Basic Probability - 1

What is Probability?

A probability is the **numeric value** representing the **chance, likelihood, or possibility** that a particular event will occur.

The **probability of an event A**, denoted by **P(A)**, is a number between 0 and 1 that represents the likelihood of A occurring.

- If $P(A)$ = 0, the event A is **impossible**.
- If $P(A)$ = 1, the event A is **certain** to occur.
- If

  $$0 \leq P(A) \leq 1$$

  the event A is possible but not guaranteed.

Comparison of Probability Types

Aspect	A Priori Probability	Empirical Probability	Subjective Probability
Definition	Based on prior knowledge or theoretical reasoning	Based on observed data from experiments or historical records	Based on personal judgment, intuition, or belief
Example	Probability of rolling a 4 on a fair die = 1/6	After flipping a coin 100 times and getting 53 heads, P(heads) = 0.53	An expert estimates 70% chance of rain based on experience

$\boxed{\text{Examples:}}$

Categorize These Scenarios into Probability Types

1. A teacher flipped a coin 200 times in class and observed 112 heads. She calculated the probability of getting heads as 112/200 = 56%.

2. A financial analyst reviews a company's performance, market trends, and economic conditions, then estimates there's a 75% chance the stock price will increase next quarter.

3. When rolling a standard six-sided die, you determine that the probability of rolling a number greater than 4 is 2/6

$\boxed{Solutions:}$

1. Empirical probability
2. Subjective probability
3. A prior probability

Formula

$$P(occurrence) = \frac{\text{Number of favourable outcomes}}{\text{Total Number of outcomes}}$$

$\boxed{Example:}$

Standard deck of cards
26 red cards, 26 black cards (52 total)

$$P(blackcard) = \frac{26}{52} = 0.50$$

Key definitions

- **Sample space**: Collection of all possible events
- **Event**: Each possible outcome of a variable
- **Simple event**: Described by a single characteristic
- **Joint event**: Has two or more characteristics
- **Complement of A (A')**: All events that are not part of A

$\boxed{Example:}$

Rolling a Die 🎲

- Sample Space
  All possible outcomes when you roll a die:

  Sample Space = {1, 2, 3, 4, 5, 6}
- Event
  Any outcome or group of outcomes
  - Event A = Getting an even number = {2, 4, 6}
  - Event B = Getting a number greater than 4 = {5, 6}
- Simple Event
  Described by ONE characteristic only

    - Simple Event 1 = Rolling a 3 = {3}
    - Simple Event 2 = Rolling a 5 = {5}
    - Simple Event 3 = Getting an even number = {2, 4, 6}
- Joint Event
  Has TWO or MORE characteristics (must satisfy multiple conditions)

  Joint Event = Getting an even number AND greater than 3
    - Even numbers: {2, 4, 6}
    - Greater than 3: {4, 5, 6}
    - **Both conditions:** {4, 6}
      So the joint event = {4, 6}
- Complement of A (A′)
  Everything that is NOT in event A

  If Event A = Getting an even number = {2, 4, 6}
  Then A′ = NOT getting an even number = {1, 3, 5}

  Notice: A + A′ = {2, 4, 6} + {1, 3, 5} = {1, 2, 3, 4, 5, 6} = Sample Space ✓

Question 1: Basic Probability

Scenario: A box contains 8 red balls and 6 blue balls. Two balls are drawn **without replacement**.

Tasks:

1. Probability both are red
2. Probability first blue, second red
3. Probability **at least one blue**

Solution:

1. **Both red**

$$P(\text{Red}_1 \cap \text{Red}_2) = \frac{8}{14} \times \frac{7}{13} = \frac{56}{182} \approx 0.3077$$

2. **First blue, second red**

$$P(\text{Blue}_1 \cap \text{Red}_2) = \frac{6}{14} \times \frac{8}{13} = \frac{48}{182} \approx 0.2637$$

3. **At least one blue**

$$P(\text{At least one blue}) = 1 - P(\text{No blue}) = 1 - P(\text{Both red}) = 1 - \frac{56}{182} \approx 0.6923$$

Venn Diagrams & Set Notations

Set Notations

Set notation is used in mathematics to essentially list numbers, objects or outcomes.

Basic Set Symbols

Notation	Meaning	Example
{ }	Set brackets	A = {1, 2, 3}
S or U or Ω	Sample space / Universal set (all possible elements)	P(Ω) = 1
∈	"Is an element of" / "belongs to"	2 ∈ A means "2 is in set A"

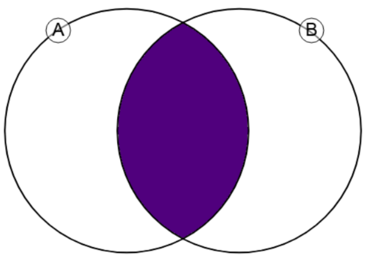
Notation	Meaning	Example
∉	"Is not an element of"	5 ∉ A means "5 is not in set A"
∅ or { }	Empty set (no elements)	B = ∅

Set Operations, Formulas and Venn Diagrams

1. Intersection

Notation	Name	Meaning
A ∩ B	AND	Elements in BOTH A and B (overlap)

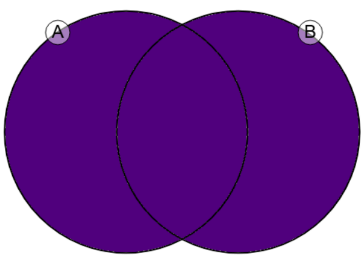
Venn Diagram



2. Union

Notation	Name	Meaning
A ∪ B	OR	Elements in A or B or both (total area)

Venn Diagram



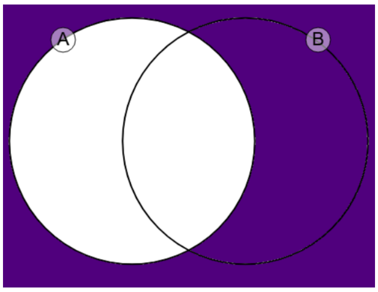
Formula (General Addition Rule)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

3. Complement

Notation	Name	Meaning
A′ or Aᶜ	NOT A	Elements NOT in A

Venn Diagram



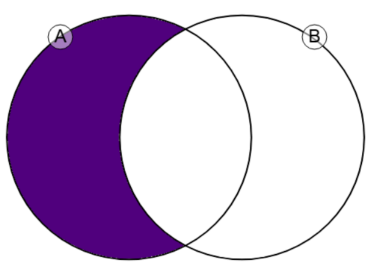
Formula

$$P(A') = 1 - P(A)$$

4. Only A

Notation	Name	Meaning
A − B or A ∩ B′	A but not B	Elements in A but NOT in B

Venn Diagram



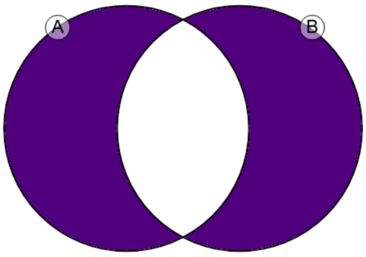
Formula

$$P(A - B) = P(A) - P(A \cap B)$$

5.Symmetric Difference

Notation	Name	Meaning
A Δ B or A ⊕ B	XOR	Elements in A or B but NOT both

Venn Diagram



Formula

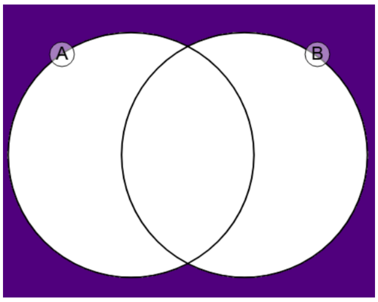
$$P(A \Delta B) = P(A) + P(B) - 2P(A \cap B)$$

$$P(A \Delta B) = P(A \cup B) - P(A \cap B)$$

6.Neither

Notation	Name	Meaning
(A ∪ B)′ or A′ ∩ B′	NOT A and NOT B	Elements in neither A nor B

