Hypergraph Tectonics

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1 Introduction

1.1 Why study hypergraphs?

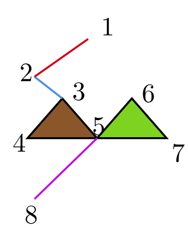
In this paper, we will be introducing a new tool for studying network structure. A straightforward example of the network structure we will be presented with in this paper is the collection of all international trade agreements, as was proposed by [**?**]. Each individual trade agreement is a particular connection between the member nations, which we may refer to as *actors*. This network structure has some interesting properties. If multiple actors are connected by a single trade agreement, then it is not necessary that any two actors are themselves connected to each other by differing two party trade agreement. Furthermore, studying large scale impacts to changes in the network, such as the imposition of tariffs from one actor to another, can become much more complicated. Not only will it involve studying the effect of the imposed tariffs on the other actor, but also how the financial impact on the actor will effect any trade agreements it is currently involved in, and how those effects may spread.

In this paper, we will be studying a global view of network structure. Rather than analyzing and identifying individual actors, we will be making observations on the overall features which are present within the network.

1.2 What is a Hypergraph?

Definition 1.1. A *hypergraph* is a pair $(V, \mathcal{H})$, where $\mathcal{H} \subset 2^V$. A *uniform hypergraph* is a hypergraph where $\mathcal{H} \subset \binom{V}{k} = \{S \subset V \mid |S| = k\}$ for some positive integer k.

Example 1.2. The following is a hypergraph consisting of edges and 3-edges (which connect three vertices in one edge).



This can be expressed as an incidence matrix

	Edge 1	Edge 2	Edge 3	Edge 4	Edge 5
Vertex 1	1	·	·	·	·
Vertex 2	1	1	·	·	·
Vertex 3	·	1	1	·	·
Vertex 4	·	·	1	·	·
Vertex 5	·	·	1	1	1
Vertex 6	·	·	·	1	·
Vertex 7	·	·	·	1	·
Vertex 8	·	·	·	·	1

For our work, we will consider all hypergraphs to be uniform. If any hypergraph is not uniform, it can be made to be uniform via the following functor

Lemma 1.3. For all hypergraphs $\mathcal{H}$, there is a functor which maps $\mathcal{H}$ to a uniform hypergraph $\overline{\mathcal{H}}$

Proof. Let k be the greatest number of vertices present in any hyperedge of $\mathcal{H}$. For each hyperedge e which contains less than $j < k$ vertices, adjoin to it $k - j$ new vertices $w_1, \ldots, w_{k-j}$, so that

$$\{v_1, v_2, \ldots, v_k\} \mapsto \{v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_{k-j}\}.$$

For each hyperedge we use distinct $w_i's$ to construct our new hypergraph. $\square$

A different approach is to add a singular point at infinity to "complete" each hyperedge, as was done by [**?**] when using an embedding of the hypergraph into a lower dimensional space to determine the closeness of communities. In this paper, we are adding multiple distinct points so that we do not accidentally further relate hyperedges beyond the connections found in the original hypergraph. However, later in this paper we will see the effect of adding a singular "point at infinity." With this lemma, we can regard all hypergraphs as uniform, so long as we retain the information of which vertices were added after applying the functor.

1.3 Current Techniques

In the current literature, many techniques focus on studying the incidence matrix of a hypergraph. Studying the rank and singular values of this matrix gives information about the connectedness of the hypergraph. In recent years more work has been done in using the eigenvalues for an adjacency tensor to a hypergraph. The work of [**?**] and [**?**] pioneered this path, as they gave a clear road map for defining eigenvalues and eigenvectors on a multilinear product. Following this, [**?**] utilizes a rescaled adjacency tensor to create an analog for spectral graph theory on hypergraphs. This is used to study bounds on average degrees and coloring of vertices. By considering the Laplacian tensor, constructed by subtracting from the diagonal multiway-array the rescaled adjacency tensor, [**?**]

to study the connectivity of a hypergraph. Similarly, [?] [?] [?] have used this eigen theory for tensors to study hypergraphs.

Other researches have taken different paths to relate hypergraphs and tensors. In [?], a one to one correspondence is made between hypergraphs with restricted probability distributions on the vertices and tensor networks which model tensor contractions. In this vein, we will be looking directly at correspondences between tensors and hypergraphs to make our analysis, as opposed to studying the eigen theory of tensors.

2 Main Theorem

This theory builds on work done by [?] and [?] in studying derivation algebras. Furthermore, we will be looking at 3-uniform hypergraphs in this paper, but this theory can be expanded to higher order hypergraphs.

2.1 Motivating Derivations

First, Let us see how we could chagne our perspective of a tensor. Consider

$$\Gamma = \left[\underset{\{1\}}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}, \underset{\{x\}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \right]$$

The labels 1 and x are used to keep track of the slices of our $2 \times 2 \times 2$ tensor. We could have stored this information in a multiplication table.

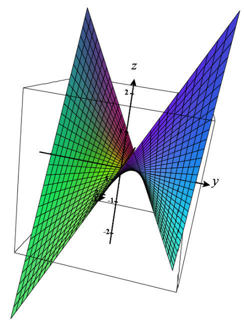
$*$	1	x
1	1	x
x	x	-1

This translation happened by viewing 1 and x as the vectors $(1, 0)$ and $(0, 1)$. Now, this multiplication table is really expressing $\mathbb{R}[x]/(x^2 + 1) = \mathbb{C}$. This tells us that our tensor was really holding onto the information of the complex numbers, which we were able to deduce by viewing it as a multiplication table. This is our motivation for studying products.

