The Orthogonal Group

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These notes contain basic information about the orthogonal group. We will assume all of our vector spaces come with a non-degenerate inner product which we will also call normed spaces. $O(V)$ will denote the orthogonal group of V.

1 Preliminaries

1.1 The null cone

Let V be a normed space, we then define the *null cone* of V, $\mathcal{N}(V)$, to be all the vectors with zero norm. This space is a cone for if γ is null and $c \in \mathbb{R}$ then $c\gamma$ is also null. Also note that any isometry preserves the null cone, i.e. if $A : V \to W$ is an isometry then $A\mathcal{N}(V) \subset \mathcal{N}(W)$.

In Lorentzian signature the null cone is easy to describe: if we write $\gamma = \gamma^0 e_0 + \gamma^i e_i$ where $-\|e_0\|^2 = \|e_i\|^2 = 1$, then γ is null if

$$(\gamma^0)^2 = \|\gamma\|^2. \tag{1}$$

So the null vectors lie in families of spheres whose radius is given by the time component.

The null cone is not a linear subspace of V but we have the following:

Lemma 1.1. *Let $\gamma \in V$ be null. Then*

1. *$\exists \bar{\gamma}$ null with $\langle \gamma, \bar{\gamma} \rangle = 1$.*
2. *If ξ is null and $\langle \gamma, \xi \rangle \neq 0$ then $\gamma + c\xi$ is non null $\forall c \in \mathbb{R}^\times$.*

Proof.

1. By the non-degeneracy of the inner product choose a v with $\langle v, \gamma \rangle = 1$. Then $\bar{\gamma} = v - \frac{\|v\|^2}{2}\gamma$ also has inner product 1 with γ and is null.
2. For null γ and ξ we have $\|\gamma + \xi\|^2 = \langle \gamma, \xi \rangle$. The result then follows as $c\xi$ is null.

$\square$

1.2 Orthogonal complements

Let $W \subset V$ be a linear subspace. We say that it has an *orthogonal complement* if $V = W \oplus W^\perp$.

Lemma 1.2. *Let $W \subset V$ have an orthogonal complement. Then*

1. *The inner product restricted to W and $W^\perp$ is non-degenerate.*
2. *$(W^\perp)^\perp = W$.*
3. *If $g \in O(V)$ and $gW \subset W$ then $gW^\perp \subset W^\perp$.*

Proof.

1. Let $w \in W$, then we can find some $w' \oplus w'' \in W \oplus W^\perp = V$ with $\langle w' \oplus w'', w \rangle = \langle w', w \rangle \neq 0$ by the non-degeneracy of the inner product in V. It follows that the inner product is non-degenerate on W. An identical argument goes through for $W^\perp$.
2. Clearly $W \subset (W^\perp)^\perp$. Conversely assume that $w \in (W^\perp)^\perp$ then by the first part we have that w must lie in W as the inner product restricted to $W^\perp$ is non degenerate.

3. As g is an automorphism we have that $AW = W$. Now let $v \in W^\perp$ and write $Av = w \oplus u$ under the direct sum. If $w \neq 0$ choose $w' \in W$ with $\langle Aw', w\rangle \neq 0$ which exists by the first part and that $AW = W$. We have
$$0 = \langle v, w'\rangle = \langle Av, Aw'\rangle = \langle w \oplus u, Aw'\rangle = \langle w, Aw'\rangle$$
which is a contradiction so $w = 0$ and $AW^\perp \subset W^\perp$.

□

For non null vectors we can use the Gram-Schmidt algorithm to construct orthognal complements as we would do for definite spaces.

Lemma 1.3. *Let $(v_i)_{i=1,\ldots,r}$ be non null linearly independent vectors and W the subspace spanned by them. Then W has an orthogonal complement.*

Proof. We shall prove the result by induction on the dimension of W. For the base case, $r = 1$, let $u \in V$ then $u - \frac{\langle u,v_1\rangle}{\langle v_1,v_1\rangle}$ is orthogonal to v_1 and this shows that $V = (v_1) \oplus (v_1)^\perp$.

For the inductive step assume that $W_{r-1} = (v_1, \ldots, v_{r-1})$ has an orthogonal complement. Then by linear independence v_r has a non-zero component in $W_{r-1}^\perp$, call it w_r, and by 1.2 we can apply the base case to w_r to get that $W_{r-1}^\perp = (w_r) \oplus (w_r)^\perp \cap W_{r-1}^\perp$. Then $W = W_{r-1} \oplus (w_r)$ has an orthogonal complement. □

For null vectors we need to take a little more care. If we have a set of orthogonal linearly independent vectors $(\gamma_i)_{i=1,\ldots,r}$ then a *dual basis* for this set is another family of orthogonal linearly independent null vectors $(\bar\gamma_i)_{i=1,\ldots,r}$ such that
$$\langle \gamma_i, \bar\gamma_j\rangle = \delta_{ij}. \tag{2}$$