Venn Diagram



Formula

$$P(A \cup B)' = 1 - P(A \cup B)$$

$\boxed{Example:}$

- **Universal Set S** = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

- **A** = {2, 4, 6, 8}
- **B** = {6, 7, 8, 9}

Operation	Notation	Result	Elements
Intersection	A ∩ B	Overlap	{6, 8}
Union	A ∪ B	Combined	{2, 4, 6, 7, 8, 9}
Complement of A	A′	Not in A	{1, 3, 5, 7, 9, 10}
Complement of B	B′	Not in B	{1, 2, 3, 4, 5, 10}
Only A	A − B or A ∩ B′	A but not B	{2, 4}
Only B	B − A or B ∩ A′	B but not A	{7, 9}
Symmetric Difference	A Δ B	Either but not both	{2, 4, 7, 9}
Neither	(A ∪ B)′ or A′ ∩ B′	Outside both	{1, 3, 5, 10}

Question 2: Venn Diagrams & Set Operations

Scenario:

Universal set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Define two sets:

$$A = \{2, 4, 6, 8, 10\}, \quad B = \{5, 6, 7, 8, 9\}$$

Tasks:

1. Find $A \cap B$
2. Find $A \cup B$
3. Find $A \Delta B$
4. Find $A - B$
5. Find $(A \cup B)'$
6. Probability that a randomly selected number from S is in **exactly one of the sets**

Solution:

1. **Intersection $A \cap B$**

$$A \cap B = \{6, 8\}$$

2. **Union $A \cup B$**

$$A \cup B = \{2, 4, 5, 6, 7, 8, 9, 10\}$$

3. **Symmetric Difference $A \Delta B$**

$$A \Delta B = (A \cup B) - (A \cap B) = \{2, 4, 5, 7, 9, 10\}$$

4. **Only A $A - B$**

$$A - B = A \cap B' = \{2, 4, 10\}$$

5. **Neither $(A \cup B)'$**

$$(A \cup B)' = S - (A \cup B) = \{1, 3, 11, 12\}$$

6. **Probability of exactly one set**

- Exactly one = Only A + Only B
- Only B = B - A = {5, 7, 9}
- Only A = {2, 4, 10}

Number of elements in exactly one = 3 + 3 = 6

Total elements in S = 12

$$P(\text{Exactly one}) = \frac{6}{12} = 0.5$$

Question 3: Venn Diagrams & Set Operations

Scenario:

In a class of 120 students:

- 50 students take **Mathematics** (A)
- 40 students take **Biology** (B)
- 20 students take **both Mathematics and Biology** $(A \cap B)$

Tasks:

1. Find the probability that a student takes **Mathematics or Biology** $(A \cup B)$ using the **union formula**.
2. Find the probability that a student takes **Mathematics but not Biology** $(A - B)$ using the **difference formula**.
3. Find the probability that a student takes **exactly one subject** $(A \Delta B)$ using the **symmetric difference formula**.
4. Find the probability that a student takes **neither subject** $((A \cup B)')$ using the **complement formula**.

Solution:

1. **Drama or Science (Union formula)**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A) = \frac{40}{100} = 0.4, \quad P(B) = \frac{30}{100} = 0.3, \quad P(A \cap B) = \frac{10}{100} = 0.1$$

$$P(A \cup B) = 0.4 + 0.3 - 0.1 = 0.6$$

---

2. **Drama but not Science (Difference formula)**

$$P(A - B) = P(A) - P(A \cap B) = 0.4 - 0.1 = 0.3$$

---

3. **Exactly one club (Symmetric Difference formula)**

$$P(A \Delta B) = P(A) + P(B) - 2P(A \cap B) = 0.4 + 0.3 - 2(0.1) = 0.5$$

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4. **Neither club (Complement of union formula)**

$$P((A \cup B)') = 1 - P(A \cup B) = 1 - 0.6 = 0.4$$

Probability Types

Contingency Table: Customer Behavior

	Purchased	Did Not Purchase	Total
Planned	200	50	250
Did Not Plan	100	650	750
Total	300	700	1000

1. Simple (Marginal) Probability

Definition: Probability of a single event occurring (found in margins/totals of table)

Example:

$$
\begin{aligned}
P(\text{Planned}) &= \frac{250}{1000} = 0.25 \\
P(\text{Did Not Plan}) &= \frac{750}{1000} = 0.75 \\
P(\text{Purchased}) &= \frac{300}{1000} = 0.30 \\
P(\text{Did Not Purchase}) &= \frac{700}{1000} = 0.70
\end{aligned}
$$

2. Joint Probability

Definition: Probability of TWO events occurring together (at the same time)

Notation: $P(A \cap B)$ or $P(A \text{ AND } B)$

Formula:

$$P(A \cap B) = \frac{\text{Frequency of both A and B}}{\text{Total}}$$

Example:

$$
\begin{aligned}
P(\text{Planned} \cap \text{Purchased}) &= \frac{200}{1000} = 0.20 \\
P(\text{Planned} \cap \text{Did Not Purchase}) &= \frac{50}{1000} = 0.05 \\
P(\text{Did Not Plan} \cap \text{Purchased}) &= \frac{100}{1000} = 0.10 \\
P(\text{Did Not Plan} \cap \text{Did Not Purchase}) &= \frac{650}{1000} = 0.65
\end{aligned}
$$

3. Marginal Probability (Using Joint Probabilities)

Definition: Total probability of an event by summing all its joint probabilities

Formula:

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) + \cdots + P(A \cap B_k)$$

Example:

$$
\begin{aligned}
P(\text{Planned}) &= P(\text{Planned} \cap \text{Purchased}) + P(\text{Planned} \cap \text{Did Not Purchase}) \\
&= 0.20 + 0.05 = 0.25 \quad \checkmark
\end{aligned}
$$

$$
\begin{aligned}
P(\text{Purchased}) &= P(\text{Planned} \cap \text{Purchased}) + P(\text{Did Not Plan} \cap \text{Purchased}) \\
&= 0.20 + 0.10 = 0.30 \quad \checkmark
\end{aligned}
$$

Question 4: Probability Types & Contingency Table

Scenario:

A store recorded the purchasing behavior of 1000 customers:

	Purchased	Did Not Purchase	Total
Planned	200	50	250
Did Not Plan	100	650	750
Total	300	700	1000

Tasks:

1. Find the **simple probability** of a customer being:

   a) Planned

   b) Purchased

2. Find the **joint probability** of:

   a) Planned **and** Purchased

   b) Did Not Plan **and** Did Not Purchase

3. Verify a **marginal probability** using joint probabilities:

   a) $P(\text{Purchased})$

   b) $P(\text{Planned})$

4. Find the **conditional probability** that a customer **purchased** given that they **planned**.
5. Determine if **planning** and **purchasing** are **independent events**.

