NUMERICAL METHODS

Gate Notes

NUMERICAL METHODS

To find solutions to

1. Algebraic and Transcendental equations
2. System of linear equations
3. Integration of a function
4. Differential equations.

① Solution of Algebraic and transcendental equations:

Algebraic equations: polynomials of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$$

Transcendental equations: combination of polynomial, exponential and trigonometric functions.

Ex: $\tan x = x$, $xe^x - 1 = 0$, $\cos x - xe^x = 0$

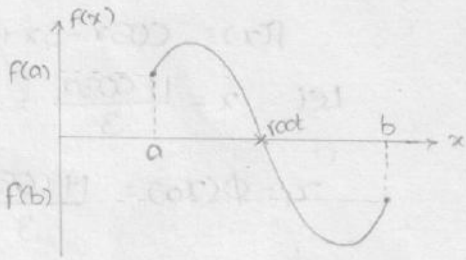
Intermediate value theorem: If $f(x)$ is a continous function defined on $[a,b]$ and $f(a)$, $f(b)$ are having opposite signs, then there exist atleast one root of $f(x)$ in $[a,b]$

Ex: $f(x) = x^3 + x - 1$

$f(0) = -1$ (-ve)

$f(1) = 1+1-1$ (+ve)

$\therefore$ one root in the interval $[0,1]$



Iterative Method (Successive approximation):

1. Express $f(x)=0$ as $x = \phi(x)$
2. choose an initial approximation x_0 to the root in the given interval
3. Find the next approximations using $x_1 = \phi(x_0)$
   $x_2 = \phi(x_1)$
   $x_3 = \phi(x_2)$ ---
4. Stop the iteration if two consecutive iterations gives the same approximation.

** The sequence of approximations converges to the root if $|\phi'(x)| < 1$

Ex: $x^3+x-1=0$: $[a,b]=[0,1]$

$x = 1-x^3$	$x=\sqrt[3]{1-x}$	$x(x^2+1)=1$
$\lvert\phi'(x)\rvert = \lvert -3x^2\rvert$	$\lvert\phi'(x)\rvert = \left\lvert \frac{1}{3}(1-x)^{-2/3}(-1)\right\rvert$	$x=\frac{1}{x^2+1}$
$= 3x^2$	$\lvert\phi'(x)\rvert = \frac{1/3}{(1-x)^{2/3}}$	$\lvert\phi'(x)\rvert = \frac{-1}{(1+x^2)^2}\cdot 2x$
$\lvert\phi'(0)\rvert = 0<1$	$\lvert\phi'(0)\rvert = 1/3 < 1$	$\lvert\phi'(0)\rvert = 0<1$
$\lvert\phi'(1)\rvert = 3>1$	$\lvert\phi'(1)\rvert = \infty$	$\lvert\phi'(1)\rvert = 1/2<1$
×	×	✓

$\therefore$ Let $\phi(x)=\frac{1}{1+x^2}$ and let $x_0=0$

$x_1=\phi(x_0)=\frac{1}{1+0^2}=1$

$x_2=\phi(x_1)=\frac{1}{1+1^2}=0.5$

$x_3=\phi(x_2)=\frac{1}{1+(0.5)^2}=0.8$

$x_4=\phi(x_3)=\frac{1}{1+(0.8)^2}=0.6097$

⋮

using calci $\frac{1}{1+(Ans)^2}$ press "="

$x_{18}=0.6822$

$x_{19}=0.6824$

$x_{20}=0.6822$

$x_{21}=0.6824$

$x_{22}=0.6822$

Ex: $\cos x = 3x-1$

$f(x)=\cos x-3x+1$: $[0,1]$

Let $x=\frac{1+\cos x}{3}=\phi(x)$ & Let $x_0=0$

$x_1=\phi(x_0)=\frac{1+\cos 0}{3}=0.6666$

using calci $\frac{1+\cos(Ans)}{3}$ press "="

⋮

$x_5=0.6071$, $x_6=0.6071$

Bi-Section Method: Suppose that a root of $f(x)=0$ lies in the interval $I_0=(a_0,b_0)$ i.e $f(a_0)\times f(b_0)<0$ (opposite signs)

1. Bi-sect the interval to obtain $c_1=\frac{a_0+b_0}{2}$
2. Root lies in the interval $I_1=(a_0,c_1)$ if $f(a_0)\times f(c_1)<0$

