Configuration spaces

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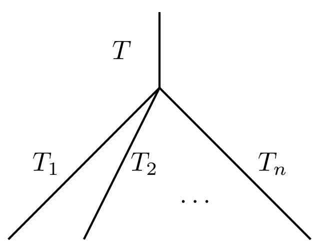
This note is based around looking into some more detail at the spaces used by Kontsevich for part of his proof of the formaility conjecture, [Kon97]. We see that there is an interesting link between some basic algebraic structures, in the shape of operads, on one hand, and the compactification of the configuration spaces on the other. The operadic structure has a particularly elegent geometric picture. There are some additional references online about these spaces, namely [Gai].

1 Operads and computation trees

In this section we set up the basic algebraic building blocks that we will be working with as well as outlining the general philosophy behind the compactification.

1.1 Typed operads and computation trees

A computation tree can be thought of a sequence of operations that one can perform in a strongly typed language. Each function in our language has some signature, which is a list of arguments each of a given type, as well as a return type. We shall generally denote this information by $T(T_1, \ldots, T_n)$ although it will be helpful to think of it in the following form as well:



We can glue functions together and their associativity means we can forget about the order in which we glued them and just consider the total structure. Additionally when we compute we should end up with a single return type and no loops, so the final structure we end up with should look like a tree with all the edges labelled by types and only certain vertices allowed. Informally this is what a computation tree is.

A simple way to construct such trees are from operads. A *typed operad* simply formalizes the typed language construction: We have sets $\mathcal{O}(T; T_1, \ldots, T_n)$, which we semantically think of as functions of the given signature, and compositions:

$$\mathcal{O}(T; T_1, \ldots, T_n) \times \mathcal{O}(T_1; T_{1,1}, \ldots, T_{1,m_1}) \times \cdots \times \mathcal{O}(T_n; T_{n,1}, \ldots, T_{n,m_n}) \to \mathcal{O}(T; T_{1,1}, \ldots, T_{n,m_n}). \tag{1}$$

Moreover we have an action of a suitable product of the symmetric group (we group the arguments of the same type in the signature and then permute them) and identity elements in $\mathcal{O}(T; T)$ where T runs over all the types of the operad. There are the usual equivariance properties that need not be spelt out here, [May].

To any operad we can then define the set of computation trees as the diagrams that can be composed for the given operad. These diagrams end up looking like simple Feynman diagrams.

1.2 The category of computation trees

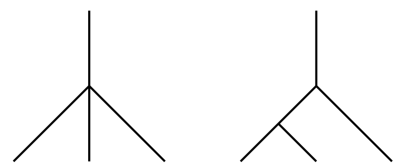
Let T be a computation tree for some operad and take some internal edge of this tree. We can concatenate out this edge using currying via the operad structure, (1), where we put the identity functions on those terms we want preserved. This gives us a way of associating to each computation tree of one where an internal edge has been collapsed of the same type. These functions generate the morphisms in a category which we shall denote by $\mathcal{T}_{T;T_1,\dots,T_n}$ whose objects are computation trees for the given operad of the prescribed type.

As in the category of trees we have that morphisms give the category the structure of a poset and any two computation trees have a smallest upper bound. Another fun exercise is to see that for a quadratic operad all the initial trees are binary trees.

1.3 The configuration and boundary operad

The two operads we are going to construct in the text we shall call the configuration operad, see also [nLa], and the boundary operad. Although the details of there construction will be found later we can still set the scene enough to construct the computation trees for the operads of a given type.

The model for the configuration operad is that of n-distinct points in $\mathbb{R}^d$ where such configurations are considered up to an equavalence of translations and dilations. We denote this space by C_n. The symmetric group simply permutes the points in this operad. This operad has a single type and composition is allowed for any arity so its computation trees are exactly the category of trees. The leaves of the tree can be simply labelled by elements C_1. For example the following give the computation trees of the C_3:



The idea behind this operad is that we can paste a sub-configuration at a given point. This would require us to have some coherent notion of a configuration with zero extension which is developed below.

The boundary operad is a more intersting beast: We consider configurations which have n-points in the interior of our space and m-points on the boundary. If we are far away from the boundary then the interior points should look like configuration points as above. We thus should be able to insert a normal configuration of the interior into an interior point of a boundary configuration as long as the size of the configuration is small (the same case as we ran into prevoiusly). For points on the boundary we should be able to insert an internal point as long as it is not too far from the boundary so we know where to insert it. The model this gives for the operad is then

$$C_{n,m} \times \prod_{i=1}^{n} C_{p_i} \times \prod_{j=1}^{m} C_{q_j,r_j} \to C_{p_1+\cdots+p_n+q_1+\cdots+q_m,r_1+\cdots+r_m}.$$

We can build up configuration trees for the above rules as before. See the final section for a more detailed explanation of the trees and plenty of diagrams.