Solution:

1. **Simple (Marginal) Probabilities**

$$P(\text{Planned}) = \frac{250}{1000} = 0.25$$

$$P(\text{Purchased}) = \frac{300}{1000} = 0.30$$

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2. **Joint Probabilities**

$$P(\text{Planned} \cap \text{Purchased}) = \frac{200}{1000} = 0.20$$

$$P(\text{Did Not Plan} \cap \text{Did Not Purchase}) = \frac{650}{1000} = 0.65$$

---

3. **Verify Marginal Probabilities using Joint Probabilities**

$$P(\text{Purchased}) = P(\text{Planned} \cap \text{Purchased}) + P(\text{Did Not Plan} \cap \text{Purchased}) = 0.20 + 0.10 = 0.30$$

$$P(\text{Planned}) = P(\text{Planned} \cap \text{Purchased}) + P(\text{Planned} \cap \text{Did Not Purchase}) = 0.20 + 0.05 = 0.25$$

---

4. **Conditional Probability**

$$P(\text{Purchased} \mid \text{Planned}) = \frac{P(\text{Planned} \cap \text{Purchased})}{P(\text{Planned})} = \frac{0.20}{0.25} = 0.8$$

5. **Check Independence**

Events A = Planned, B = Purchased

- If independent: $P(A \cap B) = P(A) \cdot P(B)$

$$P(A) \cdot P(B) = 0.25 \cdot 0.30 = 0.075$$

$$P(A \cap B) = 0.20 \neq 0.075$$

✅ **Conclusion:** Not independent

Important Concepts

Mutually Exclusive Events

Definition: Two events that CANNOT happen at the same time

Characteristic:

$$\text{If A and B are mutually exclusive:}$$

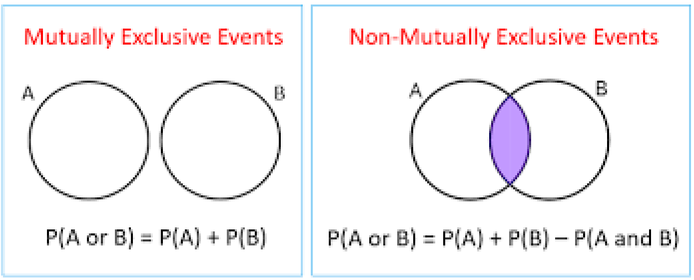
$$P(A \cap B) = 0$$

$$P(A \cup B) = P(A) + P(B)$$

Examples:

- Coin flip: Getting heads AND tails (mutually exclusive)
- Student status: Being a freshman AND being a senior (mutually exclusive)

Visual:



Collectively Exhaustive Events

Definition: Events that cover ALL possible outcomes (nothing left out)

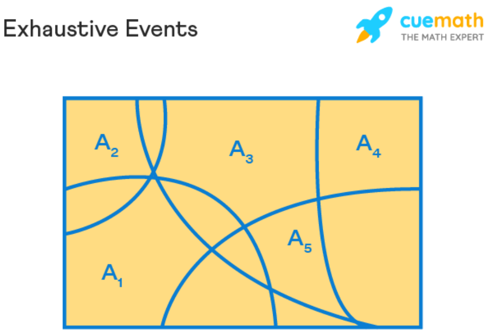
Characteristic:

$$\text{If } B_1, B_2, \ldots, B_k \text{ are collectively exhaustive:}$$
$$P(B_1) + P(B_2) + \cdots + P(B_k) = 1$$

Examples:

- {Heads, Tails} when flipping a coin - exhaustive
- {Pass, Fail} for a test - exhaustive
- {Planned, Did Not Plan} - exhaustive

Visual:



Mutually Exclusive AND Collectively Exhaustive

When BOTH conditions apply:

Events that:

1. Don't overlap (mutually exclusive)
2. Cover everything (collectively exhaustive)

Characteristic:

$$P(B_1 \cap B_2 \cap B_3 \cap \ldots \cap B_k) = 0 \quad \text{(no overlap)}$$
$$P(B_1) + P(B_2) + \cdots + P(B_k) = 1 \quad \text{(covers everything)}$$

Example:

{Planned, Did Not Plan}

- Mutually exclusive: Can't be both ✓
- Collectively exhaustive: Everyone is one or the other ✓

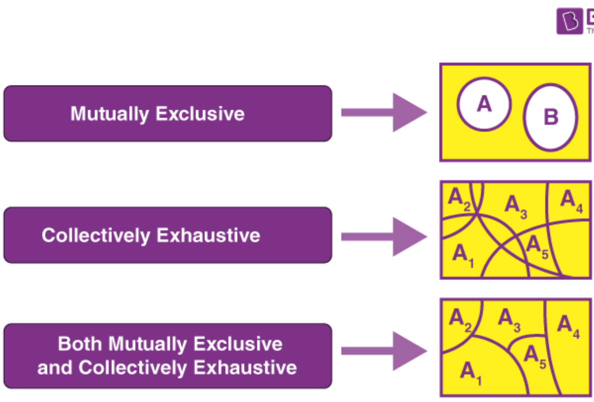
Why important for marginal probability:

When $B_1, B_2, \ldots, B_k$ are mutually exclusive and collectively exhaustive:

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_k)$$

This is guaranteed to give the total probability of A

Visual:



$\boxed{Exercise:}$

Given Table:

	Pass	Fail	Total
Studied	70	10	80
Didn't Study	15	5	20
Total	85	15	100

Questions:

1. Find $P(\text{Studied})$ - Simple probability
2. Find $P(\text{Pass})$ - Simple probability
3. Find $P(\text{Studied} \cap \text{Pass})$ - Joint probability
4. Find $P(\text{Pass})$ using joint probabilities
5. Are {Studied, Didn't Study} mutually exclusive?
6. Are {Pass, Fail} collectively exhaustive?

$\boxed{Answers:}$

1. $P(\text{Studied})$= $\frac{80}{100} = 0.80$

2. $P(\text{Pass})$ = $\frac{85}{100} = 0.85$

3. $P(\text{Studied} \cap \text{Pass})$= $\frac{70}{100} = 0.70$

4. $P(\text{Pass})$ using joints:

$$\begin{aligned} P(\text{Pass}) &= P(\text{Studied} \cap \text{Pass}) + P(\text{Didn't Study} \cap \text{Pass}) \\ &= \frac{70}{100} + \frac{15}{100} \\ &= \frac{85}{100} = 0.85 \quad \checkmark \end{aligned}$$

5. Yes, mutually exclusive - A student cannot both study and not study

$$P(\text{Studied} \cap \text{Didn't Study}) = 0$$

6. Yes, collectively exhaustive - Every student either passed or failed

$$P(\text{Pass}) + P(\text{Fail}) = 0.85 + 0.15 = 1.00 \quad \checkmark$$

Question 5: Mutually Exclusive & Collectively Exhaustive Events

Scenario:

A class of 100 students wrote a test. Their study habits and results were recorded:

	Pass	Fail	Total
Studied	70	10	80
Didn't Study	15	5	20
Total	85	15	100

Tasks:

1. Find the **simple probability** that a student studied.
2. Find the **simple probability** that a student passed.
3. Find the **joint probability** that a student both studied **and** passed.
4. Verify $P(\text{Pass})$ using **joint probabilities**.
5. Are the events **{Studied, Didn't Study}** mutually exclusive?
6. Are the events **{Pass, Fail}** collectively exhaustive?
7. Find the **conditional probability** that a student passed **given that they studied**.
8. Are **Studying** and **Passing** independent events?

