COMPLEX VARIABLES

Gate Notes

Complex variables

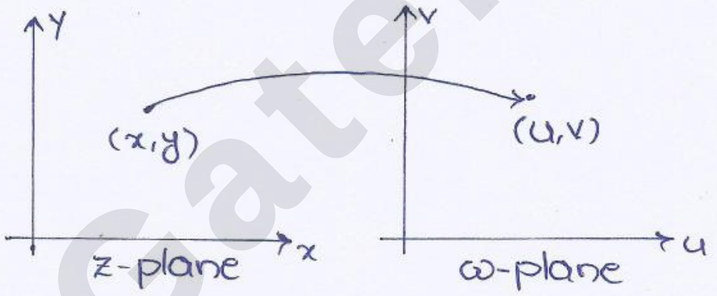
A variable of the form $z = x+iy$ where $i^2 = -1$ and x, y are real is called complex variable. By considering $x = r\cos\theta$, $y = r\sin\theta$ we have

$z = r[\cos\theta + i\sin\theta] = re^{i\theta}$ is called polar form or Mod amplitude form. where $r = |z| = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

We can associate a complex valued function to the complex variable z then we can write

$$\omega = f(z) = u(x,y) + i\,v(x,y)$$

To represent this complex function we require two planes namely z-plane & ω-plane. For each point (x,y) in z-plane there exist a point (u,v) in ω-plane with the function $\omega = f(z)$. The point (u,v) is called image of (x,y)



Every complex function $f(z)$ can always be expressable in the form $\omega = f(z) = u(x,y) + i\,v(x,y)$ where $u(x,y)$ is called real part of $f(z)$ and $v(x,y)$ is called imaginary part of $f(z)$.

Ex: $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$

$f(z) = e^z = e^{x+iy} = e^x\cos y + i e^x \sin y$

$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

$f(z) = \log_e z = \log_e(x+iy) = \log_e(re^{i\theta})$

$= \log_e r + i\theta.\log_e e = \log_e\sqrt{x^2+y^2} + i.\tan^{-1}(y/x)$

$= \frac{1}{2}\log_e(x^2+y^2) + i\tan^{-1}(y/x)$

prob: The mod amplitude form of $\frac{1+i}{1-i}$ is

sol: $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2} = \frac{2i}{2} = i$

$r = \sqrt{0^2+1^2} = 1 \qquad \theta = \tan^{-1}(1/0) = \pi/2$

$\therefore$ Mod amplitude form $= 1e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2$

prob: If $\alpha + i\beta = \frac{1}{a+ib}$ then $(\alpha^2+\beta^2)(a^2+b^2) =$

sol: $\alpha + i\beta = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$

$\therefore \alpha = \frac{a}{a^2+b^2} \qquad \beta = \frac{-b}{a^2+b^2}$

$\alpha^2+\beta^2 = \frac{a^2}{(a^2+b^2)^2} + \frac{b^2}{(a^2+b^2)^2} = \frac{1}{(a^2+b^2)}$

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$(a^2+b^2)(\alpha^2+\beta^2) = \underline{1}$

prob: The value of $i^i =$

sol: $i^i = e^{\log_e i^i} = e^{i\log_e i} = e^{i[\frac{1}{2}\log_e 1 + i\tan^{-1}(1/0)]}$

$= e^{i \cdot i\pi/2} = e^{-\pi/2}$

* $\log_e i = i\pi/2 \Rightarrow i = e^{i\pi/2}$

prob: The value of $\sqrt{2i}$

sol: $\sqrt{2i} = \sqrt{2}.\sqrt{i} = \sqrt{2}.e^{\log_e\sqrt{i}} = \sqrt{2}.e^{\frac{1}{2}\log_e i}$

$= \sqrt{2}e^{\frac{1}{2}(i\pi/2)} = \sqrt{2}e^{i\pi/4} = \sqrt{2}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = 1+i$

* $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ----$

* $e^{iy} = 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} - ---$

$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - ---\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - ---\right)$ [Maclaurin's expansion]

$= \cos y + i\sin y$ [Euler's formula]

* $e^{2n\pi i} = \cos(2n\pi) + i\sin(2n\pi) = 1$

$\Rightarrow e^z = e^{z+2n\pi i}$, n is any integer

$\therefore e^z$ is periodic with period $2\pi i$ (GATE 97 CE)