Now that we have products, lets look at the most simple product. Take

$$x = f(x, y) = xy.$$

As a first pass, we would graph this function



This graph is a saddle. The next thing we would do is study the curvature of the function. This is done by computing derivatives. Since we have a product, we need ot work with the product rule of derivatives. The product rule for derivatives is

$$\frac{d}{dx}(f(x)g(x)) = \left(\frac{d}{dx}f(x)\right)g(x) + f(x)\left(\frac{d}{dx}g(x)\right)$$

Let us now rewrite this rule. Define $f(x)g(x) := \Gamma(f(x), g(x))$, and $D := \frac{d}{dx}$. Then the product rule is

$$D\Gamma(f(x), g(x)) = \Gamma(Df(x), g(x)) + \Gamma(f(x), Dg(x))$$

This is our motivator for looking at derivations.

2.2 The Adjacency Tensor

First, we will embed the hypergraph coordinates into vector spaces over a field $\mathbb{F}$. In this paper, we will be using $\mathbb{Q}$. This embedding is by mapping the vertex i to the standard basis vector e_i.

Definition 2.1. Let V be a finite set of size m and $\mathcal{H} \subset \binom{V}{n}$, where $\binom{V}{n}$ denotes subsets of size n. Label the entries of V by $\{1, 2, \ldots, m\}$. For a given field $\mathbb{F}$, define

$$\begin{aligned} \varphi : V \to \mathbb{F}^m, \quad &\mapsto e_i \\ \tilde{\varphi} : V^n \to (\mathbb{F}^m)^n& \\ (i_1, i_2, \ldots, i_n) \mapsto (\varphi(i_1), &\varphi(i_2), \ldots, \varphi(i_n)) \end{aligned}$$

Definition 2.2. Let V be a finite set of size m and $\mathcal{H} \subset \binom{V}{n}$. We define the *adjacency tensor* corresponding to $\mathcal{H}$ as

$$\Gamma_{\mathcal{H}} : (\mathbb{F}^m)^n :=\to \mathbb{F} \tag{1}$$

$$\Gamma_{\mathcal{H}}(\tilde{\varphi}(X)) := \begin{cases} 1 & X \in \mathcal{H} \\ 0 & else \end{cases} \tag{2}$$

We will write $\Gamma := \Gamma_{\mathcal{H}}$ if the hypergraph is clear by context.

Definition 2.3. Let $\langle \Gamma \mid V_l \times \cdots V_1 \rightarrowtail V_0$ be a tensor. Its *derivation algebra* $\mathrm{Der}(\Gamma)$ is

$$Der(\Gamma) := \left\{ D \in End(V_0) \times \prod_{i=1}^{l} End(V_i) \mid \sum \langle \Gamma \mid D_\alpha v_\alpha, v_{\overline{\alpha}} \rangle = D_0 \langle \Gamma \mid v \rangle \right\}$$

The following notation will be used in proving our main theorem.

Definition 2.4. For a set $A \in V$, define

$$\{x, y, A\} := \left\{ \{x, y, a\} \in \binom{V}{3} \mid a \in A \right\}.$$

If $A \subset \binom{V}{2}$ define

$$\{x, A\} := \left\{ \{x, a, b\} \in \binom{V}{3} \mid \{a, b\} \in A \right\}$$

If an ordering is necessary, we will write such elements out more explicitly.

Definition 2.5. The following notation will be used to define our tensors

- $E_{(i,j,k)}$ is the $n \times n \times n$ tensor (n to be determined explicitly or by context) with a 1 at position (i, j, k), and a zero elsewhere.
- $E_{\{i,j,k\}} := \sum\limits_{\sigma \in Sym(i,j,k)} E_{(i^\sigma, j^\sigma, k^\sigma)}$ is the $n \times n \times n$ tensor with a 1 at position (a, b, c), where (a, b, c) is a permutation of (i, j, k), for all possible permutations of the three elements
- $E_{\{A,j,k\}} := \sum\limits_{a \in A} \sum\limits_{\sigma \in Sym(a,j,k)} E_{(a^\sigma, j^\sigma, k^\sigma)}$ is the sum of the tensors $E_{\{a,j,k\}}$, indexed by the elements $a \in A$. Similarly, $E_{(A,j,k)} := \sum\limits_{a \in A} E_{(a,j,k)}$.

Definition 2.6. Let $\mathcal{H}$ be an arbitrary hypergraph with vertices V. Let $X = \{x_1, x_2, \ldots, x_a\} \subset V$. Define the *neighborhood* of X to be

$$N_{\mathcal{H}}(X) := \{\{y_1, y_2, \ldots, y_b\} \subset V \mid \{x_1, x_2, \ldots, x_a, y_1, y_2, \ldots, y_b\} \in \mathcal{H}\}$$

Theorem 2.7. Consider a hypergraph $\mathcal{H}$ with n vertices (labeled 1-n, collectively in the set V) and its associated tensor Γ. Let $\mathcal{D}$ denote the derivation algebra of Γ. Define $D_0(\lambda)$ to be the matrix

$$D_0(\lambda)(e_k) = \begin{cases} \lambda e_1 & k = 1 \\ -\lambda e_2 & k = 2 \\ 0 & else \end{cases}$$

where we let $D_0 := D_0(1)$. Let (a, b, c) be a hyperedge in $\mathcal{H}$ such that $N_{\mathcal{H}}(a) = \{b, c\}$ and $N_{\mathcal{H}}(b) = \{a, c\}$. Then after renaming vertices so that a is labeled 1 and b is labeled 2, (D_0, D_0, D_0) is a derivation in $\mathcal{D}$.