Solution:

1. **Simple Probabilities**

$$P(\text{Studied}) = \frac{80}{100} = 0.80$$

$$P(\text{Pass}) = \frac{85}{100} = 0.85$$

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2. **Joint Probability**

$$P(\text{Studied} \cap \text{Pass}) = \frac{70}{100} = 0.70$$

---

3. **Verify $P(\text{Pass})$ using Joint Probabilities**

$$P(\text{Pass}) = P(\text{Studied} \cap \text{Pass}) + P(\text{Didn't Study} \cap \text{Pass}) = \frac{70}{100} + \frac{15}{100} = 0.85$$

---

4. **Check Mutually Exclusive**

Events = Studied (S), Didn't Study (S')

$$P(S \cap S\prime) = 0$$

✅ **Yes, mutually exclusive**

5. **Check Collectively Exhaustive**

Events = Pass (P), Fail (F)

$$P(P) + P(F) = 0.85 + 0.15 = 1$$

✅ **Yes, collectively exhaustive**

---

6. **Conditional Probability**

$$P(\text{Pass} \mid \text{Studied}) = \frac{P(\text{Studied} \cap \text{Pass})}{P(\text{Studied})} = \frac{0.70}{0.80} = 0.875$$

---

7. **Check Independence**

Events = Studied (S), Passed (P)

$$P(S) \cdot P(P) = 0.80 \cdot 0.85 = 0.68$$

$$P(S \cap P) = 0.70 \neq 0.68$$

❌ **Not independent**

Conditional Probability

Definition:

Conditional probability is the probability of an event occurring given that another event has already occurred.

Notation

$P(A|B)$ reads as "probability of A given B"

Formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Extra Formulas

$$P(A'|B) = 1 - P(A|B)$$

$$P(B'|A) = 1 - P(B|A)$$

Key Properties

Multiplication Rule:

Using $P(A \mid B)$	**Using** $P(B \mid A)$
$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$	$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$
$P(A \mid B)\, P(B) = P(A \cap B)$	$P(B \mid A)\, P(A) = P(A \cap B)$
$P(A \cap B) = P(A \mid B)\, P(B)$	$P(A \cap B) = P(B \mid A)\, P(A)$

$$P(A \cap B) = P(A|B) \times P(B) = P(B|A) \times P(A)$$

Law of Total Probability:

If $B_1, B_2, \ldots, B_n$ form a partition of the sample space:

$$P(A) = \Sigma P(A|B_i) \times P(B_i)$$

Independence:

Independent means the outcome of one event **has no effect** on the other.

Events A and B are independent if:

$$P(A|B) = P(A)$$

Knowing B occurred doesn't change the probability of A.

Alternative test for independence:

$$P(A \cap B) = P(A) \times P(B)$$

$$P(A \cap B' \cap C) = P(A) \times P(B') \times P(C)$$

Question 6: Conditional Probability & Independence

Scenario:

A store records customer behavior over 1000 transactions:

	Bought Item	Did Not Buy	Total
Promotional Ad	180	120	300
No Ad	220	480	700
Total	400	600	1000

Tasks:

1. Find the **conditional probability** that a customer **bought an item given they saw the promotional ad**.
2. Find the **conditional probability** that a customer **did not buy given they did not see the ad**.
3. Find the **probability that a customer bought an item AND saw the ad**.
4. Find the **probability that a customer bought an item OR saw the ad**.
5. Determine if **Buying** and **Seeing the Ad** are **independent events**.
6. Find $P(\text{Did Not Buy} \mid \text{Saw Ad})$.
7. Verify using the **Multiplication Rule**: $P(A \cap B) = P(A|B) \cdot P(B)$ for Buying and Ad.

Solution

1. **Conditional Probability – Bought given Saw Ad**

$$P(\text{Bought} \mid \text{Ad}) = \frac{P(\text{Bought} \cap \text{Ad})}{P(\text{Ad})} = \frac{180/1000}{300/1000} = \frac{180}{300} = 0.6$$

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2. **Conditional Probability – Did Not Buy given No Ad**

$$P(\text{Did Not Buy} \mid \text{No Ad}) = \frac{P(\text{Did Not Buy} \cap \text{No Ad})}{P(\text{No Ad})} = \frac{480}{700} \approx 0.6857$$

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3. **Joint Probability – Bought AND Saw Ad**

$$P(\text{Bought} \cap \text{Ad}) = \frac{180}{1000} = 0.18$$

---

4. **Probability – Bought OR Saw Ad**

Use general addition formula:

$$P(\text{Bought} \cup \text{Ad}) = P(\text{Bought}) + P(\text{Ad}) - P(\text{Bought} \cap \text{Ad}) = \frac{400}{1000} + \frac{300}{1000} - \frac{180}{1000} = \frac{520}{1000} = 0.52$$

---

5. **Check Independence – Buying and Ad**

$$P(\text{Bought}) \cdot P(\text{Ad}) = \frac{400}{1000} \cdot \frac{300}{1000} = 0.4 \cdot 0.3 = 0.12$$

$$P(\text{Bought} \cap \text{Ad}) = 0.18 \neq 0.12$$

❌ Not independent

---

6. **Conditional Probability – Did Not Buy given Saw Ad**

$$P(\text{Did Not Buy} \mid \text{Ad}) = 1 - P(\text{Bought} \mid \text{Ad}) = 1 - 0.6 = 0.4$$

