Differential Equations

Gate Notes

Differential Equations

Differential Equation: Equations involving differential co.efficients are called differential equations.

$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{\partial y}{\partial x}$ ... are called differential co.efficients

Ordinary differential equation: The differential equation in which all the differential co.efficient have reference to a single independent variable.

Ex: 1. $\frac{d^2y}{dt^2} + n^2x = 0$

2. $y = x\frac{dy}{dx} + \frac{x}{dy/dx}$

3. $e^x dx + e^y dy = 0$

partial differential equation: The differential equation in which there are two or more independent variables and partial differential ~~di~~ coefficients with respect to any of them.

Ex: 1. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$

2. $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$

Order of D.E: It is the order of the highest differential co.efficient that occurs in the equation.

Degree of D.E: It is the power of highest differential co.efficient provided the equation is free from fractional powers.

Ex:	order	Degree	
1. $\left[1+(dy/dx)^2\right]^{3/2} = \left(d^2y/dx^2\right)$	2	2	[PI-05]
2. $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0$	2	1	
3. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$	2	1	
4. $\left(\frac{dy}{dx}\right)^{3/2} = \left(\frac{d^2y}{dx^2}\right)^{2/3}$	2	4	

Formation of D.E: Differential equations are formed by the elemination of arbitrary constants / functions from a relation in the variables and constants. From D.E involving 'n' arbitrary constants we get the D.E of n^{th} order.

Prob: Form the D.E by eliminating the arbitrary constants present in $y = ax + bx^2$

$$y = ax + bx^2 \longrightarrow (1)$$

d.w.r. to x $\quad y' = a + 2bx \longrightarrow (2)$

d.w.r. to 'x' $\quad y'' = 2b$

$$b = y''/2 \longrightarrow (3)$$

from (2) & (3) $\quad y' = a + xy''$

$$a = y' - xy'' \longrightarrow (4)$$

from (1) & (4) $\quad y = (y' - xy'')x + \frac{y''}{2}x^2$

$$x^2y'' + 2xy' - 2x^2y'' = 2y$$

$$x^2y'' - 2xy' + 2y = 0$$

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

prob: $\quad y = a + bx$

$$y = a + bx \longrightarrow (1)$$

d.w.r. to x $\quad y' = b \longrightarrow (2)$

d.w.r. to x $\quad y'' = 0$

$$\frac{d^2y}{dx^2} = 0$$

prob: $\quad y = a\cos x + b\sin x \longrightarrow (1)$

$$y' = -a\sin x + b\cos x \longrightarrow (2)$$

$$y'' = -a\cos x - b\sin x \longrightarrow (3)$$

(2) from (1) & (3) $y'' = -y$

$$y'' + y = 0$$

$$\frac{d^2y}{dx^2} + y = 0$$

prob: $y = ae^x + be^{2x}$

$$y' = ae^x + 2be^{2x} = y + be^{2x} \rightarrow (1)$$

$$y'' = y' + 2be^{2x}$$

$$be^{2x} = \frac{y'' - y'}{2} \rightarrow (2)$$

from (1) & (2)

$$y' = y + \frac{y'' - y'}{2}$$

$$3y' = 2y + y'' \Rightarrow y'' + 2y - 3y' = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

prob: $(x-a)^2 + (y-b)^2 = 1 \rightarrow (1)$

D.w.r. to 'x' $\quad 2(x-a) + 2(y-b)y' = 0 \rightarrow (2)$

$$(x-a) + (y-b)y' = 0$$

D.w.r. to 'x' $\quad 1 + (y-b)y'' + (y')^2 = 0$

$$y - b = -\frac{[1+(y')^2]}{y''} \rightarrow (3)$$

from (2) & (3) $\quad x - a = \frac{[1+(y')^2]\,y'}{y''} \rightarrow (4)$

from (1) & (4) $\quad \left[\frac{(1+(y')^2)\,y'}{y''}\right]^2 + \left[-\frac{(1+(y')^2)}{y''}\right]^2 = 1$

$$[1+(y')^2]^2\,[(y')^2 + 1] = (y'')^2$$

$$[1+(y')^2]^3 = (y'')^2$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \left[\frac{d^2y}{dx}\right]^2$$

Solution of D.E:

An Equation in terms of the variables which satisfies the given differential equation is called solution of D.E

Ex: For the D.E $\frac{d^2y}{dx^2}+y=0$, the solution is

$$y = a\cos x + b\sin x$$

also $y = 2\cos x$, $\cos x+\sin x$, $\sin 3x$ are also solutions.

