Computation of Control Law for State Transfer Problem in Efficient Way for a Single Input

Article *in* IEEE Transactions on Industry Applications · May 2022

DOI: 10.1109/TIA.2022.3159502

2 authors:

Rajani Metri
Rajarambapu Institute Of Technology

Bhooshan Rajpathak
Visvesvaraya National Institute of Technology

Computation of Control Law for State Transfer Problem in Efficient Way for a Single Input

Rajanikant A. Metri, *Student Member, IEEE,* Bhooshan A. Rajpathak, *Senior Member, IEEE*

***Abstract*—The article proposes a modified way to compute open loop control for the state transfer problem (STP) for a linear time-invariant system. Solving STP analytically is cumbersome task. This is mainly due to tedious computations of state transition matrix and controllability Gramian matrix. The proposed modified method enables to find control law easily and with less computations. This approach is applicable even for a 3^{rd} or higher order system with single input case analytically and without use of computer. Moreover, proposed approach minimizes use of calculator and there-by reduces the computational rounding off errors. Finally, it is shown that for some special case with knowledge of mode vector, the state transfer problem can be solved very quickly without even calculating the state transition matrix and controllability Gramian matrix. Furthermore, theoretical as well as experimental evaluation of control law is provided. Experimental results are used to validate the proposed method.**

***Index Terms*—State transfer problem, controllability Gramian, control input, position control.**

I. Introduction

In recent years development in sensor, computation, control and actuators technology in industries impacted significantly on scientific and economic development in the society. Control theory has a major role in Industry 4.0 and become the mathematical backbone for various engineering disciplines. It is widely used as an analytical tool in manufacturing, operations and control areas in industry. Especially in Electrical engineering, control theory contributes in the fields like automation, electric drives, electric vehicles, power electronic circuits etc. Most of these complex non-linear dynamical system are linearized to get the linear model of the plant to design the control and analyse the parameters.

For linear time-invariant (LTI) systems the controllability Gramian plays the most vital role in control input. The LTI system is controllable *iff* the controllability Gramian matrix is positive definite and so it will be nonsingular. It is also the fundamental factor in deciding the continuous-time open loop control signal for a state transfer problem (STP) of a LTI system [1]. The eigenvalues of this matrix express the control energy required to transfer the system states in eigenvector directions. Computed control law uses the minimum amount of control energy to steer states [2], [3] also estimation is carried out [4], [5].

Initially the problem of state transfer was developed for an optimal sampling discrete time system. The notions of '*controllable state'* and '*controllability*', its criterion for a single input case are coined by Kalman for continuous and discrete time systems [6], [7]. Subsequently authors proposed solution of the state transfer problem, and computations of state-transition matrix [8]–[13]. Use of advanced software tools in solving state transfer problems is in practice. STP involves calculation of state transition matrix and controllability Gramian. In [12] authors have illustrated nineteen ways of computing state transition matrix. Gramian matrix is calculated by a simple repeated algorithm for a stable linear dynamical plant [14]. It is also obtained by solving the continuous-time Lyapunov equation [15].

A control objective is proposed for a two-degree-of-freedom system with rotating cylinder on a moving cart and optimization of an aircraft structural design using controllability Gramian. The results shows it is easier to control and it enhances the controllability of system [3]. Calculations of finite and infinite Gramians, sub-Gramians are carried out as well as uncertainty based optimal control is proposed for power grid to analyse stability, which are derived from algebraic Sylvester and Lyapunov equations [16]–[18]. Also, the optimal actuator and sensor placement based on controllability is discussed in [19]. For six different orders, errors comparison of backward differentiation methods are evaluated and analysed for calculating stiff controllability Gramian, and closed-form solutions are also calculated [20]. In order to mitigate the power oscillation and improve plant stability, the cost-effective power system stabilizers (PSSs) have been used to provide the damping. For PSS placement, full rank optimal conditions provide complete controllability [21]. Finite and infinite time Gramians and cross-Gramians [22] are expressed as bilinear [23] and quadratic matrix form in [16]. A probabilistic state transfer over a finite time instance is introduced in [24] using Hadamard product. Authors have explored on different ways of determining Gramian matrix and control law. However, from literature it is inferred that, for finding the solution of states by computing controllability Gramian is essential.

Solutions of STP in efficient way for systems having real distinct and repeated roots are presented and comparison of computations with conventional method are presented in [1]. Furthermore, in this paper we have considered the oscillatory systems i.e. plant dynamics with undamped and under-damped nature. For single input, we have illustrated the STP through 2^{nd} and 3^{rd} order examples and one real-time application is explored using Hardware-In-Loop setup. MATLAB/Simulink plots are obtained and experimental result are used to verify the proposed method for servo plant. The contributions of this paper and advantages of STP in efficient way are given below:

The authors are with the Department of Electrical Engineering, Visvesvaraya National Institute of Technology, Nagpur, 440010, INDIA. (email:rajanikant.metri@gmail.com; bhooshanar@gmail.com)

Manuscript received October 7, 2021; revised December 9, 2021, January 20, 2022, March 3, 2022; accepted March 7, 2022.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TIA.2022.3159502, IEEE Transactions on Industry Applications

- The states can be transferred to desired position within finite time with minimum control energy and is mathematically proved.
- There is no necessity of computing controllability Gramian matrix.
- For STP, this paper applies transformation which leads to significant reductions in computations.
- The proposed method is implemented on an industrial emulator setup of servo motor. Experiments show that the state transfer is successful based on proposed control law for given finite time scenario.

The paper is arranged as follows. Section II briefs about the existing and modified method for finding the required control input for the desired state transfer. Few 2^{nd} and 3^{rd} order system example results with different cases are presented to demonstrate and verify the proposed method theoretically. In Section III, comparison between existing and modified technique is discussed. Total number of mathematical operations required to compute the control energy is also worked out followed by experimental results of an industry application in Section IV. Section V lists the conclusions.

II. Methods to Find Out Required Input for State Transfer

Let us consider a dynamical system described as given in equation (1);

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\boldsymbol{u(t)} \\ y(t) &= Cx(t) + D\boldsymbol{u(t)} \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is state matrix.

Let us assume that the system is driven by a single input. For such systems, computing required control input $\boldsymbol{u(t)}$ for transferring system from given initial state x_0 to given final state x_{t_f} in finite time (t_f) is termed as State Transfer Problem (STP) [2], [8], [15], [25]. For STP, the main constraint is on time i.e., it should be finite, whereas there is no constraint on input or its magnitude or its trajectory [15].

To compute the control input $\boldsymbol{u(t)}$ for STP, the method is first discussed in reference [26]. Following steps are used to compute the $\boldsymbol{u(t)}$:

A. Conventional Method (Method-I)

Consider a LTI system as given by (1),

1) Step 1: Compute State Transition Matrix (STM) as,
$$\Phi(t) = e^{At} = \mathscr{L}^{-1}\{[sI-A]^{-1}\} \tag{2}$$
Let us denote $[sI-A]$ by Γ and the determinant of Γ as Δ_Γ. Now, the inverse of Γ is computed as:
$$\Gamma^{-1} = \frac{\text{Adjoint of } \Gamma}{\Delta_\Gamma}$$
Then Laplace inverse of n^2 (for n^{th} order system) elements of Γ^{-1} need to be calculated.
2) Step 2: Calculate Controllability Gramian
$$W_c(0,t_f) = \int_0^{t_f} e^{A\tau}BB^\intercal e^{A^\intercal\tau}d\tau \text{ for some } 0 < t_f < \infty \tag{3}$$
STM is computed in step 1. Now the integration of $\frac{n(n+1)}{2}$ terms in (3) (since $W_c(0,t_f)$ is symmetric) are to be found. Each term of $W_c(0,t_f)$ consists of at most 'n' exponential elements. Further, we need to calculate determinant and adjoint of W_c matrix to get W_c^{-1}.
3) Step 3: Compute required control input $\boldsymbol{u(t)}$
The control input required for transferring states from initial state to final state in finite time is given by:
$$\boldsymbol{u(t)} = -B^\intercal e^{A^\intercal(t_f-t)}W_c^{-1}(0,t_f)\left[e^{At_f}x(0) - x(t_f)\right] \tag{4}$$

Clearly, mathematical computations involved in this method are given in Section III & Appendix A to evaluate $\boldsymbol{u(t)}$.

