Notes on Uniform Spaces

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1 Basic definitions

Definition 1.1. Let X be a set. A *uniform structure* on X is a non-empty collection of subsets of $X \times X$, call it Φ, such that

1. Each subset contains the diagonal, $U \in \Phi \implies \Delta_X \subset \Phi$.
2. The structure is closed under supersets, $U \in \Phi, U \subset U' \implies U' \in \Phi$.
3. The structure is closed under finite intersections, $U, V \in \Phi \implies U \cap V \in \Phi$.
4. The structure is closed under inversion, $U \in \Phi \implies U^{-1} \in \Phi$.
5. $\forall U \in \Phi$, $\exists V \in \Phi$ with $V \circ V \subset U$, i.e. we can take "roots".

We shall call an element of Φ an *entourage*. If a set has a uniform structure we shall call it a *uniform space* with the structure implicit in notation. We shall also use the word *uniformity* in place of uniform structure.

Inversion of $U \subset X \times X$ is given by $\sigma^{-1}U$ where $\sigma(x, y) = (y, x)$. We call an entourage *symmetric* if $U^{-1} = U$. As $U^{-1} \cap U \subset U$ is symmetric every entourage contains a symmetric entourage.

In the final condition by $\circ$ we mean the operation that takes $A, B \subset X \times X$ and gives

$$A \circ B = \{(x, y) \in X \times X \mid \exists z \in X \text{ with } (x, z) \in A,\ (z, y) \in B\}. \tag{1}$$

One sees that this gives an associative binary operation on such subsets and we define $V^{\circ n}$ by applying this operation to n Vs. By repeatedly applying this condition we see that for any entourage U and any $n > 0$ we can find an entourage V with $V^{\circ n} \subset U$. If in addition such a V is symmetric we shall call it an n^{th} *root* of U. If $n = 2$ we shall simply call it a root and if $n = 3$ we shall call such an entourage a cube-root. As mentioned above for any entourage we can take n^{th} roots.

Example 1.2. Let X be a set and let $\Phi = \{X \times X\}$. It is easy to see that this is a uniform structure that we call the *trivial uniform structure.*

Example 1.3. At the other end of the spectrum let $\Phi = \{A \subset X \times X \mid \Delta_X \subset A\}$. This is called the *discrete uniform structure* on X.

Example 1.4. Let d be a pseudo-metric on X. We define Φ by saying A is an entourage if we can find some $\varepsilon > 0$ and $D_\varepsilon \subset A$ where $D_\varepsilon = \{(x, y) \mid d(x, y) < \varepsilon\}$. The symmetry property of the metric gives that this structure is closed under inversion, the set is closed under intersections as $D_\varepsilon \cap D_{\varepsilon'} = D_{\min(\varepsilon,\varepsilon')}$, $d(x, x) = 0$ shows that all entourages contain the diagonal, and finally taking roots follows from the triangle inequality, i.e. $D^2_{\varepsilon/2} \subset D_\varepsilon$. We shall call this the *metric uniform structure* and it guides a lot of the intuition just as notions of metric topology guide intuition in topological spaces.

Example 1.5. Let X be a set and (Y, Φ) a uniform space. Given a function $f : X \to Y$ we define $f^*\Phi$ on X to be those sets $A \subset X \times X$ such that we can find some U an entourage on Y with $U \subset (f \times f)A$. We will call this the pullback uniform structure and denote it by $f^*\Phi$.

Definition 1.6. A *fundamental system* on X is a collection of non-empty subsets of $X \times X$, $\mathfrak{B}$, such that

1. $U \in \mathfrak{B} \implies \Delta_X \subset U$.
2. $U, V \in \mathfrak{B} \implies \exists W \in \mathfrak{B},\ W \subset U \cap V$.
3. $U \in \mathfrak{B} \implies \exists V \in \mathfrak{B},\ V \subset U^{-1}$
4. $U \in \mathfrak{B} \implies \exists V \in \mathfrak{B},\ V^{\circ 2} \subset U$.

It is clear that a fundamental system uniquely defines a uniform structure by taking all supersets. It is thus sufficient to give a fundamental system to define a uniform structure and this is often convenient.

Example 1.7. For the discrete uniform structure, 1.3, we have that $\{\Delta_X\}$ is a fundamental system for the structure.

Example 1.8. We essentially defined the metric uniform structure, 1.4, using a fundamental system. Namely $\{D_\varepsilon \mid \varepsilon > 0\}$ is a fundamental system. In fact we can replace this with a countable fundamental system.

Example 1.9. Let Y be a uniform space and $f : X \to Y$ a function. Then $\{(f \times f)^{-1}U \mid U \in \Phi_Y\}$ defines a fundamental system on X. It is immediate that this defines the pull back uniformity as in 1.5.

An important example of this is when $A \subset Y$. The corresponding uniformity will be called the *subspace uniformity.*

Example 1.10. We can define the dual of the above example, namely if Z is uniform and $f : Z \to X$ is a function then $\{(f \times f)U \mid U \in \Phi_Z\}$ is a fundamental system. The uniformity it generates is called the *push-forward uniformity.*

In the case where f is surjective we will also call this the *quotient uniformity.*

Lemma 1.11. *Two fundamental systems $\mathfrak{B}_i$ define the same uniform structure iff $\forall U \in \mathfrak{B}_1$ we can find $V \in \mathfrak{B}_2$ with $V \subset U$ and vice versa.*

We have a natural partial ordering on the set of uniform structures on a given set X by inclusion. If $\Phi \subset \Phi'$ we shall say that Φ is *coarser* than Φ', or that Φ' is *finer* than Φ. We use the standard terminology by saying that Φ is an upper bound for (Φ_i) if $\Phi_i \subset \Phi\, \forall i$ and a lower bound if $\Phi \subset \Phi_i\, \forall i$. An upper bound is called a *least upper bound* if it is coarser than all other upper bounds, similarly a *greatest lower bound* is a lower bound that is finer than all other bounds - these are both unique. We will also call a least upper bound the *join* of the (Φ_i) and the *meet* for the greatest lower bound.

Lemma 1.12. *Least upper bounds exist.*

Proof. Let $(\Phi_i)_{i\in I}$ be a non-empty family of uniform structures (the least upper bound of the empty family is the trivial uniform structure 1.2). Then let $\mathfrak{B}$ be the set of all finite intersections from the family, that is

$U \in \mathfrak{B}$ if there is some finite $\{i_1, \dots, i_r\} \subset I$ and $U = U_{i_1} \cap \dots \cap U_{i_r}$ where $U_i \in \Phi_i$. A simple check shows that this defines a fundamental system by applying the properties pointwise (and taking the union of the finite sets for the intersection), and hence defines a uniform structure on X. If Ψ is any uniform structure finer than all the Φ_i then it must contain the fundamental system $\mathfrak{B}$ and hence all their supersets. $\square$

Corollary 1.13. *Greatest lower bounds exist.*

Proof. Let $(\Phi_i)_{i\in I}$ be a non-empty family of uniform structures (the greatest lower bound of the empty family is the discrete uniform structure 1.3). Then $\{\Psi \mid \Psi \subset \Phi_i \,\forall i\}$ is a non-empty family of uniform structures (it always contains the trivial uniform structure). We can thus take its least upper bound and it follows that this is indeed a greatest lower bound. $\square$

Note that in general the join, i.e. the least upper bound, is quite easy to describe in terms of the generating uniform structures. The meet however is not constructively defined and cannot be described simply from the generating uniform structures.

Example 1.14. Let X be a set and $f_i : X \to Y_i$ a family of functions into uniform spaces. Then the least upper bound of the pullback uniform structures, 1.5. We shall call this the uniform structure generated by the (f_i).

Example 1.15. Let (X_i) be a family of uniform spaces then we put a uniform structure on $\prod_i X_i$, generated by the $p_j : \prod_i X_i \to X_j$, 1.14.

1.1 Maps

Definition 1.16. Let X and Y be uniform spaces. A function $f : X \to Y$ is called *uniform* if $(f \times f)^{-1}(U)$ is an entourage on X whenever U is an entourage on Y.

Lemma 1.17. *A function between uniform spaces $f : X \to Y$ is uniform iff $f^*\Phi_Y$ is coarser than Φ_X, or equivalently iff $f_*\Phi_X$ is finer than Φ_Y.*

Example 1.18. Let $\mathbb{R}$ be given its metric uniformity. Then a function $f : \mathbb{R} \to \mathbb{R}$ is uniform iff for any $\epsilon > 0$ we can find a $\delta > 0$ such that $(f \times f)D_\delta \subset D_\epsilon$, i.e.

$$\forall x,\, y \text{ s.t. } |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

which is the normal definition of uniformity.

Example 1.19. Consider the embedding $\mathbb{R} \subset S^1$ using stereographic projection. We then give $\mathbb{R}$ the subspace uniformity where on S^1 we give it the metric uniformity. We claim that a function $f : \mathbb{R} \to \mathbb{R}$ is uniform iff we can complete the following to a commutative diagram:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[d] & \mathbb{R} \arrow[d] \\ S^1 \arrow[r, dotted, "F"] & S^1. \end{tikzcd}$$

To show that the existence of F is sufficient follows immediately from the form of the subspace entourages. We won't show the complete proof of the necessity as it will follow easily from results later in the text. We have that $\mathbb{R} \xrightarrow{f} \mathbb{R} \to S^1$ is uniform and we shall also denote this composition by f. Let (x_n) be a sequence in $\mathbb{R}$ that tends to infinity, (where we make the identification $S^1 \cong \mathbb{R}P^1$) we claim that $(f(x_n))$ is a Cauchy sequence in S^1 which follows pretty much immediately from the definitions. It thus has an adherence point in S^1 which is unique by compactness. This defines the extension of f and uniformity of this extension follows from continuity at ∞.

Although very informally constructed here the rough ideas of the above example work in far more general setting and will guide the development of the ideas over the next couple of sections.

Example 1.20. Let (X, d) be a pseudo-metric space, then the metric is uniform $d : X \times X \to \mathbb{R}$ where $\mathbb{R}$ is given the metric uniformity. This is equivalent to showing that for any $\epsilon > 0$ we can find a $\delta > 0$ such that $d(x_i, y_i) < \delta \implies |d(x_1, x_2) - d(y_1, y_2)| < \epsilon$. We have that

$$d(x_1, x_2) \le d(x_1, y_1) + d(y_1, y_2) + d(y_2, x_2) \le 2\delta + d(y_1, y_2)$$

and swapping the roles of y_i and x_i we similarly get $d(y_1, y_2) \leq d(x_1, x_2) + 2\delta$ and so $|d(x_1, x_2) - d(y_1, y_2)| < 2\delta$ and the result follows.

It is easy to see that with this definition we get a category of uniform spaces, $\mathfrak{USp}$. This is clearly a concrete category where we simply forget the uniform structure. The following is proved identically as for toplogical spaces.

Proposition 1.21. *The forgetful functor has both a left and right adjoint. The left adjoint puts the discrete uniform structure on the set and the right adjoint puts the trivial structure.*

Proposition 1.22. *Let $f_i : X \to Y_i$ be given the coarsest uniform structure making all the f_i uniform, 1.14. Then a map $Z \to X$ is uniform iff $Z \to Y_i$ is $\forall i$.*

Similarly if we have maps $g_i : Z_i \to X$ and X is given the finest uniform structure such that all the g_i are uniform, then a map $X \to Y$ is uniform iff $X_i \to Z$ is uniform $\forall i$.