The solution of D.E is of two types

General solution: The solution of D.E which contains as many arbitrary constants as that of the order of D.E is called the general solution.

Particular solution: A solution obtained from general ~~particular~~ solution by giving particular values to arbitrary cons--tants are called particular solutions.

D.E of 1st order and 1st degree

The general form of D.E of 1st order and 1st degree is $\frac{dy}{dx}=f(x,y)$. Depending on the function $f(x,y)$ these equations are divided into 4-types

1. Variable seperable form D.E
2. Homogeneous D.E
3. Linear D.E
4. Exact D.E

1. variable seperable form: If in an equation it is possible to collect all functions of 'x' and 'dx' on one side and all the functions of 'y' and 'dy' on the other side, then the variables are said to be seperable. Thus the general form of such an equation is

$$\psi(y)\,dy = \phi(x)\,dx$$

I.O.B.S $\quad \int \psi(y)\,dy = \int Q(x)\,dx + c$

Prob: $\dfrac{dy}{dx} = \dfrac{y}{x}$

$$\frac{1}{y}\,dy = \frac{1}{x}\,dx$$

I.O.B.S $\quad \log_e y = \log_e x + c \qquad$ Let $c = \log_e k$

$$= \log_e x + \log_e k$$

$$\log_e y = \log_e xk$$

$y = xk$ is the solution.

Prob: $\dfrac{dy}{dx} = 1 + x + y + xy$

$$\frac{dy}{dx} = 1 + x + y(1+x) = (1+x)(1+y)$$

$$\frac{1}{1+y}\,dy = (1+x)\,dx$$

I.O.B.S $\quad \log_e(1+y) = x + \dfrac{x^2}{2} + c$

Prob: $\quad y - x\dfrac{dy}{dx} = a\left[y^2 + \dfrac{dy}{dx}\right]$

$$y - ay^2 = (x+a)\frac{dy}{dx}$$

$$\frac{1}{y-ay^2}\,dy = \frac{1}{x+a}\,dx$$

$$\frac{1}{y(1-ay)}\,dy = \frac{1}{x+a}\,dx$$

$$\left[\frac{a}{1-ay}+\frac{1}{y}\right]dy=\frac{1}{x+a}dx$$

I.O.B.S $\quad -\log_e|1-ay|+\log_e|y| = \log_e|x+a|+c \qquad c=\log_e K$

$$\log_e\frac{|y|}{|1-ay|} = \log_e|x+a|K$$

$$\therefore \quad \frac{|y|}{|1-ay|} = |x+a|K$$

$$\frac{y}{(1-ay)} = (x+a)K$$

Prob: $(x+y+1)^2\dfrac{dy}{dx} = 1 \longrightarrow (1)$

put $\quad x+y+1 = t \quad$ say

$$1+\frac{dy}{dx} = \frac{dt}{dx}$$

from (1) $\quad t^2\left(\dfrac{dt}{dx}-1\right) = 1$

$$\frac{dt}{dx} = \frac{1+t^2}{t^2} \Rightarrow \frac{t^2}{1+t^2}dt = dx$$

$$\Rightarrow \left[1-\frac{1}{1+t^2}\right]dt = dx$$

I.O.B.S

$$t-\tan^{-1}t = x+c$$

$$(x+y+1)-\tan^{-1}(x+y+1) = x+c$$

Prob: $\dfrac{dy}{dx} = e^{3x-2y}+x^2e^{-2y}$

$$\frac{dy}{dx} = e^{3x}\cdot e^{-2y}+x^2\cdot e^{-2y}$$

$$= (x^2+e^{3x})\cdot e^{-2y}$$

$$e^{2y}dy = (x^2+e^{3x})dx$$

I.O.B.S $\quad \dfrac{e^{2y}}{2} = \dfrac{x^3}{3}+\dfrac{e^{3x}}{3}+c$

$$3e^{2y} = 2(e^{3x}+x^3)+6c$$

2. Homogeneous D.E: D.E's of the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$ where f_1, f_2 are homogeneous functions of same degree are called homogeneous D.E's

To solve these equations we take substitution

$$y = vx \quad \text{so that} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

(or)

$$x = vy \quad \text{so that} \quad \frac{dx}{dy} = v + y\frac{dv}{dy}$$

Later, seperate the variables and integrate.