---

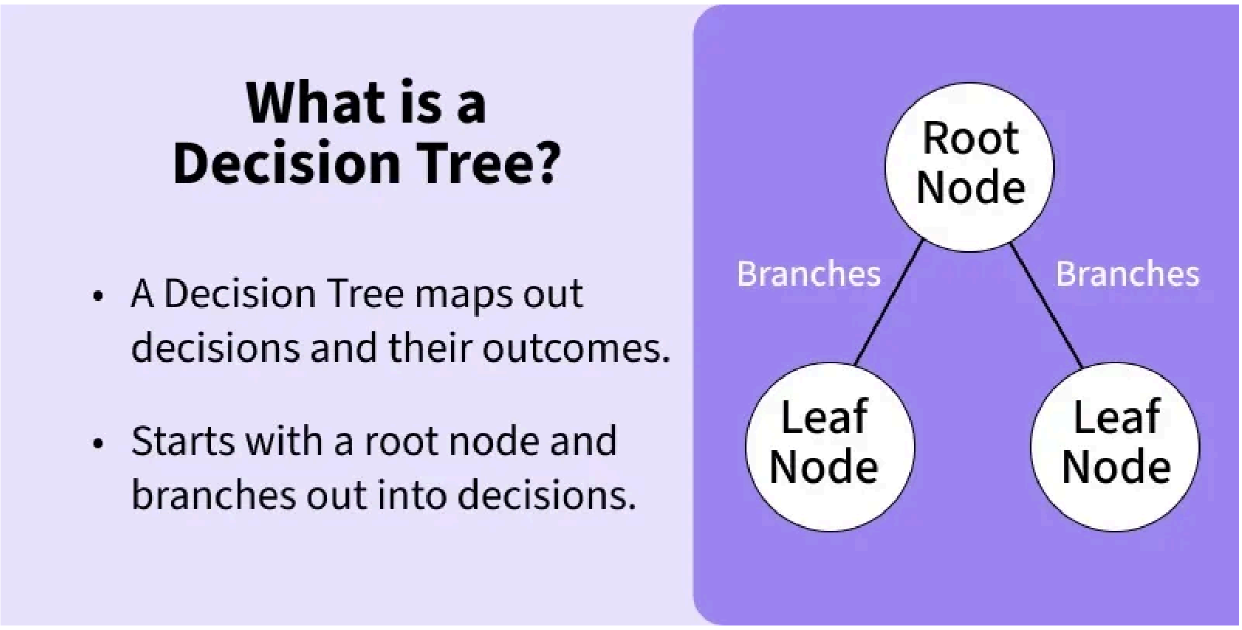
7. **Verify Multiplication Rule**

$$P(\text{Bought} \cap \text{Ad}) = P(\text{Bought} \mid \text{Ad}) \cdot P(\text{Ad}) = 0.6 \cdot 0.3 = 0.18$$

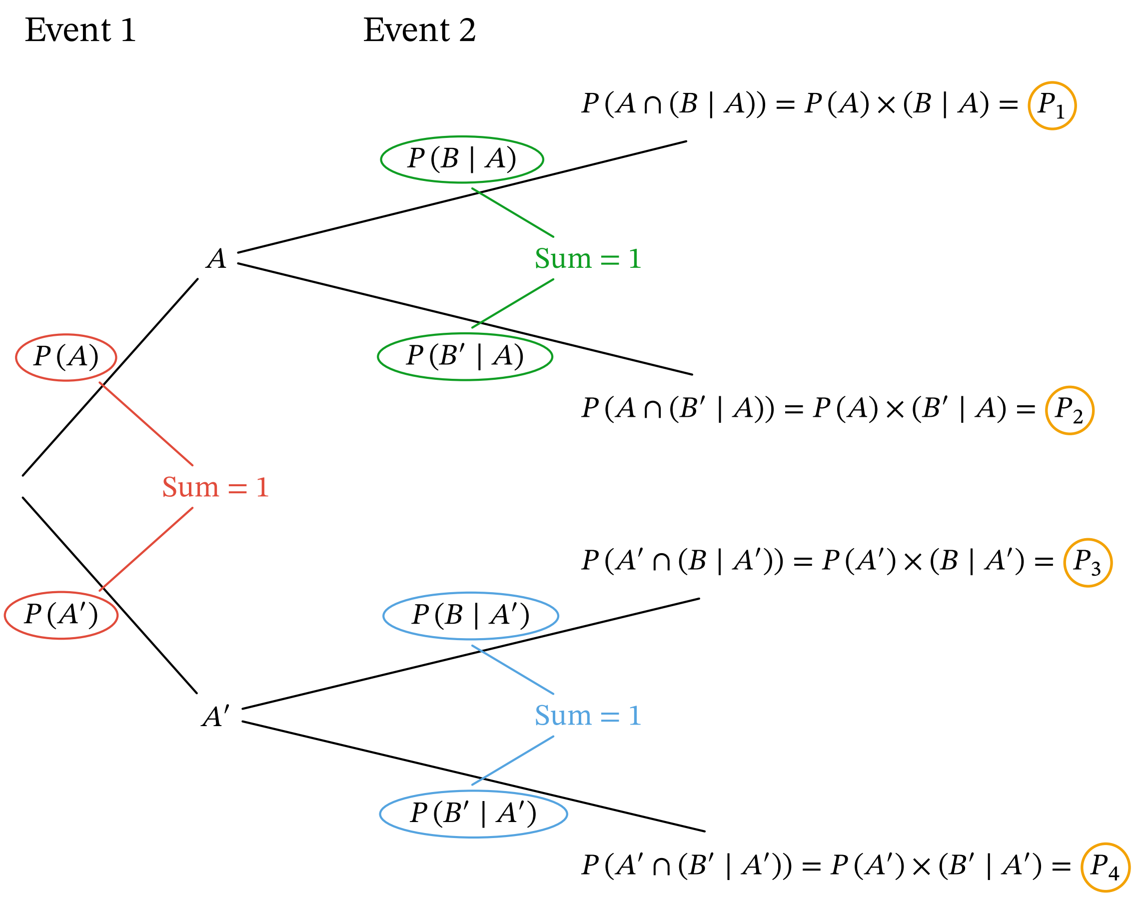
✅ Verified

Decision Tree

Definition and Structure



Alternative to contingency tables for visualizing probabilities



$$\boxed{P_1 + P_2 + P_3 + P_4 = 1}$$

Key Principles

Multiplication Rule (Along Branches)

To find the probability of a sequence of events, **multiply** the probabilities along the path:

$$P(A \cap B) = P(A) \times P(B|A)$$

Addition Rule (Across Branches)

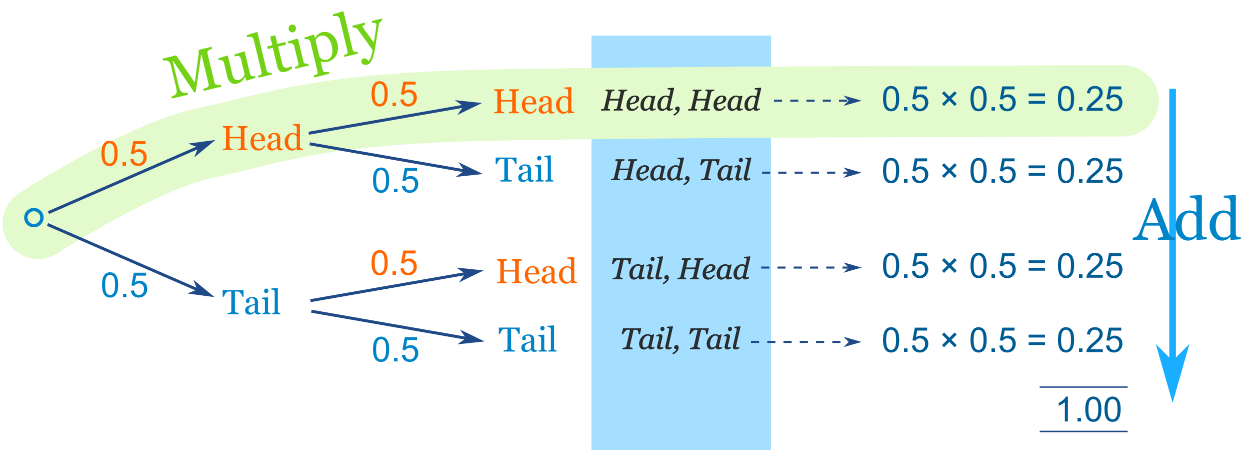
To find the probability of multiple paths leading to the same outcome, **add** the probabilities:

$$P(Outcome) = \Sigma P(\text{path to outcome})$$

Total Probability Must Equal 1

At each branch point, all probabilities leaving that node must sum to 1:

$$P(A) + P(A') = 1$$



Question 7: Decision Tree – Probability Paths

Scenario:

A company sells two types of products: **Gadgets (G)** and **Accessories (A)**.