Proof. We shall utilize the notation of the Tucker product in our proof. As a 3-valent tensor, there are three modes in which to apply the product. The translation of the derivation conditions in terms of the Tucker product is as follows:

$$\forall(u_i, u_j, u_k), \;\; \Gamma(E_1 u_i, u_j, u_k) + \Gamma(u_i, E_2 u_j, u_k) + \Gamma(u_i, u_j, E_3 u_k) = 0 \tag{3}$$

$$(\Gamma \times_1 E_1) + (\Gamma \times_2 E_2) + (\Gamma \times_3 E_3) = \sum_{m=1}^{3} \Gamma \times_m E_m = 0 \tag{4}$$

Furthermore, we will rewrite our tensor Γ more explicitly in terms of the previously defined $E_{\{i,j,k\}}$. To visualize how to read our derivation, we can imagine each vertex with label i get assigned a weight given by $(D_0)_{ii}$. So the vertex 1 gets weight 1, the vertex 2 gets weight -1, and all other vertices get weight 0. Define the sets $A = N_{\mathcal{H}}(\{1, 2\})$, and $X := \{\{i, j, k\} \in \mathcal{H} \mid \{1, 2\} \not\subset$

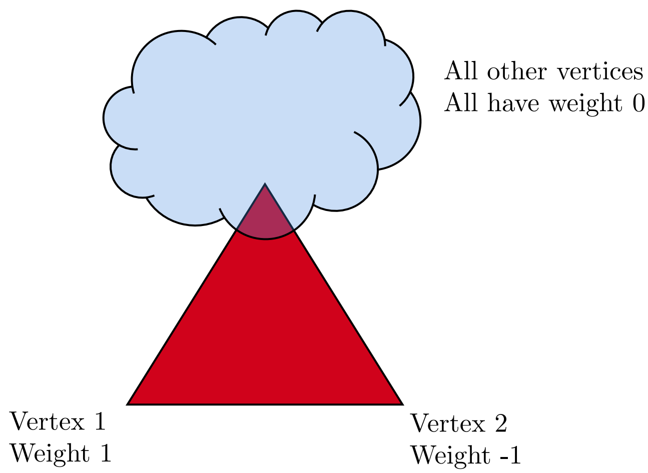


Figure 1: How derivations assign weights

$\{i, j, k\}\}$. Clearly, $\mathcal{H} = X \bigcup_{a \in A} \{a, 1, 2\}$. We can use these two sets to rewrite Γ as

$$\Gamma = E_{\{A,1,2\}} + \sum_{\{i,j,k\} \in X} E_{\{i,j,k\}}.$$

This allows us to write our Tucker product as

$$\sum_{m=1}^{3} \Gamma \times_m D_0 = \sum_{m=1}^{3} \left(E_{\{A,1,2\}} + \sum_{\{i,j,k\} \in X} E_{\{i,j,k\}} \right) \times_m (E_{11} - E_{22})$$

$$= \left(\sum_{m=1}^{3} E_{\{A,1,2\}} \times_m (E_{11} - E_{22})\right) + \left(\sum_{m=1}^{3} \sum_{\{i,j,k\}\in X} E_{\{i,j,k\}} \times_m (E_{11} - E_{22})\right)$$

Now, by construction,

$$E_{\{x,y,z\}} \times_1 E_{rr} = \begin{cases} E_{(r,y,z)} + E_{(r,z,y)} & x = r \\ 0 & else \end{cases}$$
$$E_{\{x,y,z\}} \times_2 E_{rr} = \begin{cases} E_{(y,r,z)} + E_{(z,r,y)} & x = r \\ 0 & else \end{cases}$$
$$E_{\{x,y,z\}} \times_3 E_{rr} = \begin{cases} E_{(y,z,r)} + E_{(z,y,r)} & x = r \\ 0 & else \end{cases},$$

where the condition of $x = r$ is to say that $r \in \{x, y, z\}$. While the specific equality of $x = r$, $y = r$, or $z = r$ is unknown, $\{x, y, z\}$ is a generic hyperedge. $E_{\{x,y,z\}}$ has nonzero values only at coordinates corresponding to permutations of $\{x, y, z\}$, and $(-) \times_m E_{rr}$ annihilates anything in the m^{th}-axis which is not in the r^{th} coordinate. This allows us to simplify our previous sum

$$\sum_{m=1}^{3} \Gamma \times_m D_0 = \sum_{m=1}^{3} E_{\{A,1,2\}} \times_m (E_{11} - E_{22})$$

From here, we can evaluate the sums over m and σ explicitly.

$$\begin{aligned} \sum_{m=1}^{3} \Gamma \times_m D_0 = \sum_{m=1}^{3} & E_{\{A,1,2\}} \times_m (E_{11} - E_{22}) \\ & \big((E_{(1,A,2)} + E_{1,2,A}) - (E_{(2,1,A)} + E_{(2,A,1)})\big) \\ & + \big((E_{(A,1,2)} + E_{2,1,A}) - (E_{(1,2,A)} + E_{(A,2,1)})\big) \\ & + \big((E_{(A,2,1)} + E_{2,A,1}) - (E_{(1,A,2)} + E_{(A,1,2)})\big) \end{aligned}$$

This sum is separated vertically into the three modes. We only need to utilize four permutations from $Sym(a, 1, 2)$ for each mode as the only permutations which contribute non trivially are the ones which have either 1 or 2 in the coordinate corresponding to the particular mode. Now, we can reorganize our sum (and distribute negatives)

$$\begin{aligned} & E_{(1,A,2)} + E_{1,2,A} - E_{(2,1,A)} - E_{(2,A,1)} \\ & + E_{(A,1,2)} + E_{2,1,A} - E_{(1,2,A)} - E_{(A,2,1)} \\ & + E_{(A,2,1)} + E_{2,A,1} - E_{(1,A,2)} - E_{(A,1,2)} \\ & = (E_{(1,A,2)} - E_{(1,A,2)}) + (E_{1,2,A} - E_{(1,2,A)}) \\ & + (E_{(A,1,2)} - E_{(A,1,2)}) + (E_{2,1,A} - E_{(2,1,A)}) \end{aligned}$$

$$\begin{aligned}&+(E_{(A,2,1)} - E_{(A,2,1)}) + (E_{2,A,1} - E_{(2,A,1)})\\&=0\end{aligned}$$

As $\sum_{m=1}^{3} \Gamma \times_m D = 0$, (D_0, D_0, D_0) is indeed a derivation in $\mathcal{D}$. $\square$

We can now create a family of derivations utilizing this theorem.