Proof. Necessity in both cases is clear. For sufficiency for the least upper bound uniformity let $g : Z \to X$ be such that $f_i \circ g$ is uniform. Then as any entourage on X has the form of a finite intersection $\cap(f_i \times f_i)^{-1}U_i$ the result follows. For the second case suppose that $f : X \to Y$ is given and $f \circ g_i : Z_i \to Y$ is uniform for all i. Then the pull-back uniformity on X under f, 1.5, is such that $g_i : Z_i \to X$ is uniform and hence we must have that the uniformity is coarser than the greatest lower bound. □

We will call the uniformities on X defined in the above proposition as being *generated by* (f_i).

Example 1.23. Let $f : X \to Y$ be a uniform map. Let X/f denote the equivalence relation generated by $x \sim_f y \Leftrightarrow f(x) = f(y)$. Then if we put on X/f the quotient uniformity we gaet that $X/f \to Y$ is an injective uniform map, so every uniform map can be decomposed into a quotient followed by an injection.

1.2 Pseudo-metrics

We have previously seen that any pseudo-metric naturally defines a uniform structure on a set, 1.4. It follows that any family of pseudo-metrics defines a uniform structure by taking the least upper bound. In fact we shall show in this section that every uniform structure arises in this way. The ideas here are taken from [Jam90]

Let X be a uniform space, by a *r-sequence* we mean a family of symmetric entourages (U_n) such that $U_0 = X \times X$, and $U_n^r \subset U_{n-1}$. Given such a sequence we define a function $\operatorname{ord} : X \times X \to \mathbb{N} \cup \{\infty\}$ by $\operatorname{ord}(x, y) = \sup\{n \mid (x, y) \in U_n\}$. We then set

$$d_{U_\bullet}(x, y) = \inf_{(x_0, \ldots, x_k) \in \operatorname{chain}(x,y)} \sum_{i=0}^{k-1} 2^{-\operatorname{ord}(x_i, x_{i+1})} \tag{2}$$

where $\operatorname{chain}(x, y)$ denotes the set of chains between x and y, i.e. $x_0 = x$ and $x_k = y$. We shall also refer to the $\sum 2^{-\operatorname{ord}(x_i, x_{i+1})}$ as the length of the chain $(x_0, \ldots, x_k)$.

Lemma 1.24. *$d_{U_\bullet}$ is a pseudo-metric for any r-sequence. Moreover the metric uniform structure is coarser than the the original structure.*

Proof. As $\operatorname{ord}(x, x) = \infty$ the chain (x, x) gives $d(x, x) = 0$. As the entourages are symmetric we have that $\operatorname{ord}(x, y) = \operatorname{ord}(y, x)$ and so we have that the length of the chain is equal to the length of its transpose so that $d(x, y) = d(y, x)$. Finally for the triangle inequality we see that there is an equivalence between chains from x to y that passes through z and a pair of chains, one from x to z and one from z to y. Thus we get that $d(x, z) + d(z, y)$ is the infimum of all chains from x to y that pass through z and so clearly $\geq d(x, y)$.

For the second part of the lemma we can consider the fundamental system $\{D_{1/2^n}\}$ see 1.8. We have that $U_{n+1} \subset D_{1/2^n}$ and the result follows. □

Lemma 1.25. *If (U_n) is a 3-sequence then* $\operatorname{ord}(x, y) \leq n \implies d_{U_\bullet}(x, y) \geq 1/2^{n+1}$.

Proof. The claim is equivalent to showing that if the length of a chain $(x_0, \ldots, x_k)$ is less than $1/2^{n+1}$ then $\operatorname{ord}(x_0, x_k) > n$. We shall prove this by induction on the length of the chain, the base case $k = 1$ holding trivially. For the inductive step take a chain $(x_0, \ldots, x_k)$ with $k > 1$ and length $< 1/2^{n+1}$. It follows for any split we choose, i.e. $i \in \{1, \ldots k\}$, that one of the sub-chains $(x_0, \ldots, x_i)$ and $(x_i, \ldots, x_k)$ must have length $< 1/2^{n+2}$. If we can find a split where both have length $< 1/2^{n+2}$ then by the inductive step we must have that $\mathrm{ord}(x_0, x_i), \mathrm{ord}(x_i, x_k) > n+1$ and so concatenating we get $\mathrm{ord}(x_0, x_k) > n$. If not choose a split that minimises the distance between the length of the two sub-chains - we will assume that the first chain $(x_0, \ldots, x_i)$ is shorter than $(x_i, \ldots, x_k)$ although a virtually identical argument holds if the roles are switched. Then if we consider $(x_{i+1}, \ldots, x_k)$ we must have that this chain has $< 1/2^{n+2}$, else the cut wouldn't be minimal. If we then consider the triple $(x_0, \ldots, x_i), (x_i, x_{i+1}), (x_{i+1}, \ldots, x_k)$ then we have by induction that $\mathrm{ord}(x_0, x_i) > n+1$ and $\mathrm{ord}(x_{i+1}, x_k) > n+1$. Conversely if $\mathrm{ord}(x_i, x_{i+1}) \leq n+1$ then the length of the chain is at least $1/2^{n+1}$, a contradiction, so we must have that they have order $> n+1$ as well. It follows that as we have a 3-sequence that the concatenation of the triple lies in U_{n+1}, i.e. $\mathrm{ord}(x_0, x_k) \geq n+1$. □

Corollary 1.26. *Every uniform structure is equivalent to one given by the least upper bound of a family of psuedo-metrics.*

Proof. Consider the family of all 3-sequences and take the corresponding pseudo-metrics and the least upper bound uniform structure. Then by 1.24 we have that the original uniform structure is finer than this least upper bound. Conversely we have that for any entourage U we can build a 3-sequence with $U_1 = U$ so that $D_{1/4} \subset U$ for this pseudo-metric and hence the uniform structure is coarser as well and must be equivalent. □

We shall call a uniform structure *metrizable* if it is generated by a single pseudo-metric.

Corollary 1.27. *A uniform structure is metrizable iff it has a countable fundamental system.*

Proof. Necessity follows from the fundamental system as described in 1.8. Conversely take a countable fundamental system (B_n). Then define a 3-sequence with $U_1 = B_1$ and then inductively $U_{n+1}^{\circ 3} \subset U_n \cap B_{n+1}$. For this 3-sequence we have that $D_{1/2^{n+2}} \subset B_n$ and so the result follows. □

2 The uniform topology

Let (X, Φ) be a uniform space and U an entourage. Let $U[x] = \{y \mid (y, x) \in U\}$ and similarly $[x]U = \{y \mid (x, y) \in U\}$, for a symmetric entourage these are identical. Let $\mathcal{U}$ be the set of subsets of X where $U \in \mathcal{U}$ if $\forall x \in U$ we can find an entourage V of X such that $V[x] \subset U$. In fact it is sufficient to consider symmetric entourages (for if $W \subset V$ then $W[x] \subset V[x]$) and thus this defines the same family of subsets if we had used $[x]V$ instead.

Proposition 2.1. *$\mathcal{U}$ defines a topology on X. Moreover $\{U[x] \mid U \in \Phi\}$ defines a neighbourhood base of $x \in X$ in this topology.*

Proof. The empty set lies in $\mathcal{U}$ and using the entourage $X \times X$ we see that X does as well. Being closed under arbitrary union follows immediately. For being closed under intersection let $U_i \in \mathcal{U}$, $x \in U_1 \cap U_2$, and take entourages V_i with $V_i[x] \subset U_i$. Then $(V_1 \cap V_2)[x] \subset U_1 \cap U_2$.

For the second part let us firstly define the interior of a set $A \subset X$. Let $A^{\circ} = \{x \in A \mid \exists U \in \Phi_X \; U[x] \subset A\}$. To see that this set is open under the uniform topology let $y \in A^{\circ}$, so we have some U with $U[y] \subset A$. Let V be a root of U and let $z \in V[y]$, then for $z' \in V[z]$ we have $z' \in U[y]$ hence $V[z] \subset A$ so that $z \in A^{\circ}$ and A° is open as claimed.

Now we can turn to the proof, let U be an entourage and set $W = U[x]^{\circ}$. This is an open set and, $x \in W$ and $W \subset U[x]$ hence the neighbourhood base is finer than that generated by the $U[x]$. It is immediately coarser and hence the two filters are equal. □

Given a fundamental system of entourages it is sufficient to use only such sets in the construction of the topology or the neighbourhood base.

Example 2.2. For the trivial uniform structure, 1.2, we have that $\mathcal{U}$ is the indescrete topology - we only have a single entourage to check.

Example 2.3. For the discrete uniform structure, 1.3, the induced topology on X is the discrete one. To see this note that as $\Delta_X \in \Phi$ all the singletons are in the topology, namely $\Delta_X[x] = \{x\}$.

Example 2.4. For the metric uniform structure, 1.4, we have that the induced topology is that of the metric. To see this use the description of neighbourhood basis coming from 2.1. This was used in the proof of 2.1.

Lemma 2.5. *For a uniform space X any entourage is a neighbourhood of the diagonal under the product topology.*

Proof. Let U be an entourage and V a root of U. Then $V[x] \times V[x] \subset U$ for all x hence so is their union. $\square$

Lemma 2.6. *Let U be an entourage of X and V a cube-root of U. Then $\overline{V} \subset U$.*

Proof. Let $(x, y) \in \overline{V}$. We have $V[x] \times V[y] \cap V \neq \emptyset$ so that we can find $(u, v) \in V$ with $(x, u), (y, v) \in V$. Concatenating we get the result. $\square$

Lemma 2.7. *The uniform topology is regular.*

Proof. Let $x \in X$ and $Z \subset X$ disjoint from X. Then we can find an entourage U with $U[x] \subset X \setminus Z$. By 2.6 take an entourage V such that $\overline{V} \subset U$. Then $\overline{V}[x] \subset U[x]$ is a neighbourhood of x and $Z \subset \overline{V}[x]^c$. $\square$

Lemma 2.8. *If the uniform topology is T_0 then it is T_2.*

Proof. Take two distinct points x and y and an open neighbourhood of x not containing y, call it U. Then $Z = X \setminus U$ is closed and disjoint from x hence by 2.7 they can be seperated by open sets. $\square$

Proposition 2.9. *Every uniform map is continuous for the uniform topologies.*

Proof. Let $f : X \to Y$ be uniform. It is sufficient to show that for any $x \in X$ and any neighbourhood of $f(x)$, N, we have that $f^{-1}N$ is a neighbourhood of x. Let U be an entourage on Y such that $U[f(x)] \subset N$ then $(f \times f)^{-1}U$ is an entourage on X and $\left((f \times f)^{-1}U\right)[x] \subset f^{-1}N$. $\square$

Example 2.10. Consider $p : X \to Z$ where X is uniform and Z is given the uniform structure induced by p. Then p is an open map. To see this note that for U an entourage on X we have that pU is an entourage on Z. Then $p(U[x]) = pU[px]$ and hence it maps neighbourhoods to neighbourhoods and the result follows.