1.4 Magnification

We can now describe the philosophy that we will use for constructing the compactification of the spaces as explained in [Kon97]. Each edge in the computation tree corresponds to a level of magnification of the points in the configuration such that the internal structure is too small to see from the upper level and thus we must explore the tree further to determine its structure. This allows us to bring points infinitely close together whilst still retaining enough information to build a valid configuration under a slight perturbation. This concept of magnification also explains the structure of the vertices for the boundary operad - if you are of finite distance to the boundary then any zoomed out observer will not be able to distinguish you from the boundary and this cannot be rectified by trvelling further up[1] the tree (i.e. zooming out more).

For the spaces we have described above as we descend down the tree we will always end up with a configuration where the particles are at finite distances and so can be described easily. The initial objects in the set up correspond to the spaces with two points, either both on the boundary or both internal, and both of these spaces are compact as we will see. Thus when we glue everything back together we get a compact space where we use the trees to distinguish finer detail.

[1] All our trees are drawn with the root at the top so the potentially confusing terminology where you travese the tree up to its root

2 The configuration operad

In this section we will construct the configuration operad. Let us fix some vector space $\mathbb{V}$ of dimension d which one can simply assume to be $\mathbb{R}^d$. We let Conf_n denote the space of configurations of n-points in $\mathbb{V}$, that is elements in $\mathbb{V}^n$ that are pairwise disjoint. Given such a set we define its center of mass, $\mu : \mathrm{Conf}_n \to \mathbb{V}$, as the average position of all the points, and its diameter, $\mathrm{diam} : \mathrm{Conf}_n \to \mathbb{R}^>$, as the maximum distance between any two points. The group $G_2 = \mathbb{R}^> \ltimes \mathbb{V}$ acts on $\mathbb{V}$ by translation and scaling. We extend this action to $\mathbb{V}^n$ diagonally and we see that it preserves Conf_n. The quotient space $C_n := \mathrm{Conf}_n / G_2$ is a smooth manifold that we will also loosely refer to as the configuration space. One can find global coordinates for C_n: translate the first coordiante to the origin, and then scale so that the second coordinate has length 1. This gives a diffeomorphism of C_n with some open subset of $S^{d-1} \times \mathbb{V}^{n-2}$. In particular we have that $C_2 \cong S^{d-1}$ is a compact smooth manifold whilst for higher cardinality configurations it is non-compact. We can interpret these non-compact directions as where two or more points get infinitely close to each other relative to the rest of the configuration.

Given an element $c \in \mathrm{Conf}_n$ we say it is in *standard position* if $\mu(c) = 0$ and $\mathrm{diam}(c) = 1$. This condition provides a continuous section of the projection $\mathrm{Conf}_n \to C_n$ as the maps μ and diam are continuous. Note that this section is not smooth unless $n = 2$.

2.1 Trees and concatenation

Let T be a reduced tree where each leaf is labelled uniquely from the set $\{1, \dots, n\}$. Firstly for some notation: $E_I(T)$ is the set of internal edges of T, $E_E(T)$ the external edges (which is naturally isomorphic to $\mathrm{leaf}(T)$), and $V_I(T)$ the set of internal vertices. For $v \in V(T)$ a vertex we let $\mathrm{star}(v)$ denote the set of edges leaving v and $\mathrm{r}(v)$ the (unique) edge entering v. Given any vertex v we define $T(v)$ the sub-tree rooted at v. We can act on such a tree with Σ_n by permutating the labelling on the leaves. We consider the category $\mathcal{T}_n$ of reduced trees with the labelling as given above. A morphism of such trees is a normal map of trees that preserves the labelling on the leaves. The symmetric group acts on the set of such trees by permutation of the leaf labelling and we construct the category of trees with leaves labelled in some finite set by the usual way.