Definition 2.8. Let $\overrightarrow{x} = (x_1, x_2, \ldots, x_{k+1}) \in \mathbb{F}^{k+1}$ denote an arbitrary vector where for all i, $x_i \neq 0$. Define

$$\begin{aligned}E_k(\overrightarrow{x}) &:= \sum_{i=1}^{k+1} x_i E_{2i-1} - x_i E_{2i}, \quad E_{jj} \in M_n(\mathbb{Q}),\\E_k(\overrightarrow{x}) &:= Diag(x_1, -x_1, x_2, -x_2, \ldots, x_{k+1}, -x_{k+1}, 0, 0, \ldots, 0) \in M_n(\mathbb{Q})\\F_k &:= \sum_{i=1}^{k+2} (-1)^{i-1} E_{ii}, \quad E_{ii} \in M_n(\mathbb{Q})\\F_k &:= Diag((-1)^0, (-1)^1, (-1)^2, \ldots, (-1)^k, (-1)^{k+1}, 0, 0, \ldots, 0) M_n(\mathbb{Q}),\end{aligned}$$

Note that $D_0 = \frac{1}{x_1} E_0(\overrightarrow{x}) = F_0$.

Lemma 2.9. After renaming the vertices of $\mathcal{H}$, choose $X := \{\{2i-1, 2i, A_i\}\}_{i=1}^{k+1} \subset \mathcal{H}$ such that

$$\begin{aligned}A_i :=& N_{\mathcal{H}}(\{2i-1, 2i\}),\\\forall j \neq i,\, 2j,\ 2j-1 \notin& A_i\\\forall \{x,y,z\} \in \mathcal{H}, \forall i,\, 2i \in \{x,y,z\} \iff& 2i-1 \in \{x,y,z\}\end{aligned}$$

With a different renaming of the vertices, choose $Y := \{\{i, i+1, A_i\}\}_{i=1}^{k+1} \subset \mathcal{H}$ such that

$$\begin{aligned}A_i :=& N_{\mathcal{H}}(\{i, i+1\}),\\\forall \{x,y,z\} \in \mathcal{H},\ i \in \{x,y,z\} \implies& (i+1 \in \{x,y,z\}) \text{ XOR } (i-1 \in \{x,y,z\})\\\forall i,j,\ \{i, i+1, j\}, \{i, i+1, j+1\} \notin& \mathcal{H},\end{aligned}$$

Let Γ_X (Γ_Y) be the associated tensor of $\mathcal{H}$ with respect to the labeling corresponding to the set X (the set Y), and $\mathcal{D}_X$ ($\mathcal{D}_Y$) the derivation algebra of Γ_X (Γ_Y). Then $(E_k(\overrightarrow{x}), E_k(\overrightarrow{x}), E_k(\overrightarrow{x})) \in \mathcal{D}_X$, and $(F_k, F_k, F_k) \in \mathcal{D}_Y$.

Proof. First, choices of X and Y are always possible by setting $k = 0$ and choosing a single hyperedge in $\mathcal{H}$.

We will first show that $(F_k, F_k, F_k) \in \mathcal{D}_Y$. This requires strong induction with the two previous cases, and so a direct computation of the second base case of $k = 1$ is necessary. For our second base case, our derivation is (F_1, F_1, F_1), with

$$F_1 := = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 0_{(n-3)\times(n-3)} \end{bmatrix}$$

We factor our tensor as

$$\Gamma_Y := \left(E_{\{1,2,A_1\}}\right) + \left(E_{\{2,3,A_2\}}\right) + \left(\sum_{\substack{\{x,y,z\}\in\mathcal{H} \\ \{1,2\}\not\subseteq\{x,y,z\} \\ \{2,3\}\not\subseteq\{x,y,z\}}} E_{\{x,y,z\}}\right)$$

By the conditions used to rename our vertices, these three sets partition all hyperedges in our tensor. Our Tucker product now evaluates to

$$\begin{aligned}
&\sum_{m=1}^{3} \Gamma_Y \times_m F_k = \\
&\sum_{m=1}^{3}\left(E_{\{1,2,A_1\}} + E_{\{2,3,A_2\}} + \left(\sum_{\substack{\{x,y,z\}\in\mathcal{H} \\ \{1,2\}\not\subseteq\{x,y,z\} \\ \{2,3\}\not\subseteq\{x,y,z\}}} E_{\{x,y,z\}}\right)\right) \times_m F_k \\
&= \sum_{m=1}^{3} \left(E_{\{1,2,A_1\}} \times_m (E_{11} - E_{22} + E_{33})\right) + \left(E_{\{2,3,A_2\}} \times_m (E_{11} - E_{22} + E_{33})\right) \\
&\quad + \left(\sum_{\substack{\{x,y,z\}\in\mathcal{H} \\ \{1,2\}\not\subseteq\{x,y,z\} \\ \{2,3\}\not\subseteq\{x,y,z\}}} E_{\{x,y,z\}}\right) \times_m (E_{11} - E_{22} + E_{33}) \\
&= \sum_{m=1}^{3} \left(E_{\{1,2,A_1\}} \times_m (E_{11} - E_{22} + E_{33})\right) + \left(E_{\{2,3,A_2\}} \times_m (E_{11} - E_{22} + E_{33})\right) \\
&=0 + 0 = 0
\end{aligned}$$

Now that both base cases have been verified, we may proceed with the induction. Let $A := N_{\mathcal{H}}$, i.e. the pairs of vertices which lie in a hyperedge with $k+2$, and $B := \binom{V-\{k+2\}}{3} \cap \mathcal{H}$, i.e. the hyperedges which are not adjacent to $k+2$. This allows us to partition our tensor via adjacency with $k+2$. We factor our tensor as

$$\Gamma_Y = \left(\sum_{\{r,s\}\in A} E_{\{r,s,k+3\}}\right) + \left(\sum_{\{x,y,z\}\in\binom{V-\{k+3\}}{3}\cap\mathcal{H}} E_{\{x,y,z\}}\right)$$