Prob: solve $(x^2-y^2)dx - xy\,dy = 0$

$$\frac{dy}{dx} = \frac{x^2-y^2}{xy} \quad \text{Homogeneous in } x, \& \ y$$

put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2 - v^2x^2}{x^2v} = \frac{1-v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1-2v^2}{v}$$

$$\frac{v}{1-2v^2}\,dv = \frac{dx}{x}$$

I.O.B.S $\quad -\frac{1}{4}\int \frac{4v}{1-2v^2}\,dv = \int \frac{1}{x}dx + c$

$$-\frac{1}{4}\log(1-2v^2) = \log x + c$$

$$4\log x + \log(1-2v^2) = -4c$$

$$x^4 \cdot (1-2v^2) = e^{-4c}$$

$$x^4\left(1-2\frac{y^2}{x^2}\right) = K$$

$$x^2(x^2-2y^2) = K$$

Prob: solve $(e^{x/y}+1)dx + e^{x/y}(1-x/y)dy = 0$

$$\frac{dx}{dy} = -\frac{e^{x/y}\left(1-\frac{x}{y}\right)}{e^{x/y}+1} \text{ is a homogeneous eq}^n$$

put $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

$$v + y\frac{dv}{dy} = -\frac{e^{v}(1-v)}{e^{v}+1}$$

$$y\frac{dv}{dy} = -\frac{e^{v}(1-v)}{e^{v}+1} - v = -\frac{v+e^{v}}{1+e^{v}}$$

$$-\frac{dy}{y} = \frac{1+e^{v}}{v+e^{v}}dv$$

I.O.B.S $\quad -\log y = \log_e(v+e^{v}) + c$

$$\log y + \log(v+e^{v}) = -c$$

$$y(v+e^{v}) = e^{-c} = k \text{ say}$$

$$y\left(\frac{x}{y} + e^{x/y}\right) = k$$

$$x + ye^{x/y} = k$$

Prob: solve $\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx - x\sec^2(y/x)dy = 0$

$$\frac{dy}{dx} = \left(\frac{y}{x}\sec^2\frac{y}{x} - \tan\frac{y}{x}\right)\cos^2\left(\frac{y}{x}\right)$$

put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = (v\sec^2 v - \tan v)\cos^2 v$$

$$x\frac{dv}{dx} = -\tan v \cdot \cos^2 v = \frac{-\tan v}{\sec^2 v}$$

$$\frac{\sec^2 v}{\tan v}dv = -\frac{1}{x}dx$$

$$\log(\tan v) = -\log x + c$$

$$x\tan v = k$$

$$x\tan(y/x) = k$$

⑤ Equations reducible to homogeneous form:

The equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ can be reduced to homogeneous form as follows

Case I: when $\frac{a}{a'} \neq \frac{b}{b'}$

put $x = X+h \qquad y = Y+k$

$dx = dX \qquad dy = dY$

$$\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')} \rightarrow (1)$$

Solve $\left.\begin{array}{l} ah+bk+c=0 \\ a'h+b'k+c'=0 \end{array}\right\}$ To get h & k

So that (1) becomes

$$\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y}$$ is a homogeneous in X & Y

put $Y = vX$ or $X = vY$

solve it

Case II: when $\frac{a}{a'} = \frac{b}{b'}$

Let $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m}$ say

$$\therefore \frac{dy}{dx} = \frac{(ax+by)+c}{m(ax+by)+c'} \rightarrow ①$$

put $ax+by = t$

D.W.r.to 'x' $\frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$

from ① $\frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t+c}{mt+c'}$

Sepa~~te~~rate variables 'x' and 't'

solve it.

3. Linear D.E: A D.E is said to be linear if the dependant variable and it's differential co.efficients occur only in the first degree and not multiplied together.

Prob: Which of the following is a linear D.E?