B. Efficient Method (Method-II)

The following approach is the modification of conventional method discussed earlier. In this technique, mathematical manipulations are performed to reduce number of computations. This reduces the time for manually solving state transfer problem in classroom. Steps for the modified method are:

1) Step 1: Computation of STM×B:
Instead of calculating STM, we directly calculate the product of state transition matrix and input matrix, i.e. STM×B which saves computations.
$$\begin{aligned} \Phi(t)B = e^{At}B &= \mathscr{L}^{-1}\{[sI-A]^{-1}\}B \\ &= \mathscr{L}^{-1}\{\Gamma^{-1}\}B \end{aligned} \tag{5}$$
Since B is a constant matrix, we can take it inside Laplace inverse. The modified equation (5), can be written by denoting co-factor of Γ as $\tilde{\Gamma}$, then we have,
$$\Phi(t)B = \mathscr{L}^{-1}\left\{\frac{1}{\Delta_\Gamma}\left[B^\intercal\tilde{\Gamma}\right]^\intercal\right\};$$
where, Δ_Γ is determinant of $[sI-A]$. Taking inverse Laplace we get,
$$\Phi(t)B = e^{At}B = P\Lambda(t) \tag{6}$$
Note that $\Phi(t)B$ is decomposed into two components namely; P which is a coefficient matrix and $\Lambda(t)$ which is a *mode vector* that contains $e^{\lambda t}$ terms.
It is important to note that the matrix P is a constant matrix which contains coefficients of exponential terms obtained from $e^{At}B$ and each element of $\Lambda(t)$ consists of only singleton exponential term. This simplifies the computation of controllability Gramian.
2) Step 2: Calculate Controllability Gramian:
From (6), we write controllability Gramian as,
$$\begin{aligned} &W_c(0,t_f) = PKP^\intercal \text{ for some } 0 < t_f < \infty \\ &\text{where,} \quad K = \int_0^{t_f}\Lambda(\tau)\Lambda^\intercal(\tau)d\tau \end{aligned} \tag{7}$$
is symmetric matrix consisting of integration of singleton exponential terms if roots are real, trigonometric if roots are only imaginary and combination of exponential and trigonometric terms if roots are complex conjugate with negative real part.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TIA.2022.3159502, IEEE Transactions on Industry Applications

After substituting the limits we get K as a symmetric matrix with constant terms. Clearly, due to integration of singleton exponential terms, computing K is much easier than finding $W_c(0,t_f)$ from (3) for real roots and also for any roots.

3) Step 3: Computation of required control input $\boldsymbol{u(t)}$:

$$\boldsymbol{u(t)} = \left(e^{A^\intercal(t_f-t)}B\right)^\intercal W_c^{-1}(0,t_f)\left[x(t_f) - e^{At_f}x(0)\right]$$

Substituting Eq. (7) in above equation and simplifying,

$$\boldsymbol{u(t)} = \begin{cases} \Lambda^\intercal(t_f-t)(PK)^{-1}\left[x(t_f)-e^{At_f}x(0)\right], & 0 \leqslant t \leqslant t_f \\ 0, & t > t_f \end{cases} \tag{8}$$

Expression (8) gives the desired control input for transferring the states from initial position to final position in finite time (t_f) for the given system. Moreover, once the STP is achieved, which is the main motivation of the strategy, control input is no longer required and made zero, as the states are at equilibrium.

Furthermore, the following theorem provides proof for stability of the proposed method.

Theorem 1. *For a system of the form (1), there exists a control input* $\boldsymbol{u(t)}$ *(given by (8)) drives the system state from* $x(t_0)$ *to* $x(t_f)$ *iff reachable subspace* $\left(x(t_f) - \phi(t,t_f)x(t_0)\right)$ *is in the range space of reachability Gramian* $\left(PK(t,t_f)P^\intercal\right)$ *i.e.*

$$\mathscr{R}[t_0,t_f] = Im\,(PKP^\intercal)$$

Moreover, if $x_f = \left(PK(t,t_f)P^\intercal\right)\upsilon_0 \in Im\left(PK(t,t_f)P^\intercal\right)$, *then to transfer the states from* $x(t_0)$ *to* $x(t_f)$ *the required control input is*

$$\boldsymbol{u(t)} = \Lambda^\intercal(t_f-t)(PK)^{-1}\upsilon_0, \quad t\in[t_0,t_f]$$

Proof. Consider $\theta(t) = \phi(t_0,t)x(t)$, then

$$\dot{\theta}(t) = \phi(t)B\boldsymbol{u(t)} = P\Lambda(t)\boldsymbol{u(t)}$$

and $x(t) = \phi(t,t_0)\theta(t)$. Here, $\theta(t_f)-\theta(t_0)$ lies in the range space of $PKP^\intercal$. Thus,

$$\theta(t_f)-\theta(t_0) = -\left[x(t_f)-\phi(t,t_f)x(t_0)\right]$$

ensures the transfer of states from x_0 to x_f.

Therefore, the desired state transfer is possible *iff* $\left[x(t_f)-\phi(t,t_f)x(t_0)\right]$ lies in the range space of $PKP^\intercal$. Thus, control

$$\boldsymbol{u(t)} = \Lambda^\intercal(t_f-t)(PK)^{-1}\upsilon_0, \quad \text{for} \quad t\in[t_0,t_f]$$

accomplishes the state transfer. Similar arguments are given in [27]. □

Remark 1. *Moreover, looking at control in optimal sense, value of performance index,* $J(u) = \int_0^{t_f}\boldsymbol{u(t)}^\intercal Q\boldsymbol{u(t)}dt$, *where* Q *is* $m\times m-$*dimensional constant symmetric and positive definite weighting matrix; corresponding to control input* $\boldsymbol{u(t)}$ *is found and can be proved minimum as discussed in proof of Theorem 2.9 of [28]. The control input (Eq.* (4) *or Eq.* (8)*) is generally referred as minimum input energy in the literature [2], [3], [23], [24], [29].*

1) Special Case: The input matrix B is unit vector e.g.

$$B = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}^\intercal$$

the initial state $x(0) = B$ *and final state* $x(t_f) = 0$, for n^{th} order system we have:

$$\Phi(t_f)B = e^{At_f}B = e^{At_f}x(0) = P\Lambda(t_f), \text{ and}$$

$$e^{At}B = \mathscr{L}^{-1}\left\{\frac{1}{\Delta_\Gamma}\left[1^{st} \text{ row of } \tilde{\Gamma}\right]^\intercal\right\}$$

This shows that calculating STM is simplified, as we just need to calculate Laplace inverse of 1^{st} row i.e. only n terms. Further, input expression (8) can be written as,

$$\boldsymbol{u(t)} = -\Lambda^\intercal(t_f-t)K^{-1}P^{-1}[P\Lambda(t_f)]$$

it is further modified as,

$$\begin{aligned}\boldsymbol{u(t)} &= -\Lambda^\intercal(t_f-t)K^{-1}\Lambda(t_f) \\ &= -\Lambda^\intercal(t_f-t)\left[\int_0^{t_f}\Lambda(\tau)\Lambda^\intercal(\tau)d\tau\right]^{-1}\Lambda(t_f)\end{aligned} \tag{9}$$

where, $\boldsymbol{u(t)}$ is finite in $[0,t_f]$ & for $t>t_f$, $\boldsymbol{u(t)} = 0$.

Note that Equation (9), involves only Λ i.e exponential terms which can be obtained directly from eigenvalues. Hence, for this special case, *there is no need to find STM and controllability Gramian matrix*. This method is illustrated through an example of 3^{rd} order system.

Example 1. *(Special Case: Distinct roots):* $x(0) = B$ *and* $x(t_f) = 0$

Consider a 3^{rd} *order system describe by the equation as:*

$$\dot{x}(t) = \begin{bmatrix}-1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1\end{bmatrix}x(t) + \begin{bmatrix}1\\0\\0\end{bmatrix}\boldsymbol{u(t)}$$

$$y(t) = \begin{bmatrix}0 & 1 & 0\end{bmatrix}x(t)$$

Find the control input required to drive the system states from $x(0) = \begin{bmatrix}1\\0\\0\end{bmatrix}$ *to* $x(t_f) = \begin{bmatrix}0\\0\\0\end{bmatrix}$ *in* 1 *second.*

Solution:

The solution of this problem is worked out by hand without using calculator.