Given $T \in \mathcal{T}$ we set

$$G_2(T) = (G_2)^{E_I(T)} \times (\mathbb{V})^{E_E(T)}\,,$$
$$G_2^0(T) = (G_2)^{V_I(T)}\,.$$

We have an action of $G_2^0(T)$ on $G_2(T)$ as follows: For an element g_v it is the left action of G_2 on the G_2-module indexed by $\mathrm{star}(v)$ (this could be either G_2 or $\mathbb{V}$) and it acts on the space labelled by $\mathrm{r}(v)$ as $g \mapsto g g_v^{-1}$:

$$(g_v) \cdot (m_e) = \begin{cases} g_{\mathrm{s}(e)} m_e g_{\mathrm{t}(e)}^{-1} & e \in E_I(T) \\ g_{\mathrm{s}(e)} m_e & \text{otherwise.} \end{cases}$$

We also have a function $\gamma_T : G_2(T) \dashrightarrow \mathrm{Conf}_{\mathrm{leaf}(T)}$, called *concatenation*, which we construct inductively. For a tree of depth 1 we have that $G_2(T) \cong \mathbb{V}^{\mathrm{leaf}(T)}$ and the map is just the identity where it is defined. For the inductive step let v_0 be the root of T and $\xi \in G_2(T)$. We then define an element in $\mathbb{V}^{\mathrm{leaf}(T)}$ as

$$\left(g_e \cdot \gamma_{T(\mathrm{t}(e))}(\xi|_{T(\mathrm{t}(e))})\right)_{e \in \mathrm{star}(v_0)}$$

where here the restriction means the projection onto those edges lying in the subtree and the $\gamma_{T(\mathrm{t}(e))}$ are defined by the inductive step. In words we construct the sub-configurations and then take there union after shifting and rescaling them according to the G_2 elements along the given edges. Note that this map is globally well defined with image in $\mathbb{V}^{\mathrm{leaf}(T)}$ and the open subset on where its image lies in $\mathrm{Conf}_{\mathrm{leaf}(T)}$ we shall denote by V_T.

Lemma 2.1. *The concatenation map is a smooth submersion onto its image.*

Proof. The fact that the map is smooth follows as all the actions are smooth and the concatenation is simply the application of these actions and maps into products.

It is a submersion for given any element in the final configuration we have that it is mapped from a single leaf element acted upon by the G_2 elements from the chain of internal edges lying above it. Thus perturbing the leaves clearly gives a surjective map onto the tangent space of $\mathrm{Conf}_{\mathrm{leaf}(T)}$ (and to $\mathbb{C}^{\mathrm{leaf}(T)}$ on the whole space). $\square$

Proposition 2.2. *The concatenation map is equivariant under the G_2^0 action where its action on $\mathbb{V}^{\mathrm{leaf}(T)}$ is given by the action of the G_2 term corresponding to the root of T.*

Let Γ_T denote the induced quotient map to $C_{\mathrm{leaf}(T)}$. It is a diffeomorphism onto its image.

Proof. If we look at the image of a leaf element of T, say x_i, then it is mapped to $g_{e_0}g_{e_1}\cdots g_{e_r}x_i$ where the edges $(e_0, e_1, \dots, e_r, i)$ give the unique path from the root to the leaf. The equivariance under the action is now clear.

From 2.1 we have that the map Γ is a submersion and by counting dimensions we see that it is a local diffeomorphism. Therefore it is sufficient to show that Γ is an injection. Take two points that are mapped to the same element in $C_{\mathrm{leaf}(T)}$ and representatives $\xi, \xi' \in G_2(T)$. The fact that they are mapped to the same elements tells us that we can find some $g \in G_2$ such that

$$gg_{e_0}g_{e_1}\cdots g_{e_r}x_i = g'_{e_0}g'_{e_1}\cdots g'_{e_r}x'_i$$

where the edges are given by the path as explained above. Now for each $e \in \mathrm{star}_I(\mathrm{r}(T))$ we can find a unique element h in the G_2 corresponding to $\mathrm{t}(e)$ such that $gg_eh = g'_e$. We can then act with this element on ξ' to get that in the product above g'_{e_0} is replaced by gg_{e_0} and g'_{e_1} is replaced by hg'_{e_1}. We can now continue to push these elements down the tree using the group G_2^0 until it hits the terms on the leaves. But then we have $gg_{e_0}g_{e_1}\cdots g_{e_r}x_i = gg_{e_0}g_{e_1}\cdots g_{e_r}x'_i$ which shows that $x_i = x'_i$ and so the ξ and ξ' define the same class. $\square$

We denote by U_T the projection of V_T under $G_2^0(T)$, which is the open subset on which Γ is defined[2]. In other texts it is now common to look for explicit representatives of this projection, for example choosing all leaf points to lie at the origin. We do not do this as it makes the formulae less tractable although we do generally like to think of this quotient space, $G_2(T)/G_2^0(T)$ as being built from configurations on each internal vertex with scales along the edges and we will use this idea below.

