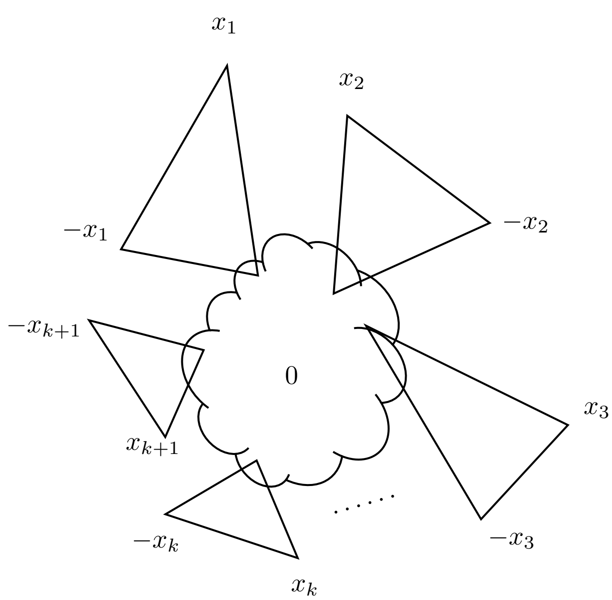

Figure 5: Weights on hyperedges in a continent with peninsulas

Working with the graphical interpretations of derivations assigning weights to vertices, this corollary says that we can build out our hypergraph structure by isolating mountain ranges, continents, and peninsulas all glued together, and that there exists a derivation which will identify this process. Figure 6 shows what this may look like.

In this example, the corresponding derivation would be (D, D, D), where

$$D := \begin{bmatrix} X' &&&&&&& \\ & -x_3 &&&&&& \\ && x_3 &&&&& \\ &&& -x_3 &&&& \\ &&&& -x_3 &&& \\ &&&&& Y' && \\ &&&&&& Z' & \\ &&&&&&& 0 \end{bmatrix},$$

$$X' := \begin{bmatrix} x_1 &&&&&&& \\ & -x_1 &&&&&& \\ && x_2 &&&&& \\ &&& -x_2 &&&& \\ &&&& -x_4 &&& \\ &&&&& \ddots && \\ &&&&&& x_{a+1} & \\ &&&&&&& -x_{a+1} \end{bmatrix}$$

$$Y' := \begin{bmatrix} y_1 &&&&&&& \\ & -y_1 &&&&&& \\ && y_2 &&&&& \\ &&& -y_2 &&&& \\ &&&& -y_4 &&& \\ &&&&& \ddots && \\ &&&&&& y_{b+1} & \\ &&&&&&& -y_{b+1} \end{bmatrix}$$

$$Z' := \begin{bmatrix} z_1 &&&&&&& \\ & -z_1 &&&&&& \\ && z_2 &&&&& \\ &&& -z_2 &&&& \\ &&&& -z_4 &&& \\ &&&&& \ddots && \\ &&&&&& z_{c+1} & \\ &&&&&&& -z_{c+1} \end{bmatrix}$$

When considering how continents and mountain ranges may be connected, we have thus far only considered trees. We must now consider loops.

Lemma 3.4. If a mountain range loops with $2n$ different mountains, then (after relabeling) F_{2n} is a derivation for the correspoding tensor. If the loop is con-