Here, it is seen that the input matrix B is in companion form, $x(0) = B$ *and* $x(t_f) = 0$*. Hence, the control input required for state transfer is calculated by equation (9). In other words, only* $\Lambda(t)$ *term is required. Eigenvalues of the given system are*

$$\mid sI - A\mid = (s+1)(s^2+4s+1) = 0$$

$$\therefore s = -1, -2\pm\sqrt{3}$$

So,

$$\Lambda(t) = \begin{bmatrix}e^{-t} \\ e^{-(2-\sqrt{3})t} \\ e^{-(2+\sqrt{3})t}\end{bmatrix}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TIA.2022.3159502, IEEE Transactions on Industry Applications

Now, using equation (9), we directly find the control input required in 2 steps. We first calculate the integral term i.e. K with limits 0 to 1 ($t_f = 1sec$):

$$K = \int_0^{t_f} \Lambda(\tau)\Lambda^{\mathsf{T}}(\tau)d\tau$$

$$K = \int_0^1 \begin{bmatrix} e^{-\tau} \\ e^{-(2-\sqrt{3})\tau} \\ e^{-(2+\sqrt{3})\tau} \end{bmatrix} \begin{bmatrix} e^{-\tau} & e^{-(2-\sqrt{3})\tau} & e^{-(2+\sqrt{3})\tau} \end{bmatrix} d\tau$$

$$K = \int_0^1 \begin{bmatrix} e^{-2\tau} & e^{-(3-\sqrt{3})\tau} & e^{-(3+\sqrt{3})\tau} \\ e^{-(3-\sqrt{3})\tau} & e^{-(4-2\sqrt{3})\tau} & e^{-4\tau} \\ e^{-(3+\sqrt{3})\tau} & e^{-4\tau} & e^{-(4+2\sqrt{3})\tau} \end{bmatrix} d\tau$$

$$K = \left. \begin{bmatrix} \frac{e^{-2\tau}}{-2} & \frac{e^{-(3-\sqrt{3})\tau}}{-(3-\sqrt{3})} & \frac{e^{-(3+\sqrt{3})\tau}}{-(3+\sqrt{3})} \\ \frac{e^{-(3-\sqrt{3})\tau}}{-(3-\sqrt{3})} & \frac{e^{-(4-2\sqrt{3})\tau}}{-(4-2\sqrt{3})} & \frac{e^{-4\tau}}{-4} \\ \frac{e^{-(3+\sqrt{3})\tau}}{-(3+\sqrt{3})} & \frac{e^{-4\tau}}{-4} & \frac{e^{-(4+2\sqrt{3})\tau}}{-(4+2\sqrt{3})} \end{bmatrix} \right|_0^1$$

Note that every term of K is singleton exponential term hence, integration is easier. Putting the limits:

$$K = \begin{bmatrix} \frac{(1-e^{-2})}{2} & \frac{(1-e^{-(3-\sqrt{3})})}{(3-\sqrt{3})} & \frac{(1-e^{-(3+\sqrt{3})})}{(3+\sqrt{3})} \\ \frac{(1e^{-(3-\sqrt{3})})}{(3-\sqrt{3})} & \frac{(1-e^{-(4-2\sqrt{3})})}{(4-2\sqrt{3})} & \frac{(1-e^{-4})}{4} \\ \frac{(1-e^{-(3+\sqrt{3})})}{(3+\sqrt{3})} & \frac{(1-e^{-4})}{4} & \frac{(1-e^{-(4+2\sqrt{3})})}{(4+2\sqrt{3})} \end{bmatrix}$$

simplifying further (using calculator) we get:

$$K = \begin{bmatrix} 0.4323 & 0.5667 & 0.2095 \\ 0.5667 & 0.7741 & 0.2454 \\ 0.2095 & 0.2454 & 0.1339 \end{bmatrix}$$

$$\therefore K^{-1} = \begin{bmatrix} 644.0272 & -362.8583 & -342.6309 \\ -362.8583 & 207.5249 & 187.3950 \\ -342.6309 & 187.3950 & 200.1079 \end{bmatrix} \tag{10}$$

Now using equation (9 & 10),

$$\boldsymbol{u(t)} = -\Lambda^{\mathsf{T}}(t_f - t)K^{-1}\Lambda(t_f)$$

$$\therefore \quad u(t) = -\begin{bmatrix} e^{-(1-t)} & e^{-(2-\sqrt{3})(1-t)} & e^{-(2+\sqrt{3})(1-t)} \end{bmatrix} \times$$

$$\begin{bmatrix} 644.0272 & -362.8583 & -342.6309 \\ -362.8583 & 207.5249 & 187.3950 \\ -342.6309 & 187.3950 & 200.1079 \end{bmatrix} \begin{bmatrix} e^{-1} \\ e^{-(2-\sqrt{3})} \\ e^{-(2+\sqrt{3})} \end{bmatrix}$$

Simplifying it,

$$\boldsymbol{u(t)} = 17.9541 \times e^{t} - 22.0671 \times e^{(2-\sqrt{3})t} - 0.5174 \times e^{(2+\sqrt{3})t} \tag{11}$$

The required minimum control input and states transfer from initial to final points are simulated as shown in Fig. 1.

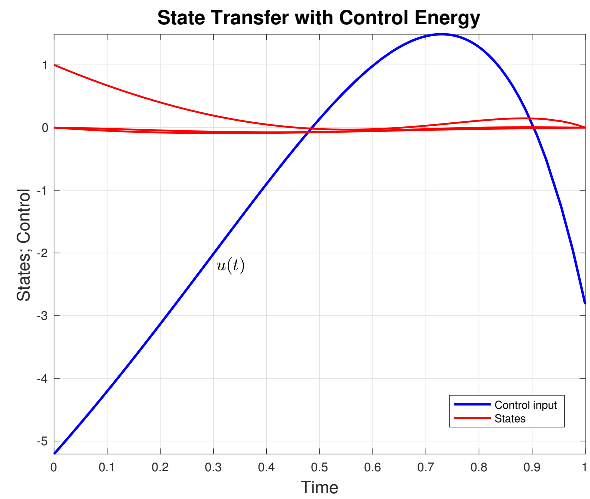

Fig. 1. Control input $\boldsymbol{u(t)}$ and states $x(t)$ for Example 1 (Distinct eigenvalues)

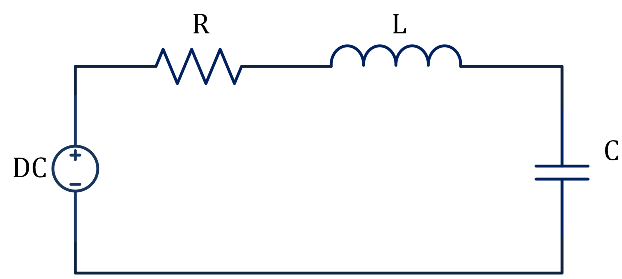

Fig. 2. Series RLC circuit

Many physical plants are modelled using RLC circuits. Therefore, a system (series RLC circuit in Fig. 2) with special case having complex roots is considered here. Example 2 in Appendix B illustrates STP for a special case having complex roots; and $\boldsymbol{u(t)}$ & states are shown in Fig. 3. In

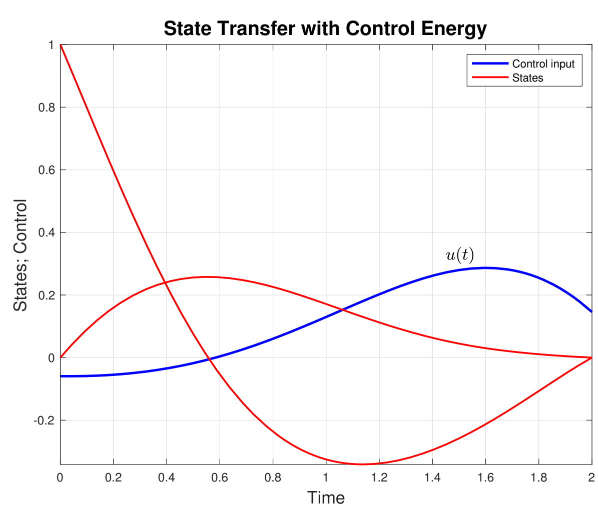

Fig. 3. Control input $\boldsymbol{u(t)}$ and states $x(t)$ for Example 2 (Complex eigenvalues)

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TIA.2022.3159502, IEEE Transactions on Industry Applications

above examples, only eigenvalues are required to obtain the control input and there is no requirement of computation of STM and controllability Gramian separately which makes the calculations very simple. For class-room teaching, generally 2^{nd} or 3^{rd} order systems can be considered and control input can be easily computed with this technique. It serves the purpose of imbibing the concept of state transfer in lesser time with the minimal use of calculator.

Furthermore, proposed technique is extended for a general case i.e. arbitrary initial and final states with repeated roots is considered, to illustrate effectiveness of the proposed method in Example 3 of Appendix B. The state transfer with minimum control input is shown in Fig. 4.