* $e^{-iy} = \cos y - i\sin y$

* $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ $\quad\quad$ $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

$\cosh z = \frac{1}{2}(e^{z} + e^{-z})$ $\quad\quad$ $\sinh z = \frac{1}{2}(e^{z} - e^{-z})$

* $\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}\left[e^{i(x+iy)} + e^{-i(x+iy)}\right]$

$= \frac{1}{2}\left[\cos x\,(e^{-y} + e^{y}) + i \sin x\,(e^{-y} - e^{y})\right]$

$= \cos x \cdot \cosh y - i \sin x \cdot \sinh y$

* $\sin z = \sin x \cdot \cosh y + i \cos x \cdot \sinh y$

* $\sin(iz) = \frac{1}{2i}\left[e^{i(iz)} - e^{-i(iz)}\right] = \frac{1}{2i}\left[e^{-z} - e^{z}\right]$

$= \frac{i}{2}\left[e^{z} - e^{-z}\right] = i \sinh z$

$\cos(iz) = \frac{1}{2}\left[e^{i(iz)} + e^{-i(iz)}\right] = \frac{1}{2}\left[e^{-z} + e^{z}\right] = \cosh z$

* $\sinh z = \frac{1}{i}\sin(iz) = -i \sin(i(x+iy))$

$= i \sin(y - ix) = \sinh x \cdot \cos y + i \cosh x \cdot \sin y$

$\cosh z = \cosh x \cdot \cos y + i \sinh x \cdot \sin y$

* In general $\ln z = \ln\sqrt{x^2+y^2} + i \tan^{-1}(y/x) + i2n\pi$. The principal value is obtained when $n=0$

prob: Find the general and principle values of $\log(1+\sqrt{3}i)$

sol: $|z| = \sqrt{1+3} = 2$ $\quad\quad$ $\text{Arg}(z) = \tan^{-1}(\sqrt{3}) = \pi/3$

$\log(1+\sqrt{3}i) = \log 2 + i\left(\frac{\pi}{3} + 2n\pi\right)$, n any integer

principal value $\text{Log}(1+\sqrt{3}i) = \log 2 + i\pi/3$ $\quad (n=0)$

prob: Find general and principal values of $(-i)^i$

sol: $(-i)^i = e^{i\log(-i)} = e^{i\left[\log 1 + i\left(-\frac{\pi}{2} + 2n\pi\right)\right]}$

$= e^{\frac{\pi}{2} - 2n\pi}$

principal value $= e^{\pi/2}$

properties of moduli:

- $|z_1 + z_2| \le |z_1| + |z_2|$

  $|z_1 - z_2| \le |z_1| + |z_2|$

  $|z_1 + z_2| \ge |z_1| - |z_2|$

  $|z_1 - z_2| \ge |z_1| - |z_2|$

- $|z_1 z_2 z_3 \dots z_n| = |z_1| \cdot |z_2| \cdot \dots \cdot |z_n|$

  $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(GATE 94 IN): The real part of the complex number $z = x + iy$ is given by (a) $Re(z) = z - z^*$ (b) $Re(z) = \frac{z - z^*}{2}$ ✓(c) $= \frac{z + z^*}{2}$ (d) $= z + z^*$

Sol: $z = x + iy \Rightarrow z^* = x - iy$

$$Re(z) = x = \frac{z + z^*}{2}$$

(GATE 97 IN): The complex number $z = x + iy$ which satisfy the equation $|z+1| = 1$ lie on

(a) Circle with (1,0) as center and radius 1

✓(b) circle with (-1,0) as center and radius 1

(c) y-axis (d) x-axis

Sol:

$$|z+1| = 1 \Rightarrow |x + iy + 1| = 1$$

$$(x+1)^2 + y^2 = 1$$

Equation of Circle with (x_0, y_0) as center and r as radius is $(x - x_0)^2 + (y - y_0)^2 = r^2$

$\therefore$ $|z+1| = 1$ is a circle with center (-1,0) and radius = 1

(GATE 2005 CE): Which one of the following is not true for the complex numbers z_1 and z_2?