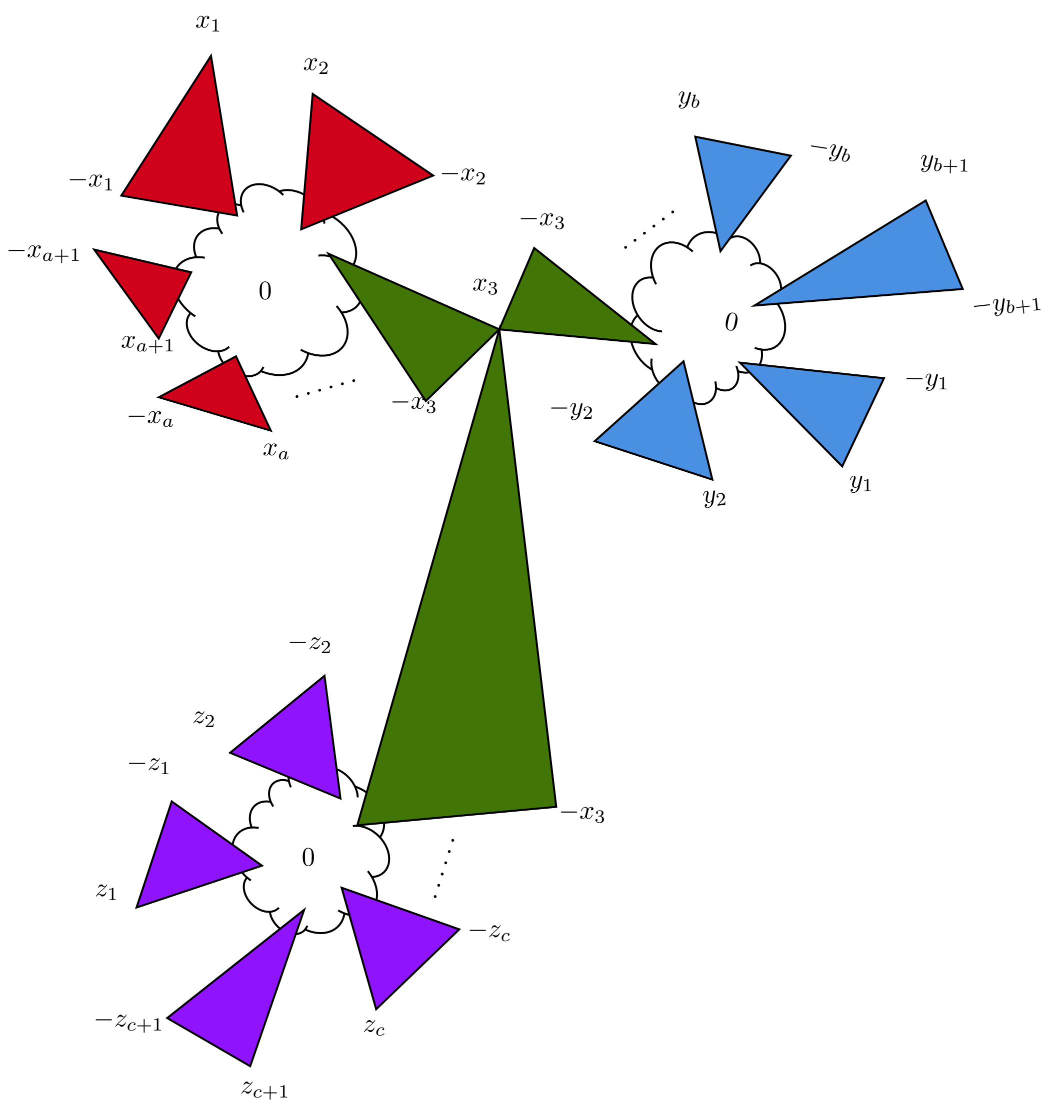

Figure 6: Combination of Continents, Mountain Ranges, and Peninsulas

structed with $2n+1$ mountains, then after a relabeling $F_{2n}+(-1)^{2n+1}2E_{2n+2,2n+2}$ is a derivation for the corresponding tensor.

Proof. With our correspondence of elements of the Cartan subalgebra of the Derivation algebra and weights on a hypergraph, we can give the proof of this lemma with an argument on the weights of hyperedges. The following two pictures show this argument: (insert tikz image here). □

Finally, the third way in which we can get a derivation is by taking the sub-hypergraph $[\{1,2,3\},\{3,4,5\},\{5,6,7\},\{7,8,9\}]$, and give the vertex 4 a weight of 1, the vertex 5 a weight -1, and the vertex 6 a weight of 1. (this might be covered already by F_k, and my future argument sying mountain peaks connecting continents cannot have weight other than zero).