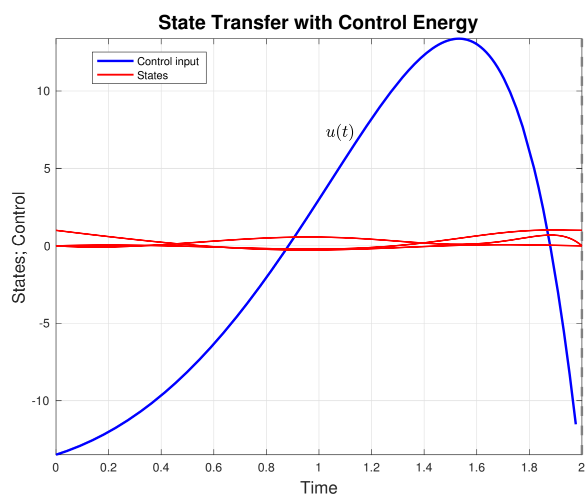

Fig. 4. Control input $\boldsymbol{u(t)}$ and states $x(t)$ for Example 3 (Repeated eigenvalues)

The proposed method which relies on mode vector is applied to an imaginary roots case (Example 4 in Appendix B) and corresponding control energy required for states to steer from initial to final state are shown in Fig. 5.

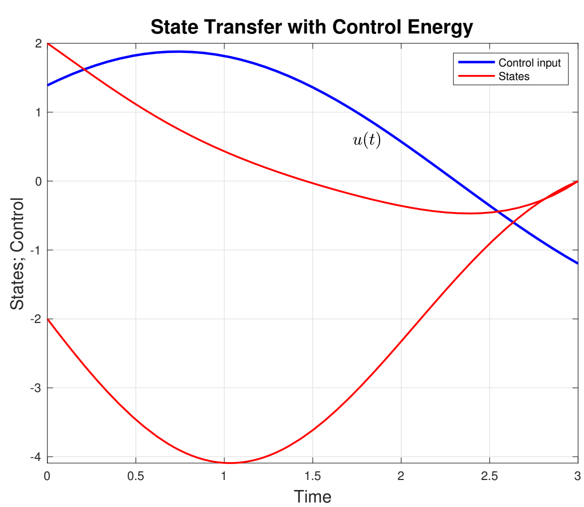

Fig. 5. Control input $\boldsymbol{u(t)}$ and states $x(t)$ for Example 4 (Imaginary roots)

Example 5 in Appendix B presents a case with real & complex eigenvalues and its state transfer is depicted in the Fig. 6.

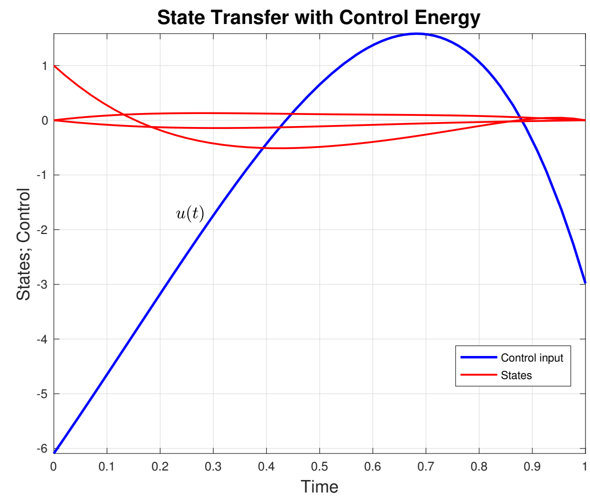

Fig. 6. Control input $\boldsymbol{u(t)}$ and states $x(t)$ for Example 5 (Real and complex roots)

Above all different eigenvalue scenarios illustrates the simplification in computations to calculate control input required for state transfer problem. Moreover, these examples represents different type of plant dynamics in real-time.

For the proposed approach, desired control energy computed is confined to finite time (t_f) duration and it is not asymptotic. If time t scale is changed, the control input would span over that time scale interval and no divergence will occur.

III. Discussions

As stated earlier, calculating control input for state transfer problem is very important task. It becomes even more challenging for 3^{rd} or higher order systems. If we consider a 3^{rd} order system with input matrix B in companion form, initial state $x(0) = B$ and $x(t_f) = 0$, then we have demonstrated that one can solve state transfer problem with reduced computations.

We have given here a table which compares number of operations involved in traditional and the modified method for calculating the control input $\boldsymbol{u(t)}$.

From Table I, it is seen that there is a drastic reduction in calculations of co-factors and inverse of Laplace terms while finding e^{At} term. In conventional method, n numbers of expressions for integration need to be taken and each expression contains algebraic sum of different exponential terms whereas, in modified method, only single exponential term will be present and taking integration of such term is simple and less time consuming.

As K is symmetric matrix, only $\frac{n(n+1)}{2}$ elements need to be computed. Other than these, there is a significant reduction in matrix operations i.e. addition and multiplication of elements while calculating either $(sI-A)^{-1}$ or $\boldsymbol{u(t)}$.

Even if the system is not in companion form then too, using equation (9), finding $\boldsymbol{u(t)}$ is much simpler as there will be

TABLE I
COMPARISON OF RESULTS FOR n^{th} ORDER, SINGLE INPUT WITH SPECIAL CASE

Operation	Method-I Traditional	Method-II Modified
Number of determinants required	02	02
Number of co-factors required	$\frac{3n^2+n}{2}$	$\frac{n^2+3n}{2}$
Number of Laplace inverse required	n^2	n
Number of integration terms[1]	$\frac{n(n+1)}{2}$	$\frac{n(n+1)}{2}$
Calculation of coefficient terms	$\frac{n(n+1)}{2}$	n
Number of matrix elements' addition operation required	$(5n^2-4n-1)$	(n^2-1)
Number of matrix elements' multiplication operation required	$\frac{11n^2+3n}{2}$	$n(n+2)$

[1] For method-I, these terms are not single terms, every term consisting of n exponential elements, forming algebraic expressions of exponential.

reduction in matrix operations compared to traditional method and hence the reduction in time. Apart from calculating control input, for computing controllability Gramian (W_c), this modified method is effective, as it involves only matrix multiplication given by equation (7).

IV. EXPERIMENTAL RESULTS

The industrial emulator hardware system supported by Hardware-In-Loop (HIL) is shown in Fig. 7. It is an electromechanical system (Quanser SRV-02 Plant [30]) that represents the important classes of systems home electronic devices (DVDs), auto-mobiles (vehicle parts assembly), commercial aircraft (push-pull mechanism), robotics (point-to-point movement of robot or positioning of robotic arm) and many more where position control [31], [32] surfaces and rotate objects at precise angles, distances and automated assembly machines. Proposed work is implemented on the aforementioned HIL setup and concerned control scheme development flow diagram is depicted in Fig. 8. The SRV-02 unit is equipped with a Faulhaber coreless DC motor model 2338S006, two optical encoders, one tachometer and one potentiometer. Quanser Q4 data acquisition (DAQ) terminal board is used to interface between SRV-02 plant and target computer. Specifications of the experimental set-up used in this paper are depicted in Appendix C.

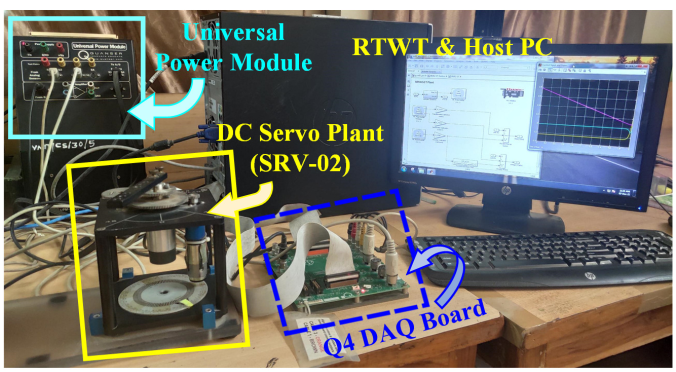

Fig. 7. Industrial emulator hardware set-up

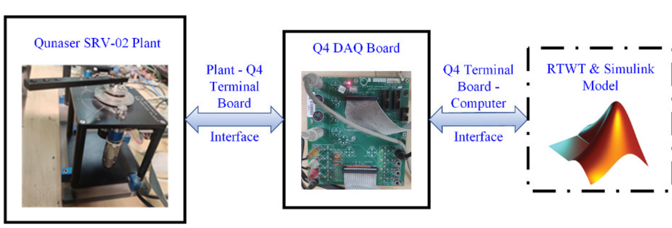

Fig. 8. Control scheme development flow diagram

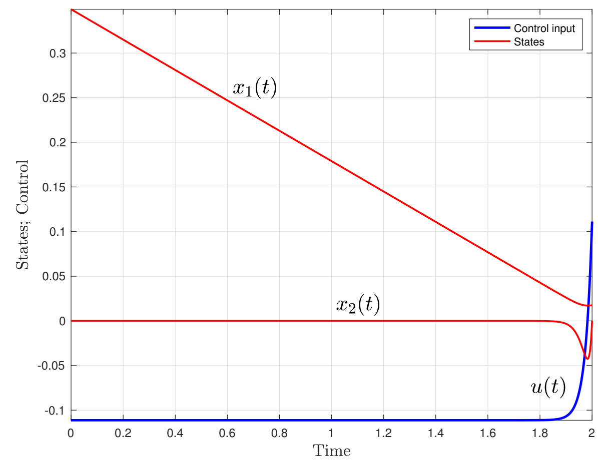

Fig. 9. Simulation results SRV-02 plant with $t_f = 2$ *sec*

To implement the control law on the computer, MATLAB 2012 is used as the application host environment. It includes the MATLAB/Simulink, Real-Time Windows Target (RTWT) & control and Simulink development environment. The plant dynamics are written in state space equation as:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -\frac{1}{\tau}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{K_1}{\tau}\end{bmatrix}\boldsymbol{u(t)}$$