Corollary 2.11. *The uniform topology gives a functor $\mathfrak{USp} \to \mathfrak{Top}$ whose image lies in the class of regular topological spaces.*

The functor is not full, though see 2.12 below. It is however faithful and it commutes with the natural forgetful functor from $\mathfrak{Top}$ to $\mathfrak{Set}$. The adjoints also commute, i.e. the left adjoint of $\mathfrak{Top} \to \mathfrak{Set}$ is $\mathfrak{Set} \to \mathfrak{USp} \to \mathfrak{Top}$ where the first map is the left adjoint of the forgetful functor on uniform spaces and the second is the functor in 2.11.

If X is a set with both a uniform and topological structure we say that the structures are *compatible* if the uniform topology is equal to the given topology. We also say that a topological space is *uniformizable* if there is a uniform structure compatible with the given topology.

Proposition 2.12. *Let X be a uniform space with compact uniform topology. Then any continuous map $f : X \to Y$ to a uniform space is uniform.*

Proof. Let U be an entourage on Y and U' a root of U. Given $x \in X$ choose an entourage V'_x such that $V'_x[x] \subset f^{-1}U'[f(x)]$ which exists by continuity and take V_x a root of V'_x. By compactness we can find finitely many of the V_i such that $V_i[x_i]$ cover X. Let $V = \cap_i V_i$, an entourage, and $(x, y) \in V$, then we can find some x_i such that $(x, x_i) \in V_i$ and as (x, y) is we get $(y, x_i) \in V'_i$. It follows that $(f(x), f(x_i)), (f(y), f(x_i)) \in U'$ and so $(f(x), f(y)) \in U$. $\square$

Theorem 2.13. *If X is a uniformizable compact space then the uniform structure is unique.*

Proof. Apply 2.12 to the identity map $1_X : X \to X$ to see that the two uniform structures are each coarser than the other. $\square$

2.1 Hausdorff space of a uniform space

We have seen above that uniform spaces have quite rich seperation properties. In fact more is true and we can associate to any uniform space a Hausdorff space naturally.

Let R be the intersection of all the entourages on X. It follows directly from the properties of the uniform structure that this is an equivalence relation. We then give X/R the quotient uniform structure.

Lemma 2.14. *The uniform structure on X is generated by the map $\pi_R : X \to X/R$.*

Proof. The induced structure is clearly coarser as the map is uniform so we need to show it is finer. Let U be an entourage on X, we shall let U/R denote $(\pi_R \times \pi_R)U$ the entourage on X/R. Let $(x, y) \in (\pi_R \times \pi_R)^{-1}(U/R)$. Then we can find some $(x', y') \in U$ with $(x, x'), (y, y') \in R$. Then as $R \subset U$ we can concatenate to get $(x, y) \in U^{\circ 3}$. As U was arbitrary it follows the induced uniform structure is finer than the original one and it is clearly coarser. □

Lemma 2.15. *X/R is Hausdorff for its uniform topology.*

Proof. Let $[x] \neq [y] \in X/R$. This implies that there is some entourage U such that $(x, y) \notin U$. Let V be a root of U then $V[x] \cap V[y] = \emptyset$ and so $\pi_R(V[x]) \cap \pi_R(V[y]) = \emptyset$. □

Corollary 2.16. *Every uniform space admits a uniform section from its Hausdorff space, $X/R \to X$.*

Proof. Let $\chi : X \to X$ be some choice function for the relationship R, i.e. it picks a unique element in each equivalence class. Then let U be an entourage and V a cube root of U. If $(x, y) \in V$ we have $(x, \chi(x))$, $(y, \chi(y)) \in V$ by definition of the relation R so that $(\chi(x), \chi(y)) \in U$ and the map χ is uniform. It follows the map $X/R \to X$ is by 1.22. □

Corollary 2.17. *Let Y be a Hausdorff uniform space and $f : X \to Y$ a uniform map. Then there exists a unique $F : X/R \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ X/R \arrow[ur, dotted] & \end{tikzcd}$$

Moreover X/R is universal with this property.

Proof. We define the map by taking the section as in 2.16. The uniqueness of such a map is immediate and universality follows. □

Corollary 2.18. *A uniform space X is Hausdorff iff $R = \Delta_X$.*

Proof. If $R = \Delta_X$ then for $x, y \in X$ we can find some V with $(x, y) \notin V$. It follows that if W is a root of V that $W[x] \cap W[y] = \emptyset$.

Conversely if X is Hausdorff then for $x \neq y$ if we take disjoint neighbourhoods seperating them we can find an entourage U with $U[x] \cap U[y] = \emptyset$ and then $(x, y) \notin U$. □

2.2 Divisibility

Definition 2.19. A topological space X is called divisible if for every neighbourhood U of the diagonal in $X \times X$ we can find a root of U.

The following follows immediately from the definition

Lemma 2.20. *If X is divisible then $N(\Delta_X)$, the neighbourhoods of the diagonal, is a uniform structure.*

Some caution should be taken with the above lemma. It doesn't say that the uniform structure is compatible with the original topology. For example the space $\mathbb{N}$ where the open sets are given by $U_n = \{m \mid m < n\}$ for $n = 0, \ldots, \infty$ is divisible (there is only the trivial neighbourhood of the diagonal) but the uniform topology is coarser. We do however have

Lemma 2.21. *If X is a divisible regular space then the uniform structure $N(\Delta_X)$ is compatible with the topology.*

Proof. The uniform topology is clearly coarser so we just need to show it is finer. Let U be a non-empty open set and let $Z = X \setminus U$. Let $x \in U$ then by regularity choose some $x \in V \subset U$ a neighbourhood with $Z \cap \overline{V} = \emptyset$. Then $\{U, \overline{V}^c\}$ gives a neighbourhood cover of X and so $W = U \times U \cup \overline{V}^c \times \overline{V}^c$ is a neighbourhood of the diagonal. If $(y, x) \in W$ then $y \in U$ so that $W[x] \subset U$. As x was arbitrary the result follows. $\square$

Example 2.22. Let (X, d) be a metric space and $U \in N(\Delta_X)$. Then for every $x \in X$ we can find some $\epsilon(x) > 0$ such that $B_{\epsilon(x)}(x) \times B_{\epsilon(x)}(x) \subset U$. It follows that

$$\bigcup_{x \in X} B_{\epsilon(x)}(x) \times B_{\epsilon(x)}(x) \subset U.$$

Then we consider V as given by the union overall $B_{\epsilon(x)/3}(x) \times B_{\epsilon(x)/3}(x)$. It follows that if $(x, y) \in V \circ V$ we can find some $z, u, v \in X$ with $x, z \in B_{\epsilon(u)/3}(u)$ and $y, z \in B_{\epsilon(v)/3}(v)$. Assume that $\epsilon(u) \leq \epsilon(v)$, then $d(x, v) \leq d(x, u) + d(u, z) + d(z, v) \leq (\epsilon(u) + \epsilon(u) + \epsilon(v))/3 \leq \epsilon(v)$. It follows that $x, y \in B_{\epsilon(v)}(v)$ so that $(x, y) \in U$.

Example 2.23. If X is a regular divisible space then any continuous map $f : X \to Y$ to a uniform space Y is uniform. This follows immediately from 2.5. It follows that we have a left adjoint to the forgetful functor $\mathfrak{USp} \to \mathfrak{Top}$ if we restrict to the category of regular divisble spaces.

Lemma 2.24. *If X is a regular space and $x, y \in X$ can be seperated then we can find some $W \in N(\Delta_X)$ such that $(x, y) \notin W \circ W$.*

Proof. By assumption we can find some open U, V containing x and y respectively with $U \cap V = \emptyset$. By regularity we can then find $Z \subset U$ a closed neighbourhood of x and $Z' \subset V$ a closed neighbourhood of y. Let $W = (Z \cup Z')^c$, then $D = U \times U \cup V \times V \cup W \times W$ is a neighbourhood of the diagonal. If $(x, y) \in D \circ D$ then we can find some $z \in X$ with $(x, z) \in D$ and $(z, y) \in D$. As $x \notin V \cup W$ we get $z \in U$ and similarly using y we have $z \in V$ a contradiction. $\square$

Proposition 2.25. *If X is regular compact then it is divisble.*

Proof. If X is not divisble then we have some U an open neighbourhood of the diagonal with $V \circ V \not\subset U$ for all neighbourhoods of Δ_X. Then $\{(V \circ V) \setminus U\}$, where V runs over the neighbourhoods of the diagonal, is a filter base on the compact space $X \times X \setminus U$ as $(A \cap B)^{\circ 2} \subset A^{\circ 2} \cap B^{\circ 2}$. By compactness this filter has an adherence point (x, y).

By 2.24 we can find some neighbourhood of the diaginal W with $(x, y) \notin W \circ W$ contradicting the fact that they are an adherence point of the filter. $\square$

We now have the dual to the uniqueness theorem above.

Theorem 2.26. *If X is regular compact then $N(\Delta_X)$ defines a uniform structure on X which is the unique one compatible with the topology by 2.13.*

3 Completeness

Definition 3.1. A filter F on a uniform space X is called *Cauchy* if for every entourage U of X we can find an $A \in F$ such that $A \times A \subset F$. We shall denote by $\mathscr{C}(X)$ the set of Cauchy filters on X.

Example 3.2. Consider the neighbourhood filter $N(x)$. Given an entourage U take a root V of U, then $V[x] \times V[x] \subset U$ hence $N(x)$ is Cauchy.

Lemma 3.3. *If a filter F converges to x then it is a Cauchy filter.*

Proof. We have $N(x) \subset F$ which is Cauchy by 3.2 and the result is then clear. $\square$

Lemma 3.4. *An adherence point of a Cauchy filter is a convergence point.*

Proof. Let F be Cauchy and x an adherence point. Take $U[x]$ a neighbourhood of x and V a root of U. Let $A \in F$ be such that $A \times A \subset V$ then as $A \cap V[x] \neq \emptyset$, by adherence, choose some a in the intersection. Then given any $b \in A$ we have $(b, a), (a, x) \in V$ hence $A \subset U[x]$ and the result follows. □

Recall that for a filter F on X and a function $f : X \to Y$ we get the filter $f_* F$ on Y which is generated by the base $\{fA \mid A \in F\}$.

Lemma 3.5. *If $f : X \to Y$ is uniform then $f_* \mathscr{C}(X) \subset \mathscr{C}(Y)$.*

Proof. Let $F \in \mathscr{C}(X)$ and V an entourage on Y. Then we can find some $A \in F$ with $A \times A \subset (f \times f)^{-1}(V)$, i.e. $fA \times fA \subset V$, so that $f_* F$ is Cauchy. □

3.1 Complete spaces

Definition 3.6. A uniform space is called *complete* if it is Hausdorff and all Cauchy filters converge.