Lemma 1.4. *Let $(\gamma_i)_{i=1,\ldots,r}$ be orthogonal linearly independent null vectors and $(\bar\gamma_i)_{i=1,\ldots,r}$ a dual basis. Then $H = (\gamma_1, \ldots, \gamma_r, \bar\gamma_1, \ldots, \bar\gamma_r)$ has an orthogonal complement.*

Proof. Let $e_i^\pm = \gamma_i \pm \bar\gamma_i$. Then by 1.1 none of th $e_i^\pm$ are null and as they span H they are linearly independent. Thus we can apply 1.3 to H. □

Lemma 1.5. *Let $(\gamma_i)_{i=1,\ldots,r}$ be a orthogonal linearly independent null vectors. Then we can find a dual basis for them.*

Proof. We prove this by induction on r. The base case $r = 1$ is handled by 1.1. For the inductive step assume that we have a dual basis $(\bar\gamma)_{i=1,\ldots,r-1}$. Let $\widetilde\gamma_i = \bar\gamma_i - \langle\gamma_r, \bar\gamma_i\rangle\gamma_r$. By the orthogonality of the (γ_i) we have that $(\widetilde\gamma_i)_{i=1,\ldots,r-1}$ is still a dual basis to $(\gamma_i)_{i=1,\ldots,r-1}$ and moreover $\langle\gamma_r, \widetilde\gamma_i\rangle = 0$ for $i = 1, \ldots, r-1$. Now by 1.4 we have $V = H_{r-1} \oplus H_{r-1}^\perp$ where here H is the subspace spanned by the γ_i and $\widetilde\gamma_i$ for $i = 1, \ldots, r-1$. We have that $\gamma_r \in H_{r-1}^\perp$ and so by 1.1 and 1.2 we can find some null $\widetilde\gamma_r \in H_{r-1}^\perp$ with $\langle\gamma_r, \widetilde\gamma_r\rangle = 1$. It follows that $(\widetilde\gamma)_{i=1,\ldots,r}$ is a dual basis to (γ_i). □

1.3 Reflections and symplectic reflections

Reflections and their symplectic cousins form, as we shall see, the basic building blocks of all orthogonal maps. In a definite signature they have a simple interpretation where one can think of flipping the sign of one if the coordinates - this holds in the non definite case. We also have the slightly more mysterious symplectic reflections which come from subspaces of the null cone.

Let $v \in V$ be non null. We then define the reflection along v, R_v, as the linear map:
$$R_v(u) = u - 2\frac{\langle u, v\rangle}{\langle v, v\rangle}v. \tag{3}$$

Lemma 1.6. *Let v be non null. Then*

1. *$R_v(v) = -v$.*
2. *$R_v \in O(V)$.*
3. *$R_v|_{(v)^\perp} = 1_{(v)^\perp}$.*

Proof. The first point is a direct calculation. For the second using 1.4 write $V = (v) \oplus (v)^\perp$. We have that any vector in V can be written as $cv \oplus w$ for $c \in \mathbb{R}$ and $w \in (v)^\perp$ and so
$$R_v(cv \oplus w) = -cv \oplus w.$$
The last two points immediately follow. □

As mentioned above we have an additional reflection which we call a *symplectic reflection.* There is no way to use a single null vector to generate an orthogonal transformation but assume we have two orthogonal null vectors, γ and ξ. We then define the symplectic reflection along them by

$$S_{\gamma,\xi}(v) = v + \langle \gamma, v\rangle \xi - \langle \xi, v\rangle \gamma. \tag{4}$$

Lemma 1.7. *Let γ and ξ be orthogonal null vectors. Then*

1. *$S_{\gamma,\xi}(\gamma) = \gamma$ and $S_{\gamma,\xi}(\xi) = \xi$.*
2. *$S_{\gamma,\xi}$ is the identity iff γ and ξ are linearly dependent.*
3. *$S_{\gamma,\xi} \in O(V)$.*
4. *Assume that γ and ξ are linearly independent and take a dual basis $(\bar{\gamma}, \bar{\xi})$ and let H be the span of the four null vectors. Then $S_{\gamma,\xi}|_{H^\perp} = 1_{H^\perp}$.*

Proof. The first point is a direct calculation from the formula recalling that γ and ξ are orthogonal. Clearly if ξ is proportional to γ (or vice versa) then S is the identity. Conversely if they are linearly independent pick a dual basis by 1.5 $(\bar{\gamma}, \bar{\xi})$. Then

$$S(\bar{\gamma}) = \bar{\gamma} + \xi, \quad S(\bar{\xi}) = \bar{\xi} - \gamma$$

hence S is not the identity which proves the second point.