We will now continue on with evaluating the Tucker product for the derivation.

$$
\begin{aligned}
&\sum_{m=1}^{3} \Gamma_Y \times_m F_k \\
=&\sum_{m=1}^{3} \left(\sum_{\{r,s\}\in A} E_{\{r,s,k+2\}} + \sum_{\{x,y,z\}\in B} E_{\{x,y,z\}} \right) \times_m (-1)^k (E_{k+1,k+1} - E_{k+2,k+2}) \\
+&\sum_{m=1}^{3} \Gamma \times_m F_{k-2,k-2} \\
=&(-1)^k \sum_{m=1}^{3} \left(\sum_{\{r,s\}\in A} E_{\{r,s,k+2\}} \right) \times_m (E_{k+1,k+1} - E_{k+2,k+2})
\end{aligned}
$$

Where $\sum_{m=1}^{3} \left(\sum_{\{x,y,z\}\in B} E_{\{x,y,z\}} \right) \times_m (E_{k+1,k+1} - E_{k+2,k+2}) = 0$ because by construction neither $k+1$ nor $k+2$ are in any $\{x,y,z\} \in B$, and so the tucker product will always evaluate to zero. We proceed with our calculations. For the following we factor out $(-1)^k$ for brevity, as our end result is to show that the sum evaluates to zero.

$$
\begin{aligned}
&\sum_{m=1}^{3} \left(\sum_{\{r,s\}\in A} E_{\{r,s,k+2\}} \right) \times_m (E_{k+1,k+1} - E_{k+2,k+2}) \\
=&\sum_{m=1}^{3} \sum_{\substack{\{r,s\}\in A \\ k+1=s}} \big(E_{\{r,k+1,k+2\}} \times_m (E_{k+1,k+1} - E_{k+2,k+2}) \big) \\
+&\sum_{m=1}^{3} \sum_{\substack{\{r,s\}\in A \\ k+1\notin\{r,s\}}} \big(E_{\{r,s,k+2\}} \times_m (E_{k+1,k+1} - E_{k+2,k+2}) \big)
\end{aligned}
$$

By the construction of Y we have that the second summand evaluates to zero as there is no pair of vertices adjacent to $k+2$ which does not include $k+1$. We define $A' := N_{\mathcal{H}}(\{k+1,k+2\})$. Our sum thus further simplifies as

$$
\begin{aligned}
&\sum_{m=1}^{3} \sum_{a\in A'} \big(E_{\{a,k+1,k+2\}} \times_m (E_{k+1,k+1} - E_{k+2,k+2}) \big) \\
=&\big((E_{(k+1,A',k+2)} + E_{(k+1,k+2,A')} - E_{(k+2,A',k+1)} - E_{(k+2,k+1,A')}) \\
&+ (E_{(A',k+1,k+2)} + E_{(k+2,k+1,A')} - E_{(A',k+2,k+1)} - E_{(k+1,k+2,A')}) \\
&+ (E_{(A',k+2,k+1)} + E_{(k+2,A',k+1)} - E_{(A',k+1,k+2,)} - E_{(k+1,A',k+2)}) \big) \\
=&0
\end{aligned}
$$

Now, we must show that $(E_k(\overrightarrow{x}), E_k(\overrightarrow{x}), E_k(\overrightarrow{x})) \in \mathcal{D}_X$. Unlike the previous family of derivations, we do not need to rely on strong induction. Define $A := N_{\mathcal{H}}(\{2k+1, 2k+3\})$. Our evaluation of the derivation is as follows:

$$\begin{aligned}
&\sum_{m=1}^{3} \Gamma \times_m E_k(\overrightarrow{x}) \\
&= \sum_{m=1}^{3} \left(\sum_{\substack{\{r,s,t\}\in\mathcal{H} \\ 2k+2\notin\{r,s,t\}}} E_{\{r,s,t\}} + E_{\{2k+1,2k+2,A\}} \right) \times_m \left(\sum_{i=1}^{k+1} x_i E_{2i-1} - x_i E_{2i} \right) \\
&= \sum_{m=1}^{3} \Gamma \times_m E_{k-1}((x_1, x_2, \ldots, x_k)) \\
&+ \sum_{m=1}^{3} \left(\sum_{\substack{\{r,s,t\}\in\mathcal{H} \\ 2k+2\notin\{r,s,t\}}} E_{\{r,s,t\}} + E_{\{2k+1,2k+2,A\}} \right) \times_m (x_{k+1} E_{2k+1} - x_{k+1} E_{2k+2}) \\
&\quad = \sum_{m=1}^{3} \big(E_{\{2k+1,2k+2,A\}} \times_m (x_{k+1} E_{2k+1})\big) - \big(E_{\{2k+1,2k+2,A\}} \times_m (x_{k+1} E_{2k+2})\big) \\
&\quad = 0
\end{aligned}$$

□

3 Interpreting our Derivations

We will now aim to better understand what thee theorems tell us about our hypergraphs. The renaming coming from Y is used to view our hypergraph as it is in the following figures.