- 60% of customers buy Gadgets; 40% buy Accessories.
- Among Gadget buyers: 30% buy the extended warranty.
- Among Accessory buyers: 20% buy the extended warranty.

Tasks:

1. Draw a **decision tree** showing the paths for buying Gadgets or Accessories and buying or not buying the warranty.
2. Find the probability that a customer **buys a Gadget AND purchases the extended warranty**.
3. Find the probability that a customer **buys an Accessory AND purchases the extended warranty**.
4. Find the probability that a customer **buys the extended warranty (any product)**.
5. Verify that the sum of all final path probabilities = 1.
6. Find the probability that a customer **buys a Gadget given that they purchased the extended warranty**.

Solution

1. **Decision Tree Setup**

- **First Branch:** Product type
  - $P(G) = 0.6$
  - $P(A) = 0.4$
- **Second Branch:** Warranty purchase
  - Gadget: $P(\text{Warranty}|G) = 0.3$, $P(\text{No Warranty}|G) = 0.7$
  - Accessory: $P(\text{Warranty}|A) = 0.2$, $P(\text{No Warranty}|A) = 0.8$

---

2. **Probability – Gadget AND Warranty**

$$P(G \cap \text{Warranty}) = P(G) \times P(\text{Warranty}|G) = 0.6 \times 0.3 = 0.18$$

---

3. **Probability – Accessory AND Warranty**

$$P(A \cap \text{Warranty}) = P(A) \times P(\text{Warranty}|A) = 0.4 \times 0.2 = 0.08$$

---

4. **Probability – Any Warranty Purchase**

Use **Addition Rule across branches:**

$$P(\text{Warranty}) = P(G \cap \text{Warranty}) + P(A \cap \text{Warranty}) = 0.18 + 0.08 = 0.26$$

5. **Verify Sum of All Paths = 1**

Compute **all four path probabilities**:

- G + Warranty = 0.18
- G + No Warranty = 0.6 × 0.7 = 0.42
- A + Warranty = 0.08
- A + No Warranty = 0.4 × 0.8 = 0.32

Sum: 0.18 + 0.42 + 0.08 + 0.32 = 1 ✅

---

6. **Conditional Probability – Gadget given Warranty**

$$P(G \mid \text{Warranty}) = \frac{P(G \cap \text{Warranty})}{P(\text{Warranty})} = \frac{0.18}{0.26} \approx 0.692$$

Bayes' Theorem

Purpose

describes how to update our beliefs about the likelihood of an event based on new evidence.

Basic Formula

For two events A and B, Bayes' Theorem states:

$$\text{Posterior} = \frac{\text{Likelihood} \times \text{Prior}}{\text{Evidence}}$$

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}$$

Where:

- **P(A|B)** = Posterior probability (probability of A given that B has occurred)
- **P(B|A)** = Likelihood (probability of B given that A has occurred)
- **P(A)** = Prior probability (initial probability of A)
- **P(B)** = Marginal probability (total probability of B)

Derivation of Bayes' Theorem from Conditional Probability

$$\begin{array}{c|c} \textbf{Using } P(A \mid B) & \textbf{Using } P(B \mid A) \\ \hline P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} & P(B \mid A) = \dfrac{P(A \cap B)}{P(A)} \\ \\ P(A \cap B) = P(A \mid B)\, P(B) & P(A \cap B) = P(B \mid A)\, P(A) \end{array}$$

$$P(A \mid B)\, P(B) = P(B \mid A)\, P(A)$$

$$\frac{P(A \mid B)\, P(B)}{P(B)} = \frac{P(B \mid A)\, P(A)}{P(B)}$$

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)}$$

Extended Form

When event B can occur with either A or not-A (denoted A'):

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B|A) \times P(A) + P(B|A') \times P(A')}$$

Also:

where $B_1, B_2, \ldots, B_k$ are mutually exclusive and collectively exhaustive

$$P(B_i|A) = \frac{P(A|B_i) \times P(B_i)}{P(A|B_1) \times P(B_1) + P(A|B_2) \times P(B_2) + \ldots + P(A|B_k) \times P(B_k)}$$

Examples

$\boxed{Example 1:}$

Problem: A disease affects 1% of the population. A test for the disease is 95% accurate (correctly identifies 95% of sick people and correctly identifies 95% of healthy people). If you test positive, what's the probability you actually have the disease?

Given:

$$\begin{aligned}
&P(\text{Disease}) = 0.01 \\
&P(\text{Healthy}) = 0.99 \\
&P(\text{Positive}|\text{Disease}) = 0.95 \\
&P(\text{Positive}|\text{Healthy}) = 0.05 \text{ (false positive rate)}
\end{aligned}$$

$\boxed{Solution 1:}$

$$\begin{aligned}
P(\text{Disease}|\text{Positive}) &= \frac{P(\text{Positive}|\text{Disease}) \cdot P(\text{Disease})}{P(\text{Positive}|\text{Disease}) \cdot P(\text{Disease}) + P(\text{Positive}|\text{Healthy}) \cdot P(\text{Healthy})} \\
&= \frac{0.95 \times 0.01}{(0.95 \times 0.01) + (0.05 \times 0.99)} \\
&= \frac{0.0095}{0.0095 + 0.0495} \\
&= \frac{0.0095}{0.059} \\
&\approx 0.161 \text{ or } 16.1\%
\end{aligned}$$

$\boxed{Example 2:}$

Problem: 60% of emails are spam. The word "free" appears in 80% of spam emails but only 10% of legitimate emails. If an email contains the word "free," what's the probability it's spam?

Given:

$$\begin{aligned}
P(\text{Spam}) &= 0.60 \\
P(\text{Legitimate}) &= 0.40 \\
P(\text{"free"}|\text{Spam}) &= 0.80 \\
P(\text{"free"}|\text{Legitimate}) &= 0.10
\end{aligned}$$

$\boxed{Solution 2:}$

$$P(\text{Spam}|\text{"free"}) = \frac{P(\text{"free"}|\text{Spam}) \times P(\text{Spam})}{P(\text{"free"}|\text{Spam}) \times P(\text{Spam}) + P(\text{"free"}|\text{Legitimate}) \times P(\text{Legitimate})}$$

$$= \frac{0.80 \times 0.60}{(0.80 \times 0.60) + (0.10 \times 0.40)}$$

$$= \frac{0.48}{0.48 + 0.04}$$

$$= \frac{0.48}{0.52}$$

$$\approx 0.923 \text{ or } 92.3\%$$

$\boxed{Example 3:}$

Problem: A factory has three machines (A, B, C) that produce 30%, 45%, and 25% of the total output respectively. The defect rates are 2%, 3%, and 4% for machines A, B, and C. If a randomly selected item is defective, what's the probability it came from machine B?