If γ and ξ are linearly dependent that S is the identity by above so is orthogonal. We will thus assume they are linearly independent for the rest of the proof of the last two points. Let $(\bar{\gamma}, \bar{\xi})$ be any dual basis and H the span. Then H has an orthogonal complement by 1.4. It follows directly from the formula of S that $S|_{H^\perp}$ is the identity. We have also calculated its action on H above and using the properties of the dual basis we see that $S|_H \in O(H)$ hence $S \in O(V)$. □

1.4 Generators of the orthogonal group

We now have all the ingredients to prove the main result, a version of the Cartan-Dieudonné theorem:

Proposition 1.8. *The group $O(V)$ is generated by reflections and symplectic reflections. Moreover if V has signature (n, m) then any element is the product of at most p reflections and q symplectic reflections where $p + 2q \leq n + m$.*

Proof. We will prove the result by induction on $\dim(V)$. In the base case, when the dimension is one, we have that the space is definite and so any orthogonal map is just multiplication by ± 1. The non-trivial map is just reflection along any non-zero (and hence necessarily non null) vector.

Now to the inductive step: let $g \in O(V)$. Assume that we can find a $v \in V$ with $gv - v$ non null. We find that $R_{gv-v}(v) = gv$. If v is non null then the orthogonal transformation $g' = R_{gv-v}^{-1} \circ g$ fixes v and hence is described by an element of $O((v)^\perp)$. By the inductive step we have that g' is generated by reflections and symplectic reflections and so $g = R_{gv-v} \circ g'$ is as well. Now if v is in fact null then choose a dual basis $\bar{v}$. We then have the non null orthogonal vectors $v_\pm = v \pm \bar{v}$ - if either of these are such that $gv_\pm - v_\pm$ is non null then we can use either of them to reflect. If both are null then we have that $(gv_+ - v_+) + g(v_- - v_-) = 2(gv - v)$ is non null and so by 1.1 so is $g(v_+ - v_+) + cg(v_- - v_-)$ for any $c \in \mathbb{R}^\times$. But this is just $g(v_+ - cv_-) - (v_+ - cv_-)$ and if we choose $c \neq \pm 1$ then $v_+ - cv_-$ is non null as well. It follows that we can use this non null vector to generate the reflection which completes the proof if we can find a v with $gv - v$ is non null.

Conversely assume that $gv - v$ is null for all $v \in V$. Let $a = g - 1$ the image of a is a linear subspace of the null cone of V. Let (γ_i) be a basis of this image. If any $\langle \gamma_i, \gamma_j \rangle \neq 0$then $\gamma_i + \gamma_j$ is non null by 1.1 and so cannot lie in the image of a which is a contradiction. Hence (γ_i) is a set of linearly independent orthogonal null vectors. Pick a dual basis $(\bar{\gamma}_i)$ which exists by 1.5. We have that

$$\delta_{ij} = \langle \gamma_i, \bar{\gamma}_j \rangle = \langle g\gamma_i, g\bar{\gamma}_j \rangle = \langle \gamma_i + a\gamma_i, \bar{\gamma}_j + a\bar{\gamma}_j \rangle = \delta_{ij} + \langle a\gamma_i, \bar{\gamma}_j \rangle.$$

By the non-degeneracy of the dual basis we must have that $a\gamma_i = 0$ so that g fixes each γ_i. Let $\xi = a\bar{\gamma}_1$. As (γ_1, ξ) are orthogonal we can construct the symplectic reflection $S_{\gamma_1,\xi}$. We have from 1.7 that

$$S_{\gamma_1,\xi}(\gamma_1) = \gamma_1 = g\gamma_1, \quad S_{\gamma_1,\xi}(\bar{\gamma}_1) = \bar{\gamma}_1 + \xi = g\bar{\gamma}.$$

If we let $H = (\gamma_1, \bar{\gamma}_1)$ then we see that $g' = S^{-1}_{\gamma_1,\xi} \circ g$ fixes H and hence is described by an element of $O(H^{\perp})$ and so applying the inductive step again we see that g' and so g can be written as a product of reflections and symplectic reflections.

We see that the reflection reduced the dimension by one whilst the symplectic reflection reduced it by two which gives the result in terms of the number of reflections and symplectic reflections that are needed to construct an element of $O(V)$. $\square$

Corollary 1.9. *The groups $O(n)$, $O(n, 1)$, and $O(1, m)$ are generated by reflections.*

Proof. It is sufficient to show the maximal linear subspace of orthogonal null vectors has dimension ≤ 1 then by 1.7 any symplectic reflection will be trivial. This is immediate in the case of a definite space as there are no null vectors.

Now if there are two linearly independent orthogonal null vectors (γ, ξ) pick a dual basis $(\bar{\gamma}, \bar{\xi})$ and let H be the basis generated by these vectors. Then we have that the signature of H is $(2, 2)$ and it follows that there can be no such subspace in the spaces given in the claim by Sylvester's law of inertia. $\square$