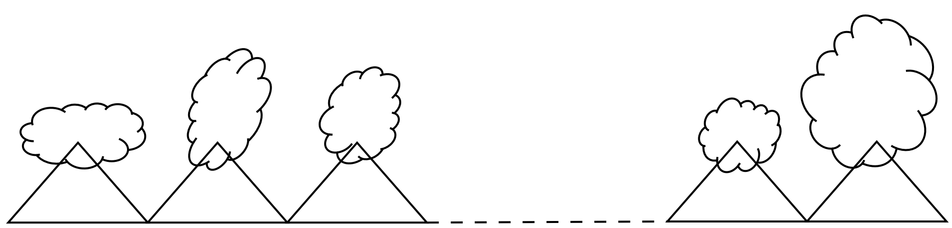


Figure 2: Mountain Ranges

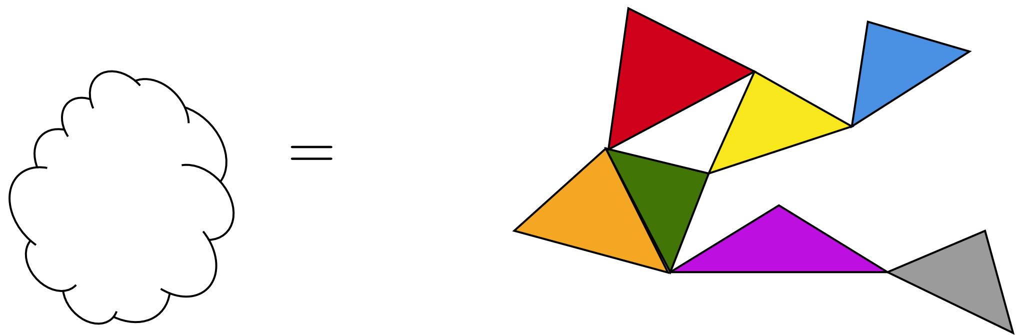


Figure 3: Continents

In Figure 2, we can imagine certain distinguished hyperedges, which we call *mountains*, forming a *mountain range*, where each mountain connects to an adjacent mountain at a vertex. At the top of the mountains are *continents*, which in the Figure 3 we can see represent an arbitrary conglomerate of hyperedges. Continents may intersect with each other, but the key is that the bases of the mountain range forms the given structure. What our theorem says is that if we assign weights to each vertex as in Figure 4, if we take any hyperedge in our hypergraph and add together the weights on each vertex, they sum to zero.

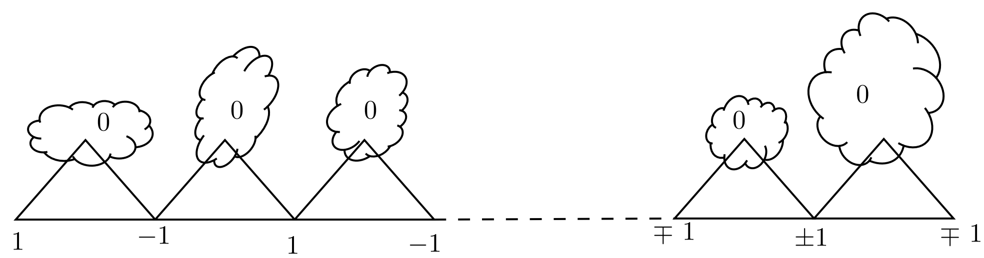


Figure 4: Weights on hyperedges in a mountain range

While clouds may intersect, it is relatively easy to check if this occurs. If we were to create a new hypergraph by deleting the mountain range, we can check for connected components.

Definition 3.1. Given a hypergraph $\mathcal{H} = (V, E)$, the *incidence graph* of $\mathcal{H}$ is a bipartite graph $\mathcal{G} = (V_V \bigcup V_E, E_H)$ constructed as follows:

- V_V are the vertices of $\mathcal{H}$.
- V_E are vertices corresponding to the hyperedges e of $\mathcal{H}$.

- There is an edge from a vertex $v' \in V_V$ to a vertex in $e \in V_E$ if the vertex v' in $\mathcal{H}$ is contained in the hyperedge e in $\mathcal{H}$.

Lemma 3.2. A hypergraph $\mathcal{H}$ is connected if and only if its incidence graph $\mathcal{G}$ is connected.

Proof. If a hypergraph is connected, then between any two vertices there is a path connecting them where the path follows adjacency through hyperedges. In the incidence graph, this same path can be followed. If vertex v_1 and v_2 are connected via shared adjacency in hyperedge e_1, on the incidence matrix v_1 and v_2 are both adjacent to e_1, and so a path connects them. Therefore, any path in $\mathcal{H}$ corresponds to a path in $\mathcal{G}$, and vice versa. $\square$

By this lemma, checking if a hypergraph is connected is equivalent to checking if a bipartite graph is connected.

For the labeling of X in our theorem, we refer to figures 5 and 6:

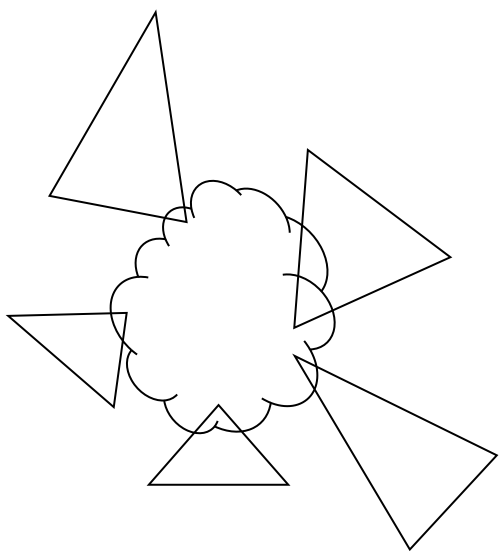


Figure 5: Peninsulas off the Mainland

In Figure 5, we have a central continent which has coming from it various *peninsulas*. Our theorem says that if we assign weights to each vertex as in Figure 6, on any hyperedge the sum of the weights on the vertices equals to zero.