Let $e \in E_I(T)$ be an internal edge, we then can define a map $\gamma_{T,T/e} : G_2(T) \to G_2(T/e)$ that is the identity on all edges except $\mathrm{star}(\mathrm{t}(e))$ where its action is given by the left action of g, that is if $\xi = (g_e) \in G_2(T)$ we have

$$\gamma_{T,T/e}(\xi)_{e'} = \begin{cases} g_eg_{e'} & e' \in \mathrm{star}(\mathrm{t}(e)) \\ g_{e'} & \text{otherwise.} \end{cases}$$

Chasing the definitions we have that $\gamma_{T/e} \circ \gamma_{T,T/e} = \gamma_T$, and that it is equivariant. We thus get a functor from the space of trees to manifolds with diffeomorphisms.

To construct the configuration operad we will let some of the scales by which we insert configurations up the tree to formally go to zero. Let $G_2^+(T)$ denote the space where now at each internal edge we stick a copy of $G_2^+ = \mathbb{R}^{\geq} \times \mathbb{V}$ rather than G_2. This space is a manifold with corners rather than a smooth manifold. The group G_2^0 still acts on this space.

Lemma 2.3. *The action G_2^0 on $G_2^+(T)$ is the action of a Lie group on a manifold with corners and the quotient naturally has the structure of a manifold with corners.*

2.2 Construction of the operad

Let T be a tree and $v \in V_I(T)$ be some internal vertex. We can define a map from $\tau_v : G_2^+(T) \to \mathbb{V}^{\mathrm{star}(v)} \times (\mathbb{R}^{\geq})^{\mathrm{star}_I(v)}$ which simply breaks down the elements in $\mathrm{star}(v)$ into its translations and scale components. Given $\varepsilon > 0$ we define the set $V_T(\varepsilon)$ as those points $\xi \in V_T$ such that for all vertices $v \in V_I(T)$:

1. $|\mu_v(\xi)| < 1$,
2. $|x_i - x_j| > \varepsilon$ for each $i, j \in \mathrm{star}(v)$,
3. $s_i < 1/\varepsilon$ for $i \in \mathrm{star}_I(v)$,
4. and $\mathrm{diam}_v(\xi) \in (1/2, 3/2)$

[2]When we quotient out by the action we do need to restrict to the given U_T to discuss smoothness as the quotient itself is not smooth at the other points.

where μ_v is the center of mass of the points in $\mathbb{V}^{\mathrm{star}(v)}$ as described under τ_v, and diam_v is their diameter. This is an open set and its image under the quotient by $G^0(T)$ we denote by $W_T(\varepsilon) \subset U_T$. As we restrict the location of the center of mass and the diameter these spaces cannot cover the full configuration space, we do however have

Lemma 2.4. *There exists an $r > 0$ such that for any $\varepsilon \in (0, r)$ the sets $W_T(\varepsilon)$ cover C_n.*

Proof. Let c be a configuration in standard position and take two points of maximal distance, $x, y \in c$. Then if we project each point onto the interval $[0, 1]$ by looking at their component in the $x - y$ direction we can find some point in this interval such that it has a distance $> 1/n$ to the next point. This splits the points into a non-trivial partition and taking the center of masses we get that they define a configuration of two elements (as they are split by a hyperplane). Then if we consider the scale required to move the subconfigurations from standard position we see that it is $< n$ as the center of masses are at least $1/n$ distance apart and the subconfigurations must have diameter ≤ 1 as it is a subconfiguration of one in standard position. Thus if we choose $\varepsilon < 1/n$ we see that at this vertex the conditions to lie in $V_T(\varepsilon)$ are all satisfied. We can then proceed by induction on the sub-configurations (which are in standard position). The base case $n = 1$ holding trivially we get the result. ☐

Let $\overline{V}_T(\varepsilon)$ denote the closure of this set in $G_2^+(T)$.

Lemma 2.5. *$\overline{V}_T(\varepsilon)$ is compact and defines a configuration at each vertex under the map $G_2(T) \to \mathbb{V}^{\mathrm{star}(v)}$.*

Proof. In $\mathbb{V}^{\mathrm{star}(v)} \times (\mathbb{R}^{\geq})^{\mathrm{star}_I(v)}$ the image of $V_T(\varepsilon)$ lies in a compact subset. This is clear for the scales, the set is $[0, 1/\varepsilon]$, and the condition on the center of mass and diameter imply that all the points must lie within a ball of radius 3 from the origin. It follows that the closure is a subspace of a compact set and hence closed.

