Linear Algebra

Gate Notes

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LINEAR ALGEBRA

Matrix: A matrix is a rectangular array of numbers or functions arranged in 'm' rows and 'n' columns such that each row has same no. of elements (n) and each column has same no. of elements (m).

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}.$$

a_{ij} denotes element in i^{th} row and j^{th} column.

Row matrix: Matrix having single row. Ex: $[2 \ 4 \ 6 \ 8]$

Column matrix: Matrix having single column. Ex: $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

Square matrix: Matrix having equal no. of rows and columns.

Ex: $\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$

* The elements along diagnol of a square matrix (a_{ij}, $i=j$) are called leading or principle diagonal elements.

* The sum of diagonal elements of a square matrix A is called Trace of A.

Diagonal matrix: A square matrix, except the leading diagonal elements are equal to zero is called diagonal matrix.

Ex: $\begin{bmatrix} 9 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Scalar matrix: A diagonal matrix, whose leading diagonal elements are equal is called Scalar matrix

Ex: $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Unit matrix: A diagonal matrix, whose leading diagonal elements all equal to '1' is called unit matrix.

Ex: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

Null matrix: All the elements of matrix are zero Ex: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Symmetic matrix: A square matrix with $a_{ij} = a_{ji}$ for all i and j.

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

Skew symmetric matrix: A square matrix with $a_{ij} = -a_{ji}$ for all i & j

Ex: $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

* For a skew symmetrix matrix the leading diagonal Elements all are equal to zero.

Matrix transpose: Intechanging of elements in rows with the corresponding elements in the columns. resulting matrix is denoted by 'A^T' or 'A^1'

Ex: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{m\times n} \Rightarrow A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}_{n\times m}$

* If $A = A^T$ then it is symmetric * $(A^T)^T = A$
* If $A = -A^T$ then it is skew - symmetric. * $(AB)^T = B^T . A^T$
* Every given square matrix can be expressed as sum of symmetric & skew-symmetric matrices.

$$A = \underbrace{\frac{1}{2}(A+A^T)}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(A-A^T)}_{\text{Skew Symmetric}}$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$\frac{1}{2}(A+A^T) = \underbrace{\begin{bmatrix} 1 & 5/2 \\ 5/2 & 4 \end{bmatrix}}_{\text{Symmetric}}$ $\frac{1}{2}(A-A^T) = \underbrace{\begin{bmatrix} 0 & -1/2 \\ +1/2 & 0 \end{bmatrix}}_{\text{Skew-Symmetric}}$

Triangular matrix:

* A square matrix, whose elements below the leading diagonal are zero is called upper triangular matrix.

Ex: $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ upper triangular.

* A square matrix whose elements above the leading diagonal are equal to zero is called lower triangular matrix.

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ Lower triangular

Multiplication by scalar: $K\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} Ka_1 & Kb_1 & Kc_1 \\ Ka_2 & Kb_2 & Kc_2 \end{bmatrix}$

Multiplication of two matrices:

$$[A]_{m\times n} \cdot [B]_{n\times p} = [AB]_{m\times p}$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \\ n_1 & n_2 \end{bmatrix} = \begin{bmatrix} a_1l_1+b_1m_1+c_1n_1 & a_1l_2+b_1m_2+c_1n_2 \\ a_2l_1+b_2m_1+c_2n_1 & a_2l_2+b_2m_2+c_2n_2 \\ a_3l_1+b_3m_1+c_3n_1 & a_3l_2+b_3m_2+c_3n_2 \end{bmatrix}$$

* Multiplication is possible only if no. of columns in first matrix is equal to no. of rows in second matrix.
* $AB \neq BA$.
* $(AB)^T = B^T \cdot A^T$

Determinant of matrix:

* For 1×1 matrix, the number itself is determinent.
* For 2×2 matrix of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the determinent is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Minor: Minor of an element is the determinent obtained by deleting the row and column in which the element is present.

Ex: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow$ a_{11} minor is $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

a_{23} minor is $\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$

Cofactor: Co.factor of any element 'a_{ij}' in a matrix is equal to $(-1)^{i+j}\cdot m_{ij}$. where 'm_{ij}' is minor of 'a_{ij}'.