$$y(t) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

where, $x_1(t)$, $x_2(t)$ are angular position and angular velocity of load disk of SRV-02 respectively; $\boldsymbol{u(t)}$ is input energy to drive the motor and model time constant $\tau = 0.0254$ & system gain $K_1 = 1.5286$.

Control law objective is to transfer the load disk from initial position to the desired position in finite time. For experimentation purpose the selected position is 20 degree as the initial position and one degree as desired position and control energy calculated for finite time of 2 *seconds*.

The control input required to drive the system states from $x(0) = \begin{bmatrix}0.3491\ rad\\ 0\end{bmatrix}$ to $x(t_f) = \begin{bmatrix}0.0175\ rad\\ 0\end{bmatrix}$ in 2 *sec* is calculated as:

$$\boldsymbol{u(t)} = \begin{cases}1.416\times 10^{-35}e^{39.37*t} - 0.1113, & 0 \leqslant t \leqslant 2,\\ 0, & t > 2\end{cases} \tag{12}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TIA.2022.3159502, IEEE Transactions on Industry Applications

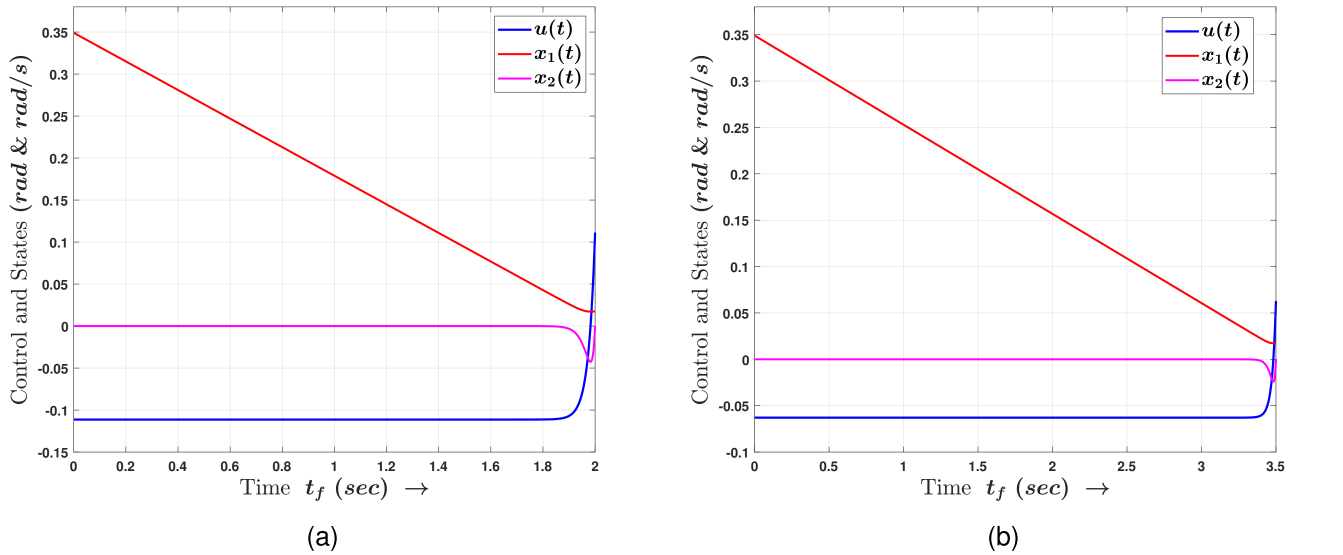

Fig. 10. Experimental results: Angular position, velocity and control law for SRV-02 plant with fixed initial and final conditions. (a) Case-I: with $t_f = 2$ *sec* (b) Case-II: with $t_f = 3.5$ *sec*

The results obtained from the HIL are shown in Fig. 10a, which shows that both states ($x_1(t)$ in red & $x_2(t)$ in magenta colour) are transferred to desired position in finite time with given calculated control input ($\boldsymbol{u(t)}$ in blue colour) voltage. The simulation and hardware results comparison of state transfer and control are illustrated in the Fig. 9 and Fig. 10a. Thus, simulation results are verified with experimental tests.

Moreover, to demonstrate the non divergence nature of control input, we determined $\boldsymbol{u(t)}$ for aforementioned initial and final positions of states with $t_f = 3.5$ *sec* and it is found out that $\boldsymbol{u(t)} = 1.803 \times 10^{-61} e^{39.37*t} - 0.0629$ is applied for $t \in [0, 3.5]$, and states are converged to final positions and once states are at equilibrium, thereafter $\boldsymbol{u(t)}$ is switched to zero and is not divergent. The corresponding control input and states are plotted in Fig. 10b. Similarly, to showcase adaptability of proposed methodology, it is tested against various end points and finite time. The corresponding control input is calculated for respective case and is depicted in the Table II.

TABLE II
DIFFERENT TEST CASES OF STATES & FINITE TIME AND CORRESPONDING CONTROL INPUT

Case	$\boldsymbol{x(0)}$	$\boldsymbol{x(t_f)}$	$\boldsymbol{t_f}$	$\boldsymbol{u(t)}$
I	$\begin{bmatrix}20^\circ & 0\end{bmatrix}^\intercal$	$\begin{bmatrix}1^\circ & 0\end{bmatrix}^\intercal$	2	$1.416 \times 10^{-35} e^{39.37*t} - 0.1113$
II	$\begin{bmatrix}20^\circ & 0\end{bmatrix}^\intercal$	$\begin{bmatrix}1^\circ & 0\end{bmatrix}^\intercal$	3.5	$1.803 \times 10^{-61} e^{39.37*t} - 0.0629$
III	$\begin{bmatrix}0^\circ & 0.15\end{bmatrix}^\intercal$	$\begin{bmatrix}20^\circ & 0.1\end{bmatrix}^\intercal$	3	$0.07602 - 1.076 \times 10^{-70} e^{39.37*t}$
IV	$\begin{bmatrix}40^\circ & 0.1\end{bmatrix}^\intercal$	$\begin{bmatrix}25^\circ & 0.05\end{bmatrix}^\intercal$	4	$6.209 \times 10^{-70} e^{39.37*t} - 0.044$
V	$\begin{bmatrix}25^\circ & 0.1\end{bmatrix}^\intercal$	$\begin{bmatrix}0^\circ & 0\end{bmatrix}^\intercal$	1	$4.825 \times 10^{-18} e^{39.37*t} - 0.3025$
VI	$\begin{bmatrix}0^\circ & 0\end{bmatrix}^\intercal$	$\begin{bmatrix}15^\circ & 0.1\end{bmatrix}^\intercal$	0.5	$0.3776 - 1.763 \times 10^{-9} e^{39.37*t}$

The test conditions I & II are shown in Fig.10 and remaining HIL results are presented in Fig. 11. States, angular position is in degrees & converted to *radians* (in Fig. 11 shown by red colour) and angular velocity in rad/s (shown by magenta colour). Finite time t_f is in *seconds* and respective control input is shown by blue colour in results.

In cases III & IV, load disk is steered from some arbitrary initial positions to desired states in finite time for computed input energy. Furthermore, cases V & VI have one of the extreme positions to be $\begin{bmatrix}0^\circ & 0\end{bmatrix}$. Moreover, the arm movement in cases III & VI is clockwise, so the control input decreases as t tends to t_f, whereas in other two cases it increases as t reaches finite time.

