Dynamic Structural Models

1. Hansen and Singleton (1982)

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Introduction

- Welcome everyone!
- In these afternoon sessions, we will cover the empirical implementation of structural models.
- You can access/download all material at the following page: https://github.com/ambarkley/Dynamic-Structural-Models
- If you want to follow along without running the code, you can view the workbooks in a web browser by going to https://ambarkley.github.io/Dynamic-Structural-Models/index.html

Goal for today

- Start with a basic economic problem: how much to consume and how much to save?
- House this problem within a dynamic model of consumer choice.
  - Two parameters: discount factor β and utility parameter γ.
- Assume optimal behavior $\rightarrow$ first-order conditions characterize behavior.
- Euler equations ($\approx$ first-order conditions over time) are the equations used in estimation.
  - Choose β and γ to make the Euler equations "as correct as possible."

The model

- Representative consumer with CARA utility over consumption C_t

$$U = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \, u(C_t) \right], \qquad 0 < \beta < 1,$$

  where $u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$ for $\gamma > 0$.

- Consumer faces budget constraint:

$$C_t + P_t Q_t \leq (P_t + D_t) Q_{t-1} + W_t$$

  where P_t is the price of an asset in period t, Q_t is the quantity of the asset held, and W_t is wage income.

First-order conditions

- Consumer's problem: maximize utility subject to a budget constraint. Characterize solution via Lagrangian:

$$\mathcal{L} = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(C_t) \right] - \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \lambda_t (C_t + P_t Q_t - (P_t + D_t) Q_{t-1} - W_t) \right]$$

- Two first order conditions (wrt C_t and Q_t):

$$\mathbb{E}_0 \left[\beta^\tau u'(C_\tau) \right] - \lambda_\tau = 0, \quad -\lambda_\tau P_\tau + \lambda_{\tau+1}(P_{\tau+1} + D_{\tau+1}) = 0$$

- After defining $R_t \equiv (P_{t+1} + D_{t+1})/P_t$, we rearrange and get:

$$\mathbb{E}_t \left[\beta \frac{u'(C_{t+1)}}{u'(C_t)} \, R_t \; - \; 1 \right] \; = \; 0$$

Estimating equations

- Plugging in $u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$:

$$\mathbb{E}_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_t - 1 \right] = 0$$

- Note that $\left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$ is the marginal rate of substitution for consumption at t vs $t+1$, and R_t is the return on the asset.

- This equation has to hold *for all* t and after conditioning on any *known information at time* t.

- Let Z_t be a vector containing information known at t (e.g., last period's returns).

$$\mathbb{E}_t \left[Z_t \left\{ \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_t - 1 \right\} \right] = 0$$

Results

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TABLE I

INSTRUMENTAL VARIABLE ESTIMATES FOR THE PERIOD 1959:2–1978:12

Cons	Return	*NLAG*	$\hat{\alpha}$	$\widehat{\mathrm{SE}}(\hat{\alpha})$	$\hat{\beta}$	$\widehat{\mathrm{SE}}(\hat{\beta})$	χ^2	DF	Prob
NDS	EWR	1	– 0.9360	2.5550	.9930	.0060	5.226	1	.9774
NDS	EWR	2	0.1529	2.3468	.9906	.0056	7.378	3	.9392
NDS	EWR	4	1.2605	2.2669	.9891	.0059	9.146	7	.7577
NDS	EWR	6	0.1209	2.0455	.9928	.0054	14.556	11	.7963
NDS	VWR	1	– 1.0350	1.8765	.9982	.0045	1.071	1	.6993
NDS	VWR	2	0.1426	1.7002	.9965	.0044	3.467	3	.6749
NDS	VWR	4	– 0.0210	1.6525	.9969	.0043	5.718	7	.4270
NDS	VWR	6	– 1.1643	1.5104	.9997	.0041	11.040	11	.5601
ND	EWR	1	– 1.5906	1.0941	.9930	.0034	7.186	1	.9926
ND	EWR	2	– 0.7127	0.9916	.9918	.0034	12.040	3	.9928
ND	EWR	4	– 0.1261	0.8917	.9921	.0035	14.638	7	.9591
ND	EWR	6	– 0.4193	0.8256	.9936	.0033	18.016	11	.9188
ND	VWR	1	– 1.2028	0.7789	.9976	.0027	1.457	1	.7726
ND	VWR	2	– 0.5761	0.7067	.9975	.0027	5.819	3	.8792
ND	VWR	4	– 0.6565	0.6896	.9978	.0027	7.923	7	.6606
ND	VWR	6	– 0.9638	0.6425	.9985	.0027	10.522	11	.5159

Results

TABLE III

INSTRUMENTAL VARIABLES ESTIMATION WITH MULTIPLE RETURNS

Equally- and Value-Weighted Aggregate Returns 1959:2–1978:12

Cons	$NLAG$	$\hat{\alpha}$	$\widehat{SE}(\hat{\alpha})$	$\hat{\beta}$	$\widehat{SE}(\hat{\beta})$	χ^2	DF	Prob.
NDS	1	− 0.5901	1.7331	.9989	.0041	18.309	6	.9945
NDS	2	1.0945	1.4907	.9961	.0040	24.412	12	.9821
NDS	4	0.3835	1.4208	.9975	.0039	40.234	24	.9798
ND	1	− 0.6494	0.6838	.9982	.0025	19.976	6	.9972
ND	2	− 0.0200	0.6071	.9982	.0025	27.089	12	.9925
ND	4	− 0.1793	0.5928	.9986	.0025	42.005	24	.9871

Value-Weighted Aggregate Stock Returns and Risk-Free Bonds Returns 1959:2–1978:12

Cons	$NLAG$	$\hat{\alpha}$	$\widehat{SE}(\hat{\alpha})$	$\hat{\beta}$	$\widehat{SE}(\hat{\beta})$	χ^2	DF	Prob.
NDS	1	−.1405	.0420	.9998	.0001	31.800	8	.9999
NDS	2	−.1472	.0376	.9998	.0001	44.083	16	.9998
NDS	4	−.1405	.0320	.9996	.0001	65.250	32	.9995
ND	1	−.0962	.0461	.9995	.0001	25.623	8	.9988
ND	2	−.1150	.0377	.9995	.0001	39.874	16	.9991
ND	4	−.1611	.0364	.9994	.0001	60.846	32	.9985

Three Industry-Average Stock Returns 1959:2–1977:12

Cons	$NLAG$	$\hat{\alpha}$	$\widehat{SE}(\hat{\alpha})$	$\hat{\beta}$	$\widehat{SE}(\hat{\beta})$	χ^2	DF	Prob.
NDS	1	1.5517	1.8006	.9906	.0046	13.840	13	.6147
NDS	4	0.6713	1.2466	.9940	.0035	88.211	49	.9995
ND	1	0.7555	0.7899	.9924	.0029	13.580	13	.5959
ND	4	0.5312	0.5512	.9939	.0024	89.501	49	.9996

Exercises

1. Modify the estimation code to use only one asset each period instead of two. How do the results (point estimates and SEs) compare?
2. One can use additional lags for the instrumental variables. Add another lag to consumption and asset returns. Again, compare the estimates to the ones previously obtained.
3. (*Advanced*) Two-step GMM often performs poorly in small samples. Instead, an iterated GMM estimator, in which the weighting matrix is updated until the change in parameter estimates falls below some tolerance, is preferred. Alter the code to perform iterated GMM instead of two-stage GMM.