Ex: $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then co.factor of b_3 i.e $B_3 = (-1)^{3+2}\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Determinant: Determinant of a matrix is defined as sum of product of elements of any row or column with corresponding co.factor.

$$\Delta = a_1 A_1 + b_1 B_1 + c_1 C_1$$
$$= a_1 (-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1(-1)^{1+2}\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1(-1)^{1+3}\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$
$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Tips for GATE:

* While calculating determinant, always select a row or column with more number of elements equal to '0'

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 5 & 0 & 6 \end{vmatrix} = 4[6-15] = -36$ [select 2nd row]

* Try to remember co.factor signs $(-1)^{i+j}$ for each element as

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}_{3\times3} \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}_{4\times4}$$

Properties of determinants:

1. Determinant remains unaltered by changing its rows into columns and columns into rows. $|A| = |A^T|$
2. If two parellel lines of determinant are interchanged, then the sign of determinant changes (same numerical value)
3. If two parellel lines are identical then det = 0

   If all the elements in a row or column are zeros then det = 0

* 4. If each element of a row or column is multiplied by same factor, then determinant also multiplied by same factor.

$$\begin{vmatrix} a_1 & Kb_1 & c_1 \\ a_2 & Kb_2 & c_2 \\ a_3 & Kb_3 & c_3 \end{vmatrix} = K\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ $|KA| = K^n |A|$

5. The determinant of upper or lower triangular matrix is equal to product of leading diagonal elements.

Ex: $\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 1[18-0] = 18$
$= \underbrace{1 \times 3 \times 6}_{\text{diagonal elements.}}$

6. If the elements of determinant Δ are function of 'x' and if 'K' parellel lines becomes equal when $x=a$ then $(x-a)^{K-1}$ is a factor of Δ.

Ex: $\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = (a-b)(a-c)(b-c)$ by putting $a=b, a=c, b=c$.

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7. Product of determinants:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} a_1l_1+b_1m_1+c_1n_1 & a_1l_2+b_1m_2+c_1n_2 & a_1l_3+b_1m_3+c_1n_3 \\ a_2l_1+b_2m_1+c_2n_1 & a_2l_2+b_2m_2+c_2n_2 & a_2l_3+b_2m_3+c_2n_3 \\ a_3l_1+b_3m_1+c_3n_1 & a_3l_2+b_3m_2+c_3n_2 & a_3l_3+b_3m_3+c_3n_3 \end{vmatrix}$$

8. $|\text{co.factor matrix of } A| = |A|^{n-1}$ n = order

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ co.factor matrix $= \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

$|A| = 4-6 = -2$ $|\text{co.factor matrix}| = 4-6 = -2$

9. If $|A| = 0$, then A is called Singular matrix.

10. If $|A| \neq 0$, then A is called non-Singular matrix.

<u>Adjoint matrix</u>: Transpose of the co.factor matrix.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ co.factor matrix $= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$\text{Adj}[A] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

* $A(\text{Adj } A) = (\det A) I$
* $\det(\text{Adj } A) = (\det A)^{n-1}$ $\quad n = \text{order}$

$|\text{adj} A| = \big||A|. A^{-1}\big|$
$= |A|^n |A^{-1}|$
$= |A|^n |A|^{-1} = |A|^{n-1}$

<u>Inverse of square matrix:</u> A matrix "B" is said to be inverse of a non-singular matrix "A" if $AB = BA = I$.

$$A^{-1} = \frac{\text{Adj}[A]}{|A|} \qquad A \cdot A^{-1} = I$$

$(\text{Adj}[A])^{-1} = \frac{A}{|A|}$

Ex: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

* $\text{adj}(\text{adj}[A]) = |A|^{n-2} A$

we know
$A.\text{adj}(A) = |A|. I$
replace A by adj(A)
$\text{adj} A.[\text{adj}(\text{adj} A)] = |\text{Adj} A|.I$
$A^{-1}|A| [\text{adj}(\text{adj} A)] = |\text{Adj} A|. I$
$\text{adj}(\text{adj} A) = \frac{|A|^{n-1}}{|A|}. A$
$= |A|^{n-2}. A$

<u>Properties:</u>

* $(AB)^{-1} = B^{-1}. A^{-1}$
* If $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} 1/d_1 & 0 & 0 \\ 0 & 1/d_2 & 0 \\ 0 & 0 & 1/d_3 \end{bmatrix}$
* If a non-singular matrix A is symmetric then A^{-1} is also symmetric.

* Every odd order symmetric matrix is singular i.e $|A| = 0 \Rightarrow A^{-1}$ does not exist for that.

<u>Problems</u>

1. consider matrix $X_{4\times3}$, $Y_{4\times3}$, $P_{2\times3}$ then order of $[(P.(X^TY)^{-1})P^T]^T$

$$\left[\left(P_{2\times3} \cdot (X^T_{4\times3} \cdot Y_{4\times3})^{-1}\right). P^T_{2\times3}\right]^T = \left[\left(P_{2\times3} \cdot (X_{3\times4} \cdot Y_{4\times3})^{-1}\right) P_{3\times2}\right]^T$$
$$= \left[(P_{2\times3} \cdot (3\times3)^{-1}. P_{3\times2}\right]^T$$
$$= \left[P_{2\times3} \cdot P_{3\times2}\right]^T = (2\times2)^T = (2\times2)$$

Tip for GATE:

Determinant calculation [for matrix with less no. of zero elements]

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, (Applicable for 3x3)

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

$$|A| = a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12} a_{23}a_{31}$$
$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