   Root lies in the interval $I_2=(c_1,b_0)$ otherwise.
3. Repeat above procedure for 'n' times. finally the root is given by midpoint of the interval

Ex: x^3+x-1 : $[0,1]$

$c_1=\frac{0+1}{2}=0.5$

$f(c_1)=(0.5)^3+(0.5)-1=-0.37$ (-ve)

$f(a_0)\times f(c_1)=(-1)\times(-0.37)=0.37>0$ $\therefore$ New Interval $=(c_1,b_0)$ $=(0.5,1)$

(2)

$C_2 = \frac{0.5+1}{2} = 0.75$

$f(0.5) \times f(0.75) = -0.064 < 0$ $\therefore$ New interval $= [0.5, 0.75]$

$\vdots$

$C_{12} = 0.68225$ $\therefore$ New interval $= [0.6822, 0.6825]$

Regula falsi Method / Method of false position:

1. Assume two initial approximations x_0, x_1 such that $f(x_0) \times f(x_1) < 0$
2. Find next approximation using $x_{k+1} = \frac{x_{k-1} f_k - x_k f_{k-1}}{f_k - f_{k-1}}$
3. Root lies in the interval (x_0, x_2) if $f(x_0) \times f(x_2) < 1$

   Root lies in the interval (x_2, x_1) otherwise.
4. Continue the process untill desired accuracy is achieved.

If $f(x)$ in the interval $[a, b]$, for simplicity select $x_0 = a$, $x_1 = b$

then $x_{n+1} = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

Ex: $x^3 + x - 1 = 0$: $[0 \ 1]$

$x_0 = \frac{0 \times f(1) - 1 \times f(0)}{f(1) - f(0)} = 0.5$ $x_1 = \frac{0.5 f(1) - 1 f(0.5)}{f(1) - f(0.5)} = 0.6363$

$f(x_0) = -0.37$ (-ve) $\vdots$

New interval $= [0.5, 1]$

Another way: Fix one end of interval (say b) $f(b) > 0$ and apply iteration formula to find new value for other end of the interval. After desired iterations, other end of interval itself gives the root.

Secant Method / Chord Method:

1. It is same as regula falsi method. The only difference is, while selecting initial approximation, we don't consider the condition $f(x_0) \times f(x_1) < 0$
2. So, this method may or may not converge
3. If it converge, it converges faster than regula falsi method.

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Newton raphson method: The iterative formula in Newton raphson method is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

→ Intially assume x_0

→ Newton raphson method fails if $f'(x)=0$

→ If $f'(x)$ is very large then root can be find very rapidly

(99CS) → condition for the convergence of newton raphson method is $|f(x)\cdot f''(x)| < |f'(x)|^2$

Ex: $x^3+x-1=0$: $[0,1]$

Sol: $f(x) = x^3+x-1 \Rightarrow f'(x) = 3x^2+1$

Let $x_0 = 0.5$

$$x_1 = 0.5 - \frac{(0.5)^3+(0.5)-1}{3(0.5)^2+1} = 0.714$$

$x_2 = 0.6831 \qquad x_3 = 0.6823 \qquad x_4 = 0.6823$

(95CS)

prob: Find iterative formula for $\sqrt{N}$ using Newton Raphson method?

Sol

$x = \sqrt{N} \Rightarrow x^2 = N \Rightarrow x^2-N=0$

$f(x_n) = x_n^2 - N \Rightarrow f'(x_n) = 2x_n$

From N.R Method $\quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{x_n^2-N}{2x_n} = \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{x_n^2+N}{2x_n} = \frac{1}{2}\left[\frac{N}{x_n} + x_n\right]$$

(96-CS)

prob: Iterative formula for $\sqrt[3]{N}$

$x = \sqrt[3]{N} \Rightarrow x^3 = N \Rightarrow x^3 - N = 0$

$f(x_n) = x_n^3 - N \Rightarrow f'(x_n) = 3x_n^2$

From N.R Method $\quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{x_n^3-N}{3x_n^2} = \frac{2x_n^3+N}{3x_n^2}$$

$$x_{n+1} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right]$$

(3)

(GATE-95): Let $f(x) = x - \cos x$. using NR method find x_{n+1} using x_n

Sol: $f(x) = x - \cos x \Rightarrow f'(x) = 1 + \sin x$

$$x_{n+1} = x_n - \frac{[x_n - \cos x_n]}{1 + \sin x_n}$$

(GATE-97 CS): N.R method is used to find the root of the equation $x^2 - 2 = 0$. If the iterations are started from -1, then the iteration will

a) converges to -1 b) converges to $\sqrt{2}$ c) converges to $-\sqrt{2}$ ✓ d) not conv.

