Let k be a field of characteristic zero, $W \in k[x,y]$ a polynomial such that

$$R := k[[x,y]]/W$$

has an isolated singularity. It is always possible to effect a change of variables so that W is x-general of some order $m \geqslant 1$ (i.e. the monomial x^m has a nonzero coefficient), and if we assume that this has been done, then $\{\partial_x W, W\}$ forms a system of parameters for $k[[x,y]]$. Hence it makes sense to use this pair of polynomials in the denominator of generalised fractions in local cohomology. Let us define $\mathbf{D}_{\mathrm{Sg}}(R)$ to be the stable category of matrix factorisations of W. It is a theorem of Auslander that the identity functor on $\mathbf{D}_{\mathrm{Sg}}(R)$ is the Serre functor, which is therefore determined by a family of k-linear trace maps

$$\mathrm{Tr}^{\mathrm{sg}}_{(\varphi,\psi)} : \mathrm{End}_{\mathbf{D}_{\mathrm{Sg}}(R)}((\varphi,\psi)) \longrightarrow k. \tag{12}$$

indexed by the matrix factorisations (φ,ψ) of W.

Theorem 2. *Let (φ,ψ) be a matrix factorisation of W, and $\alpha = (\alpha_1,\alpha_2)$ an endomorphism of this matrix factorisation. Then the k-linear maps*

$$\mathrm{Tr}^{\mathrm{sg}}_{(\varphi,\psi)}(\alpha) = \mathrm{Res}\begin{bmatrix}\mathrm{Tr}(\alpha_1 \partial_x(\varphi)\psi)\; dx \wedge dy \\ \partial_x W, \quad W\end{bmatrix} \tag{13}$$

give rise to the Serre functor on $\mathbf{D}_{\mathrm{Sg}}(R)$.

If W belongs to the ideal of $k[[x,y]]$ generated by $\partial_x W, \partial_y W$, say

$$W = W_x \partial_x W + W_y \partial_y Y,$$

then we say that W is *quasi-homogeneous*, and in this case we set $E = W_x\partial_x W + W_y \partial_y Y$. Let Q be the odd matrix $\left(\begin{smallmatrix} 0 & \varphi \\ \psi & 0\end{smallmatrix}\right)$. We say that (φ,ψ) is *quasi-homogeneous* if there is an even matrix T such that $Q - 2EQ = [Q,T]$. If both W and (φ,ψ) are quasi-homogeneous, then the above trace agrees with the Kapustin-Li formula. Note that the residue in (13) should be interpreted as an artifact of local duality, so the thing with square brackets is an element of the injective envelope $E(k)$ (known as a generalised fraction in the theory of local duality) and Res is some k-linear map $E(k) \longrightarrow k$.

1. Residues

Generalised fractions arise as follows: let $S = k[[x_1,\dots,x_n]]$ be some power series ring over a field, and set $\omega_S = \Omega^n_{S/k}$. Then the local cohomology module $H^n_{\mathfrak{m}}(\omega_S)$ gives an injective envelope of the residue field k as an S-module. If $t_1,\dots,t_n$ is a system of parameters with associated "stable" Koszul complex

$$K := \bigotimes_{i=1}^{n}(S \longrightarrow S[t_i^{-1}])$$

then one can use K to compute the local cohomology as

$$\varphi : \omega_S \otimes_S S[t_1^{-1},\dots,t_n^{-1}] \twoheadrightarrow H^n(K \otimes_S \omega_S) =: H^n_{\mathfrak{m}}(\omega_S).$$

One day way to define generalised fractions (there are several) is as the images of "true" fractions under this surjection, so given $\gamma \in \omega_S$ (e.g. $dx_1 \wedge \cdots \wedge dx_n$) and $e_1,\dots,e_n \geqslant 1$ we define

$$\begin{bmatrix} \gamma \\ t_1^{e_1},\dots,t_n^{e_n}\end{bmatrix} := \varphi\left(\frac{\gamma}{t_1^{e_1}\cdots t_n^{e_n}}\right)$$

Note that the denominator of the generalised fraction is actually a sequence; the order matters, because if you interchange say $t_i^{e_i}$ with $t_{i+1}^{e_{i+1}}$ then you pick up a sign factor of -1. If you think of what the cohomology of the Koszul complex means, it is clear that the t_i act on generalised fractions by reducing the order of the exponents in the denominator, and if any of the denominators e_i are not positive, then the generalised fraction is zero.