For the second part as all the points are $> \varepsilon$ apart their closure must also have a non-zero distance between each point. ☐

Lemma 2.6. *The concatenation maps $\gamma_{T,T/e}|_{V_T(\varepsilon)}$ (resp. $\Gamma_{T,T/e}|_{W_T(\varepsilon)}$) extend to the closures.*

Proof. Recall the structure of the concatenation, it simply acted by the element $g \in G_2$ on all elements in $\mathrm{star}(\mathrm{t}(e))$. So now take a sequence of elements $(\xi_n) \in W_{T,T/e}$. We then claim that the scale $s_e(\xi_n)$ is bounded below by some constant. To see this we note that the concatenation can be seen as

$$\mathbb{V}^{\mathrm{star}(\mathrm{s}(e))} \times \mathbb{R}^{>} \times \mathbb{V}^{\mathrm{star}(\mathrm{t}(e))} \to \mathbb{V}^{\mathrm{star}(\mathrm{s}(e)) \cup \mathrm{star}(\mathrm{t}(e)) \setminus \{e\}}$$

which just multiplies the elements of $\mathbb{V}^{\mathrm{star}(\mathrm{t}(e))}$ by the corresponding scale and then offsetting them by x_e. The condition that this lies in $W_{T/e}$ forces $s > \varepsilon \,\mathrm{diam}(c')$ where c' is the configuration defined at the vertex $\mathrm{t}(e)$. As this diameter is bounded below by $1/2$ this implies that $s > \varepsilon/2$ and thus in the limit we get a positive value for s_e i.e. an actual group element. We can then apply this group element to the corresponding spaces to get the map. ☐

We can now define the space $\overline{C}_n$, it is taken as the limit over the gluing data:

$$\overline{C}_n = \varinjlim \overline{W}_T = \bigsqcup \overline{W}_T / \sim .$$

Lemma 2.7. *There is a natual embedding $C_n \to \overline{C}_n$ and its closure is the whole space.*

Proof. The existence of the embedding comes from 2.4: Take any configuration c and find some T so that it lies in W_T in the sense that there is some $\xi \in W_T$ with $\Gamma_T(\xi) = c$. Then this class in $\overline{C}_n$ is well defined by the definition of the gluing data and so this gives the map. As the map Γ_T is a diffeomorphism, 2.2, we have that this is continuous and open and hence an embedding. The fact that is closure is the whole space is immediate. ☐

Lemma 2.8. *The space $\overline{C}_n$ is independent of the choice of ε, i.e. for any other choice ε' the spaces are homeomorphic.*

Proof. We note that there is the inclusion $\overline{W}_T(\varepsilon) \subset \overline{W}_T(\varepsilon')$ for $\varepsilon' < \varepsilon$. We then get the map as defined in 2.7 factors as $C_n \to \overline{C}_n(\varepsilon) \to \overline{C}_n(\varepsilon')$. Thus the corresponding map is a homeomorphism. ☐

We can use this fact to *concatenate out* any non-zero scales. Take some $\xi \in W_T(\varepsilon)$ and consider the set of its non-zero scales. When we concatenate this into $G_{T/e}/G^0_{T/e}$ we saw in the proof of 2.6 that this may not lie in $W_{T/e}(\varepsilon)$ for if s_e is small it might bring the points too close together. However it will always keep them a finite distance apart so it will define an element in $W_{T/e}(\varepsilon')$ for some other ε' and from 2.8 we can just use this cutoff instead. Proceeding along all the non-zero edges we end up with an element where all the scales are zero on some new tree T'.

Lemma 2.9. *In the above construction the tree T' and the value of the configuration classes at each vertex are uniquely defined. This gives a stratification*

$$\overline{C}_n = \bigsqcup_T C_T$$

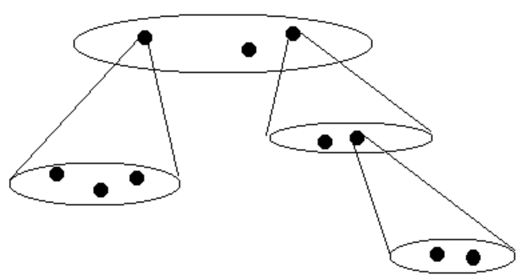
where the union is over all computation trees and $C_T = \prod_{v \in V_I(T)} C_{\mathrm{star}(v)}$.

Proof. The fact that the tree is uniquely defined is clear for in the proof of 2.6 we saw that a scale being zero or not is preserved under the concatenation and hence any of the identifications.