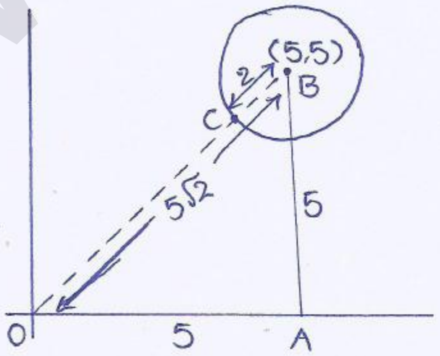
(a) $\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}$ (b) $|z_1 + z_2| \le |z_1| + |z_2|$ ✓(c) $|z_1 + z_2| \le ||z_1| - |z_2||$

(d) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

$$|z_1 + z_2| \ge |z_1| - |z_2|$$

(GATE 2005 IN): consider the circle $|z - 5 - 5i| = 2$ in the complex number plane (x, y) with $z = x + iy$. The minimum ~~number~~ distance from origin to circle is?

Sol:



$$OB^2 = OA^2 + AB^2 = 50$$

$$OB = 5\sqrt{2}$$

$$OC = OB - CB = 5\sqrt{2} - 2$$

(3)

(GATE-06 EC): For the function of a complex variable $\omega = \ln z$ (where $\omega = u + iv$ and $z = x + iy$) the u = constant lines get mapped in the z-plane as

Sol: $\omega = \ln z = \ln(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}(y/x)$

$$u = \frac{1}{2}\ln(x^2+y^2) = C \quad \text{(constant)}$$

$$\ln(x^2+y^2) = 2C$$

$$x^2+y^2 = e^{2C} \Rightarrow x^2+y^2 = (e^C)^2$$

Represent concentric circles with radius e^C.

(GATE-07 PI): If a complex number $z = \frac{\sqrt{3}}{2} + i\frac{1}{2}$ then z^4 is

Sol: $z = \frac{\sqrt{3}}{2} + i\frac{1}{2} \Rightarrow iz = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \omega$ say

[ω is a root of $x^3 = 1$]

$$i^4 z^4 = \omega^4 = \omega^3.\omega \quad [\omega^3 = 1]$$

$$z^4 = \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

(GATE-08 PI): The value of the expression $\frac{-5+i10}{3+4i}$

Sol: $\frac{(-5+i10)}{(3+4i)} \times \frac{(3-4i)}{(3-4i)} = \frac{25+50i}{25} = 1+2i$

(GATE-08 EC): The equation $\sin(z) = 10$ has

(a) no real or complex solution (b) Exactly two distinct complex sol's

(c) a unique solution ✓(d) an infinite number of complex sol's

Sol: $\sin z = 10 \Rightarrow \frac{e^{iz} - e^{-iz}}{2i} = 10$

$$\Rightarrow (e^{iz}) - \frac{1}{e^{iz}} = 20i$$

$$\Rightarrow (e^{iz})^2 - 20i(e^{iz}) - 1 = 0$$

$$e^{iz} = \frac{-(-20i) \pm \sqrt{-400+4}}{2} = 10i \pm 3\sqrt{11}i = i[10 \pm 3\sqrt{11}]$$

$$iz = \log_e i[10 \pm 3\sqrt{11}]$$

$$= \log_e i + \log_e(10 \pm 3\sqrt{11})$$

$$iz = \log 1 + i\left(\frac{\pi}{2} \pm 2n\pi\right) + \log(10 \pm 3\sqrt{11})$$

$$z = \frac{\pi}{2} \pm 2n\pi - i\log(10 \pm 3\sqrt{11})$$

$\therefore$ z has infinite no of complex sol's

(GATE - 2013 EE): Square roots of $-i$, where $i = \sqrt{-1}$ are

(a) $i, -i$ (b) $\cos(-\pi/4) + i\sin(-\pi/4)$, $\cos(3\pi/4) + i\sin(3\pi/4)$

(c) $\cos(\pi/4) + i\sin(3\pi/4)$, $\cos(3\pi/4) + i\sin(\pi/4)$

(d) $\cos(3\pi/4) + i\sin(-3\pi/4)$, $\cos(-3\pi/4) + i\sin(3\pi/4)$

Sol: $i = e^{i\pi/2} \Rightarrow -i = \frac{1}{i} = e^{-i\pi/2}$

$\sqrt{-i} = (-i)^{1/2} = e^{-i\pi/4}$

and $-i = i^3 = e^{i3\pi/2} \Rightarrow \sqrt{-i} = e^{i3\pi/4}$

$\therefore$ square roots of $-i$ are $e^{-i\pi/4}$, $e^{i3\pi/4}$

(GATE - 14 EE): All the values of the multi valued complex function 1^i, where $i = \sqrt{-1}$ are.