TABLE III
QUANTITATIVE COMPARISON OF CONTROL LAW COMPUTATIONAL OPERATIONS OF SRV-02 PLANT

Computational operation	Conventional [2], [3], [24]	Proposed
Number of coefficient terms	3	2
Matrix elements additions	11	3
Matrix elements multiplications	25	8
Number of cofactors	7	5
Number of Laplace inverses	4	2

The performance comparison of proposed method with conventional one for SRV-02 plant is given in terms of number of computations required to calculate the control input is given in Table III.

Furthermore, these significant reduction in computations is shown in terms of time required for the processor (CPU time) and executing the commands (Elapsed time) in Table IV. From

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TIA.2022.3159502, IEEE Transactions on Industry Applications

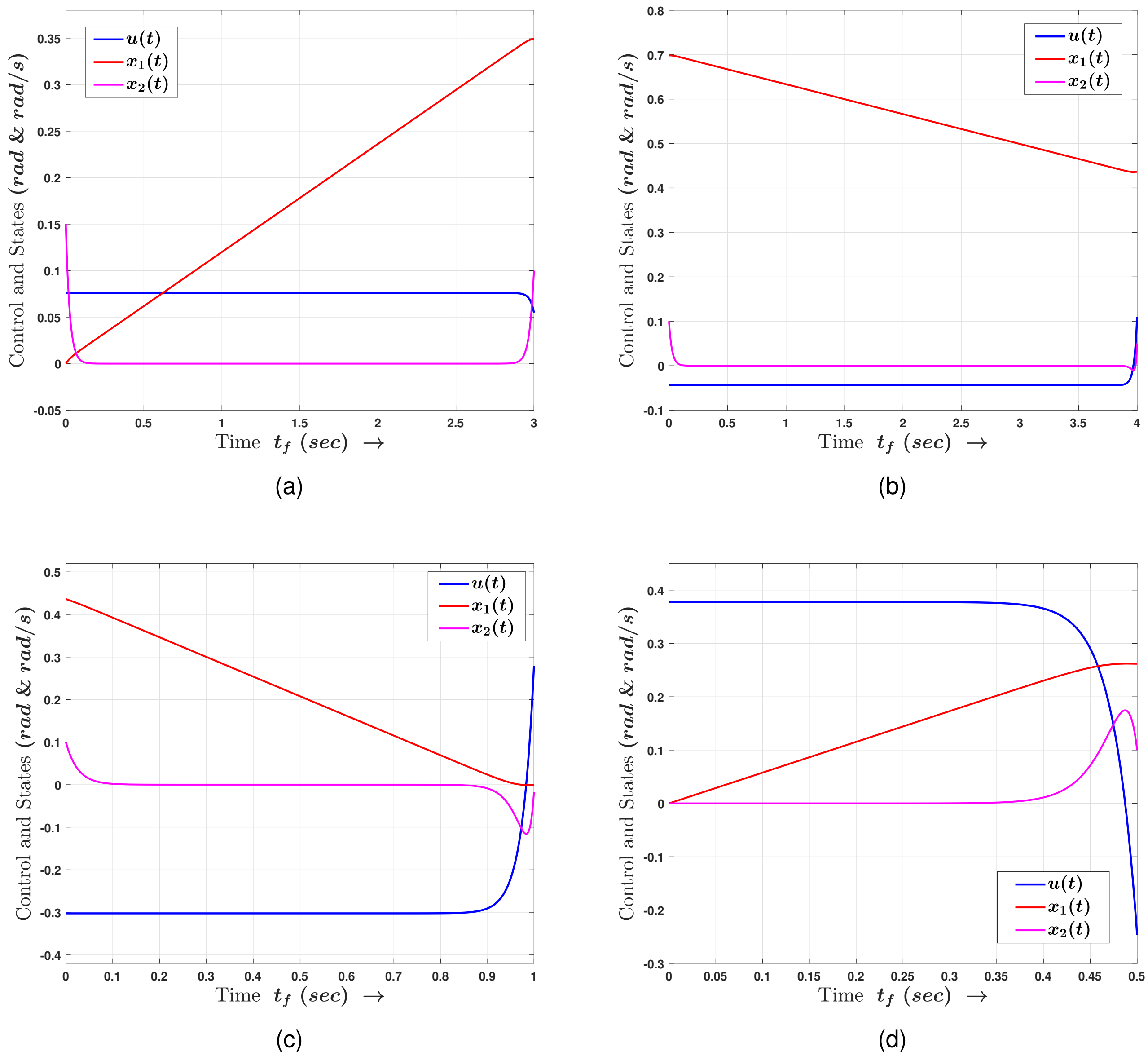

Fig. 11. Experimental test results: Angular position (red), velocity (magenta) and control input (blue) for SRV-02 plant with different initial & final conditions and finite times. (a) Case-III: with $t_f = 3$ *sec* (b) Case-IV: with $t_f = 4$ *sec* (c) Case-V: with $t_f = 1$ *sec* (d) Case-VI: with $t_f = 0.5$ *sec*

TABLE IV
COMPARISON OF CONTROL LAW COMPUTATION TIME OF SRV-02 PLANT

SRV-02	CPU time (sec)		Elapsed time (sec)	
Test Cases	Conventional*	Proposed	Conventional*	Proposed
I	2.6563	2.4063	2.4852	2.3368
II	2.5938	2.4531	2.4594	2.3964
III	2.6256	2.3750	2.4249	2.3444
IV	2.5156	2.4219	2.4331	2.3688
V	2.8594	2.5625	2.5071	2.4693
VI	2.5313	2.3729	2.4411	2.3615

* Ref.s [2], [3], [24]

the relative analysis given in Table IV, it is inferred that, compared to conventional control strategy [2], [3], [24], the proposed strategy has achieved reduced computational time with low number of processor cycles, resulting in enhancement of computational efficiency.

Also, from results of case-I & II it is evident that when t_f is small, the required control effort is more and vice versa for STP. The profile of minimum energy (E_{min}) with finite time (t_f) of SRV-02 plant is shown in Fig. 12, which infers that for case-I initial and final conditions, as t_f increases, E_{min} reduces. For $t_f = 0.5$ *sec* maximum effort is needed, whereas for $t_f = 5$ *sec* it takes lesser energy to steer the states.

From experimental results, one can see the proposed method achieves desired load disk position in finite time, it confirms the STP is successful. Also, test cases shows that the servo

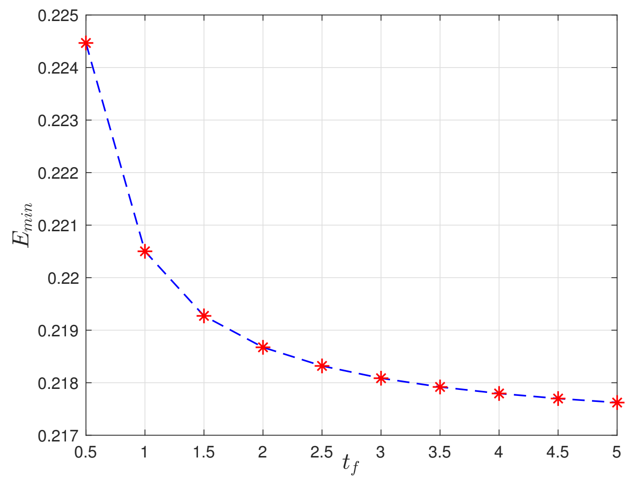

Fig. 12. Minimum control energy (E_{min}) vs finite time (t_f)

motor's states transfer are successful under different finite time conditions.

V. Conclusions

Computation of controllability Gramian and control law for state transfer problem with single input case is proposed in this article. The proposed method is useful in solving state transfer problem for an industrial application. The modified approach reduces calculations in computation of $\boldsymbol{u(t)}$ and W_c. Furthermore, it is shown that for a special case, the control energy required is only dependent on eigenvalues, and computation of controllability Gramian is not necessary. In this case, vector $\Lambda(t)$ is the only term which need to be computed. Effectiveness of proposed strategy is illustrated through distinct eigenvalue examples. Theoretical proof of proposed method and experimental validation for an industrial emulator of position control are also presented. It is inferred that MATLAB Simulink and HIL results regarding the states and control are consistent. Moreover, experimental tests with different extreme points and end time validates the theoretical results. Proposed method is found to give better computational time performance when compared to conventional methods.

The comparison with standard technique given in literature shows that the proposed method is efficient for single input system. Further, the proposed work can be extended to compute control laws for multi input systems and can be implemented on Hardware-In-Loop setup, forms the next step in this work.