Lemma 3.7. *Let X be a uniform space such that all Cauchy filters converge, and let X/R be the corresponding Hasudorff space. Then X/R is complete.*

Proof. Using the existence of the section 2.16 and 3.5 we can lift any Cauchy sequence to X. By assumption this converges to some element so we claim that the image of this element in X/R is a point of convergence for the original Cauchy sequence. If not then we can take a neighbourhood of $[x]$, W, that has an empty intersection with some element in the filter, A. Then $\pi_R^{-1} W \cap \pi_R A = \emptyset$ contradicting the fact that x was a limit of $(\pi_R)_* F$. □

Proposition 3.8. *A subset $A \subset X$ of a complete space is complete iff its closed.*

Proof. For the necessity suppose that $x \in \overline{A} \setminus A$. Then $N(x) \cap A$ is a Cauchy filter on A that does not contain a convergence point hence A is not complete. For sufficiency suppose that A is closed and F be a Cauchy filter on A. Then $\iota_* F$ is Cauchy in X and so converges to some $x \in X$. As X is regular, 2.7, if $x \notin A$ we can find a $U \in N(x)$ with $U \cap A = \emptyset$ contradicting the fact that it is a filter. □

Proposition 3.9. *Any compact uniform space is complete.*

Proof. Let F be a Cauchy filter. Then it has an adherence point and hence converges by 3.4. □

Example 3.10. $\mathbb{R}$ with the metric uniformity is complete.

3.2 Minimal filters

Let $A \subset X$ and take an entourage U. We let $A \cdot U$ denote the U-translates of A, that is:

$$A \cdot U = \{y \mid \exists a \in A, (y, a) \in U\} \tag{3}$$

From the definition we see that

$$A \cdot U = \bigcup_{a \in A} U[a] \tag{4}$$

and thus is a neighbourhood of A given the uniform topology. We also see that if V is a root of U then $V[x] \cdot V \subset U[x]$ for all x.

Lemma 3.11. *Let F be a filter then the set of all its translates by entourages gives a filter base and we denote the corresponding filter by F°. Moreover $F^{\circ} \subset F$ and it is Cauchy iff F is.*

Proof. We will show that the family of translates of F form a filter base. For any set A we have $A \subset A \cdot U$ and so the translates are not empty and they don't contain the empty set. If $A \cdot U$ and $B \cdot V$ belong to F° then $A \cap B \in F$ and $U \cap V$ is an entourage and we clearly have $(A \cap B) \cdot (U \cap V) \subset (A \cdot U) \cap (B \cdot V)$ hence they define a filter base. Moreover as $A \subset A \cdot U$ we clearly have that the associated filter, which is F°, is a subfilter of F.

Let F be a Cauchy filter and U an arbitrary entourage. Take V a cube-root of U and choose $A \in F$ such that $A \times A \subset V$. Let $x, y \in A \cdot V$ then $\exists a, b \in A$ with $(a, x), (b, y) \in V$ and we also have $(a, b) \in V$ by choice of A. It follows $(x, y) \in U$ and so $A \cdot V \times A \cdot V \subset U$. □

Lemma 3.12. *If F is Cauchy and $G \subset F$ is a Cauchy subfilter then $F^{\mathrm{o}} \subset G$.*

Proof. Let $A \in F$ and U an entourage on X. Choose $B \in G$ with $B \times B \subset U$. As $A \cap B \neq \emptyset$ pick any $a \in A \cap B$ then for all $b \in B$ we have $(a, b) \in U$, i.e. $b \in U[a] \subset A \cdot U$. □

It follows that F^{o} is the smallest Cauchy subfilter of F. In particular $(F^{\mathrm{o}})^{\mathrm{o}} = F^{\mathrm{o}}$ and thus we can "fatten" any element of F^{o}, i.e. given $A \in F^{\mathrm{o}}$ we can find a $B \in F^{\mathrm{o}}$ and an entourage U with

$$B \cdot U \subset A. \tag{5}$$

In fact this is a defining property for a Cauchy filter for if F satisfies this requirement then $F \subset F^{\mathrm{o}}$.

Definition 3.13. A Cauchy filter F such that $F^{\mathrm{o}} = F$ will be called *minimal*.

We denote $\mathscr{C}^{\min}(X) \subset \mathscr{C}(X)$ the set of minimal Cauchy filters on X.

Example 3.14. Consider the nighbourhood Cauchy filter, $N(x)$. Given any neighbourhood $U[x]$ then for a root of U, V, we have $V[x] \cdot V \subset U[x]$ as alluded to earlier. Thus $N(x) \subset N(x)^{\mathrm{o}}$, hence $N(x)$ is minimal.

Lemma 3.15. *Let F be Cauchy. Then F converges to x iff F^{o} does.*

Proof. As $F^{\mathrm{o}} \subset F$ the condition is clearly sufficient. For the converse as $N(x) \subset F$ is minimal by 3.14 we have $F^{\mathrm{o}} = N(x)$. □

Remark 3.16. It is important to note that although the pushforward of a Cauchy filter is Cauchy it is not the case that minimality is preserved. For example we can consider $(0, 1)$ with two subspace uniformities from its embedding in the interval and the circle. One can see that the embedding in the interval is finer and so defines a uniform map $((0, 1), \Phi_I) \to ((0, 1), \Phi_{S^1})$. Then there are two minimal Cauchy filters on $((0, 1), \Phi_I)$, the two endpoints, that map to the same filter on $((0, 1), \Phi_{S^1})$.

In fact there is a nice interplay between on the one hand getting more filters by having a coarser uniformity and the coarseness of the uniformity merging together more Cauchy filters to minimal ones. See the example in the precompact section for a case where the merging completely dominates.

3.3 Pull backs

In general the pullback of a filter is not a filter. However under certain circumstances, which are those thare are needed for the proofs around completeness, we can.

Lemma 3.17. *Let $f : X \to Y$ with Y uniform and f having a dense image. Then for a filter F on Y, $f^! F := \{f^{-1}A \mid A \in F^{\mathrm{o}}\}$ is a filter base on X.*

Proof. The set is closed under intersections as $f^{-1}A \cap f^{-1}B = f^{-1}(A \cap B)$ so we just need to show that we have no empty sets in the filter. Given $A \in F^{\mathrm{o}}$ we can find some $B \in F$ and entourage U with

$$B \cdot U = \bigcup_{b \in B} U[b] \subset A.$$

Then as f has a dense image $f^{-1}U[b] \neq \emptyset$ as it is a neighbourhood of b and it follows $f^{-1}A$ is not empty. □

Note that to define $f^!$ we need to be in the realm of uniform spaces, or at least the codomain of f must be.

Corollary 3.18. $F \subset f_* f^! F$.

Lemma 3.19. *Let $f : X \to Y$ be a uniform map with dense image. If the uniform structure on X is equal to the induced uniform structure by f and if F is Cauchy so is $f^! F$.*

Proof. Taking the notation in the claim let U be an entourage on X. Then by the condition on the uniform structure we can find some V an entourage on Y with $(f \times f)^{-1} V \subset U$. By Cauchyness of F we can find a $A \in F$ with $A \times A \subset V$, and so $f^{-1}A \times f^{-1}A \subset (f \times f)^{-1} V \subset U$. □

3.4 Completion of a uniform space

We begin by giving $\mathscr{C}(X)$ a natural uniform structure: For an entourage U on X let $\mathscr{C}(U)$ denote the pairs of Cauchy filters, (F_1, F_2), such that we can find an $A_i \in F_i$ with $A_1 \times A_2 \subset U$.

Proposition 3.20. *The family $\{\mathscr{C}(U) \mid U \in \Phi_X\}$ is a fundamental system for a uniform structure.*

Proof. By the defining property of a Cauchy filter that $\Delta_{\mathscr{C}(X)} \subset \mathscr{C}(U)$ for all entourages U.

For closure under intersection we clearly have $\mathscr{C}(U \cap V) \subset \mathscr{C}(U) \cap \mathscr{C}(V)$ (in fact with not much work one can see they are equal). Similarly $\mathscr{C}(U)^{-1} = \mathscr{C}(U^{-1})$.

Finally let V be a root of U. Then for $(F_1, F_2), (F_2, F_3) \in \mathscr{C}(V)$ choose $A_1 \in F_1$, $A_2, B_2 \in F_2$ and $B_3 \in F_3$ such that $A_1 \times A_2 \subset V$ and $B_2 \times B_3 \subset V$. Then as $A_2 \cap B_2 \neq \emptyset$ choose some a in the intersection and we have that $A_1 \times \{a\}$ and $\{a\} \times B_3$ are subsets of V and so by concatenation $A_1 \times B_3 \subset U$, i.e. $\mathscr{C}(V)^{\circ 2} \subset U$. $\square$

We give $\mathscr{C}^{\min}(X)$ the subset uniform structure.

Lemma 3.21. *Under the above uniform structure $\mathscr{C}^{min}(X)$ is a Hausdorff space.*

Proof. Let F_i be two minimal Cauchy filters on X and assume that they cannot be seperated, i.e. $(F_1, F_2) \in \mathscr{C}(U)$ for all entourages U. Then let $A_i \in F_i$ and find $A_i' \cdot U \subset A_i$ by the structure of the minimal filters. By assumption we can find $B_i \in F_i$ with $B_1 \times B_2 \subset U$. Choose $a_i \in A_i' \cap B_i$ so that $(a_1, a_2) \in U$ and in particular $a_2 \in A_1$ so that $A_1 \cap A_2 \neq \emptyset$. As the A_i where arbitrary it follows that $F_1 \cap F_2$ defines a filter base and hence a filter which contains both F_i and thus is Cauchy. It then follows that $F_1 = F_2$ using minimality. This shows the space is T_0 and using 2.8 the result follows. $\square$

Lemma 3.22. *$\mathscr{C}^{min}(X)$ is isomorphic to the Hausdorff space related to $\mathscr{C}(X)$.*

Proof. Firstly note that $(F, F^{\text{o}}) \in \mathscr{C}(U)$ for all entourages U. To see this let V be a root of U and choose $A \in F$ with $A \times A \subset V$. Then $A \cdot V \in F^{\text{o}}$ so that $A \times A \cdot V \subset U$.

Next we claim that $F \sim_R G$ iff $F^{\text{o}} = G^{\text{o}}$. This is sufficient from what we have just shown. Necessity follows from 3.21.

From this it follows that the universal map $\mathscr{C}(X)/R \to \mathscr{C}^{\min}(X)$ is a uniform bijection and the map $\mathscr{C}^{\min}(X) \to \mathscr{C}(X) \to \mathscr{C}(X)/R$ is its inverse. $\square$

Using this result and the existence of the section 2.16, which in this case is simply the subspace, we will liberally move between the two spaces of Cauchy filters without making explicit reference to the application of taking the corresponding minimal section. Note however care needs to be taken when considering multiple uniformities by the remark at the end of the section on minimal filters.

Lemma 3.23. *Let $(F, G) \in \mathscr{C}(U)$ then $(F^{\text{o}}, G^{\text{o}}) \in \mathscr{C}(U \circ U \circ U^{-1})$.*

Proof. Choose $A \in F$ and $B \in G$ with $A \times B \subset U$. Then $(x, y) \in A \cdot U \times B \cdot U$ if we can find some $a \in A$ and $b \in B$ with (x, a), $(y, b) \in U$. The result follows. $\square$

We have a natural function $N : X \to \mathscr{C}^{\min}(X)$ that maps each point to its neighbourhood filter by 3.14.

Lemma 3.24. *The map $N : X \to \mathscr{C}^{min}(X)$ is uniform. Moreover it has dense image and the uniform structure on X is induced from that on $\mathscr{C}^{min}(X)$.*

Proof. Let U be an entourage on X and choose V a cube-root of U. Then if $(x, y) \in V$ we have $V[x] \times V[y] \subset U$ and hence $V \subset (N \times N)^{-1} U$ so that N is uniform.