If we take our system of parameters to be the canonical one $t_i = x_i$ and set $dV = dx_1 \wedge \cdots \wedge dx_n$ then as a k-vector space $E(k) = H^n_{\mathfrak{m}}(\omega_S)$ has a basis given by the generalised fractions

$$\begin{bmatrix} dV \\ x_1^{e_1}, \ldots, x_n^{e_n} \end{bmatrix}, \quad e_1, \ldots, e_n \geqslant 1$$

and the *residue* is the k-linear map

$$\mathrm{Res} : H^n_{\mathfrak{m}}(\omega_S) \longrightarrow k,$$
$$\mathrm{Res}\begin{bmatrix} dV \\ x_1^{e_1}, \ldots, x_n^{e_n} \end{bmatrix} = \begin{cases} 1 & e_1 = \cdots = e_n = 1, \\ 0 & \text{otherwise} \end{cases}.$$

So it remains to explain how to take the residue of a generalised fraction whose denominator is expressed in terms of a system of parameters which is *not* the canonical one. There is a simple transformation rule: if $t_1, \ldots, t_n$ is a system of parameters then there are exponents q_i for which we can express $x_i^{q_i} = \sum_{j=1}^n a_{ij} t_j$, and this case

$$\begin{bmatrix} \gamma \\ t_1, \ldots, t_n \end{bmatrix} = \begin{bmatrix} \det(a_{ij})\gamma \\ x_1^{q_1}, \ldots, x_n^{q_n} \end{bmatrix}$$

There is a whole calculus of these residue symbols, you can find some information in Conrad's book on Grothendieck duality and base change, for example. But what you need to calculate the Kapustin-Li formula is only the very basic stuff, one sufficient list I happen to have on my desk is [CH07, Definition 2.1].

2. Example 1: A_n

Let us calculate the trace formula for A_n singularities. Set $W = x^2 + y^{n+1}$ for $n \geqslant 1$ odd, and for $1 \leqslant j \leqslant (n-1)/2$ consider the matrix factorisation

$$\varphi_j = \psi_j = \begin{pmatrix} x & y^j \\ y^{n+1-j} & -x \end{pmatrix}.$$

In this case both $\{\partial_x W, W\}$ and $\{\partial_y W, W\}$ are systems of parameters for $k[[x, y]]$ and we can use either, but let us stick with the first. Given an arbitrary polynomial f we want to calculate a residue

$$\mathrm{Res}\begin{bmatrix} f dx \wedge dy \\ \partial_x W, W \end{bmatrix} = \frac{1}{2} \cdot \mathrm{Res}\begin{bmatrix} f dx \wedge dy \\ x, x^2 + y^{n+1} \end{bmatrix}$$

Either directly using the transformation rule, or using the fact that in a generalised fraction you can add any multiple of one element of the sequence of denominators to any other without changing the fraction, we have

$$\frac{1}{2} \cdot \mathrm{Res}\begin{bmatrix} f dx \wedge dy \\ x, x^2 + y^{n+1} \end{bmatrix} = \frac{1}{2} \cdot \mathrm{Res}\begin{bmatrix} f dx \wedge dy \\ x, y^{n+1} \end{bmatrix}$$

If we now look at the definition of the residue map, then for any $i, j \geqslant 1$ and polynomial f, we have

$$\mathrm{Res}\begin{bmatrix} f dx \wedge dy \\ x^i, y^j \end{bmatrix} = C_{x^{i-1}y^{j-1}}(f)$$

where for a monomial m, $C_m(f)$ means the scalar coefficient of m in f. Hence

$$\mathrm{Res}\begin{bmatrix} f dx \wedge dy \\ \partial_x W, W \end{bmatrix} = \frac{1}{2} \cdot \mathrm{Res}\begin{bmatrix} f dx \wedge dy \\ x, x^2 + y^{n+1} \end{bmatrix} = \frac{1}{2} \cdot C_{y^n}(f).$$

We are interested in the case $f = \mathrm{Tr}(\alpha_1 \partial_x(\varphi)\psi)$ where $\alpha = (\alpha_1, \alpha_2)$ is an endomorphism of our matrix factorisation. One calculates that

$$\mathrm{Tr}(\alpha_1 \partial_x(\varphi_j)\psi_j) = \alpha_1^{11} x - y^{n+1-j}\alpha_1^{12} + \alpha_1^{21} y^j + x\alpha_1^{22},$$

whence

$$\begin{aligned}\mathrm{Tr}^{\mathrm{sg}}(\alpha) &= \mathrm{Res}\begin{bmatrix}\mathrm{Tr}(\alpha_1\partial_x(\varphi_j)\psi_j)dx \wedge dy \\ \partial_x W, W\end{bmatrix} = \frac{1}{2}\cdot C_{y^n}(\alpha_1^{11}x - y^{n+1-j}\alpha_1^{12} + \alpha_1^{21}y^j + x\alpha_1^{22}) \\ &= \frac{1}{2}\left\{C_{y^{n-j}}(\alpha_1^{21}) - C_{y^{j-1}}(\alpha_1^{12})\right\}.\end{aligned}$$