3.1 Generating Examples

There are two ways in which we can construct standard hypergraph structures from a graph. Both enlists the use of cones. In the first, we take an arbitrary graph $\mathcal{G}$ with vertex set V and edge set E. We then construct a hypergraph $\mathcal{H}$ with vertex set $V \cup \{\infty\}$ and edge set $E' := \{\{v_i, v_j, \infty\} \mid \{v_i, v_j\} \in E\}$. This is equivalent to drawing our graph $\mathcal{G}$, including a "point at infinity," and connecting each edge to this point at infinity.

When looking at our adjacency tensor, the evaluations are as follows

$$\Gamma(x,y,z) := \begin{cases} 1 & \{(x,y) \in E\} \wedge \{z = \infty\} \\ 0 & else \end{cases}, \tag{5}$$

where we account for symmetry in our evaluations. Once again, we will look for symmetric subalgebras of the Cartan subalgebra (diagonal matrices). For ease of our explanations, we will utilize our interpretation of weights on vertices, and that a triple of the same diagonal matrix are in the derivation algebra if and only if their corresponding weight assignments sum to zero on each hyperedge.

In this interpretation, the weight on ∞ is a_∞, and a triple of matrices is a derivation if and only if the weight assignments on the base graph $\mathcal{G}$ has the property that the sum of weights on adjacent vertices (adjacency being defined as lying on the same hyperedge) is zero. This holds if and only if $\mathcal{G}$ is bipartite.

Proof. Let a_∞ be the weight on ∞. Beginning with any pair of vertices which lie on a hyperedge with ∞, one can be assigned the weight x_i and may be given the label of 1. The other is necessarily assigned the weight of $-x_i - a_\infty$, and may be given the label of 2. If ∞ and 1 lie on a hyperedge with a third vertex 0 which is not 2 then 0 must be given a weight of $-x_i - a_\infty$. Equivalently, If ∞ and 2 lie on a hyperedge with a third vertex 3 which is not 1, then 3 must be given a weight of x_i. Furthermore, $\{1,3,\infty\}$ cannot be a hyperedge, and neither can $\{0,2,\infty\}$. This consturction provides us with a bipartite graph.

Likewise, if we were to begin with a bipartite 2-graph, then we can assign each side of the bipartie graph its own weight x_i and y_i, and then the point at infinity will receive a weight of $-x_i - y_i$. □

In this proof, i is a number of the number of connected components in our graph $\mathcal{G}$, whcih will determine how many different weight-pairs we need. Our derivations are thus of the form $a_\infty E_{\infty,\infty} + x_i E_{11} + (-x_i - a_\infty)E_{22}$.

In the second way of constructing a hypergraph from a graph, we instead generate a cone for each edge in the graph $\mathcal{G}$. We do this by assigning a distinct point at infinity ∞_{xy} for each edge $(x, y) \in E$. In this scenario, we are less restricted in our weight assignment. We can assign any weight to the vertices in $\mathcal{G}$. All we require is that if a_x and a_y are weights of adjacent vertices x and y, the weight on ∞_{xy} is $-a_x - a_y$. This gives us derivations with matrices of the form $a_x E_{11} + b_y E_{22} + (-a_x - b_y)E_{33}$.

4 Community Detection in a hypergraph

One way in which we can use these results is in detecting community structure. Typically, "communities" are defined as "collections of vertices in a graph which are 'more' connected to each other than 'outside' vertices." This kind of definition leaves a very vague notion of distinct communities, which allows for a more fluid identification of communities, as well as the use of eigentheory and SVD to obtain identifications. Our use of derivations provides a stricter and more easily identifiable definition of community structure.

Definition 4.1. Let $\mathcal{H}$ be a hypergraph. A *continental structure* on $\mathcal{H}$ is an relabeling and identification of vertices so that a linear combination of $(E_a(\overrightarrow{x_a}), E_a(\overrightarrow{x_a}), E_a(\overrightarrow{x_a}))'s$ and $(F_b, F_b, F_b)'s$ with coefficients of 1 form a derivation which is interpretable as a combination of continents, mountain ranges, and peninsulas, where the continents do not intersect. The number of continents is $\sum_{i=1}^{j}(b_i + 1)$, where j is the number of mountain ranges, and $b_i + 1$ is the number of mountains in the i^{th} mountain range.

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