Given:

$$\begin{aligned} P(A) &= 0.30 \\ P(B) &= 0.45 \\ P(C) &= 0.25 \\ P(\text{Defective}|A) &= 0.02 \\ P(\text{Defective}|B) &= 0.03 \\ P(\text{Defective}|C) &= 0.04 \end{aligned}$$

$\boxed{Solution 3:}$

First, find P(Defective):

$$P(\text{Defective}) = P(\text{Defective}|A) \times P(A) + P(\text{Defective}|B) \times P(B) + P(\text{Defective}|C) \times P(C)$$

$$= (0.02 \times 0.30) + (0.03 \times 0.45) + (0.04 \times 0.25)$$

$$= 0.006 + 0.0135 + 0.01$$

$$= 0.0295$$

Now apply Bayes' Theorem:

$$
\begin{aligned}
P(B|\text{Defective}) &= \frac{P(\text{Defective}|B) \times P(B)}{P(\text{Defective})} \\
&= \frac{0.03 \times 0.45}{0.0295} \\
&= \frac{0.0135}{0.0295} \\
&\approx 0.458 \text{ or } 45.8\%
\end{aligned}
$$

Question 8: Bayes' Theorem – Multiple Machines

Scenario:

A factory has **three machines (A, B, C)** producing items:

- Machine A: 30% of items, defect rate 2%
- Machine B: 45% of items, defect rate 3%
- Machine C: 25% of items, defect rate 4%

A randomly selected item from the total output is found to be **defective**.

Tasks:

1. Find the total probability that an item is defective.
2. Using Bayes' Theorem, find the probability that the defective item **came from Machine B**.
3. Find the probability that the defective item **did NOT come from Machine C**.
4. If the factory introduces a **Machine D** producing 10% of items with a defect rate of 5%, recalculate the **probability that a defective item came from Machine B**.

Solution

1. **Total Probability of Defective Item**

Use **Law of Total Probability**:

$$\begin{aligned} P(\text{Defective}) &= P(\text{Defective}|A)P(A) + P(\text{Defective}|B)P(B) + P(\text{Defective}|C)P(C) \\ &= (0.02 \times 0.30) + (0.03 \times 0.45) + (0.04 \times 0.25) \\ &= 0.006 + 0.0135 + 0.01 \\ &= 0.0295 \end{aligned}$$

---

2. **Probability Defective Item Came from Machine B**

Use **Bayes' Theorem**:

$$\begin{aligned} P(B|\text{Defective}) &= \frac{P(\text{Defective}|B) \cdot P(B)}{P(\text{Defective})} \\ &= \frac{0.03 \times 0.45}{0.0295} \\ &= \frac{0.0135}{0.0295} \\ &\approx 0.458 \text{ or } 45.8\% \end{aligned}$$

3. **Probability Defective Item Did NOT Come from Machine C**

Use complement rule:

$$P(\text{Not C}|\text{Defective}) = 1 - P(C|\text{Defective})$$

First, find $P(C|\text{Defective})$:

$$P(C|\text{Defective}) = \frac{P(\text{Defective}|C) \cdot P(C)}{P(\text{Defective})} = \frac{0.04 \times 0.25}{0.0295} = \frac{0.01}{0.0295} \approx 0.339$$

Then:

$$P(\text{Not C}|\text{Defective}) = 1 - 0.339 = 0.661 \text{ or } 66.1\%$$

---

4. **Adding Machine D (10% production, 5% defect rate)**

- Update total probability of defective item:

$$\begin{aligned} P(\text{Defective}) &= (0.02 \times 0.30) + (0.03 \times 0.45) + (0.04 \times 0.25) + (0.05 \times 0.10) \\ &= 0.006 + 0.0135 + 0.01 + 0.005 \\ &= 0.0345 \end{aligned}$$

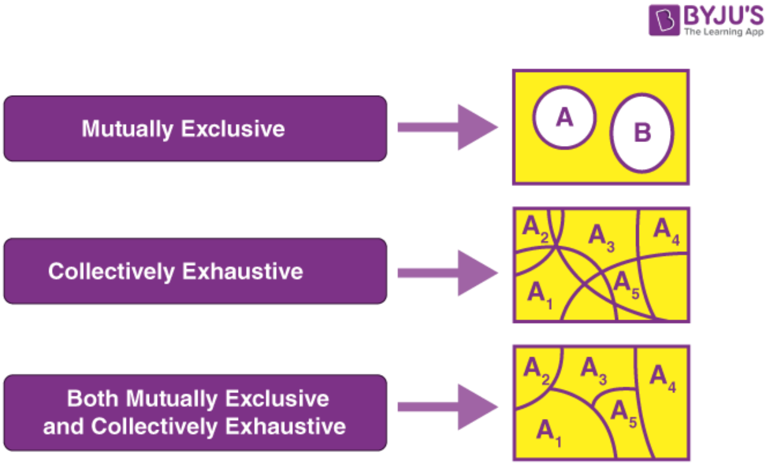
- Updated probability that defective item came from Machine B:

$$\begin{aligned} P(B|\text{Defective}) &= \frac{0.03 \times 0.45}{0.0345} = \frac{0.0135}{0.0345} \\ &\approx 0.391 \text{ or } 39.1\% \end{aligned}$$

Counting Rules

Counting rule 1

For Mutually exclusive and collectively exhaustive events:



And **same number of events are coming in each trial**.

$$\text{Number of outcomes} = k^n$$

where,

- k = number of different outcomes per trial
- n = number of trials

$\boxed{Example}$

Tossing a coin 5 times

- k = 2 (heads or tails)
- n = 5 (five tosses)

$$\text{Number of outcomes} = 2^5 = 32$$

Counting rule 2

Number of events differs from trial to trial:

$$\text{Number of outcomes} = (k_1) \times (k_2) \ldots (k_n)$$

where:

$$k_i = \text{number of events on trial i}$$

$\boxed{Example}$

License plates (3 letters, 3 numbers)

- First letter: 26 choices
- Second letter: 26 choices
- Third letter: 26 choices
- First number: 10 choices (0-9)
- Second number: 10 choices
- Third number: 10 choices

$$Total: (26)(26)(26)(10)(10)(10) = 17,576,000$$

Counting rule 3 - Factorials

Number of ways to arrange n items in order:
n factorial

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$3! = 3 \times 2 \times 1 = 6$$

$$0! = 1$$

$\boxed{Example}$

Arranging 6 books on a shelf

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \text{ different arrangements}$$