(a) $\frac{d^2y}{dx^2}+\underline{\left(\frac{dy}{dx}\right)^2}+y=0$ (b) $\frac{d^2y}{dx^2}+\underline{y\frac{dy}{dx}}+y=0$

(c) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+\underline{y^2}=0$ ✓(d) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$

The standard form of a 1st order linear D.E (Leibnitz linear Equation) is (CE-97)

$$\frac{dy}{dx}+Py=Q$$ where P, Q are function of 'x' or 'constants'

It's solution is $y(I.F)=\int Q(I.F)dx+C$

where $I.F=e^{\int P dx}$

Prob: solve $(x+1)\frac{dy}{dx}-y=e^{3x}(1+x)^2$

$$\frac{dy}{dx}-\frac{1}{(x+1)}y=e^{3x}(1+x)$$

$$P=\frac{-1}{x+1} \qquad Q=e^{3x}(1+x)$$

$$I.F=e^{\int -\frac{1}{x+1}dx}=e^{-\log_e(1+x)}=\frac{1}{1+x}$$

Solution is $y\cdot\frac{1}{x+1}=\int \frac{e^{3x}(1+x)}{1+x}dx+C$

$$\frac{y}{x+1}=\frac{e^{3x}}{3}+C$$

Prob: Solve $y(\log y)dx+(x-\log y)dy=0$

$$\frac{dx}{dy}+\frac{x}{y\log y}=\frac{1}{y}$$

$$P=\frac{1/y}{\log y} \qquad Q=\frac{1}{y}$$

$$I.F = e^{\int \frac{1/y}{\log y} dy} = e^{\log(\log y)} = \log y$$

solution is $x \log y = \int \frac{1}{y} \log y \, dy + c$

$$x \log y = \frac{1}{2} (\log y)^2 + c$$

$$x = \frac{1}{2} \log y + c.(\log y)^{-1}$$

Prob: $\cos^2 x \frac{dy}{dx} + y = \tan x$

$$\frac{dy}{dx} + \frac{y}{\cos^2 x} = \tan x \cdot \sec^2 x$$

$$I.F = e^{\int \sec^2 x \, dx} = e^{\tan x}$$

solution is $y(e^{\tan x}) = \int \tan x \cdot \sec^2 x \cdot e^{\tan x} dx + c$

$$= \int t e^t dt + c \qquad \tan x = t$$
$$\sec^2 dx = dt$$

$$= te^t - e^t + c$$

$$y(e^{\tan x}) = e^{\tan x}(\tan x - 1) + c$$

Bernaulli's equation:

The equation $\frac{dy}{dx} + Py = Qy^n$ where P, Q are functions of x, is reducible to the Leibnitz's linear equation and is usually called bernaulli's equation.

Divide b.s by $y^n \Rightarrow y^{-n} \frac{dy}{dx} + Py^{1-n} = Q \rightarrow (1)$

Let $y^{1-n} = z \Rightarrow (1-n) y^{-n} \frac{dy}{dx} = \frac{dz}{dx}$

$$y^{-n} \frac{dy}{dx} = \frac{1}{(1-n)} \frac{dz}{dx}$$

from(1) $\frac{1}{(1-n)} \frac{dz}{dx} + Pz = Q$

$$\frac{dz}{dx} + P(1-n)z = Q(1-n) \qquad \text{(CE-05)}$$

which is a Linear D.E in 'z'

Prob: Solve $x\frac{dy}{dx}+y=x^3y^6$

$$y^{-6}\frac{dy}{dx}+\frac{y^{-5}}{x}=x^2 \rightarrow (1)$$

Let $y^{-5}=z \Rightarrow -5y^{-6}\frac{dy}{dz}=\frac{dz}{dx}$

from (1) $$-\frac{1}{5}\frac{dz}{dx}+\frac{z}{x}=x^2$$

$$\frac{dz}{dx}-\frac{5}{x}\cdot z=-5x^2 \quad \text{Leibnitz eqn}$$

$$I.F = e^{-\int\frac{5}{x}dx}=e^{-5\log x}=x^{-5}$$

Solution is $$z\cdot e^{-5}=\int(-5x^2)\cdot x^{-5}dx+c$$

$$z\cdot e^{-5}=-5\cdot\frac{x^{-3+1}}{-3+1}+c$$

$$y^{-5}\cdot e^{-5}=-5\,\frac{x^{-2}}{-2}+c$$

4. Exact D.E: D.E's which can be expressable as an exact differential of some function of x,y are called exact D.E

Ex: $y\,dx+x\,dy=0 \Rightarrow d(xy)=0$

$$\Rightarrow xy=c$$

$\frac{1}{y}dx-\frac{x}{y^2}dy=0 \Rightarrow d\left(\frac{x}{y}\right)=0$

$$\Rightarrow \frac{x}{y}=c$$

In general, for an equation of the form $M\,dx+N\,dy=0$ to be exact, the required condition is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. If it is satisfied then the solution of the equation is

$$\int_{y\ \text{const}} M\,dx+\int(\text{terms of } N \text{ independent of } x)\,dy=c$$

