Dynamic Structural Models

3. Dynamic Games

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Introduction

- Two changes when going from single-agent to multi-agent settings:
  (i) State transitions depend on **all players'** actions.
  (ii) Flow payoffs depend on **all players'** actions.
- Example: entry/exit game of market interaction with two state variables
  - Market demand state (evolves exogenously), and
  - Number of firms (evolves endogenously).
- We estimate choice probabilities conditional on the state and integrate over rivals' choices to get "single-agent" versions of utility and state transitions.

Introduction

- What we observe in the data:
  - Conditional choice probabilities of each firm conditional on the *state*.
  - State transitions conditional on the *action profile* of all firms' choices.
- Use the CCPs to transform the observed state transitions conditional on the action profile into an ex-ante version that depends only on firm i's action:

$$f^(i)_j(x'|x) = \sum_{\mathbf{a}_{-i} \in A_{-i}} P(a_{-i}|x) F(x'|x, a_{-i}, a_i = j)$$

- That is, we integrate over the distribution of choices by i's rivals to estimate state transitions from the perspective of firm i.
- Once that is done, we can apply the same single-agent estimator.

Exploiting terminal decisions

- Games with an "exit" decision allow us to exploit finite dependence.
- Representation means that we can express the value function with respect to a chosen decision j:
$$V(x) = v_j(x) + \Psi_j(p(x))$$
- If the chosen decision is exit, the game ends: no more dynamics to worry about!
- We can use this representation to "force" the game to end in $t+1$ for a firm that chooses to continue in period t.
- Parameterization: linear utility, set exit payoff to zero in all states.

$$u_0(x) = x\theta, \qquad , u_1(x) = 0 \quad \forall x$$

Introduction

- Notation: $j = 1$ is exit, $j = 0$ is remain in market.

$$
\begin{aligned}
v_1(x) &= 0 \qquad\qquad\qquad\qquad \text{(Exit} \Rightarrow \text{no future payoffs)} \\
v_0(x) &= x\theta + \beta \sum_{x' \in \mathcal{X}} V(x') f_0(x'|x) \\
&= x\theta + \beta \sum_{x' \in \mathcal{X}} [v_1(x') + \gamma - \ln(p_1(x'))] f_0(x'|x) \\
&= x\theta + \beta \sum_{x' \in \mathcal{X}} [0 + \gamma - \ln(p_1(x'))]\, f_0(x'|x)
\end{aligned}
$$

- Therefore, the difference in the conditional value functions is

$$
v_1(x) - v_0(x) = x\theta \; \underbrace{-\beta \sum_{x' \in \mathcal{X}} [\gamma - \ln(p_1(x'))]\, f_0(x'|x)}_{\equiv \Upsilon(x)}
$$

Estimation steps

1. Estimate CCPs $\hat{p}(a_i = j|x)$ and state transitions $\hat{F}(x'|x, a_i)$.
2. Calculate "big P"

$$\hat{P}(a_{-i}|x) = \prod_{i' \neq i} \hat{p}(a_{i'} = a_{-i}(i')|x)$$

3. Use $\hat{P}$ to estimate distribution over next period's state:

$$\hat{f}_j^{(i)}(x'|x) = \sum_{a_{-i} \in A_{-i}} \hat{P}(a_{-i}|x)\hat{F}(x'|x, a_{-i}, a_i = j)$$

4. Calculate expected flow payoffs:

$$u_j^{(i)}(x) = \sum_{a_{-i} \in A_{-i}} \hat{P}(a_{-i}|x)U_j^{(i)}(x|a_{-i})$$

Estimation steps

5. Calculate $\Upsilon(x)$:

$$\hat{\Upsilon} = \beta \sum_{x' \in \mathcal{X}} \left[\gamma - \ln(\hat{p}_1(x'))\right] \hat{f}_0(x'|x)$$

6. Use logit estimator to maximize likelihood of observed decisions:

$$\hat{\theta} = \arg\max_{\theta} \sum_{i=1}^{I} \sum_{t=1}^{T} \mathbf{1}\{d_{it} = 0\}(X_t\theta + \hat{\Upsilon}) - \log(1 + \exp(X_t\theta + \hat{\Upsilon}))$$