To show that the unform structure is induced by N as we have that N is uniform it is sufficient to show that the induced structure is finer than that on X. So take V an entourage on X then for $(x, y) \in (N \times N)^{-1} \mathscr{C}(V)$ we have that we can find an entourage W with $W[x] \times W[y] \subset V$ in particular $(x, y) \in V$.

Finally we need to show that the image of N is dense. Let F be a minimal Cauchy filter, U an entourage on X. Choose V a root of U and $A \in F$ with $A \times A \subset V$. Then for $a \in A$ we have that $V[a] \times A \subset U$ for if $x \in V[a], y \in A$ then $(a, x) \in V$ and $(y, a) \in V$ and so $N(a) \in \mathscr{C}^{\min}(U)[F]$. $\square$

Lemma 3.25. *Let F be a Cauchy filter on X. Then N_*F converges to F^{o} in $\mathscr{C}^{min}(X)$.*

Proof. We show that F^{o} is an adherence point of N_*F and the result follows by 3.4. So let U be an entourage on X and $A \in F$. Take V a root of U and $B \in F$ with $B \times B \subset V$. Then for $x \in B \cap A$ we have $V[x] \times B \subset U$ so that $N(x) \in \mathscr{C}(U)[F] \cap NA$. $\square$

Theorem 3.26. *$\mathscr{C}^{min}(X)$ is complete.*

Proof. By the properties of N as given in 3.24 we satisfy the assumptions in 3.19 and thus for F a Cauchy sequence on $\mathscr{C}^{\min}(X)$ we have $N^!F$ is a Cauchy sequence on X. It follows that $N_*N^!F$ converges to $(N^!F)^{\text{o}}$ by 3.25 and as $F \subset N_*N^!F$ we have that $F^{\text{o}} = N_*N^!F^{\text{o}}$ and by 3.15 the result follows. $\square$

We thus have defined a natural dense embedding of a uniform space into a complete Hausdorff space. This construction is in fact universal.

Lemma 3.27. *The operation $\mathscr{C}^{min}$ is a functor from the category of uniform spaces into that of complete spaces. If X is complete then it is isomorphic to $\mathscr{C}^{min}(X)$.*

Proof. The fact that it is a functor is almost an immediate consequence of 3.5, we need to show that the map $\mathscr{C}(f) = [F \mapsto (f_*F)^{\text{o}}]$ is uniform. This follows for if we let U be an entourage on Y then choose some entourage $V \subset (f \times f)^{-1}(U)$. If $(F_1, F_2) \in \mathscr{C}(V)$ it follows that we can find some $A_i \in F_i$ with $fA_1 \times fA_2 \subset U$ and using 3.23 we get the result.

For the second part define the map $\mathscr{C}^{\min}(X) \to X$ that takes each minimal Cauchy sequence to its limit. This is clearly a bijection and an inverse to N so we just need to show its uniform. This is immediate as if $(N(x), N(y)) \in \mathscr{C}(U)$ then $(x, y) \in U$. $\square$

Proposition 3.28. *Let $X \to Y$ be any uniform map of X into a complete space. Then there exists a unique map $\mathscr{C}^{min}(X) \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ \mathscr{C}^{min}(X) \arrow[ur, dotted] & \end{tikzcd}$$

Proof. Apply the Cauchy functor as in 3.27 and compose with the uniform isomorphism $\mathscr{C}^{\min}(Y) \to Y$ described there for existence. Uniqueness follows as any uniform map is continuous. $\square$

Corollary 3.29. *For $A \subset X$ with the subspace uniform structure we have that $\mathscr{C}^{min}(A)$ is isomorphic to the closure of A in $\mathscr{C}^{min}(X)$.*

Corollary 3.30. *Let X and Y be uniform spaces. Then $\mathscr{C}^{min}(X \times Y) \cong \mathscr{C}^{min}(X) \times \mathscr{C}^{min}(Y)$.*

Proof. One can check that we have a natural uniform map $\mathscr{C}^{\min}(X) \times \mathscr{C}^{\min}(Y) \to \mathscr{C}^{\min}(X \times Y)$ where we take as a base the product of elements in each filter. The map in the other direction comes from applying the functor $\mathscr{C}^{\min}$ and applying the universal property. $\square$

4 Precompactness

Definition 4.1. We say that a uniform space X is *precompact* if its completion is compact.

Example 4.2. Let $X \hookrightarrow K$ be a dense embedding of a topological space X into a Hausdorff compact space. K has the unique neighbourhood uniformity, 2.26, and we give X the subspace uniformity (which is compatible with the topology). It follows from 3.29 that the completion of X with respect to this uniformity is K and hence all precompact uniformities on X compatible with its topology arise in this fashion.

Now let βX denote the Stone-Čech compactification of X. We shall denote the corresponding uniformity by Φ_β. Then for any precompact uniformity Φ on X compatible with its topology we get the following diagram

$$\begin{tikzcd} (X, \Phi_\beta) \arrow[r, "1_X"] \arrow[d] & (X, \Phi) \arrow[d] \\ \beta X \arrow[r] & K \end{tikzcd}$$

where K denotes the compact completion of (X, Φ). The two vertical arrows are uniform as is the lower horizontal one by 2.12. Moreover by 3.24 we now that the uniformity (X, Φ) is induced by the map $X \to K$ so to check if the top arrow is uniform it is sufficient to check that its composition to K which follows as the other two arrows are. It follows that the Stone-Čech uniformity is the finest precompact uniformity on a space that is compatible with the topology.

Lemma 4.3. *$A \subset X$ with the subspace uniformity is precompact iff its closure in X is.*

Proof. We have from 3.29 that the completion of A is isomorphic to the closure of A in $\mathscr{C}^{\min}(X)$ and this closure is clearly isomorphic to the closure in $\mathscr{C}^{\min}(X)$ of the closure of A in X. □

Definition 4.4. Given an entourage V we say that a set $W \subset X$ is *V-small* if $W \times W \subset V$.

We say that a uniform space X is uniformally-small if given any entourage V of X we can find a finite cover of X with V-small sets.

Lemma 4.5. *If $f : X \to Y$ is uniform with dense image, then if X is uniformally-small so is Y.*

Proof. Take V an entourage on Y and W a cube-root. Then take a finite $(f \times f)^{-1}W$-small cover of X, (A_i). Then $\overline{f(A_i)} \times \overline{f(A_i)} \subset V$ by 2.6 and hence gives a V-small cover of Y. □

Lemma 4.6. *If X is uniformally-small then every ultrafilter on X is Cauchy.*

Proof. Let F be an ultrafilter on X and U an entourage. Take a finite U-small cover (W_i). Then one of them must lie in F else the (finite) intersection of their complements does which is the empty set. □

Proposition 4.7. *A uniform space X is precompact iff it is uniformally-small.*

Proof. For necessity recall that the map $N : X \to \mathscr{C}^{\min}(X)$ defines the uniformity on X, 3.24. Let V be an entourage on X and we can find some entourage W on $\mathscr{C}^{\min}(X)$ such that $(N \times N)^{-1}(W) \subset V$. Then we can find some finite W-small cover (W_i) and pulling back gives a cover of X with V-small sets.

For sufficiency let X be a Hausdorff complete uniform space that is uniformally-small and let us take an arbitrary cover (U_i) of X. Assume that there is no finite subcover, then $\{X \setminus U_i\}$ defines a subbasis for a filter F on X. By the ultrafilter theorem we can find some ultrafilter E such that $F \subset E$ which is Cauchy by 4.6. Then by completeness we have $N(x) \subset E$ for some $x \in X$ but this is absurd for $\exists i$ with $x \in U_i$.

Finally if X is an arbitrary uniformally-small space then its completion is uniformally-small by 4.5 and then the result follows. □

Corollary 4.8. *The image of any precompact space under a uniformally continuous map is precompact.*

Proof. Let C be precompact and $f : C \to Y$ a uniform map. Then let V be an entourage on Y, take a finite $(f \times f)^{-1}V$-small cover of C, this then defines a V-small cover of $f(C)$. □

Corollary 4.9. *Let X be a uniform space whose uniformity is generated by a collection of maps $f_i : X \to X_i$ to uniform spaces. Then X is precompact iff $f_i(X)$ is $\forall i$.*

Proof. Necessity comes from 4.8 and sufficienct follows from the explicit form of the least upper bound uniformity. □

Corollary 4.10. *Let X be a complete uniform space. Then $C \subset X$ is relatively compact iff it is uniformally-small.*

Proof. The result follows for X by 4.7 and 3.8. $\square$

5 Topological groups

Topological groups hold a wealth of interesting examples of uniform spaces and have a particularly rich structure. Let G be a topological group and U a neighbourhood of the identity. We then define the left and right entourage, U_L and U_R, by

$$U_L = \{(x, y) \mid y \in xU\} \quad U_R = \{(x, y) \mid y \in Ux\} \tag{6}$$

If we let this run over all neighbourhoods of the identity we see that this generates two fundamental systems. To take roots use that for any neighbourhood U of the identity we can find some V with $V^2 \subset U$ then $V_L \circ V_L \subset U_L$ and similarly for the right uniform structure. We will denote G with its left (resp. right) uniform structure as G_L (resp. G_R).

Clearly if the group is abelian then the left and right uniform structures are equal. Moreover by 2.13 if the group is compact then the two uniformities are equal.

As before we can define the join and meet of these uniformities which we denote by $G_{L\vee R}$ and $G_{L\wedge R}$. The meet of the two uniformities is also known as the *Roelcke uniformity*. As noted earlier the meet of two uniformities is usually hard to describe but in this case we have the following:

Proposition 5.1. *$G_{L\wedge R}$ has a fundamental system given by the sets of the form*

$$U_{L\wedge R} = \{(x, y) \mid y \in UxU\}$$

where U runs over $N(e)$.

Proof. It is simple to check using the properties of the neighbourhood of the identity that this does define a fundamental system and it is clearer coarser than both the left and right uniformities and hence coarser than $\Phi_{L\wedge R}$. For the converse let $\Psi \subset \Phi_L, \Phi_R$. So let $U \in \Psi$ and V a root of U. Then choose a $W \in N(e)$ with $W_R, W_L \subset V$ so that $W_R \circ W_L \subset U$. Now let $(x, y) \in W_{L\wedge R}$, then $y = uxv$ for $u, v \in W$ and so $(x, ux) \in W_R$, $(ux, uxv) \in W_L$ hence $(x, y) \in U$ so that the given uniformity is finer than Ψ. $\square$

Proposition 5.2. *All four uniformities are compatibly with the topology on G.*

Proof. The fact that Φ_L and Φ_R follow as gU and Ug are both neighbourhood bases of g. For the join of the two consider $(V_L \cap V_R)[g]$, it is sufficient to find W a neighbourhood of e such that $gW \subset (V_L \cap V_R)[g]$. We have $(V_L \cap V_R)[g] = gV \cap Vg = g\left(V \cap g^{-1}Vg\right)$ and the latter is a neighbourhood of the identity so we get compatability. Finally for the Roelcke uniformity again using the continuity of Ad_g given some $U \in N(e)$ we can find some V with $g^{-1}VgV \subset U$. Then it follows $[g]V_{L\wedge R} \subset gU$ and the result follows. $\square$

Lemma 5.3. *For a fixed $g \in G$ left and right multiplication by g is uniform $G \to G$ for all four uniformities.*

Proof. Let us consider $U \in N(e)$ then we can find $V \in N(e)$ such that $g^{-1}Vg \subset U$ and $gWg^{-1} \subset U$ by continuity of Ad. It immediately follows that

$$\begin{array}{rr}
U_L = (l_g \times l_g)^{-1}(U_L), & V_L \subset (r_g \times r_g)^{-1}U_L \\
W_R \subset (l_g \times l_g)^{-1}(U_R) & U_R = (r_g \times r_g)^{-1}U_R \\
(W \cap V)_{L\wedge R} \subset (l_g \times l_g)^{-1}(U_{L\wedge R}) & (W \cap V)_{L\wedge R} \subset (r_g \times r_g)^{-1}(U_{L\wedge R}) \\
(U \cap V)_{L\vee R} \subset (l_g \times l_g)^{-1}(U_{L\vee R}) & (U \cap W)_{L\vee R} \subset (r_g \times r_g)^{-1}(U_{L\vee R})
\end{array}$$

$\square$

The natural next question is which of the group operations are uniformly continuous. We start by looking at the multiplication - here we find an obstruction.