2. The following represents equation of straight line $\begin{vmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$ The line passes through ?

a) (0, 0) b) (3, 4) ✓ c) (4, 3) d) (4, 4)

$$x(8) - 2(y) + 4(y-8) = 0$$

$$4x + 4 = 16 \qquad \text{option b} ✓$$

3. If $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$ then top row of A^{-1} is ?

$|A| = 1 \qquad A^{-1} = \dfrac{\text{Adj } A}{|A|} = \dfrac{(\text{co. factor})^T}{|A|}$

co. factors of 1st column $= \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

1st row of $A^{-1} = [5 \;\; -3 \;\; 1]$

4. If $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 0.5 & a \\ 0 & b \end{bmatrix}$ then $a+b$?

$A \cdot A^{-1} = I \Rightarrow \begin{bmatrix} 1 & 2a - 0.1b \\ 0 & 3b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$3b = 1 \Rightarrow b = 1/3$ & $2a = 0.1/3 \Rightarrow a = 1/60 \Rightarrow (a+b) = \frac{7}{20}$

5. If $A = (a_{ij})_{m \times n}$ such that $a_{ij} = i + j$, $\forall i, j$; then sum of the elements of A is

$$A = \begin{bmatrix} 1+1 & 1+2 & 1+3 & \cdots & 1+n \\ 2+1 & 2+2 & 2+3 & \cdots & 2+n \\ \vdots & \vdots & \vdots & & \vdots \\ m+1 & m+2 & m+3 & \cdots & m+n \end{bmatrix}_{m \times n}$$

$$1 \times m \quad 2 \times m \quad 3 \times m \quad \cdots \quad n \times m \rightarrow n \text{ terms}$$

$$\frac{m(m+1)}{2} \quad \frac{m(m+1)}{2} \quad \frac{m(m+1)}{2} \quad \cdots \quad \frac{m(m+1)}{2} \rightarrow n \text{ terms}$$

$$\text{Sum of elements} = n \times \frac{m(m+1)}{2} + [m + 2m + 3m + \cdots + nm]$$

$$= \frac{mn(m+1)}{2} + m[1 + 2 + 3 + \cdots + n]$$

$$= \frac{mn(m+1)}{2} + \frac{mn(n+1)}{2}$$

$$= \frac{mn}{2}(m+n+2)$$

6. If $A=(a_{ij})_{3\times3}$, $B=(b_{ij})_{3\times3}$ such that $b_{ij}=2^{i+j}\cdot a_{ij}\ \forall\ i,j$; If $|A|=2$ then $|B|=?$ (a) 2^{10} (b) 2^{11} (c) 2^{12} (d) 2^{13} ✓

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 2$$

$$|B| = \begin{vmatrix} 2^2a_{11} & 2^3a_{12} & 2^4a_{13} \\ 2^3a_{21} & 2^4a_{22} & 2^5a_{23} \\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33} \end{vmatrix} = 2^2\cdot2^3\cdot2^4 \begin{vmatrix} a_{11} & 2a_{12} & 2^2a_{13} \\ a_{21} & 2a_{22} & 2^2a_{23} \\ a_{31} & 2a_{32} & 2^2a_{33} \end{vmatrix}$$

$$= 2^9\cdot2\cdot2^2\cdot|A| = 2^{13}$$

7. If $A=(a_{ij})_{n\times n}$ such that $a_{ij}=i^2-j^2\ \forall\ i,j$. Then find sum of all the elements of A?

$[a_{ij}=i^n-j^n]$

$$A = \begin{bmatrix} 0 & -3 & -8 & \cdots & (1^2-n^2) \\ 3 & 0 & -5 & \cdots & (2^2-n^2) \\ 8 & 5 & 0 & \cdots & (3^2-n^2) \\ \vdots & \vdots & \vdots & & \\ (n^2-1^2) & (n^2-2^2) & (n^2-3^2) & \cdots & 0 \end{bmatrix}_{n\times n}$$

Above matrix is a skew symmetric matrix. It has all the diagonal elements equal to zero. When added up, non diagonal elements cancel out each other, resulting in final sum = 0. [always]

8. The value of $\begin{vmatrix} 1+b & b & 1 \\ b & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix} = ?$

(or) Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ then find Δ with 3^{rd} column

$C_1 \to C_1 + C_2$

$$= \begin{vmatrix} 1+2b & b & 1 \\ 1+2b & 1+b & 1 \\ 1+2b & 2b & 1 \end{vmatrix}$$

$$= \underbrace{\begin{vmatrix} 1 & b & 1 \\ 1 & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix}}_{\text{same lines}} + \begin{vmatrix} 2b & b & 1 \\ 2b & 1+b & 1 \\ 2b & 2b & 1 \end{vmatrix}$$

$$= 0 + 2b\underbrace{\begin{vmatrix} 1 & b & 1 \\ 1 & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix}}_{\text{same lines}}$$

$= 0$