(a) purely imaginary (b) real & non negative (c) on the unit circle

(d) equal in real and imaginary parts.

Sol: $1^i = (\cos 2n\pi + i\sin 2n\pi)^i$

$= (e^{i2n\pi})^i = e^{-2n\pi} = \frac{1}{e^{2n\pi}}$ It is real & non-negative.

(GATE - 14 ME): The argument of complex number $\frac{1+i}{1-i}$

Sol: $\frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{2i}{2} = i = 0 + 1.i$

Argument $= \tan^{-1}(1/0) = \tan^{-1}(\infty) = \pi/2$

Analytic function: A function $f(z)$ is said to be analytic in a region R of z-plane if the derivative of $f(z)$ exists at each and every point in that region.

Necessary and sufficient condition: For a function $f(z) = u + iv$ to be analytic in the region 'R'

(i) Partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ must exist in 'R'

(ii) They should satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(or)

$$\underbrace{u_x = v_y, \quad u_y = -v_x}_{\text{Cauchy Reiman or C-R equations}}$$

prob: $f(z) = z^2$.

$$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$

$u = x^2 - y^2$ $\qquad$ $v = 2xy$

$u_x = 2x$ $\qquad$ $v_x = 2y$

$u_y = -2y$ $\qquad$ $v_y = 2x$

$$u_x = v_y \qquad u_y = -v_x$$

$\therefore f(z) = z^2$ is analytic throughout the z-plane.

prob: $f(z) = \bar{z}$

$\bar{z} = x - iy \Rightarrow u = x \quad v = -y$

$u_x = 1$ $\qquad$ $v_x = 0$

$u_y = 0$ $\qquad$ $v_y = -1$

$$u_x \neq v_y$$

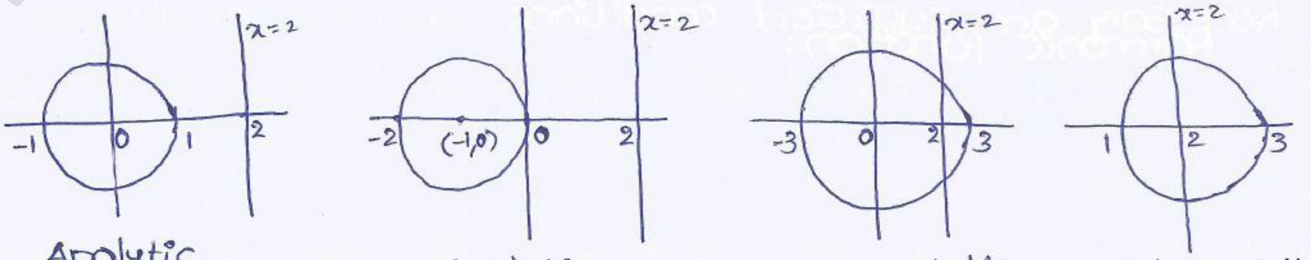
$\therefore f(z) = \bar{z}$ is not analytic throughout the z-plane.

Note: 1. Any function which involves $\bar{z}$ is always not analytic throughout the z-plane.

2. Every polynomial in z is always analytic throughout the z-plane.

Ex: $f(x) = u + iv$ suppose if $u_x = \frac{1}{x-2}$, $u_y = 1$, $v_x = -1$, $v_y = \frac{1}{x-2}$

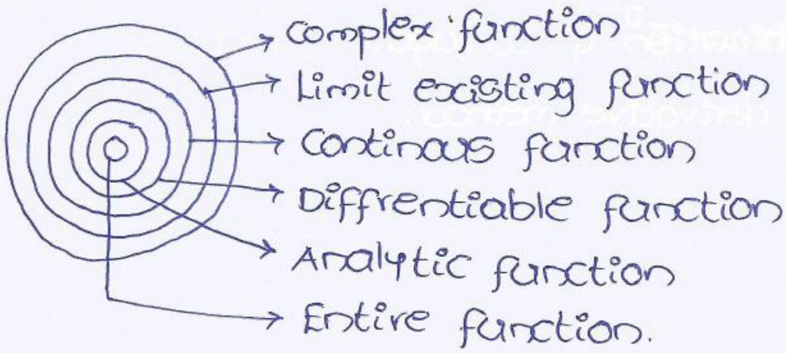
This function is analytic in the regions which does not contain $x=2$ like $|z|=1$, $|z+1|=1$ and this function is not analytic in the regions which contains $x=2$ like $|z|=3$, $|z-2|=1$



Analytic $\qquad$ Analytic $\qquad$ Not analytic $\qquad$ Not analytic

Entire function (or) Holomorphic (or) Regular function:

A function which is analytic in the entire z-plane is called entire function. Ex: Every polynomial in z.