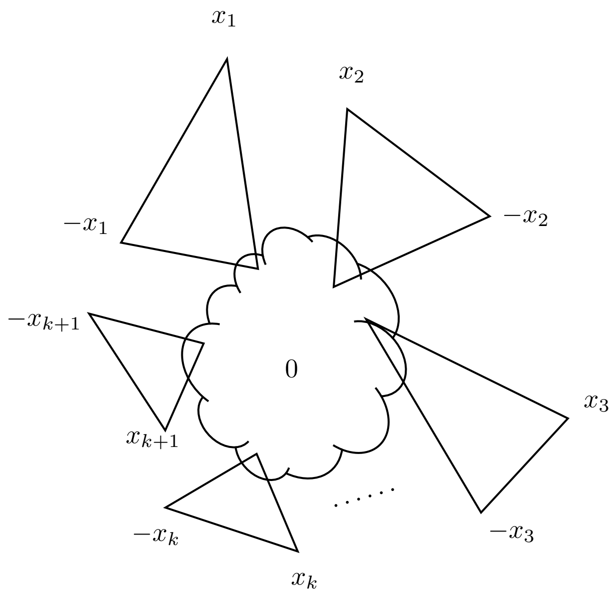


Figure 6: Weights on hyperedges in a continent with peninsulas

Corollary 3.3. With an appropriate relabeling to avoid unwanted overlaps, linear combinations of $(E_a(\overrightarrow{x_a}), E_a(\overrightarrow{x_a}), E_a(\overrightarrow{x_a}))'s$ and $(F_b, F_b, F_b)'s$ (with varying $a's$ and $b's$) form a derivation which is interpretable as a combination of continents, mountain ranges, and peninsulas.

Working with the graphical interpretations of derivations assigning weights to vertices, this corollary says that we can build out our hypergraph structure by isolating mountain ranges, continents, and peninsulas all glued together, and that there exists a derivation which will identify this process. Figure 7 shows what this may look like.

In this example, the corresponding derivation would be (D, D, D), where

$$D := \begin{bmatrix} X' & & & & & & & \\ & -x_3 & & & & & & \\ & & x_3 & & & & & \\ & & & -x_3 & & & & \\ & & & & -x_3 & & & \\ & & & & & Y' & & \\ & & & & & & Z' & \\ & & & & & & & 0 \end{bmatrix},$$

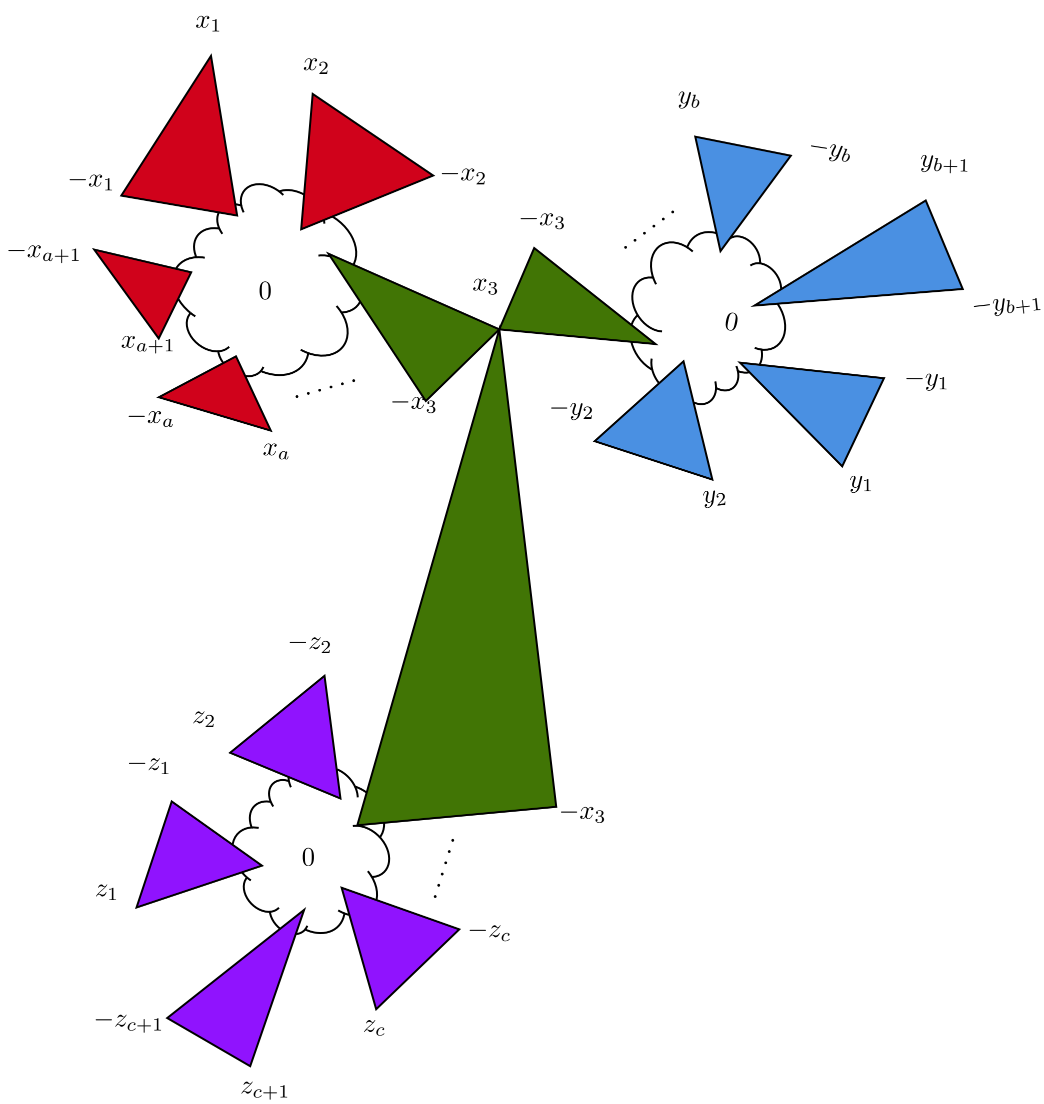


Figure 7: Combination of Continents, Mountain Ranges, and Peninsulas

$$X' := \begin{bmatrix} x_1 & & & & & & & \\ & -x_1 & & & & & & \\ & & x_2 & & & & & \\ & & & -x_2 & & & & \\ & & & & -x_4 & & & \\ & & & & & \ddots & & \\ & & & & & & x_{a+1} & \\ & & & & & & & -x_{a+1} \end{bmatrix}$$