Counting rule 4 - Permutations

Number of ways to arrange x objects from n objects in order:

$${}_nP_r = \frac{n!}{(n-r)!}$$

Expanded Form:

$${}_nP_r = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

$\boxed{Example}$

6 books, but only room for 4 on shelf

$${}_6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 360$$

Counting Rule 5 - Combinations

Number of ways to arrange x objects from n objects , order does not matter:

$$ {}_nC_r = \frac{n!}{r!(n-r)!} $$

$$ {}_nC_r = \frac{{}_nP_r}{r!} $$

$\boxed{Example}$

Selecting 4 books from 6, order irrelevant

$$ {}_6C_4 = \frac{6!}{4!(6-4)!} = \frac{6!}{4! \cdot 2!} = \frac{6 \times 5}{2 \times 1} = 15 $$

Permutations vs. Combinations

Aspect	Permutations	Combinations
Order	Order **MATTERS**	Order **DOES NOT MATTER**
Formula	$${}_nP_r = \frac{n!}{(n-r)!}$$	$${}_nC_r = \frac{n!}{r!(n-r)!}$$
Alternative Notation	$${}_nP_r$$	$${}_nC_r$$
Expanded Form	$$n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$	$$\frac{n \times (n-1) \times \cdots \times (n-r+1)}{r!}$$
Result Size	Larger	Smaller

Question 9: Counting Rules & Arrangements

Scenario:

A school is arranging a **mini talent show**. There are **8 participants**, and they need to select **4 participants** to perform in a sequence. Additionally, they want to create **a 6-character password** for online voting: 3 letters followed by 3 digits.

Tasks:

1. How many possible sequences of performers are there if **order matters**?
2. How many ways can the 4 performers be chosen if **order does NOT matter**?
3. Using **Counting Rule 1**, if a coin is tossed 6 times, how many possible outcomes exist?
4. Using **Counting Rule 2**, how many different license plates are possible with **2 letters followed by 4 digits**, where letters and digits can repeat?
5. Arrange **5 books on a shelf** – how many different orders are possible?
6. Select **3 books from 7** – how many ways if order does NOT matter?

Solution

1. **Sequences of 4 performers (order matters) → Permutations**

Use:

$$_8P_4 = \frac{8!}{(8-4)!} = \frac{40320}{24} = 1680$$

Answer: 1680 sequences

---

2. **Choosing 4 performers (order irrelevant) → Combinations**

Use:

$$_8C_4 = \frac{8!}{4!(8-4)!} = \frac{40320}{24 \cdot 24} = 70$$

Answer: 70 ways

---

3. **Coin tossed 6 times → Counting Rule 1**

$$\text{Number of outcomes} = 2^6 = 64$$

Answer: 64 outcomes

4. **License plates: 2 letters + 4 digits → Counting Rule 2**

- Letters: 26 choices each
- Digits: 10 choices each

$$\text{Total outcomes} = 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 26^2 \cdot 10^4 = 6,760,000$$

Answer: 6,760,000 possible license plates

---

5. **Arranging 5 books → Factorials**

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

Answer: 120 different arrangements

---

6. **Selecting 3 books from 7 → Combinations**

$$_7C_3 = \frac{7!}{3!(7-3)!} = \frac{5040}{6 \cdot 24} = 35$$

Answer: 35 ways

Summary

1. Key Probability Concepts

Basic Probability Rules

$$0 \leq P(A) \leq 1$$

Types of Probabilities:

- **Simple Probability**: Probability of a single event
- **Joint Probability**: Probability of two or more events occurring together
- **Marginal Probability**: Total probability of an event across all conditions

General Addition Rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

When to use:

- Finding probability of "A **OR** B"
- Union of events
- Subtract intersection to avoid double-counting

2. Conditional Probability

Formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Reads as: "Probability of A given B"

Independence

Events A and B are independent if:

$$P(A|B) = P(A)$$

OR equivalently:

$$P(A \cap B) = P(A) \times P(B)$$

Multiplication Rules

For Independent Events:

$$P(A \cap B) = P(A) \times P(B)$$

For Dependent Events:

$$P(A \cap B) = P(A) \times P(B|A)$$

3. Bayes' Theorem

Purpose: Revise probabilities based on new information

Process: Prior Probability → Posterior Probability

$$P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}$$

4. Counting Rules Summary

Rule	Formula	When to Use	Example
Rule 1	k^n	Same # of outcomes each trial	Coin flips, dice rolls
Rule 2	$(k_1)(k_2)\cdots(k_n)$	Different # of outcomes per trial	Menu choices, outfits
Rule 3	$n!$	Arrange all n items	Arranging books

Rule	Formula	When to Use	Example
Rule 4	$_nP_x = \frac{n!}{(n-x)!}$	Arrange x from n (order matters)	Race positions, passwords
Rule 5	$_nC_x = \frac{n!}{x!(n-x)!}$	Select x from n (order doesn't matter)	Committees, teams

Key Decision: Does Order Matter?

```
Does order matter?
├─ YES → Use Permutations (Rule 4)
└─ NO  → Use Combinations (Rule 5)
```

5. Permutations vs. Combinations

Aspect	Permutations	Combinations
Order	Matters	Doesn't matter
Formula	$_nP_x = \frac{n!}{(n-x)!}$	$_nC_x = \frac{n!}{x!(n-x)!}$
Result	Larger number	Smaller number
Keywords	Arrange, order, sequence, rank	Choose, select, group, committee
Example	PIN codes, race positions	Lottery, team selection

Relationship:

$$_nC_x = \frac{_nP_x}{x!}$$

6. Common Mistakes to Avoid

Mistake 1: Confusing AND vs. OR

- **AND** = Intersection (∩) → Multiply if independent
- **OR** = Union (∪) → Add then subtract intersection

$$P(A \text{ AND } B) = P(A \cap B)$$
$$P(A \text{ OR } B) = P(A) + P(B) - P(A \cap B)$$

Mistake 2: Assuming Independence

Always verify:

$$P(A|B) = P(A) \quad ?$$

$$P(A \cap B) = P(A) \times P(B) \quad ?$$

Mistake 3: Conditional Probability Confusion

$$P(A|B) \neq P(B|A)$$

Always identify the condition properly!

Mistake 4: Permutations vs. Combinations

- Read carefully: does order matter?
- Remember:

  $${}_nP_x >_n C_x$$

  for same n and x

7. Problem-Solving Strategy

Step-by-Step Approach:

Step 1: Identify the type of probability question

- Simple? Joint? Conditional?

Step 2: Organize the data

- Contingency table
- Decision tree
- Venn diagram

Step 3: Select appropriate formula

- Addition rule?
- Multiplication rule?
- Bayes' theorem?

Step 4: Calculate carefully

- Show all steps
- Check that probabilities sum to 1

Step 5: Interpret results in context

- What does the answer mean?
- Does it make intuitive sense?

8. Practice Tips

Visual Tools:

- **Venn diagrams** for visualizing relationships
- **Decision trees** for sequential events
- **Contingency tables** for organizing data

Check Your Work:

- Do probabilities sum to 1?
- Is answer between 0 and 1?
- Does it make intuitive sense?

Practice Different Types:

- Simple and joint probabilities
- Conditional probabilities
- Bayes' theorem problems
- Counting problems

Understand the Logic:

- Don't just memorize formulas
- Know **when** to use each formula
- Understand **why** formulas work