⑦ prob: solve $y\sin 2x\,dx - (1+y^2+\cos^2 x)dy = 0$

$$M = y\sin 2x \qquad N = -(1+y^2+\cos^2 x)$$

$$\frac{\partial M}{\partial y} = \sin 2x \qquad \frac{\partial N}{\partial x} = 2\cos x\cdot\sin x = \sin 2x$$

Solution is $\int\limits_{y\ const} y\sin 2x + \int -(1+y^2)dy + c$

$$-y\frac{\cos 2x}{2} - y - \frac{y^3}{3} = c$$

prob: $(x^2-4xy-2y^2)dx + (y^2-4xy-2x^2)dy = 0$

$$\frac{\partial M}{\partial y} = -4x-4y \qquad \frac{\partial N}{\partial y} = -4x-4y$$

Solution is $\int\limits_{y\ const}(x^2-4xy-2y^2)\,dx + \int y^2 dy = 0$

$$\frac{x^3}{3} - 4y\cdot\frac{x^2}{2} - 2y^2x + \frac{y^3}{3} = c$$

$$x^3 - 6x^2y - 6xy^2 + y^3 = K$$

prob: $(2xy+y-\tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0$

$$\frac{\partial M}{\partial y} = 2x - \sec^2 y + 1 \qquad \frac{\partial N}{\partial x} = 2x - \tan^2 y$$

$$= 2x - \tan^2 y$$

Solution is $\int\limits_{y\ const}(2xy+y-\tan y)dx + \int \sec^2 y\,dx = 0$

$$2y\cdot\frac{x^2}{2} + xy - x\cdot\tan y + \tan y = c$$

Previous problems

prob: (GATE 94): The necessary & sufficient conditions for the D.E of the form $M(x,y)dx + N(x,y)dy$ to be exact is

Sol: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

CE-99: If C is a constant then the solution of $\frac{dy}{dx} = 1+y^2$ is

Sol:
$$\frac{1}{1+y^2} dy = dx$$

IN-08:
Particular Soln
$y = \tan(x+3)$

I.O.B.S $\tan^{-1} y = x + C$

$$y = \tan(x+C)$$

ME-03: The solution of D.E $\frac{dy}{dx} + y^2 = 0$ is

Sol:
$$\frac{1}{y^2} dy = -dx$$

I.O.B.S $-\frac{1}{y} = -x + C$

$$\frac{1}{y} = x - C \Rightarrow y = \frac{1}{x+K} \quad (K = -C)$$

CE-04: Bio transformation of an organic compound having concentration (x) can be modeled using an ordinary differential eqn $\frac{dx}{dt} + Kx^2 = 0$, where K is the reaction constant. If $x = a$ at $t = 0$ then solution of the eqn is

Sol:
$$\frac{dx}{dt} = -Kx^2$$

$$-\frac{dx}{x^2} = K\,dt$$

I.O.B.S $+\frac{1}{x} = +Kt + C \quad \rightarrow ①$

Given $x = a$ at $t = 0 \Rightarrow \frac{1}{a} = C$

$\therefore$ Soln is $\frac{1}{x} = Kt + \frac{1}{a}$

EE-05: The solution of 1st order D.E $\dot{x}(t) = -3x(t)$, $x(0) = x_0$ is

EC-08: $x(0) = 2$

Sol:
$$\frac{dx}{dt} = -3x$$

$$\frac{1}{x} dx = -3\,dt$$

I.O.B.S $\log x = -3t + C$

$$x = e^{-3t} e^{C} \rightarrow ①$$

Given $x(0) = x_0$

from ① $x_0 = e^0 \cdot e^C$

$e^C = x_0$

$\therefore$ Solution is $x = e^{-3t} x_0$

$$I.F = e^{\int \frac{1}{x}dx} = e^{\log_e x} = x$$

Solution is $y.x = \int x^3.x\,dx + c$

$$y.x = \frac{x^5}{5} + c$$

Given $y(1) = \frac{6}{5} \Rightarrow \frac{6}{5} = \frac{1}{5} + c \Rightarrow c = 1$

