Dynamic Structural Models

Unobserved Heterogeneity

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Latent types in the data generating process

- We observe data $\{Y_i\}_{i=1}^N$.
- Data comes from two different distributions:
  - With probability π_1, $Y_i \sim F_1$
  - With probability π_2, $Y_i \sim F_2$.
- We assume we know F_1, F_2 up to a finite set of parameters θ_1, θ_2.
- We do not assume that we know the mixing parameters π_k.
- Problem is to estimate $\Xi \equiv \{\theta_1, \theta_2, \pi_1, \pi_2\}$ using $\{Y_n\}$.

Introduction

- We could, in principle, estimate Ξ using maximum likelihood.
  - Problematic in practice. Slow with unreliable convergence.
- Instead, we use the Expectation-Maximization algorithm. This sequentially updates π and θ.
- We define Z_{ik} as the random variable over the *latent type* of observation i.
  - $Z_{i1} = 1$ if observation i is drawn from F_1, and $Z_{i2} = 1$ if it is drawn from F_2.
  - Note that $\pi_k = \mathbb{E}[Z_{ik}]$.

EM algorithm: Initialization and Expectation

0. Fix initial guesses for Ξ and set convergence tolerance.
   Then, at each iteration m, perform the following steps:

1. Update π – the Expectation step. Given current parameters π, θ, calculate the expected value of Z_{ik} for all i, k.

$$E[Z_{ij}|Y, \theta^{(m)}, \pi^{(m)}] = \frac{\pi_j^{(m)} f(Y_i|\theta_j^{(m)})}{\pi_1^{(m)} f(Y_i|\theta_1^{(m)}) + \pi_2^{(m)} f(Y_i|\theta_2^{(m)})}$$

   We can also update the mixing parameters at this stage:

$$\pi_k^{(m+1)} = \frac{1}{N} \sum_{i=1}^{N} E[Z_{ik}|Y, \theta^{(m)}, \pi^{(m)}]$$

EM algorithm: Maximization

2. We maximize the likelihood *with respect to θ only*. The log-likelihood is:

$$\mathcal{L}(\theta) = \sum_{i=1}^{N}\sum_{k=1}^{2} \underbrace{E[Z_{ik}|Y, \theta^{(m)}, \pi^{(m)}]}_{\Pr(Z_{ik}=1)} \log(f(Y_i|\theta_j) + \sum_{i=1}^{N}\sum_{k=1}^{2} \underbrace{E[Z_{ik}|Y, \theta^{(m)}, \pi^{(m)}]}_{\Pr(Z_{ik}=1)} \log(\pi_j)$$

Note that this is just a *weighted likelihood*.

- Each observation has weights on the likelihood of being a "type one" observation or a "type two" observation.
- The weights were determined in Step 1.

- Update distribution parameters: $\theta^{(m+1)} = \arg\max_\theta \mathcal{L}(\theta)$.
- Repeat Steps 1-2 until overall likelihood converges.

Application to dynamic discrete choice models

- Now we return to bus engine replacement. We assume that the permanent type of bus n, s_n, is *unobserved to the econometrician* and can take two values $s_n \in \{1, 2\}$.
- We have already seen how to construct the likelihood if s_n was observable.
- To account for unobserved heterogeneity, we repeat this calculation twice:
  - Calculate likelihood of observing choice d_{nt} assuming the bus type is $s_n = 1$
  - Calculate the likelihood of observing choice d_{nt} assuming the bus type is $s_n = 2$.
- With these calculations and initial guesses for θ (utility parameters) and π (proportion of bus types), we can estimate the model with the EM algorithm.

Exploiting renewal decisions

- Games with a "reset" decision allow us to exploit finite dependence.
- Representation means that we can express the value function with respect to a chosen decision j:
$$V(x) = v_j(x) + \Psi_j(p(x))$$
- If the chosen decision is replace, the state "resets."
- We specify choices to reset the state starting from two different choices, canceling out the future values.
- Parameterization: linear utility, set replace payoff to zero in all states.

$$u_0(x) = x\theta, \qquad , u_1(x) = 0 \ \ \forall x$$

Exploiting renewal decisions

- Notation: $j = 1$ is replace, $j = 0$ is continue.

$$
\begin{aligned}
v_1(x) &= 0 + \beta V(\underline{x}) \\
&= \beta(\gamma - \ln(p_1(\underline{x}))) + \beta^2 V(\underline{x}) \\
v_0(x) &= x\theta + \beta \sum_{x' \in \mathcal{X}} V(x') f_0(x'|x) \\
&= x\theta + \beta \sum_{x' \in \mathcal{X}} [v_1(x') + \gamma - \ln(p_1(x'))] f_0(x'|x) \\
&= x\theta + \beta \sum_{x' \in \mathcal{X}} [\gamma - \ln(p_1(x'))]\, f_0(x'|x) + \beta^2 V(\underline{x}) \\
\Rightarrow \quad v_1(x) - v_0(x) &= x\theta \ - \beta \sum_{x' \in \mathcal{X}} [\gamma - \ln(p_1(x'))]\, f_0(x'|x)
\end{aligned}
$$

EM steps for bus engine replacement problem

- Define $\Delta v(x_{nt}, s) \equiv v_2(x_{nt}, s) - v_1(x_{nt}, s)$. The likelihood given bus type s is

$$\ell_{nt}(d_{nt}|x_{nt}, s, p_1, \theta) = \frac{d_{1nt} + d_{2nt}\exp(\Delta v(x_{nt}, s))}{1 + \exp(\Delta v(x_{nt}, s))}$$

where

$$\Delta v(x_{nt}, s) = \theta_0 + \theta_1 x_{nt} + \theta_2 s + \beta[\log(p_1(0, s)) - \sum_{x' \in \mathcal{X}} \log(p_1(x', s)) f_2(x'|x_{nt})]$$

- We calculate the likelihood for each observation-bus type pair and then perform the Expectation step

$$q_{ns}^{(m+1)} = \frac{\pi_s^{(m)} \prod_{t=1}^{T} \ell_{nt}(d_{nt}|x_{nt}, s, p_1^{(m)}, \theta^{(m)})}{\sum_{s'=1}^{S} \pi_{s'}^{(m)} \prod_{t=1}^{T} \ell_{nt}(d_{nt}|x_{nt}, s', p_1^{(m)}, \theta^{(m)})}$$

where q_{ns} is the probability that bus n is type s.

EM steps for bus engine replacement problem

- Next we update the CCPs:

$$p_1^{(m+1)}(x,s) = \frac{\sum_{n=1}^{N}\sum_{t=1}^{T} d_{1nt} q_{ns}^{(m+1)} 1\{x_{nt}=x\}}{\sum_{n=1}^{N}\sum_{t=1}^{T} q_{ns}^{(m+1)} 1\{x_{nt}=x\}}$$

- Finally, we maximize the likelihood with respect to θ:

$$\theta^{(m+1)} = \arg\max_{\theta} \sum_{n=1}^{N}\sum_{s=1}^{S}\sum_{t=1}^{T} q_{ns}^{(m+1)} \log\left(\ell_{nt}(d_{nt}|x_{nt}, s_n, p_1^{(m+1)}, \theta)\right)$$

and update π for the next iteration:

$$\pi_s^{(m+1)} = \frac{1}{N}\sum_{n=1}^{N} q_{ns}^{(m+1)}$$