$$Y' := \begin{bmatrix} y_1 & & & & & & & \\ & -y_1 & & & & & & \\ & & y_2 & & & & & \\ & & & -y_2 & & & & \\ & & & & -y_4 & & & \\ & & & & & \ddots & & \\ & & & & & & y_{b+1} & \\ & & & & & & & -y_{b+1} \end{bmatrix}$$

$$Z' := \begin{bmatrix} z_1 & & & & & & & \\ & -z_1 & & & & & & \\ & & z_2 & & & & & \\ & & & -z_2 & & & & \\ & & & & -z_4 & & & \\ & & & & & \ddots & & \\ & & & & & & z_{c+1} & \\ & & & & & & & -z_{c+1} \end{bmatrix}$$

When considering how continents and mountain ranges may be connected, we have thus far only considered trees. We must now consider loops.

Lemma 3.4. If a mountain range loops with $2n$ different mountains, then (after relabeling) F_{2n} is a derivation for the correspoding tensor. If the loop is constructed with $2n+1$ mountains, then after a relabeling $F_{2n}+(-1)^{2n+1}2E_{2n+2,2n+2}$ is a derivation for the corresponding tensor.

Proof. With our correspondence of elements of the Cartan subalgebra of the Derivation algebra and weights on a hypergraph, we can give the proof of this lemma with an argument on the weights of hyperedges. The following two pictures show this argument: (insert tikz image here). □

Finally, the third way in which we can get a derivation is by taking the sub-hypergraph $[\{1,2,3\},\{3,4,5\},\{5,6,7\},\{7,8,9\}]$, and give the vertex 4 a weight of 1, the vertex 5 a weight -1, and the vertex 6 a weight of 1. (this might be covered already by F_k, and my future argument sying mountain peaks connecting continents cannot have weight other than zero).

3.1 Generating Examples

There are two ways in which we can construct standard hypergraph structures from a graph. Both enlists the use of cones. In the first, we take an arbitrary graph $\mathcal{G}$ with vertex set V and edge set E. We then construct a hypergraph $\mathcal{H}$ with vertex set $V \cup \{\infty\}$ and edge set $E' := \{\{v_i, v_j, \infty\} \mid \{v_i, v_j\} \in E\}$. This is equivalent to drawing our graph $\mathcal{G}$, including a "point at infinity," and connecting each edge to this point at infinity.

When looking at our adjacency tensor, the evaluations are as follows

$$\Gamma(x, y, z) := \begin{cases} 1 & \{(x, y) \in E\} \wedge \{z = \infty\} \\ 0 & else \end{cases}, \tag{5}$$

where we account for symmetry in our evaluations. Once again, we will look for symmetric subalgebras of the Cartan subalgebra (diagonal matrices). For ease of our explanations, we will utilize our interpretation of weights on vertices, and that a triple of the same diagonal matrix are in the derivation algebra if and only if their corresponding weight assignments sum to zero on each hyperedge.

In this interpretation, the weight on ∞ is a_∞, and a triple of matrices is a derivation if and only if the weight assignments on the base graph $\mathcal{G}$ has the property that the sum of weights on adjacent vertices (adjacency being defined as lying on the same hyperedge) is zero. This holds if and only if $\mathcal{G}$ is bipartite.

Proof. Let a_∞ be the weight on ∞. Beginning with any pair of vertices which lie on a hyperedge with ∞, one can be assigned the weight x_i and may be given the label of 1. The other is necessarily assigned the weight of $-x_i - a_\infty$, and may be given the label of 2. If ∞ and 1 lie on a hyperedge with a third vertex 0 which is not 2 then 0 must be given a weight of $-x_i - a_\infty$. Equivalently, If ∞ and 2 lie on a hyperedge with a third vertex 3 which is not 1, then 3 must be given a weight of x_i. Furthermore, $\{1, 3, \infty\}$ cannot be a hyperedge, and neither can $\{0, 2, \infty\}$. This consturction provides us with a bipartite graph.

Likewise, if we were to begin with a bipartite 2-graph, then we can assign each side of the bipartie graph its own weight x_i and y_i, and then the point at infinity will receive a weight of $-x_i - y_i$. $\square$

In this proof, i is a number of the number of connected components in our graph $\mathcal{G}$, whcih will determine how many different weight-pairs we need. Our derivations are thus of the form $a_\infty E_{\infty,\infty} + x_i E_{11} + (-x_i - a_\infty)E_{22}$.

In the second way of constructing a hypergraph from a graph, we instead generate a cone for each edge in the graph $\mathcal{G}$. We do this by assigning a distinct point at infinity ∞_{xy} for each edge $(x, y) \in E$. In this scenario, we are less restricted in our weight assignment. We can assign any weight to the vertices in $\mathcal{G}$. All we require is that if a_x and a_y are weights of adjacent vertices x and y, the weight on ∞_{xy} is $-a_x - a_y$. This gives us derivations with matrices of the form $a_x E_{11} + b_y E_{22} + (-a_x - b_y)E_{33}$.

4 Future Work

For my final thesis, I will completing the following work. Most immediately, I will be verifying that the derivations which I have defined span all symmetric and diagonal derivations. Following this, I will be writing code to automate these calculations and interpretations. I will then collaborate with other mathematicians to use this new tool to study different hypergraph structures. My thesis will consist of both the theory underlying this tool set, as well as the resulting data analysis which will utilize the tool set.

An open question which I will also be studying is how these diagonal derivations interact with cumulant tensors. This will involve both interpreting the action of the derivations, as well providing the appropriate analogy to the associated random variables.