$\therefore$ Solution is $xy = \frac{x^5}{5} + 1$

$$y = \frac{x^4}{5} + \frac{1}{x}$$

PI-10: The solution of the D.E $\frac{dy}{dx} - y^2 = 1$ satisfying the condition $y(0) = 1$ is

Sol:

$$\frac{dy}{dx} = 1 + y^2$$

$$\int \frac{1}{1+y^2}dy = \int dx + c$$

$$\tan^{-1}y = x + c$$

Given $y(0) = 1 \Rightarrow \tan^{-1}(1) = 0 + c \Rightarrow c = \pi/4$

$\therefore$ solution is $\tan^{-1}y = x + \pi/4$

$$y = \tan(x + \pi/4)$$

PI-10: Which of the following D.E has a solution given by the function $y = 5\sin(3x + \pi/3)$

Sol:

$$y = 5\sin 3x.\cos\pi/3 + 5\cos 3x.\sin\pi/3$$

$$= \underbrace{5\cos\pi/3}_{a}.\sin 3x + \underbrace{5\sin\pi/3}_{b}\cos 3x$$

$y = a\sin 3x + b\cos 3x$ a&b are arbitrary constants

$$y' = 3a\cos 3x - 3b\sin 3x$$

$$y'' = -9a\sin 3x - 9b\cos 3x$$

$$= -9y$$

$$y'' + 9y = 0$$

(06 - ME): The solution of D.E $\frac{dy}{dx} + 2xy = e^{-x^2}$ with $y(0) = 1$ is

Sol:

$$P = 2x \qquad Q = e^{-x^2}$$

$$I.F = e^{\int P dx} = e^{x^2}$$

Solution is $y(e^{x^2}) = \int e^{-x^2} \cdot e^{x^2} dx + C$

$$y \cdot e^{x^2} = x + C$$

Given $y(0) = 1 \Rightarrow 1 \cdot e^0 = 0 + C \Rightarrow C = 1$

$$\therefore y e^{x^2} = x + 1$$

$$y = (x+1) e^{-x^2}$$

(CE-07): The solution of D.E $\frac{dy}{dx} = x^2 y$ with the condition that $y = 1$ at $x = 0$ is

Sol:

$$\frac{1}{y} dy = x^2 dx$$

I.O.B.S $\quad \log y = \frac{x^3}{3} + C$

$$y = e^{x^3/3} \cdot e^C$$

Given $y = 1$ at $x = 0 \Rightarrow 1 = e^0 \cdot e^C \Rightarrow e^C = 1$

$\therefore$ solution is $y = e^{x^3/3}$

(CE-09): Solution of D.E $3y \frac{dy}{dx} + 2x = 0$ represents a family of

Sol:

$$3y\, dy + 2x\, dx = 0$$

I.O.B.S $\quad 3 \frac{y^2}{2} + 2 \frac{x^2}{2} = C$

$$\frac{x^2}{1} + \frac{y^2}{(2/3)} = C$$

Represents a family of ellipses

(ME-09): The solution of $x \frac{dy}{dx} + y = x^4$ with $y(1) = \frac{6}{5}$ is

Sol:

$$\frac{dy}{dx} + \frac{1}{x} \cdot y = x^3$$

9 (IN-10): consider the differential equation $\frac{dy}{dx} + y = e^x$ with $y(0) = 1$. Then the value of $y(1)$ is

Sol: $I.F = e^{\int dx} = e^x$

Solution is $y \cdot e^x = \int e^x \cdot e^x dx + C$

$$ye^x = \frac{e^{2x}}{2} + C$$

Given $y(0) = 1 \Rightarrow 1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$

$$\therefore ye^x = \frac{e^{2x}}{2} + \frac{1}{2}$$

$$y = \frac{e^x}{2} + \frac{1}{2}e^{-x} = \frac{1}{2}(e^x + e^{-x})$$

$$y(1) = \frac{e^1 + e^{-1}}{2}$$

(EE-11): With K as constant, the possible solution for the 1st order D.E $\frac{dy}{dx} = e^{-3x}$ is

