GSV trick for Solomon's knot VERSION 2

We want to see whether we can use [GSV, eq. (5.5)] to compute unreduced KR homology for the two-component link "Solomon's knot VERSION 2", aka L4a1. Its linking number is 2, and our results for its n=2 and n=3 KR invariants are as follows:

```
lk = 2; KR2 = Expand[(t^4 q^2 + t^4 + t^3 q^2 + t^2 q^6 + q^(10) + q^8) / (q^(12))];
KR3 = Expand[(t^4 q^6 + 2 t^4 q^4 + 2 t^4 q^2 + t^4 + t^3 q^4 +
      t^3 q^2 + t^2 q^(10) + t^2 q^8 + q^(14) + q^(12) + q^(10)) / (q^(18))];
```

Thus the "universal" terms Ui := \sum_{Q,s,r\in\Z} D_{Q,s,r} q^{iQ+s} t^r for i=2,3 on the right-hand side of [GSV, eq. (5.5)] are given by (where we leave the parameter a:=alpha undetermined for now):

```
U2 = Expand[(q - 1 / q) (q^(4 lk) KR2 - t^a (q + 1 / q)^2)]
```

$$-q^3 + q^7 - q\,t^2 + q^3\,t^2 - \frac{t^3}{q^3} + \frac{t^3}{q} - \frac{t^4}{q^5} + \frac{t^4}{q} + \frac{t^a}{q^3} + \frac{t^a}{q} - q\,t^a - q^3\,t^a$$

```
U3 = Expand[Simplify[(q - 1 / q) (q^(6 lk) KR3 - t^a ((q^3 - q^(-3)) / (q - 1 / q))^2)]]
```

$$-q^3 + q^9 - q\,t^2 + q^5\,t^2 - \frac{t^3}{q^5} + \frac{t^3}{q} - \frac{t^4}{q^7} - \frac{t^4}{q^5} + \frac{t^4}{q} + q\,t^4 + \frac{t^a}{q^5} + \frac{t^a}{q^3} + \frac{t^a}{q} - q\,t^a - q^3\,t^a - q^5\,t^a$$

Now we can try to use [GSV, eq. (5.5)] to reproduce our n=3 KR invariant (note that the two components involved are just two unknots by themselves). It works:

```
KR3 - Expand[
  Simplify[q^(-6 lk) (t^a Simplify[((q^3 - q^(-3)) / (q - 1 / q))^2] + U3 / (q - 1 / q))]]
```

```
0
```

Now try to determine the constants D_{Q,s,r} from U2 and U3 above. If we set a=alpha=4, then U2 and U3 have the same number of terms:

```
U2 /. {a → 4}
```

$$-q^3 + q^7 - q\,t^2 + q^3\,t^2 - \frac{t^3}{q^3} + \frac{t^3}{q} - \frac{t^4}{q^5} + \frac{t^4}{q^3} + \frac{2\,t^4}{q} - q\,t^4 - q^3\,t^4$$

```
U3 /. {a → 4}
```

$$-q^3 + q^9 - q\,t^2 + q^5\,t^2 - \frac{t^3}{q^5} + \frac{t^3}{q} - \frac{t^4}{q^7} + \frac{t^4}{q^3} + \frac{2\,t^4}{q} - q^3\,t^4 - q^5\,t^4$$

From this we can read off the constants D_{Q,s,r}:

```
UU[n_] := - q^(0 n + 3) + q^(2 n + 3) - q^(0 n + 1) t^2 +
  q^(2 n - 1) t^2 - q^(-2 n + 1) t^3 + q^(0 n - 1) t^3 - q^(-2 n - 1) t^4 +
  q^(-3) t^4 + 2 q^(0 n - 1) t^4 - q^(2 n - 3) t^4 - q^(2 n - 1) t^4 ; UU[n]
```

$$-q^3 + q^{3+2n} - q\,t^2 + q^{-1+2n}\,t^2 + \frac{t^3}{q} - q^{1-2n}\,t^3 + \frac{t^4}{q^3} + \frac{2\,t^4}{q} - q^{-1-2n}\,t^4 - q^{-3+2n}\,t^4 - q^{-1+2n}\,t^4$$

Now we use [GSV, eq. (5.5)] with a=alpha=4 to define a candidate for the KR invariants for arbitrary n:

```
KR[n_] := Expand[FullSimplify[Expand[Simplify[(q^(-2 n lk)
      (t^a Simplify[((q^n - q^(-n)) / (q - 1 / q))^2] + UU[n] / (q - 1 / q))) /. {a → 4}]]]]
```

Check that we have not made a mistake so far:

```
{KR[2] - KR2, KR[3] - KR3}
```

```
{0, 0}
```

Our results for the n=4 and n=5 KR invariants are:

```
KR4 =
  Expand[(t^4 q^(10) + 2 t^4 q^8 + 3 t^4 q^6 + 3 t^4 q^4 + 2 t^4 q^2 + t^4 + t^3 q^6 + t^3 q^4 +
      t^3 q^2 + t^2 q^(14) + t^2 q^(12) + t^2 q^(10) + q^(18) +
      q^(16) + q^(14) + q^(12)) / (q^(24))];
KR5 = Expand[(t^4 q^(14) + 2 t^4 q^(12) + 3 t^4 q^(10) + 4 t^4 q^8 + 4 t^4 q^6 + 3 t^4 q^4 +
      2 t^4 q^2 + t^4 + t^3 q^8 + t^3 q^6 + t^3 q^4 + t^3 q^2 + t^2 q^(18) + t^2 q^(16) +
      t^2 q^(14) + t^2 q^(12) + q^(22) + q^(20) + q^(18) + q^(16) + q^(14)) / (q^(30))];
```

They agree with the GSV prediction:

```
{KR[4] - KR4, KR[5] - KR5}
```

```
{0, 0}
```

The general expression for the GSV predicion is:

```
FullSimplify[
 (q^(-2 n lk) ( t^a Simplify[((q^n - q^(-n)) / (q - 1 / q))^2] + UU[n] / (q - 1 / q) )) /. {a → 4}]
```

$$\frac{1}{\left(-1+q^2\right)^2} q^{-2-6 n} \left(q^2 t^3 \left(q^2-q^4+t\right)+q^{4 n} \left(q^6 \left(-1+q^2\right)+q^2 \left(-1+q^2\right) t^2+t^4\right)-q^{2 n} \left(q^8+t^4-q^4 t^2 (1+t)+q^2 t^3 (1+t)+q^6 \left(-1+t^2\right)\right)\right)$$