Definition 5.4. Given a topological group G we say that the *adjoint is uniformally bounded* if for any $U \in N(e)$ we can find some $V \in N(e)$ such that $g^{-1}Vg \subset U$ for all $g \in G$.

Lemma 5.5. *If multiplication is uniform for the left uniformity then it is for the right (and vice versa) and this holds iff the adjoint is uniformally bounded in the group.*

Proof. Let us first show that having a uniformally bounded adjoint is sufficient for the multiplcation to be uniform. Focusing on left uniformity given a neighbourhood U of the identity we are looking for some V such that $g_i^{-1}h_i \in V \implies g_2^{-1}g_1^{-1}h_1h_2 \in U$. Take W a root of U and then V a uniform bound for the adjoint in W, i.e. $\forall g : \; g^{-1}Vg \subset W$. Then we have that for $(g_i, h_i) \in V_L$ that

$$g_2^{-1}g_1^{-1}h_1h_2 \in g_2^{-1}Vh_2 = g_2^{-1}Vg_2g_2^{-1}h_2 \subset W \cdot W \subset U.$$

An identical proof holds for right uniformity showing sufficiency.

For the converse assume left uniformity. Then for $U \in N(e)$ pick $V \in N(e)$ such that $(\mu \times \mu)(V_L \times V_L)^\sigma \subset U_L$, where $(-)^\sigma$ denotes swapping the second and third coordinates. Let $(g', g) \in G \times G$ then $\forall v \in V$: $(g', g, g'v, g) \in (V_L \times V_L)^\sigma$. It follows that

$$g^{-1}vg \in U \implies g^{-1}Vg \subset U$$

and as g was arbitrary this provides a bound for the adjoint. □

Lemma 5.6. *The adjoint of G is uniformally bounded iff the left and right uniformities on G are equal.*

Proof. Firstly suppose that the left and right uniformities are equal. Choose $V_L \subset U_R$ for some $U, V \in N(e)$. Then for $z \in gVg^{-1}$ we have that

$$zg \in gV \implies (g, zg) \in U_R \implies z \in U.$$

For the converse assume that G is uniformally bounded. Then if $gVg^{-1} \subset U$ for all $g \in G$ we have $V_L \subset U_R$. □

Corollary 5.7. *The multiplcation map is uniform for the joint uniformity iff the left and right uniformities are equal.*

Proof. From 5.6 and 5.5 we get that the condition is sufficient. For necessity assume that μ is uniform for the joint uniformity. Then as $G_{L\vee R} \to G_L$ is uniform we can apply an identical argument to the proof in 5.5 to generate uniform bounds for the adjoint. □

Corollary 5.8. *The multiplication map is uniform for the Roelcke uniformity iff the left and right uniformities are equal.*

Proof. As above from 5.6 and 5.5 we get the condition is sufficient. For necessity consider the uniform map

$$G_L \times G_R \to G_{L\wedge R} \times G_{L\wedge R} \to G_{L\wedge R}$$

where the first map is the identity on the undelrying sets. So suppose that $V_L \times V_R$ is mapped into $U_{L\wedge R}$ under this map. Then choosing $(g, g^{-1}) \in G_L \times G_R$ we find that $gV^2g^{-1} \subset U^2$. As U and g were arbitrary we see that G has a uniformally bounded adjoint. □

One should additionally note that the map

$$G_R \times G_L \to G_{L\wedge R} \tag{7}$$

given by the multiplication is uniform as can easily be checked. I am not sure if this has any real application, potentially defines an interesting pairing on the completions.

We have now looked at the uniform structure of the multiplication in some detail. The other map, the inversion $a : G \to G$, has simpler properties.

Proposition 5.9. *$a : G_L \to G_R$ is uniform and hence defines an uniform isomorphism between the left and right uniformity. Additionally a is uniform for both the Roelcke and joint uniformity.*

Proof. For the first statement we have that $(a \times a)^{-1}U_L = U_R^{-1}$.

Next to the joint uniformity where direct computation shows $(a \times a)^{-1}(U_{L\vee R}) = U_{L\vee R}^{-1}$ and similarly for the Roelcke uniformity we have that $(a \times a)^{-1}(U_{L\wedge R}) = U_{L\wedge R}^{-1}$. □

5.1 Completion

We now turn to the question of completion of a topological group. In the case that the two uniformities agree we see that all the group operations are uniform. In that case we can simply apply the functor $\mathscr{C}^{\min}$ to the corresponding diagram to get the structure of a complete Hausdorff group in which G is a dense subset. In the general case we need to do more work.

Recall from the uniformities we have constructed above that we have the following uniform maps (which are all the identity on the underlying set):

$$\begin{tikzcd} & G_L \arrow[dr] & \\ G_{L\vee R} \arrow[ur] \arrow[dr] & & G_{L\wedge R} \\ & G_R \arrow[ur] & \end{tikzcd} \tag{8}$$

Lemma 5.10. $\mathscr{C}(G_{L\vee R}) = \mathscr{C}(G_L) \cap \mathscr{C}(G_R)$.

Proof. $\subset$ follows from the diagram (8). Conversely if F lies in the intersection and $V \in N(e)$ then find $A, B \in F$ with $A \times A \subset V_L$ and $B \times B \subset V_R$ so that $A \cap B \in F$ satisfies $A \cap B \times A \cap B \subset V_L \cap V_R$. □

As we have noted earlier the failure for uniformity of the multiplication comes from the fact that we cannot provide uniform bounds for the action of the adjoint. We can however provide local bounds so it is possible that we can extend the multiplication to locally defined structures like Cauchy sequence. Given $F, F' \in \mathrm{Filter}(G)$ we can define $FF' \in \mathrm{Filter}(G)$ by

$$FF' = \{AB \mid A \in F, B \in F'\}. \tag{9}$$

This is just μ_*.

Proposition 5.11. *For the left, right and joint uniformity $\mu_* : \mathscr{C}(G) \times \mathscr{C}(G) \to \mathscr{C}(G)$, that is the multiplication preserves the Cauchy sequences.*

Proof. Let $F, F' \in \mathscr{C}(G_L)$ and $U \in N(e)$. Take a cube-root V of U, then by the Cauchyness of F' we can find some $B \in F'$ such that $B \times B \subset V_L$. Then choose some $b \in B$ so we have $B \subset bV$. We can then find some W such that $b^{-1}Wb \subset V$, we then claim that $AB \times AB \subset U_L$: we can write any element in this product as (a_1bv_1, a_2bv_2) for $a_i \in A$ and $v_i \in V$. Then

$$(a_1bv_1)^{-1}a_2bv_2 = v_1^{-1}b^{-1}a_1^{-1}a_2bv_2 \in Vb^{-1}WbV \in V^3 \subset U.$$

The proof for the right uniformity works in a similar fashion.

The result for the joint uniformity follows immediately using the result for the left and right uniformities and 5.10. □

Proposition 5.12. *For the left, right and joint uniformity the map μ_* is continuous.*

Proof. The proof has a similar flavour to the previous one, possibly they can be mapped together...

Lets start by looking at the left uniformity: let $E, F \in \mathscr{C}(G_L)$ and $U \in N(e)$. Choose a fourth root of U, call it V and choose a $B \in F$ such that $B \times B \subset V$. Choose some $b \in B$ so that $B \subset bV$. Then choose $W \in N(e)$ such that $b^{-1}Wb \subset V$. Now let $(\widetilde{E}, \widetilde{F}) \in \mathscr{C}(W)[E] \times \mathscr{C}(V)[F]$ and choose $\alpha \in E$, $\widetilde{\alpha} \in \widetilde{E}$, $\beta \in F$, $\widetilde{\beta} \in \widetilde{F}$, such that

$$\widetilde{\alpha} \times \alpha \subset W_L \quad \widetilde{\beta} \times \beta \subset V_L \quad \beta \subset B$$

Then

$$(\widetilde{\alpha}\widetilde{\beta})^{-1}\alpha\beta = \widetilde{\beta}^{-1}\widetilde{\alpha}^{-1}\alpha\beta \subset \widetilde{\beta}^{-1}W\beta \subset V\beta^{-1}W\beta \subset V^4$$

Continuity for the right uniformity follows by an identical proof with the roles reversed. For the joint case the proof is the same again except we must condition on tboth the left and right side at the same time. □

Theorem 5.13. *$\mathscr{C}^{min}(G_{L\vee R})$ is a topological group. $\mathscr{C}^{min}(G_L)$ and $\mathscr{C}^{min}(G_R)$ are topological monoids with a given isomorphism between the two of them.*

6 Function uniformities

In this section we look at some additional examples of uniform spaces coming from functions. This is mainly a write up of the topological introduction in [Gro73] in more detail.

Let S be a set and E a uniform space. Let $A \subset S$ and U an entourage on E, and define

$$W(A, U) = \{(f, g) \in F(S, E) \times F(S, E) \mid \forall x \in A : (f(x), g(x)) \in U\} \tag{10}$$

Lemma 6.1. *For fixed A the sets $W(A, U)$ where U runs over a fundamental system on E give a fundamental system for a uniform structure on $F(S, E)$.*

We shall call the uniform structure generated by this fundamental system the *uniformity of uniform convergence on A*. We shall denote $F_A(S, E)$ the set of functions $F(S, E)$ with this uniform structure.

Let $x \in S$ and consider the evaluation map $\mathrm{ev}_x : F(S, E) \to E,\ f \mapsto f(x)$. We see that

$$W(A, U) = \bigcap_{x \in A} (\mathrm{ev}_x \times \mathrm{ev}_x)^{-1} U. \tag{11}$$

It follows that if $B \subset A$ then uniform converge on A is finer than that on B. Moreover for $x \in A$ we have ev_x is uniform.

Let $\mathfrak{A}$ be a family of subsets of S. The least upper bound uniform structure of the uniform structures of uniform convergence on $A \in \mathfrak{A}$ will be called the *uniformity of uniform convergence on $\mathfrak{A}$*. We denote $F_{\mathfrak{A}}(S, E)$ the set of functions $F(S, E)$ with this uniform structure. If $A_i \in \mathfrak{A}$ is a finite set of subsets then we have

$$\bigcap_i W(A_i, U) = W\left(\bigcup_i A_i, U\right) \tag{12}$$

so that the uniform structure generated by $\mathfrak{A}$ is the same as that generated by the collection of finite unions of elements in $\mathfrak{A}$. The uniform structure is also clearly preserved if we include all subsets of elements in $\mathfrak{A}$.