Sol: $dy = e^{-3x} dx$

I.O.B.S $\quad y = \frac{e^{-3x}}{-3} + K$

(EC-11): The solution of the D.E $\frac{dy}{dx} = Ky$, $y(0) = C$ is

Sol: $\frac{1}{y} dy = K dx$

$$\log y = Kx + C_1$$

$$y = e^{Kx} \cdot e^{C_1}$$

Given $y(0) = C \Rightarrow C = e^{C_1}$

$$\therefore y = e^{Kx} \cdot C$$

(ME-11): Conside the D.E, $\frac{dy}{dx} = (1+y^2)x$. The general solution with constant 'c' is

Sol: $\frac{1}{1+y^2} dy = x\, dx$

I.O.B.S $\quad \tan^{-1} y = \frac{x^2}{2} + C \Rightarrow y = \tan\left(\frac{x^2}{2} + C\right)$

(CE-11): The solution of D.E $\frac{dy}{dx}+\frac{y}{x}=x$ with condition that $y=1$ at $x=1$ is

Sol:
$$I.F = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

solution is $y.x = \int x.x\,dx + C$

$$xy = \frac{x^3}{3}+C$$

Given $y=1$ at $x=1$

$$1 = \frac{1}{3}+C \Rightarrow C=\frac{2}{3}$$

$$\therefore xy = \frac{x^3}{3}+\frac{2}{3} \Rightarrow y = \frac{x^2}{3}+\frac{2}{3x}$$

EC,EE, IN-12: With initial condition $x(1)=0.5$, the solution of the D.E $t\frac{dx}{dt}+x=t$ is

Sol:
$$\frac{dx}{dt}+\frac{x}{t}=1$$

$$I.F = e^{\int \frac{1}{t}dt} = e^{\log_e t} = t$$

solution is $x.t = \int t.1\,dt + C$

$$xt = \frac{t^2}{2}+C$$

Given $x(1)=0.5 \Rightarrow 0.5(1) = \frac{1}{2}(1)+C \Rightarrow C=0$

$$\therefore xt = \frac{t^2}{2} \Rightarrow x = \frac{t}{2}$$

(CE-12): The solution of O.D.E $\frac{dy}{dx}+2y=0$ for the boundary condition, $y=5$ at $x=1$ is

Sol:
$$I.F = e^{\int P dx} = e^{2x}$$

Sol^n is $y.e^{2x} = \int 0\,dx + C$

$$y = C.e^{-2x}$$

Given $y=5$ at $x=1 \Rightarrow 5 = Ce^{-2}$

$$C = \frac{5}{e^2}$$

$$y = \frac{5}{e^2}e^{-2x}$$

(10) Q (EC-13): A system described by a linear, constant co-efficient ordinary, first ODE has an exact solution given by $y(t)$ for $t>0$, when the forcing function is $x(t)$ and the initial condition is $y(0)$. If one wishes to modify the system so that the solution becomes $-2y(t)$ for $t>0$, we need to

Sol: Solution is $y(t) = \frac{1}{I.F}\int \underbrace{x(t)}_{\text{forcing fun}}(I.F)dt + \underbrace{y(0)}_{\text{Initial cond}}$

To get $-2\,y(t)$ multiply on b.s with '-2'

$$-2\,y(t) = \frac{1}{I.F}\int (-2\,x(t))(I.F)\,dt + (-2\,y(0))$$

∴ Forcing function changes to $-2\,x(t)$

Initial condition " " $-2\,y(0)$

(ME-14): The general solution of the D.E $\frac{dy}{dx} = \cos(x+y)$, with 'c' as a constant is

Sol: Let $x+y=t \Rightarrow 1+\frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{dt}{dx} - 1 = \cos t$$

$$\frac{dt}{dx} = 1+\cos t = 2\cos^2\frac{t}{2}$$

$$\frac{1}{2}\sec^2\frac{t}{2}\,dt = dx$$

$$\tan\left(\frac{t}{2}\right) = x+c \Rightarrow \tan\left(\frac{x+y}{2}\right) = x+c$$

(ME-14): The solution of the initial value problem $\frac{dy}{dx} = -2xy$, $y(0)=2$ is

Sol: $\frac{1}{y}dy = -2x\,dx$

$$\log y = -x^2 + c$$

$$y = e^{-x^2}\cdot e^{c}$$

Given $y(0) = 2 \Rightarrow 2 = e^{c}$

$$y = 2e^{-x^2}$$

Higher order linear differential equation:

Linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. The general linear differential equation of n^{th} order is of the form

$$\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+P_ny=X$$

Where $P_1, P_2, \ldots, P_n$ are functions of x or constants

Linear differential equations with constant co.efficients are of the form

$$\frac{d^ny}{dx^n}+K_1\frac{d^{n-1}y}{dx^{n-1}}+K_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+K_ny=X \rightarrow (1)$$

Where $K_1, K_2, \ldots K_n$ are constants.

operator D:

Denote $\frac{d}{dx}=D,\ \frac{d^2}{dx^2}=D^2,\ \frac{d^3}{dx^3}=D^3, \ldots$

from (1) $\quad D^ny+K_1D^{n-1}y+K_2D^{n-2}y+\cdots+K_ny=X$

$$(D^n+K_1D^{n-1}+K_2D^{n-2}+\cdots+K_n)y=X$$

$$f(D)\cdot y=X \rightarrow (2)$$

Where $f(D)=D^n+K_1D^{n-1}+K_2D^{n-2}+\cdots+K_n$

i.e a polynomial in D

The solution of eq (2) is in the form

$y=$ complementing function + particular integral

$= C.F + P.I$

Complementary function: It is the solution of a differential equation of the form $f(D)y=0$

particular solution: It is the one particular solution of the given equation

Rules for finding complementing function:

In the D.E $f(D)y=0$ the equation $f(D)=0$ is called **auxilary equation (A.E)**. Depending on the roots of A.E, we have following cases for complementing function.

case I: When roots of the A.E are real and distinct say $m_1, m_2, m_3, \ldots, m_n$

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

case II: When roots of the A.E are real and repeated say $m_1, m_1, m_1, m_4, m_5, \ldots, m_n$

$$C.F = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + c_5 e^{m_5 x} + \cdots + c_n e^{m_n x}$$

case III: When roots of the A.E are complex and distinct say $\alpha + i\beta, m_3, m_4, \ldots, m_n$

$$C.F = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x] + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$$

case IV: when roots of A.E are complex and repeated say $\alpha \pm i\beta, \alpha \pm i\beta, m_5, m_6, \ldots, m_n$

$$C.F = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \cdots + c_n e^{m_n x}$$

prob: Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

Sol: A.E $\Rightarrow D^2 + 5D + 6 = 0$

$$(D+2)(D+3) = 0$$

$$D = -2, -3$$

C.F is $y = c_1 e^{-2x} + c_2 e^{-3x}$

prob: Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

Sol: A.E $\Rightarrow D^2 + 2D + 1 = 0$

$$(D+1)^2=0 \Rightarrow D=-1,-1.$$

$$C.F = (C_1+C_2x)e^{-x}$$

prob: solve D.E $\frac{d^2y}{dx^2}+\frac{dy}{dx}+1=0$

Sol: $D^2+D+1=0 \Rightarrow D=\frac{-1}{2}\pm i\frac{\sqrt{3}}{2}$

$$C.F = e^{-\frac{1}{2}x}\left[C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x\right]$$

prob: solve $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+3x=0$

Sol: A.E $D^2+2D+3=0 \Rightarrow D=-1\pm\sqrt{2}i$

$$C.F\ x=e^{-t}[C_1\cos\sqrt{2}t+C_2\sin\sqrt{2}t]$$

prob: solve $\frac{d^3y}{dx^3}+y=0$

Sol: A.E $D^3+1=0 \Rightarrow (D+1)(D^2-D+1)=0$

$$D=-1,\ \frac{-1}{2}\pm\frac{\sqrt{3}}{2}i$$

$$C.F = e^{-\frac{1}{2}x}\left[C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x\right]+C_3e^{-x}$$

prob: solve $\frac{d^3x}{dt^3}-6\frac{d^2x}{dt^2}+11\frac{dx}{dt}-6x=0$

Sol: A.E $D^3-6D^2+11D-6=0$

$$\begin{array}{r|rrrr} & 1 & -6 & 11 & -6 \\ 1 & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$(D-1)(D^2-5D+6)=0$$

$$D=1,2,3$$

$$C.F\quad x=C_1e^{t}+C_2e^{2t}+C_3e^{3t}$$

Inverse operator: An operator $\frac{1}{f(D)}$ which is operating with $f(D)\,y$ gives 'y'. Here $f(D)$ & $\frac{1}{f(D)}$ are called inverse operators to each other.

For the D.E $f(D).y=x$, the particular Integral given by $P.I=\frac{1}{f(D)}x$

Depending on the function 'x' we have the following cases.