Acknowledgments

The authors would like to thank the reviewers for their constructive comments. Also, thank the AICTE-MHRD, the Government of India's quality improvement program (QIP).

Appendix A
Computation Complexity

Computations involved in Method-I are finding inverse of $(sI - A)$ then calculating Laplace inverse which needs finding partial fractions. To find W_c, need to find integration of exponential and trigonometric terms and again find its determinant and inverse. Control law calculations again need matrix multiplications, coefficients of finite time terms.

Appendix B
Illustrative Examples of Proposed Method

Example 2. *(Special Case: Complex roots): $x(0) = B$ & $x(t_f) = 0$*
Consider a RLC circuit shown in Fig. 2 which represents many second order systems:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \boldsymbol{u(t)}$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Find the control input required to drive the system states from $x(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^\intercal$ to $x(t_f) = \begin{bmatrix} 0 & 0 \end{bmatrix}^\intercal$ in 2 seconds with $R = 2\Omega, L = 1H, C = 0.2F$.
Solution: *First calculating $\Lambda(t)$. Eigenvalues of the given system are*

$$|\, sI - A \,| = (s^2 + 2s + 5) = 0 \implies \therefore\ s = -1 \pm 2i$$
$$So,\ \Lambda(t) = \begin{bmatrix} e^{-t} sin(2t) \\ e^{-t} cos(2t) \end{bmatrix}$$

Now, using equation (7), the integral term i.e. K is computed with limits 0 to 2 ($t_f = 2sec$):

$$K = \int_0^2 \begin{bmatrix} e^{-\tau} sin(2\tau) \\ e^{-\tau} cos(2\tau) \end{bmatrix} \begin{bmatrix} e^{-\tau} sin(2\tau) & e^{-\tau} cos(2\tau) \end{bmatrix} d\tau$$

$$K = \begin{bmatrix} 0.19241 & 0.09883 \\ 0.09883 & 0.29736 \end{bmatrix} \therefore\ K^{-1} = \begin{bmatrix} 6.26712 & -2.08293 \\ -2.08293 & 4.05521 \end{bmatrix}$$

Putting above equation in (9), we get:

$$u(t) = -\begin{bmatrix} e^{-(2-t)} sin(2(2-t)) & e^{-(2-t)} cos(2(2-t)) \end{bmatrix} \times \begin{bmatrix} 6.26712 & -2.08293 \\ -2.08293 & 4.05521 \end{bmatrix} \begin{bmatrix} e^{-2} sin(4) \\ e^{-2} cos(4) \end{bmatrix}$$

After simplifying control input is,

$$\boldsymbol{u(t)} = 0.02559 \times e^t sin(2t) - 0.059734 \times e^t cos(2t)$$

Example 3. *(Normal Case: Repeated roots): $x(0) \neq B$ & $x(t_f) \neq 0$*
Consider a 3^{rd} order system describe by the equation as:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \boldsymbol{u(t)}$$
$$y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x(t)$$

Find the control input required to drive the system states from $x(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\intercal$ to $x(t_f) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\intercal$ in 2 seconds.
Solution:

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TIA.2022.3159502, IEEE Transactions on Industry Applications

First we calculate $e^{At}B$ as,

$$\Phi(t)B = e^{At}B = \mathscr{L}^{-1}\{[sI-A]^{-1}\}B$$

$$= \mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2(s+2)}\begin{bmatrix}1\\ s\\ s^2\end{bmatrix}\right\}$$

Simplifying it, we get P and mode vector

$$e^{At}B = \begin{bmatrix}-e^{-t}+te^{-t}+e^{-2t}\\ 2e^{-t}-te^{-t}-2e^{-2t}\\ -3e^{-t}+te^{-t}+4e^{-2t}\end{bmatrix} = \underbrace{\begin{bmatrix}-1 & 1 & 1\\ 2 & -1 & -2\\ -3 & 1 & 4\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}e^{-t}\\ te^{-t}\\ e^{-2t}\end{bmatrix}}_{\Lambda(t)}$$

Now, calculating the integral term i.e. K with limits 0 to 2 and solving the integration:

$$K = \begin{bmatrix}0.4908 & 0.2271 & 0.3325\\ 0.2271 & 0.1905 & 0.1092\\ 0.3325 & 0.1092 & 0.2499\end{bmatrix}$$

Now, $x(t_f) - e^{At_f}x(0)$ is,

$$x(t_f) - e^{At_f}x(0) = \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}e^{-2t_f}+2t_fe^{-t_f}\\ 2e^{-t_f}-2e^{-2t_f}-2t_fe^{-t_f}\\ 4e^{-2t_f}-4e^{-t_f}+2t_fe^{-t_f}\end{bmatrix}$$

$$\text{for } t_f = 2, \quad \text{it is } \begin{bmatrix}-0.5597 & 1.3073 & -0.0733\end{bmatrix}^\mathsf{T}$$

Control input required is computed using equation (8) as,

$$\boldsymbol{u(t)} = \begin{bmatrix}e^{-(2-t)} & te^{-(2-t)} & e^{-2(2-t)}\end{bmatrix} \times \begin{bmatrix}-147.2286 & 173.3706 & -52.0396\\ 95.3415 & -98.4940 & 26.7698\\ 158.2326 & -179.6326 & 61.5436\end{bmatrix}\begin{bmatrix}-0.5597\\ 1.3073\\ -0.0733\end{bmatrix}$$

$$= -7.483e^t + 24.9068te^t - 6.0007e^{2t}$$

Example 4. *(Imaginary roots): $x(0) \neq B$ & $x(t_f) = 0$*
Consider a 2^{nd} order system describe by the equation as:

$$\dot{x}(t) = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}x(t) + \begin{bmatrix}0\\1\end{bmatrix}\boldsymbol{u(t)}$$
$$y(t) = \begin{bmatrix}0 & 1\end{bmatrix}x(t)$$

Find the control input required to drive the system states from $x(0) = \begin{bmatrix}2\\-2\end{bmatrix}$ to $x(t_f) = \begin{bmatrix}0\\0\end{bmatrix}$ in 3 seconds.
Solution:

First we calculate $e^{At}B$ as,

$$\Phi(t)B = e^{At}B = \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\begin{bmatrix}-1\\ s\end{bmatrix}\right\}$$

Simplifying & decomposing it into P and $\Lambda(t)$ we get:

$$e^{At}B = \begin{bmatrix}sin(t)\\ cos(t)\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}sin(t)\\ cos(t)\end{bmatrix}}_{\Lambda(t)}$$

$$\therefore \quad K = \begin{bmatrix}1.56985 & 0.00995\\ 0.00995 & 1.43014\end{bmatrix}$$

Now, $x(t_f) - e^{At_f}x(0)$ is,

$$x(t_f) - e^{At_f}x(0) = \begin{bmatrix}2.26222\\ -1.69774\end{bmatrix} \quad (\textit{for } t_f = 3)$$

$$\textit{then,} \quad \boldsymbol{u(t)} = 1.26518 \times sin(t) + 1.38964 \times cos(t)$$

Example 5. *(Mixed Case: Real and complex roots): $x(0) = B$ & $x(t_f) \neq 0$*
Consider a 3^{rd} order system describe by the equation as:

$$\dot{x}(t) = \begin{bmatrix}1 & 1 & 0\\ -7 & 0 & 1\\ 25 & -1 & -4\end{bmatrix}x(t) + \begin{bmatrix}0\\0\\1\end{bmatrix}\boldsymbol{u(t)}$$

$$y(t) = \begin{bmatrix}1 & 0 & 0\end{bmatrix}x(t)$$

Find the control input required to drive the system states from $x(0) = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^\mathsf{T}$ to $x(t_f) = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^\mathsf{T}$ in 1 seconds.
Solution:

Here, $e^{At}B$ after simplification is,

$$e^{At}B = \begin{bmatrix}e^{-t}-e^{-t}cos(t)\\ -e^{-t}+e^{-t}cos(t)+e^{-t}sin(t)\\ e^{-t}-2e^{-t}sin(t)\end{bmatrix}$$

Decomposing it into P and $\Lambda(t)$ we get:

$$e^{At}B = \underbrace{\begin{bmatrix}1 & -1 & 0\\ -1 & 1 & 1\\ 1 & 0 & -2\end{bmatrix}}_{P}\underbrace{\begin{bmatrix}e^{-t}\\ e^{-t}cos(t)\\ e^{-t}sin(t)\end{bmatrix}}_{\Lambda(t)}$$