We can thus represent any element by a tree T and $\xi \in G_2^+(T)$ where all the scales on the edges are zero. So at each edge we only have the information of the offset. We can then construct an element in C_T as follows: for any internal vertex choose a group elements such that the offsets at the edges are mapped to standard position under this action. We apply this from the deepest vertices up to the root to get an element in C_T.

To complete the proof we now need to show that any element $\xi \in C_T$ does lie in some $\overline{W}_T$. Take the corresponding representations in standard position and let ε be less than the minimal distance between points in all configurations. Then if we consider the scales positive but less than $\varepsilon/2$ we see that the concatenation map is well defined. We can then take the limits as these scales go to zero to get the element ξ. □

This stratification makes the meaning of the magnifications precise and we can then read off the computation graphs:



Proposition 2.10. *$\overline{C}_n$ is a compact manifold with corners.*

Proof. Compactness follows as it is the image of a compact space using 2.5 and second countability follows from a similar argument.

Using that $C_n \to \overline{C}_n$ is an embedding by 2.7 we have that any point lying in this space has a neighbourhood isomorphic to some open subset of Euclidean space. Note that a point lies in this space iff all its scales are non-zero. If some of the scales are zero then we get an open neighbourhood isomorphic to the basic cornered neighbourhoods in Euclidean space.

Hasudorffness follows from the stratification if they lie in the same tree and if they don't then we must be able to find some subset of points that lie close under one of the trees and not the other so we can just take the closure of some seperating neighbourhoods based off the distance of the corresponding points. □

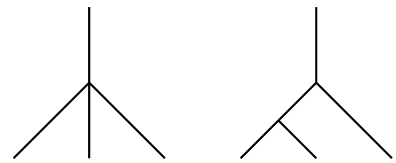
Proposition 2.11. *We have natural smooth submersions*

$$\overline{C}_n \times \overline{C}_{p_1} \times \cdots \times \overline{C}_{p_n} \to \overline{C}_{p_1 + \cdots + p_n}$$

that give the family of spaces $(\overline{C}_n)$ the structure of an operad.

2.3 Low dimensional examples

We have already seen that $\mathrm{Conf}_2 \cong S^{d-1}$. So lets turn to Conf_3. We have four trees:



where the labelling of the leaves gives three non-isomorphic trees in the second case. Thus the boundary where two points come infinitely close together is given by a product of spheres $S^{d-1} \times S^{d-1}$ corresponding to the relative location of the two points and the point a finite distance from them, and then the infinitely close internal configuration.

3 The upper half space and the boundary operad

We now want to apply the same techniques to the boundary operad as described above. If we consider the upper half space $\mathbb{H}^d = \{x_i \in \mathbb{R}^d \mid x_d \geq 0$ with the boundary $\mathbb{R}^{d-1}$ given by those points where $x_d = 0$. This space is acted on by the group $G_1 = \mathbb{R}^> \ltimes \mathbb{R}^{d-1}$. In the generic set up we consider a vector space $\mathbb{V}$ with an orientated hyperplane $\mathbb{W}$ and the pair $\mathbb{V}, \mathbb{W}$ and here $G_1 = \mathbb{R}^> \ltimes \mathbb{W}$.

3.1 Boundary trees

We will now describe boundary trees, which will play the role of the normal trees. We have two types in the tree which we refer to as the internal or boundary type. The trees are now no longer required to be reduced but they must be generated by vertices of the following form:

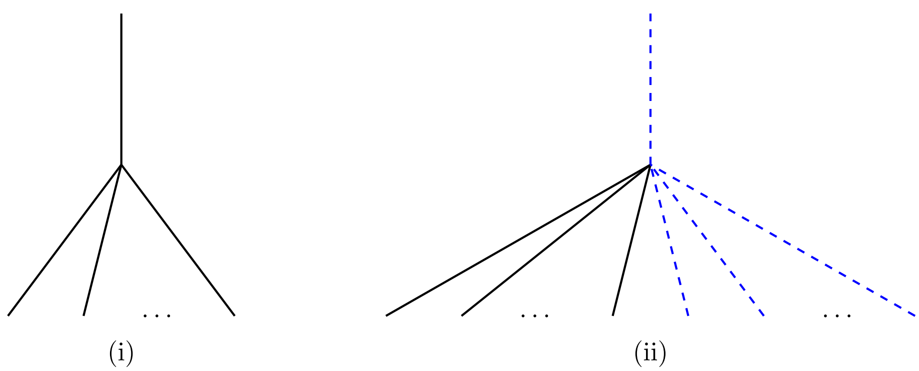


Figure 1: Valid vertices in boundary trees

The first of these vertices, type (i), corresponds to a normal tree vertex that we have been using for configurations, there must be at least two edges leaving the vertex. The second type of vertex are ones that produce boudnary terms. We now have the condition that $2n + m > 1$ where n is the number of incoming internal edges and m the incoming number of boundary edges.