C-R equations in polar form:

$$f(z) = u + iv \Rightarrow f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$$

P.d.w.r. to 'r' $\quad f'(re^{i\theta}) \times e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \rightarrow (1)$

P.d.w.r. to 'θ' $\quad f'(re^{i\theta}) \times r \times i \times e^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} \rightarrow (2)$

from (1) & (2) $\quad ri\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$

$$-r.\frac{\partial v}{\partial r} + i.r.\frac{\partial u}{\partial r} = \frac{\partial u}{\partial \theta} + i.\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r.\frac{\partial v}{\partial r} \qquad \frac{\partial v}{\partial \theta} = r.\frac{\partial u}{\partial r}$$

$$\Rightarrow \quad v_r = -\frac{1}{r}u_\theta \qquad u_r = \frac{1}{r}v_\theta$$

properties of analytic function: If $f(z) = u + iv$ is analytic function then

1. $u(x,y) = c_1$ & $v(x,y) = c_2$ are orthogonal to eachother

   Ex: $f(z) = z^2 = x^2 - y^2 + i\,2xy$

   $x^2 - y^2 = c_1$ & $2xy = c_2$ are $\perp$'ar to each other.

2. If $u(x,y)$ is a harmonic function then $v(x,y)$ is also a harmonic function.

   Harmonic function: A function $H(x,y)$ which satisfies the laplace equation $\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0$ is called harmonic function.

$$u_x = v_y \qquad u_y = -v_x$$

$$u_{xx} = v_{xy} \qquad u_{yy} = -v_{yx}$$

$$\left.\begin{array}{l} u_{xx} + u_{yy} = 0 \\ \text{lly} \quad v_{xx} + v_{yy} = 0 \end{array}\right\} \begin{array}{l} \text{If } f(z) = u + iv \text{ is analytic then} \\ u \,\&\, v \text{ satisfies laplace equation} \end{array}$$

Determination of Conjugate function:

Total derivative method: Let the real part of $u(x,y)$ of an analytic function $f(z) = u+iv$ be given. Then to find $v(x,y)$ we proceed as follows. The total derivative of v is given by

$$dv = \frac{\partial v}{\partial x}\cdot dx + \frac{\partial v}{\partial y}\cdot dy$$

By using C-R eqn's $\quad dv = -\frac{\partial u}{\partial y}\cdot dx + \frac{\partial u}{\partial x}\cdot dy$

$$v = \underbrace{\int(-u_y)\,dx}_{**y\text{ const}} + \underbrace{\int u_x\,dy}_{***\text{terms independent of } 'x'} + C$$

Similarly, If $v(x,y)$ is given. Then to find $u(x,y)$.

$$du = \frac{\partial u}{\partial x}\cdot dx + \frac{\partial u}{\partial y}\cdot dy = v_y\,dx - v_x\,dy \quad [\text{using C-R}$$

$$u = \int v_y\,dx - \int v_x\,dy + C$$

Milne - Thompson method:

1. If $u(x,y)$ is given

   Take $f'(z) = u_x - iu_y$

   Replace x by z and y by '0' in $f'(z)$

   Then integrate $f'(z)$ with respect to z.

2. If $v(x,y)$ is given

   Take $f'(z) = v_y + iv_x$

   Replace x by z and y by '0' in $f'(z)$

   Then integrate $f'(z)$ with respect to z.

(GATE-05 PI): The function $\omega = u+iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$ is not analytic at the point

✓(a) $(0,0)$ (b) $(0,1)$ (c) $(1,0)$ (d) $(2,\infty)$

Sol: $\omega = \log(x+iy)$ log is not defined at origin

(GATE-07 CE): potential function ϕ is given as $\phi = x^2-y^2$. what will be the stream function ψ with the condition $\psi = 0$ at $x=0, y=0$?

Sol: Given $\phi(x,y) = x^2-y^2$

Take $d\psi = \psi_x\,dx + \psi_y\cdot dy$