3. Example 2: D_n

Set $W = x^2 y + y^{n-1}$ for $n \geqslant 4$ odd. Note that in this case $\{\partial_x W, W\}$ is not a system of parameters but $\{\partial_y W, W\}$ is, so throughout we use y instead of x. Given a polynomial f, we need to be able to compute

$$\mathrm{Res}\begin{bmatrix} f dx \wedge dy \\ \partial_y W, W\end{bmatrix} = \frac{1}{2-n}\cdot\begin{bmatrix} f dx \wedge dy \\ x^2 + (n-1)y^{n-2}, y^{n-1}\end{bmatrix}.$$

We use the transformation

$$\begin{aligned} x^4 &= (x^2 - (n-1)y^{n-2})\cdot(x^2 + (n-1)y^{n-2}) + (n-1)^2 y^{n-3}\cdot y^{n-1}, \\ y^{n-1} &= 0\cdot(x^2 + (n-1)y^{n-2}) + 1\cdot y^{n-1}\end{aligned}$$

which has determinant $(x^2 - (n-1)y^{n-2})$, so

$$\begin{aligned}\frac{1}{2-n}\cdot\begin{bmatrix} f dx \wedge dy \\ x^2 + (n-1)y^{n-2}, y^{n-1}\end{bmatrix} &= \frac{1}{2-n}\cdot C_{x^3 y^{n-2}}\left((x^2 - (n-1)y^{n-2})f\right) \\ &= \frac{1}{2-n}\left\{C_{xy^{n-2}}(f) - (n-1)C_{x^3}(f)\right\}.\end{aligned}$$

Consider the matrix factorisation defined for $1 \leqslant j \leqslant (n-3)/2$ by (φ_j, ψ_j) where

$$\varphi_j = \begin{pmatrix} x & y^j \\ y^{n-j-2} & -x\end{pmatrix}, \quad \psi_j = \begin{pmatrix} xy & y^{j+1} \\ y^{n-j-1} & -xy\end{pmatrix}.$$

Then one computes that

$$\mathrm{Tr}(\alpha_1\partial_y(\varphi_j)\psi_j) = jy^{n-2}\alpha_1^{11} + (n-j-2)xy^{n-j-2}\alpha_1^{12} - jxy^j\alpha_1^{21} + (n-j-2)y^{n-2}\alpha_1^{22},$$

and consequently for any endomorphism $\alpha = (\alpha_1, \alpha_2)$ of (φ_j, ψ_j) we have

$$\begin{aligned}\mathrm{Tr}^{\mathrm{sg}}(\alpha) &= \mathrm{Res}\begin{bmatrix}\mathrm{Tr}(\alpha_1\partial_y(\varphi_j)\psi_j)dx \wedge dy \\ \partial_y W, W\end{bmatrix} \\ &= \frac{1}{2-n}\cdot\left\{jC_x(\alpha_1^{11}) + (n-j-2)C_{y^j}(\alpha_1^{12}) - jC_{y^{n-j-2}}(\alpha_1^{21}) + (n-j-2)C_x(\alpha_1^{22})\right\}.\end{aligned}$$

4. Form matrix

I wrote some Singular code to help calculate with these trace maps, for example to automatically use your algorithm to calculate a basis and test non-degeneracy of the above traces. In the code I refer to the *form matrix*, which is a way of packing the trace formulas into a matrix. For example, in the A_n case considered above with trace formula

$$\mathrm{Tr}^{\mathrm{sg}}(\alpha) = \frac{1}{2}\left\{C_{y^{n-j}}(\alpha_1^{21}) - C_{y^{j-1}}(\alpha_1^{12})\right\}$$

the form matrix is

$$\begin{pmatrix} 0 & -\frac{1}{2}y^{j-1} \\ \frac{1}{2}y^{n-j} & 0 \end{pmatrix},$$

and in the D_n case with trace formula

$$\mathrm{Tr}^{\mathrm{sg}}(\alpha) = \frac{1}{2-n} \cdot \left\{ jC_x(\alpha_1^{11}) + (n-j-2)C_{y^j}(\alpha_1^{12}) - jC_{y^{n-j-2}}(\alpha_1^{21}) + (n-j-2)C_x(\alpha_1^{22}) \right\}$$

the form matrix is

$$\begin{pmatrix} \frac{j}{2-n} \cdot x & \frac{n-j-2}{2-n} \cdot y^j \\ \frac{-j}{2-n} y^{n-j-2} & \frac{n-j-2}{2-n} \cdot x \end{pmatrix}.$$

The algorithm from going from the trace formula to the matrix is now obvious, I hope. The Singular code for curve singularities uses a form matrix like this as input; there are routines for the higher dimensional traces which take as input the form matrix in a block matrix format.

References

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