We will label the types as C_1 and $C_{0,1}$ and as before we assume a labelling on the leaves over a given set. The category of such trees with n internal leaves and m boundary leaves we denote by $\mathcal{T}_{n,m}$.

Given $T \in \mathcal{T}_{n,m}$ we define $G(T)$ in a similar fashion to before but now if the type of an internal edge is boundary we insert a G_1 rather than a G_2. Similarly for $G^0(T)$ the type of the edge $\mathrm{r}(v)$ determines which group we place (we will always place G_1 at the root verex).

We can define the concatenation as before by applying the elements on the edge and taking the union to work up through the tree. The subset on which this gives a valid boundary configuration we shall denote V_T as before and U_T the prejection to $G(T)/G^0(T)$. An identical argument shows that $gamma_T$ is a submersion and that Γ_T is a diffeomorphism.

To compactify the boundary configurations we are going to want to take the limit for small scales as before. Thus to any of the internal edges we let the scales of either G_1 or G_2 go to zero. We denote the corresponding

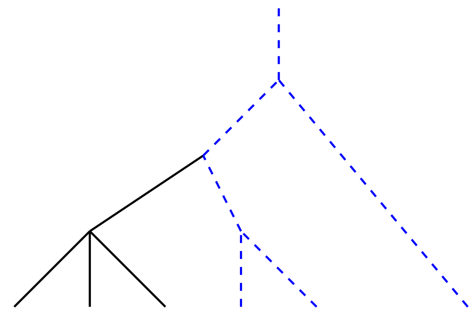


Figure 2: An example tree in $\mathcal{T}_{3,3}$

extended space as $G^+(T)$. As before we see that $G^0(T)$ acts on this space preserving the corners and so we get the quotient manifold with corners $G^+(T)/G^0(T)$.

3.2 The boundary operad

Let us define $V_T(\varepsilon)$ in a similar fashion: if the vertex v is of internal type then we apply the same requirements as for the configurations. Otherwise we require that:

1. $|\pi_{\mathbb{W}}(\mu_v)| < 1$,
2. $|x_i - x_j| > \varepsilon$ for any $i \neq j \in \mathrm{star}(v)$,
3. $s_i < 1/\varepsilon$ for $i \in \mathrm{star}_I(v)$.
4. $\mathrm{diam}(x_i), \max_i \mathrm{dist}(x_i, \mathbb{W}) \in (1/2, 3/2)$

where $\pi_{\mathbb{W}}$ denotes the orthogonal projection onto the hyperplane. In words we require that the center of mass of the configuration is a bounded distance from the origin when projected to $\mathbb{W}$, the configuration points are suitably far apart, the scales aren't too large and both the diameter and the distance of the configuration from the boundary are bounded.

We used the standard position previously in a lot of arguments and it will be useful here. Given a configuration $c \in \mathrm{Conf}_{n,m}$ we say it is in standard position if $\pi_{\mathbb{W}}(\mu(c)) = 0$ and $\max(\mathrm{diam}(c), \max_i(\mathrm{dist}(x_i, \mathbb{W}))) = 1$. Clearly any configuration can be brought into standard position by an element of G_1 and it defines a unique representative of the class in $C_{n,m}$.

Lemma 3.1. *There exits an $r > 0$ such that if $\varepsilon \in (0, r)$ then the $W_T(\varepsilon)$ cover $C_{n,m}$.*

Proof. We will proceed as before but with some minor changes. Take a configuration c in standard position. We now have two cases to consider - the diameter of the configuration is 1 or it is less than 1. In the first case we perform a similar argument to what we did for 2.4 in that we project onto the interval and take the split. The corresponding configuration now can be one of $C_{2,0}$, $C_{1,1}$, or $C_{0,2}$. In any case if we choose ε suitably we are fine, $< 1/n$, and we can proceed by induction as the partition is proper.

In the second case one should imagine a cluster of points all being quite close together and pushing to be a distance of 1 from the boundary. In this case we split the configuration into $1/\varepsilon$ buckets corresponding to the height of the points from the boundary. We again get a map onto $[0, 1]$ in this case although it may not be the case that the ε split provides a proper partition (if it does we proceed by induction). If this is not the case then all the points must lie off the boundary by at least a distance ε (where we have assumed $\varepsilon < 1/n$) and this generates a vertex with purely internal edges coming in and a boundary vertex leaving. Then the points are discernibly different from the boundary at our level of magnification and so we can consider them as an element of Conf_n. Then we can just proceed to apply 2.4 directly. $\square$

From the description of $V_T(\varepsilon)$ It follows that the points of the configuration all lie in a cube around 0 and so if we take the closure of this set in $G^+(T)$ we end up with a compact set as before.