Sol: from N.R method $x_{n+1} = \dfrac{x_n^2 + 2}{2x_n}$

Iteration started from $-1 \Rightarrow x_0 = -1$

$\therefore x_1 = \dfrac{x_0^2 + 2}{2x_0} = -1.5$ $\quad x_3 = \dfrac{x_2^2 + 2}{2x_2} = -1.4141$

$x_2 = \dfrac{x_1^2 + 2}{2x_1} = -1.4166$ $\quad x_4 = \dfrac{x_3^2 + 2}{2x_3} = -1.4141$

$\therefore$ Iterations converge to $-\sqrt{2}$

(GATE-05 CE): Given $a > 0$, we wish to calculate its reciprocal value $\frac{1}{a}$ by using NR method for $f(x) = 0$. For $a = 7$ and starting with $x_0 = 0.2$ the first two iterations will be

Sol: $x = \dfrac{1}{a} \Rightarrow x - \dfrac{1}{a} = 0$ (or) $\dfrac{1}{x} - a = 0$

$f(x) = \dfrac{1}{x} - a \Rightarrow f'(x) = -\dfrac{1}{x^2}$

$$x_{n+1} = x_n - \frac{\left(\frac{1}{x_n} - a\right)}{\left(-\frac{1}{x_n^2}\right)} = 2x_n - a x_n^2$$

Given $a = 7$, $x_0 = 0.2$

$x_1 = 2(0.2) - 7(0.2)^2 = 0.12$

$x_2 = 2(0.12) - 7(0.12)^2 = 0.1392$

(GATE-05 ME): Starting from $x_0 = 1$, one step of NR method in solving the equation $x^3 + 3x - 7 = 0$ gives the next value x_1 as

$f'(x) = 3x^2 + 3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(1) + 3(1) - 7}{3(1) + 3} = 1 + \frac{1}{2} = 1.5$$

(GATE-05 PI): The real root of the equation $xe^x = 2$ is evaluated using NR method. If the first approximation of the value of x is 0.8679, the 2nd approximation of x correct to 3 decimal places is

Sol: $f(x) = xe^x - 2 \Rightarrow f'(x) = xe^x + e^x$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.8679 - \frac{(0.8679)e^{0.8679} - 2}{(0.8679)e^{0.8679} + e^{0.8679}}$$

$$= 0.853$$

(GATE-07 CE): The following equation need to be numerically solved using NR method $x^3 + 4x - 9 = 0$. The iterative equation for this purpose is

Sol: $f(x) = x^3 + 4x - 9 \Rightarrow f'(x) = 3x^2 + 4$

$$x_{k+1} = x_k - \frac{(x_k^3 + 4x_k - 9)}{(3x_k^2 + 4)}$$

$$= \frac{3x_k^3 + 4x_k - x_k^3 - 4x_k + 9}{3x_k^2 + 4} = \frac{2x_k^3 + 9}{3x_k^2 + 4}$$

(GATE-07 EC): The equation $x^3 - x^2 + 4x - 4 = 0$ is to be solved using NR method. If $x = 2$ taken as the initial approx. of the solution then the next approx. using this method will be

Sol: $f'(x) = 3x^2 - 2x + 4$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{8-4+8+4}{12-4+4} = 4/3$$

(GATE-08 EE): Equation $e^x - 1 = 0$ is required to be solved using NR method with an initial guess $x_0 = -1$. Then after one step of NR method estimate x_1 of the solution will be given

Sol: $f(x) = e^x - 1 \Rightarrow f'(x) = e^x$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1 - \frac{e^{-1} - 1}{e^{-1}} = 0.71828$$

(GATE-08 EC): The recursion relation to solve $x = e^{-x}$ using NR method is

Sol: $f(x) = x - e^{-x} \Rightarrow f'(x) = 1 + e^{-x}$

$$x_{n+1} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}} = \frac{(1 + x_n)e^{-x_n}}{(1 + e^{-x_n})}$$