GSV trick for L6a1

We want to see whether we can use [GSV, eq. (5.5)] to compute unreduced KR homology for the two-component link L6a1. **For its linking number I find either 1 or 3 (depending on the "version"), but with these values the GSV formula does not work; so we simply set lk=2 below.** Our results for its n=2 and n=3 KR invariants are as follows:

```
lk = 2; KR2 = Expand[(t^4 q^2 + t^4 + t^3 q^2 + t^2 q^6 + q^(10) + q^8) / (q^(12))];
KR3 = Expand[(t^4 q^6 + 2 t^4 q^4 + 2 t^4 q^2 + t^4 + t^3 q^4 +
     t^3 q^2 + t^2 q^(10) + t^2 q^8 + q^(14) + q^(12) + q^(10)) / (q^(18))]
```

$$\frac{1}{q^8}+\frac{1}{q^6}+\frac{1}{q^4}+\frac{t^2}{q^{10}}+\frac{t^2}{q^8}+\frac{t^3}{q^{16}}+\frac{t^3}{q^{14}}+\frac{t^4}{q^{18}}+\frac{2\,t^4}{q^{16}}+\frac{2\,t^4}{q^{14}}+\frac{t^4}{q^{12}}$$

Thus the "universal" terms Ui := \sum_{Q,s,r\in\Z} D_{Q,s,r} q^{iQ+s} t^r for i=2,3 on the right-hand side of [GSV, eq. (5.5)] are given by (where we leave the parameter a:=alpha undetermined for now):

```
U2 = Expand[(q - 1 / q) (q^(4 lk) KR2 - t^a (q + 1 / q)^2 )]
```

$$-q^3+q^7-q\,t^2+q^3\,t^2-\frac{t^3}{q^3}+\frac{t^3}{q}-\frac{t^4}{q^5}+\frac{t^4}{q}+\frac{t^a}{q^3}+\frac{t^a}{q}-q\,t^a-q^3\,t^a$$

```
U3 = Expand[Simplify[(q - 1 / q) (q^(6 lk) KR3 - t^a ((q^3 - q^(-3)) / (q - 1 / q))^2 )]]
```

$$-q^3+q^9-q\,t^2+q^5\,t^2-\frac{t^3}{q^5}+\frac{t^3}{q}-\frac{t^4}{q^7}-\frac{t^4}{q^5}+\frac{t^4}{q}+q\,t^4+\frac{t^a}{q^5}+\frac{t^a}{q^3}+\frac{t^a}{q}-q\,t^a-q^3\,t^a-q^5\,t^a$$

Now we can try to use [GSV, eq. (5.5)] to reproduce our n=3 KR invariant (note that the two components involved are just two unknots by themselves). It works:

```
KR3 - Expand[
  Simplify[q^(-6 lk) (t^a Simplify[((q^3 - q^(-3)) / (q - 1 / q))^2] + U3 / (q - 1 / q) )]]
```

$$0$$

Now try to determine the constants D_{Q,s,r} from U2 and U3 above. If we set a=alpha=4, then U2 and U3 have the same number of terms:

```
U2 /. {a → 4}
```

$$-q^3+q^7-q\,t^2+q^3\,t^2-\frac{t^3}{q^3}+\frac{t^3}{q}-\frac{t^4}{q^5}+\frac{t^4}{q^3}+\frac{2\,t^4}{q}-q\,t^4-q^3\,t^4$$

```
U3 /. {a → 4}
```

$$-q^3+q^9-q\,t^2+q^5\,t^2-\frac{t^3}{q^5}+\frac{t^3}{q}-\frac{t^4}{q^7}+\frac{t^4}{q^3}+\frac{2\,t^4}{q}-q^3\,t^4-q^5\,t^4$$

From this we can read off the constants D_{Q,s,r}:

```
UU[n_] := - q^(3) + q^(2 n + 3) - q^(1) t^2 + q^(2 n - 1) t^2 -
  q^(-2 n + 1 ) t^3 + q^(-1 ) t^3 - q^(-2 n - 1 ) t^4 + q^(-3 ) t^4 + 2 q^(-1 ) t^4 -
  q^(2 n - 3 ) t^4 - q^(2 n - 1 ) t^4; {UU[3] - U3, UU[2] - U2} /. {a → 4}
```

$$\{0,\ 0\}$$

Now we use [GSV, eq. (5.5)] with a=alpha=0 to define a candidate for the KR invariants for arbitrary n:

```
KR[n_] := Expand[FullSimplify[Expand[Simplify[(q^(-2 n lk)
      ( t^a Simplify[((q^n - q^(-n)) / (q - 1 / q))^2] + UU[n] / (q - 1 / q) )) /. {a → 4}]]]]
```

Check that we have not made a mistake so far:

```
{KR[2] - KR2, KR[3] - KR3}
```

```
{0, 0}
```

Our result for the n=4 invariant is:

```
FullSimplify[
 (q^(-2 n lk) (t^a Simplify[((q^n - q^(-n)) / (q - 1 / q))^2] + UU[n] / (q - 1 / q) )) /. {a → 4}]
```

$$\frac{1}{\left(-1+q^2\right)^2} q^{-2-6n} \left(q^2 t^3 \left(q^2 - q^4 + t\right) + q^{4n} \left(q^6 \left(-1+q^2\right) + q^2 \left(-1+q^2\right) t^2 + t^4\right) - q^{2n} \left(q^8 + t^4 - q^4 t^2 (1+t) + q^2 t^3 (1+t) + q^6 \left(-1+t^2\right)\right)\right)$$