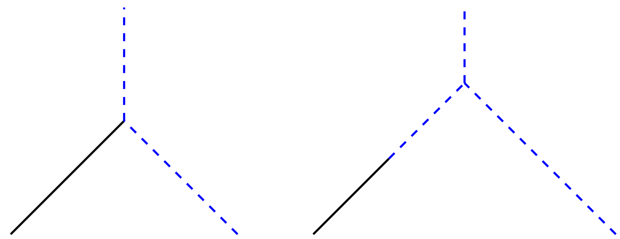
The proof of the construction of the space now proceeds in a similar fashion to that for the configuration spaces:

Theorem 3.2. *$\overline{C}_{n,m}$ is a compact manifold with corners. It has a natural stratification into computation trees and the family of such spaces has an operad structure:*

$$\overline{C}_{n,m} \times \overline{C}_{p_1} \times \cdots \overline{C}_{p_n} \times \overline{C}_{q_1,r_1} \times \cdots \overline{C}_{q_m,r_m} \to \overline{C}_{p_1+\cdots+p_n+q_1+\cdots+q_m,r_1+\cdots+r_m}$$

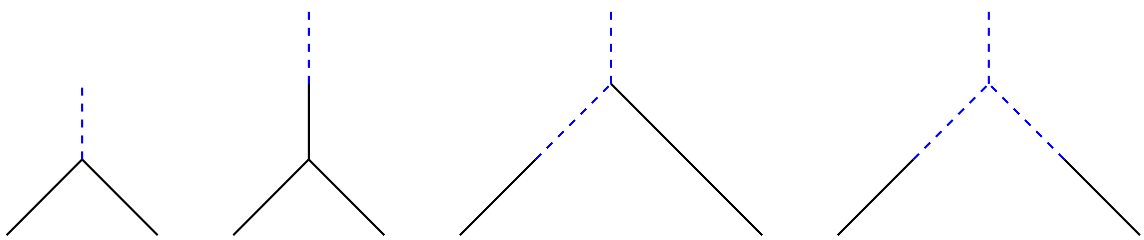
3.3 Low dimensional examples

We have more low dimensional examples that are quite easy to see. Consider $C_{1,1}$ this can be thought of as the open upper hemisphere lying above the hyperplane $\mathbb{W}$. We have two computation trees:



The compactification points, from the second tree, correspond to those configurations where the point in the interior of the manifold is far away from that on the boundary and comes close to the boundary. This gives $C_{0,2}$ which is exactly the configuration of two points on the $d-1$ dimensional vector space and is given by the sphere S^{d-2} - the equator of the upper hemisphere.

The next example to consider is $C_{2,0}$: the computation trees are:



The third of of these trees comes in a pair corresponding to the non-trivial action of Σ_2 on the leaf labels. The first of these corresponds to the partition in general position. The second to the two interior points being infinitely far away from the boundary, the third where one of the points goes to the boundary, and the final one where the two points go to the boundary but far away from each other. The corresponding topological spaces are $\mathbb{V}_+ - \{*\}$, S^{d-1} which is glued to the extracted point, and then two upper hemispheres and the sphere S^{d-2} correspond to gluing the points as they go to infinity or to the boundary. Note that topologically this space is homoemorphic to a d-dimensional annulus but the smooth structure introduces corners.

We can continue the above procedure to define a corner operad for a manifold with corners. Now the number of types depends on the dimension of the configuration space (we have sort of seen this above where the boundary operad is not of the same type for $d = 1$). The operad is described in a similar and one can construct trees to describe it using the same philosophy of magnification.

References

[Kon97] Maxim Kontsevich. "Deformation quantization of Poisson manifolds, I". In: (1997). DOI: `arXiv:q-alg/9709040v1`.

[Gai] Giovanni Gaiffi. *Compactifications of configuration spaces*. URL: `http://people.dm.unipi.it/~gaiffi/papers/giov.pdf`.

[May] Jon Peter May. *Operads, algebras and modules*. URL: `http://math.uchicago.edu/~may/PAPERS/handout.pdf`.

[nLa] nLab. *Fulton-MacPherson operad in nLab*. URL: `https://ncatlab.org/nlab/show/Fulton-MacPherson+operad`.