Now, calculating the integral term i.e. K with limits 0 to 1:

$$K = \int_0^1 \begin{bmatrix}e^{-2\tau} & e^{-2\tau}cos(\tau) & e^{-2\tau}sin(\tau)\\ e^{-2\tau}cos(\tau) & e^{-2\tau}cos^2(\tau) & e^{-2\tau}sin(\tau)cos(\tau)\\ e^{-2\tau}sin(\tau) & e^{-2\tau}sin(\tau)cos(\tau) & e^{-2\tau}sin^2(\tau)\end{bmatrix}d\tau$$

Solving the integration:

$$K = \begin{bmatrix}0.4323 & 0.3935 & 0.1398\\ 0.3935 & 0.3636 & 0.1167\\ 0.1398 & 0.1167 & 0.06874\end{bmatrix}$$

Using control equation,

$$\boldsymbol{u(t)} = -32.33e^t + 26.22e^tcos(t) + 20.28e^tsin(t)$$

Appendix C
Quanser SRV-02 Parameters

TABLE V
QUANSER SRV-02 PLANT SPECIFICATIONS

Motor nominal input voltage & max. current	6 V & 1A
Motor maximum speed	6,000 RPM
Potentiometer measurement range	±5 V
Encoder resolution (in quadrature)	4096 counts/rev

REFERENCES

[1] R. A. Metri and B. A. Rajpathak, "Computing the control input for state transfer problem in efficient way for classroom teaching," in *2020 IEEE First International Conference on Smart Technologies for Power, Energy and Control (STPEC)*. IEEE, 2020, pp. 1–5.

[2] J. P. Hespanha, *Linear systems theory*. Princeton university press, 2018.

[3] R. Gupta, W. Zhao, and R. K. Kapania, "Controllability gramian as control design objective in aircraft structural design optimization," *AIAA Journal*, vol. 58, no. 7, pp. 3199–3220, 2020.

[4] S. Cong, Y. Tang, and K. Li, "State transfer of two-level quantum system feedback control based on online state estimation," in *2021 40th Chinese Control Conference (CCC)*. IEEE, 2021, pp. 6301–6306.

[5] B. R. Gudur, G. Poddar, and B. P. Muni, "Parameter estimation and online adaptation of rotor time constant for induction motor drive," *IEEE Transactions on Industry Applications*, pp. 1–1, 2022.

[6] R. E. Kalman and J. Bertram, "General synthesis procedure for computer control of single-loop and multiloop linear systems (an optimal sampling system)," *Transactions of the American Institute of Electrical Engineers, Part II: Applications and Industry*, vol. 77, no. 6, pp. 602–609, 1959.

[7] R. E. Kalman, "On the general theory of control systems," in *Proceedings First International Conference on Automatic Control, Moscow, USSR*, 1960, pp. 481–492.

[8] R. Kalman, *Lectures on Controllability and Observability*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011, pp. 1–149.

[9] E. Kreindler and P. Sarachik, "On the concepts of controllability and observability of linear systems," *IEEE Transactions on Automatic Control*, vol. 9, no. 2, pp. 129–136, 1964.

[10] M. Vidyasagar, "A characterization of e at and a constructive proof of the controllability criterion," *IEEE Transactions on Automatic Control*, vol. 16, no. 4, pp. 370–371, 1971.

[11] M. Tarokh, "Measures for controllability, observability and fixed modes," *IEEE Transactions on Automatic Control*, vol. 37, no. 8, pp. 1268–1273, 1992.

[12] C. Moler and C. Van Loan, "Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later," *SIAM review*, vol. 45, no. 1, pp. 3–49, 2003.

[13] S. Agashe and B. Lande, "A new approach to the state-transfer problem," *Journal of the Franklin Institute*, vol. 333, no. 1, pp. 15–21, 1996.

[14] V. Sreeram, "Recursive technique for computation of gramians," *IEE Proceedings D - Control Theory and Applications*, vol. 140, no. 3, pp. 160–166, May 1993.

[15] C. T. Chen, *Linear system theory and design*. Holt, Rinehart and Winston New York, 1984, vol. 301.

[16] I. Yadykin, V. Afanasyev, and A. Grobovoy, "Gramians method in researches of the model order reduction and dynamic lti system stability problems," *IFAC Proceedings Volumes*, vol. 46, no. 2, pp. 178 – 183, 2013, 5th IFAC Symposium on System Structure and Control.

[17] I. B. Yadykin, A. A. Grobovoy, A. B. Iskakov, D. E. Kataev, and M. S. Khmelik, "Stability analysis of electric power systems using finite gramians," *IFAC-PapersOnLine*, vol. 48, no. 30, pp. 548–553, 2015.

[18] A. Joshi and S. Kamalasadan, "Optimal robust control based on partial derivative plant approximation for frequency support using multiple dc micro-grids," *IEEE Transactions on Industry Applications*, 2021.

[19] T. H. Summers, F. L. Cortesi, and J. Lygeros, "On submodularity and controllability in complex dynamical networks," *IEEE Transactions on Control of Network Systems*, vol. 3, no. 1, pp. 91–101, 2016.

[20] C. H. Choi, "Evaluation of backward differentiation methods for computing controllability gramians," in *2020 American Control Conference (ACC)*, 2020, pp. 4606–4611.

[21] N. Kulkarni, S. Kamalasadan, and S. Ghosh, "An integrated method for optimal placement and tuning of a power system stabilizer based on full controllability index and generator participation," *IEEE Transactions on Industry Applications*, vol. 51, no. 5, pp. 4201–4211, 2015.

[22] D. Kumar and V. Sreeram, "Model reduction via generalized frequency interval cross gramian," *IFAC-PapersOnLine*, vol. 51, no. 1, pp. 25 – 29, 2018, 5th IFAC Conference on Advances in Control and Optimization of Dynamical Systems ACODS 2018.

[23] Y. Zhao and J. Cortés, "Gramian-based reachability metrics for bilinear networks," *IEEE Transactions on Control of Network Systems*, vol. 4, no. 3, pp. 620–631, Sep. 2017.

[24] A. S. A. Dilip, "Probabilistic state transfer, estimation and measures for optimal actuator/sensor placement for linear systems with packet dropouts," *IEEE Control Systems Letters*, vol. 3, no. 4, pp. 865–870, Oct 2019.

[25] M. Gopal, *Modern Control System Theory*. New York, NY, USA: Halsted Press, 1984.

[26] R. Kalman, Y. Ho, and K. Narendra, "Controllability of linear dynamical systems, volume 1 of contributions to differential equations," 1963.

[27] R. W. Brockett, *Finite dimensional linear systems*. SIAM, 2015.

[28] J. Klamka, *Controllability and minimum energy control*. Springer, 2019.

[29] G. Lindmark and C. Altafini, "Minimum energy control for complex networks," *Scientific reports*, vol. 8, no. 1, pp. 1–14, 2018.

[30] J. Apkarian, M. Levis, and H. Gurocak, "User manual - srv02 rotary servo base unit: Set up and configuration," 2011.

[31] L. Zhou, W. Gruber, and D. L. Trumper, "Position control for hysteresis motors: Transient-time model and field-oriented control," *IEEE Transactions on Industry Applications*, vol. 54, no. 4, pp. 3197–3207, 2018.

[32] Q. Ni, M. Yang, S. A. Odhano, M. Tang, P. Zanchetta, X. Liu, and D. Xu, "A new position and speed estimation scheme for position control of pmsm drives using low-resolution position sensors," *IEEE Transactions on Industry Applications*, vol. 55, no. 4, pp. 3747–3758, 2019.

Rajanikant A. Metri (Student Member, IEEE) received the B.E. degree in Electrical Engineering from Government College of Engineering, Karad, India, in 2007 and the M.Tech. degree in Electrical - Control Systems from College of Engineering, Pune India, in 2010. He is currently working towards the Ph.D. degree in Electrical Engineering with Visvesvaraya National Institute of Technology, Nagpur, India.

He also work as an Assistant Professor with Department of Electrical Engineering, Rajarambapu Institute of Technology, Rajaramnagar, Islampur. His research interests include linear system theory, nonlinear systems, piecewise smooth maps and its applications to real time systems.

Bhooshan Rajpathak (Senior Member, IEEE) received the B.E. degree in Instrumentation Engineering from Mumbai University, India, in 2002 and the M.Tech. degree in Electrical - Control & Computing, and the Ph.D. degree in Electrical Engineering from Indian Institute of Technology Bombay, India, in 2006 and 2015 respectively.

He is currently an Assistant Professor with the Department of Electrical Engineering, Visvesvaraya National Institute of Technology, Nagpur, India. His background includes control, instrumentation & embedded system. His research interests include nonlinear control theory, piecewise smooth systems and micro-grid stability.