Lemma 6.2. *The map $E \to F_{\mathfrak{A}}(S, E)$ which sends each element to its constant function is uniform for any choice of $\mathfrak{A}$.*

Proof. Let us denote the map by $c : E \to F(S, E)$. Then $(c \times c)^{-1} W(A, U) = U$. □

Lemma 6.3. *Let E_i be uniform spaces and $g : E_1 \to E_2$ uniform. Then for $\mathcal{A}$ a set of subsets of S we have that the map:*

$$g_* : F_{\mathcal{A}}(S, E_1) \to F_{\mathcal{A}}(S, E_2)$$

is uniform.

Lemma 6.4. *Let $S' \subset S$ and let $\mathcal{A} \cap S' = \{A \cap S' \mid A \in \mathcal{A}\}$. Then the map $F_{\mathcal{A}}(S, E) \to F_{\mathcal{A} \cap S'}(S', E)$ given by $f \mapsto f|_{S'}$ is uniform.*

Example 6.5. Let $\mathfrak{A}$ be all the one point sets in S. We call the corresponding structure the *uniform structure of pointwise convergence*. We denote by $F_s(S, E)$ the set of functions with this uniform structure.

Consider the identification of $F(S, E)$ with the set $E^S = \prod_{x \in S} E$. Then the topology genereated by this uniformity is the product topology. To see this let us take a basic neighbourhood of f of the form $W(x_1, \ldots, x_n; U)$. Under the above identification we see that this corresponds to

$$\bigcap_{i=1}^{n} p_{x_i}^{-1}(U[f(x_i)])$$

which showa the result.

Example 6.6. Let $\mathfrak{A} = S$. The corresponding structure is called the *uniform structure of uniform convergence* and we denote it by $F_u(S, E)$.

Example 6.7. If X is a topological space let $\mathfrak{A}$ be the compact subsets of X. The corresponding uniform structure is called the *uniform structure of compact convergence* and we deonte it as $F_c(X, E)$.

There is a slight generalisation of this to the case where X is a uniform space. Then we can let $\mathfrak{A}$ denote the set of precompact sets. This gives the *uniform structure of precompact convergence* and we denote this uniform space by $F_{pc}(X, E)$.

The induced topology of the unifomity of compact convergence is the compact open topology.

The following result follows by chasing definitions:

Proposition 6.8. *Let S, T be sets, $\mathfrak{A}$ a set of subsets of S, and $\mathfrak{B}$ a set of subsets of T. Then there is an isomorphism of uniform spaces*

$$F_{\mathfrak{A}\times\mathfrak{B}}(S \times T, E) \cong F_{\mathfrak{A}}(F_{\mathfrak{B}}(T, E))$$

where the map is given by the normal currying operation.

Recall that for a uniform space X we let $X_0 = X/R$ denote the associated Hausdorff space.

Lemma 6.9. *Let $\mathfrak{A}$ be a set of subsets of S and suppose $|E| > 1$. Then $F_{\mathfrak{A}}(S, E)$ is Hausdorff iff $S = \cup\mathfrak{A}$ and E is Hausdorff.*

Proof. The fact that E needs to be Hausdorff follows from the uniform map $E \to F_{\mathfrak{A}}(S, E)$ sending each point to the corresponding constant function, see 6.2. Next if $x \notin \cup\mathfrak{A}$ then let f and g be two functions that are equal on all points of S except x where they take different values. Then we clearly have $(f, g) \in W(A; U)$ for any entourage U and $A \in \mathfrak{A}$ so that they cannot be seperated hence both conditions are necessary.

For sufficiency let $f \neq g$ and choose some $x \in S$ where they differ. Then we can find some U an entourage on E such that $(f(x), g(x)) \notin U$. It follows that $(f, g) \notin W(x; U)$ and $W(x; U)$ is an entourage on $F_{\mathfrak{A}}(S, E)$ as $x \in \cup\mathfrak{A}$. ☐

Proposition 6.10. $F_{\mathfrak{A}}(S, E)/R \cong F_{\mathfrak{A}}(S', E/R)$ *where* $S' = \cup A \in \mathfrak{A}A$.

Proof. Given $f, g \in F_{\mathfrak{A}}(S, E)$ then $f \sim_R g \implies \forall x \in S', \forall U \in \Phi_E$ we have $f(x) \sim_R g(x)$ and so they have the same image in $F_{\mathfrak{A}}(S', E/R)$. Conversely if they are not related then we can find some $x \in S'$ and $U \in \Phi_E$ such that $(f(x), g(x)) \notin U$. Then they have different images in $F_{\mathfrak{A}}(S', E/R)$ so that the map is a uniform bijection.

For the other direction pick any uniform section $E/R \to E$ by 2.16 to get a map $F_{\mathfrak{A}}(S', E/R) \to F_{\mathfrak{A}}(S', E)$. We see that if we extend such a function arbitrarily it gives a uniform mapping from $F_{\mathfrak{A}}(S', E) \to F_{\mathfrak{A}}(S, E)$ as the entourages do not "see" the points outside of S'. This shows the inverse map is uniform hence the result. ☐

6.1 Completion

Example 6.11. A filter F is Cauchy in $F_s(S, E)$ iff $\forall x \in S$ $F(x) := (\mathrm{ev}_x)_* F$ is Cauchy on E. The only if is clear as we have that ev_x is uniform. Conversely assume that $F(x)$ is Cauchy for all x. Then given an entourage $W(x; U)$ we can find some $A \in F(x)$ such that $A \times A \subset U$, then $(\mathrm{ev}_x \times \mathrm{ev}_x)^{-1} A \subset F$ gives a subset whose product lies in $W(x; U)$ and the result follows.

Lemma 6.12. *Let F be a filter in $F(S, E)$. Then F converges uniformally to f iff it is uniformally convergent and it converges pointwise to f.*

Proof. Necessity comes from 3.3 and the fact that the the pointwise topology is coarser than the uniform one. For sufficiency let F converge pointwise to f and let U be an entourage on E. Let V be a 4^{th}-root of U and choose $A \in F$ such that $A \times A \subset W(S; V)$ by Cauchyness. Then for every $s \in A$ we can find some $g_s \in A$ with $(g_s(s), f(s)) \in V$ by pointwise convergence. Let $g = [s \mapsto g_s(s)]$, by construction we have that $(g, f) \in W(S, V)$. Moreover let $s, t \in S$ then we have

$$(g(t), g_s(t)) = (g(t), g_t(t)) \circ (g_t(t), g_s(t)) \in V^{\circ 2}$$

so that $g(t) \in W(S; V^{\circ 2})[g_s]$. Then $(f, g_s) \in W(S; V^{\circ 3})$ hence we have that f is an adherence point for F and so by 3.4 it is a convergence point. ☐

Corollary 6.13. *A filter F in $F_{\mathfrak{A}}(S,E)$ converges to f iff it is Cauchy for $\mathfrak{A}$ uniform convergence and $F|_{S_0}$ converges to $f|_{S_0}$ pointwise on $S_0 = \cup\mathfrak{A}$.*

Proposition 6.14. *The space $F_{\mathfrak{A}}(S,E)$ is complete iff E is.*

Proof. Necessity comes from considering the constant functions. For sufficiency let E be complete and F a Cauchy filter on $F_{\mathfrak{A}}(S,E)$. Then by 6.13 it is sufficient to show that F converges to some function pointwise. By 6.11 it is sufficient that the filter $F(x)$ converges $\forall x \in X$ which follows by the completeness of E. $\square$

Corollary 6.15. *If $\mathfrak{A}_i$ are sets of subsets of S and the $\Phi_{\mathfrak{A}_2}$ is finer than $\Phi_{\mathfrak{A}_1}$ with both covering S then:*

1. *If H is complete with respect to with respect to the $\mathfrak{A}_1$ uniformity then it is with respect to the $\mathfrak{A}_2$ uniformity.*
2. *If $H \subset F(S,E)$ then its completion with respect to the $\mathfrak{A}_2$ is a subset of its completion with respect to $\mathfrak{A}_1$.*

Proof. We have the uniform map $E \to \mathscr{C}^{\min}(E)$ so we may assume that E is complete. Then $F_s(X,E)$ is complete and each of these uniformities is finer than the uniformity of pointwise convergence as they cover S. The two claims now follow immediately. $\square$

Lemma 6.16. *Let H be the set of functions that map each element $A \in \mathfrak{A}$ to a precompact set in E. Then H is closed in $F_{\mathfrak{A}}(S,E)$.*

Proof. Let f lie in the closure of H and $A \in \mathfrak{A}$. By 4.7 we need to show that $f(A)$ is uniformally small. Let U be an entourage on E and V a cube root of U. We can find some $g \in H$ with $(f,g) \in W(A;V)$ and by the V-smallness of $g(A)$ we can find some finite set B_i with $g(A) \subset \cup B_i$. Let $C_i = A \cap g^{-1}B_i$, these give a finite cover of A. For $(x,y) \in C_i$ we have that $(f(x),g(x))$, $(g(y),f(y)) \in V$ as they lie in the given entourage and, $(g(x),g(y)) \in V$ by the V-smallness of the cover. It follows that $(f(x),f(y)) \in U$ and so the $f(C_i)$ provide a finite U-small cover. $\square$

6.2 Continuous and uniform functions

For the space of continuous the classic results on metric spaces come through to uniform spaces with no alterations.

Theorem 6.17. *$C_u(X,E)$ is closed in $F_u(X,E)$, i.e. the limit of uniform functions is uniform.*

Proof. Let $f : X \to E$ lie in the closure of $C(X,E)$, $x \in X$ and U some entourage on E. Then take a cube-root V of U and choose some $g \in C(X,E)$ such that $(f,g) \in W(X;V)$. Then by continuity of f we can find some $V' \subset f^{-1}V[f(x)]$. So let $y \in V'$ then $(g(y),f(y))$, $(f(y),f(x))$, $(f(x),f(y)) \in V$ and so concatenating $g(y) \in U[g(x)]$, i.e. $g(V') \subset U$. This shows continuity at x and as x was arbitrary we get continuity of g. $\square$

Let X be a uniform space. Then we let $U(X,E) \subset C(X,E)$ denote the space of uniform functions. We shall decorate it as usual to indicate specific subspace uniformities.

Proposition 6.18. *$U_u(X,E)$ is closed in $F_u(X,E)$.*

Proof. The proof is almost identical to that for 6.17. Let g lie in the closure of $U_u(X,E)$ and let $U \in \Phi_E$. Choose $f \in U_u(X,E)$ such that $(g,f) \in W(X,U)$. Then $\exists V \in \Phi_X$ with $(f \times f)V \subset U$. Let $(x,y) \in V$, then $(g(x),f(x))$, $(g(y),f(y)) \in U$ and hence $(g(x),g(y)) \in U^{\circ 3}$. As U was arbitrary the result follows. $\square$

Corollary 6.19. *$C_{\mathfrak{A}}(X,E)$ is closed in $F_{\mathfrak{A}}(X,E)$ iff $\forall A \in \mathfrak{A}, f|_A \in C(A,E) \implies f \in C(X,E)$.*

Proof. If f lies in the closure of $C_{\mathfrak{A}}(X,E)$ then by 6.17 $f|_A : A \to E$ is continuous for all $A \in \mathfrak{A}$ which shows the equivalence. $\square$

Proposition 6.20. *The map $\mathrm{ev} : C_u(X,E) \times X \to E$, $\mathrm{ev}(f,x) = f(x)$ is continuous.*

Proof. Let $(f, x) \in C_u(X, E) \times X$ and U an entourage on X. Let V be a root of U and by the continuity of f choose some V' a neighbourhood of x in X such that $fV' \subset U[f(x)]$. Now let $(g, y) \in W(X; V) \times V'$ then $(g(y), f(y)) \in V$ and $f(y), f(x)) \in V$ hence $g(y) \in U[f(x)]$ and so the map is continuous at (f, x) and as they were arbitrary it is continuous. □

6.3 Equicontinuity and precompactness

Definition 6.21. Let $H \subset F(X, E)$ with X a topological space and E uniform. Then we say that H is *equicontinuous at* $x_0 \in X$ if $\forall U \in \Phi_E$ we can find some $V \in N(x_0)$ such that $(f(x), f(x_0)) \in U\ \forall x \in V, \forall f \in H$. We say that H is *equicontinuous* if for every $x \in X$ H is equicontinuous at x.

If X is a uniform space say that H is *uniformally equicontinuous* if $\forall U \in \Phi_E$ we can find a $V \in \Phi_X$ with $V \subset (f \times f)^{-1}U\ \forall f \in H$.

Proposition 6.22. *Let H be a set and E be uniform spaces, Z a topological space and $f : H \times Z \to E$ a function. For $h \in H$ we let $f_h : Z \to E$ denote the function $z \mapsto f(h, z)$, and similarly $f^z : H \to E$ is the function $h \mapsto f(h, z)$.*

1. *$\{f_h : Z \to E \mid h \in H\}$ is equicontinuous at $z_0 \in Z$ iff the map $Z \to C_u(H, E)$ is continuous at z_0.*
2. *If Z is uniform then $\{f_h \mid h \in H\}$ is uniformally equicontinuous iff $Z \to C_u(H, E)$ is uniform.*

Proof. Let U be an entoruage on E. Then $\{f_h\}$ is equicontinuous at $z_0 \in Z \Leftrightarrow$ we can find some $V \in N(z_0)$ such that $(f_h(z), f_h(z_0)) \in U\ \forall h \in H, \forall z \in V \Leftrightarrow (f^z, f^{z_0}) \in W(H; U)\ \forall z \in V \Leftrightarrow z \mapsto f^z$ is continuous at z_0.

Similar definition chasing occurs for the second point. Namely if the map is uniform then for any $U \in \Phi_E$ we can find some entourage V on Z such that $(f^z, f^w) \in W(H; U)\ \forall (z, w) \in V \Leftrightarrow (f(h, z), f(h, w)) \in U$ $\forall z,\ w \in Z\ \forall h \in H \Leftrightarrow (f_h \times f_h)(V) \subset U$. □

Corollary 6.23. *Let X be a topological space, E a uniform space and $H \subset F(X, E)$. Then*

1. *H is equicontinuous iff the map $X \to C_u(H, E)$ given by $x \mapsto [f \mapsto f(x)]$ is continuous.*
2. *If additionally X is uniform then H is uniformally equicontinuous if $X \to C_u(H, E)$ is uniform.*

Proof. We consider the map $F(X, E) \times X \to E$ given by the evaluation function and apply 6.22. □

Remark 6.24. We didn't need to make any assumptions about the continuity of the map $F : H \times Z \to E$ in 6.22 however we know that if $\{f_h\}$ is equicontinuous then each f_h is indeed continuous so the function F is continuous in its second argument.

As stated the requirements of the proposition have no continuity requirements on F in its first argument (indeed the proposition is stated with no topological structure on H). However let us give H the coarsest topology such that $f^z : H \to E$ is continuous $\forall z \in Z$. Then we claim that $f : H \times Z \to E$ is continuous if H is equicontinuous. To see this pick an entourage $U \in \Phi_E$ and take a root V. Then for $(h_0, z_0) \in H \times Z$ choose a $W \in N(h_0)$ such that $f^{z_0}W \subset V[f(h_0, z_0)]$ by the topology we just gave H and let $W' \subset Z$ be such that $(f(h, z), f(h, z_0)) \in V\ \forall h \in H$ by equicontinuity. Then if $(h, z) \in W \times W'$ we have $(f(h, z_0), f(h_0, z_0)) \in V$ by the choice of W, and $(f(h, z), f(h, z_0)) \in V$ by the choice of W'. It follows that $(f(h, z), f(h_0, z_0)) \in U$ so that f is continuous at (h_0, z_0) and hence continuous everywhere as they were arbitrary.

One nice example of this comes from the fact that $C_s(X, E) \times X \to E$ is not continuous, compare with 6.20, but its restriction to an equicontinuous set of functions is. In fact this implies some sort of "single topology" on equicontinuous functions which we now explore further.

Lemma 6.25. *Let $H \subset F(X, E)$ be equicontinuous. Then on H the uniformity of pointwise convergence on a dense subset, $X_0 \subset X$, and of compact convergence coincide.*

Proof. We need to show that for any compact $K \subset X$ and U an entourage on E we can find some entourage V on E and a finite set of points (x_i) such that $W(x_1, \ldots, x_n; V) \cap H \subset W(K; U) \cap H$. So let K and U be given and take $x \in K$. By the equicontinuity of H we can find some $V_x \in N(x)$ such that $(f(y), f(x)) \in U'$ for $f \in H$ and $y \in V_x$ where U' is an arbitrary entourage on E. By compactness of K finitely many of the V_x cover it, and by denseness of X_0 we can find a $z_i \in X_0$ such that the given V_x is a neighbourhood of z_i as well, i.e. we can find finitely many z_i, $V_i \in N(z_i)$, with $K \subset \cup V_i$ and H with the property as above. Now consider $W(z_1, \ldots, z_n; U')$ and let $f, g \in H$ lie in this entourage. Then for $y \in K$ we can find a V_i with

$y \in V_i$ so that $(f(y), f(z_i)), (g(y), g(z_i)), (g(z_i), f(z_i)) \in U'$, the first two by equicontinuity and the last by the choice of the entourage. If we choose U' to be a cube-root of U then we get the result. □

Corollary 6.26. *If $H \subset F(X, E)$ is equicontinuous then $\overline{H} \subset F_s(X, E)$ is equicontinuous.*

Proof. Let g lie in the closure of H for the uniformity of pointwise convergence. Then for any entourage U of E and $x \in X$ we can find some $V \in N(x)$ with $(f(y), f(x)) \in U$ for all $f \in H$ and $y \in V$. Then given $y, x \in V$ we can find $f \in H$ with $(f, g) \in W(x, y; U)$ and it follows that $(g(y), g(x)) \in V^{\circ 3}$. □

We now turn to the other area where we have some form of uniqeness of the given uniformity, precompact spaces.

Lemma 6.27. *$H \subset F_{\mathcal{A}}(X, E)$ is precompact iff $H|A$ is precompact in $F_u(A, E)$ for all $A \in \mathcal{A}$.*

Proof. The restriction is uniform, 6.4, so the image of H is precompact by 4.8 which shows necessity. For sufficiency let $W(A, U)$ be an entourage of $F_{\mathcal{A}}(X, E)$ then as $H|A$ is precompact we can find a finite $W(A, U)$-small cover of $H|A$. Pulling this back gives a $W(A, U)$-small cover of H and hence H is precompact. □

Example 6.28. It is instructive to look at an example of the above. Let us consider $F_s(X, K)$ where K is a compact space. Then as $F_s(X, K)|\{x\} \cong K$ is compact we can apply the above lemma and so we have that $F_s(X, K)$ is precompact and as it is complete it is compact. Making the identification $F_s(X, K) \cong K^X$ with the product topology we see that this is Tychnoff's theorem.

Proposition 6.29. *Let $H \subset C_u(X, E)$ be precompact, then it is equicontinuous.*

Proof. Let U be an entourage on E and let $x \in X$. As H is U-small we can find a finite cover (A_i) with $A_i \times A_i \subset W(X; U)$. Choose some $f_i \in A_i$. As f_i is continuous we can find some $V_i \in N(x)$ such that $f(V_i) \subset U[f(x)]$. Then $V = \cap V_i \in N(x)$. If $f \in H$ we can find some f_i such that $(f, f_i) \in W(X; U)$ so that $\forall y \in X$ $(f(y), f_i(y)) \in U$. Then for $y \in V$ we have that $(f(y), f(x)) \in U^{\circ 3}$. As U and x were arbitrary equicontinuity follows. □

We have the celebrated converse to the above results

Theorem 6.30 (Ascoli-Arzela)**.** *Let K be a compact set, E a Hausdorff uniform space, $H \subset C(K, E)$. Then H is relatively compact in $C_u(K, E)$ iff H is equicontinuous and $H(x)$ is relatively compact for all $x \in K$.*

Proof. ($\Rightarrow$) If H is precompact then by 6.29 we know that it is equicontinuous so we just need to show that $H(x)$ is relatively compact. But $\mathrm{ev}_x : C_u(K, E) \to E$ is uniformally continuous so $\mathrm{ev}_x(H) = H(x)$ is precompact by 4.8 and hence relatively compact.

($\Leftarrow$) Let us show that H is uniformally small. Take $U \in \Phi_E$. By equicontinuity for $x \in X$ we can find $V_x \in N(x)$ with $f(y) \in U[f(x)]$ for all $f \in H$ and $y \in V_x$. By compactness finitely many of the V_x cover K, call them $(V_i \in N(x_i))$. Then by the relative compactness of $H(x_i)$ we can find finitely many U-small elements covering $H(x_i)$, call them U_{ij}. Let $A_{ij} = \{f \mid f(x_i) \in U_{ij}\}$, for each i these provide a cover. Then given a finite sequence $j_1, \cdots, j_n$ we set

$$A_{j_1,\ldots,j_n} = A_{1j_1} \cap \cdots A_{nj_n}$$

These provide a finite cover of H and we claim that they are all small. To see this let $f, g \in A_{j_1,\ldots,f_n}$ and let $y \in K$. Then we can find some V_i with $y \in V_i$ and then $(f(y), f(x_i))$, $(g(y), g(x_i)) \in U$ by equicontinuity and $(f(x_i), g(x_i)) \in U$ by the condition on the $A_{j_1,\ldots,j_r}$. It follows that $(f(y), g(y)) \in U^{\circ 3}$, i.e. $(f, g) \in W(K; U^{\circ 3})$. As U was an arbitrary entourage the result follows. □

Corollary 6.31. *If X is locally compact or metrizable, E a Hausdorff uniform space and $H \subset C(X, E)$, then H is relatively compact in $C_c(X, E)$ iff H is equicontinuous and $H(x)$ is relatively compact for all $x \in X$.*

Proof. If X is locally compact or metrizable then the map $X \to C_u(H, E)$ is continuous iff it is for all $K \subset X$ compact. Then applying 6.27, 6.30, and 6.23 we get the result. □

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