# Vector Calculus

Gate Notes

# Vector calculus

| vector differentials | vector integrals |
|---|---|
| 1. Gradiant | 1. Line integral |
| 2. Divergence | 2. Surface integral |
| 3. curl | 3. volume integral |

Scalar function: A Scalar function $f(x,y,z)$ is a function defined at each point in a certain domain D in space. Its value is real and depends on the point $p(x,y,z)$ in space, but not on any particular co.ordinate system being used.

Ex: $x^2+y^2+z^2=C$

vector function: A function $\bar{v} = v_1\bar{i}+v_2\bar{j}+v_3\bar{k}$ defined at each point $p \in D$ is called a vector function.

Ex: $\bar{v} = 3x^2yz\,\bar{i} + 3xy^2z\,\bar{j} + 3xyz^2\,\bar{k}$

vector differential operator: It is denoted by '$\nabla$' (del)

In 2-dimension $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y}$

In 3-dimension $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

$$\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \quad \text{(Scalar)}$$

* The equation $\nabla^2\phi = 0$ is called laplace equation. Any function which satisfies laplace eq$^n$ is called <u>Harmonic</u> function

vector operations: $\bar{a} = a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$ , $\bar{b} = b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$

$$\bar{a}\pm\bar{b} = (a_1\pm b_1)\bar{i} + (a_2\pm b_2)\bar{j} + (a_3\pm b_3)\bar{k}$$

$\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$ Scalar product**

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$ vector product**

→ Magnitude of $\bar{a}$ ⇒ $|\bar{a}| = \sqrt{a_1^2+a_2^2+a_3^2}$

→ unit vector along the direction of $\bar{a} = \frac{\bar{a}}{|\bar{a}|}$

→ Angle between two vectors $\bar{a}$ & $\bar{b}$ ⇒ $\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$

→ $[\bar{a}\ \bar{b}\ \bar{c}] = \bar{a}\cdot(\bar{b}\times\bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

→ $\bar{r} = f_1(t)\,\bar{i} + f_2(t)\,\bar{j} + f_3(t)\,\bar{k}$ → position vector

$\frac{d\bar{r}}{dt}$ → velocity / Tangent vector

$\frac{d^2\bar{r}}{dt}$ → Accelaration vector

unit tangent vector = $\frac{(d\bar{r}/dt)}{|d\bar{r}/dt|}$

Gradient of a scalar function: Let $\phi(x,y,z)$ is a scalar function then

$$\text{grad}\,\phi = \nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

* Gradient of a scalar point function is a vector function
* Gradient gives normal to the surface
* unit normal = $\frac{\nabla\phi}{|\nabla\phi|}$

prob: Gradient of $\phi = 3x^2y - y^3z^2$ at $(-1,-1,2)$ is?

sol: $\nabla\phi = \bar{i}[6xy] + \bar{j}[3x^2 - 3y^2z^2] + \bar{k}[-2y^3z]$

$\nabla\phi|_{(1,-1,2)} = -6\bar{i} - 9\bar{j} + 4\bar{k}$ (Normal)

unit normal = $\frac{\nabla\phi}{|\nabla\phi|} = \frac{-6\bar{i}-9\bar{j}+4\bar{k}}{\sqrt{36+81+16}} = \frac{1}{\sqrt{133}}(-6\bar{i}-9\bar{j}+4\bar{k})$

②

prob: $U = x+y+z$ , $V = x^2+y^2+z^2$ , $W = xy+yz+zx$. Find $[\nabla U \; \nabla V \; \nabla W]$

Sol: $\nabla U = \bar{i}+\bar{j}+\bar{k}$

$\nabla V = 2x\,\bar{i} + 2y\,\bar{j} + 2z\,\bar{k}$

$\nabla W = (y+z)\,\bar{i} + (x+z)\,\bar{j} + (x+y)\,\bar{k}$

$$[\nabla U \;\; \nabla V \;\; \nabla W] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix} = 0$$

★ Angle between two curves is the angle between their tangents at the common point

★ Angle between two surfaces is the angle between their normals at the common point

prob: Find angle between two surfaces $x^2+y^2+z^2=9$ & $z=x^2+y^2-3$ at $(2,-1,+2)$

Sol: $\nabla\phi_1 = 2x\,\bar{i} + 2y\,\bar{j} + 2z\,\bar{k}$ $\qquad$ $\nabla\phi_2 = 2x\,\bar{i} + 2y\,\bar{j} - \bar{k}$

$= 4\bar{i} - 2\bar{j} + 4\bar{k}$ $\qquad$ $= 4\bar{i} - 2\bar{j} - \bar{k}$

$\cos\theta$ = Angle b/w normals = $\dfrac{16+4-4}{\sqrt{36}\cdot\sqrt{21}} = \dfrac{8}{3\sqrt{21}}$

properties of Gradient:

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(c_1 f + c_2 g) = c_1 \nabla f + c_2 \nabla g$ ; $c_1, c_2$ arbitrary constants.
3. $\nabla(fg) = f\cdot\nabla g + g\,\nabla f$
4. $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\cdot\nabla f - f\,\nabla g}{g^2}$ , $g \neq 0$

→ Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$|\bar{r}| = \sqrt{x^2+y^2+z^2} = r$

$r^2 = x^2+y^2+z^2$

$2r\,\partial r = 2x\,\partial x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ , $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

1. $\text{Grad } r = \nabla r = \frac{\partial r}{\partial x}\bar{i} + \frac{\partial r}{\partial y}\bar{j} + \frac{\partial r}{\partial z}\bar{k}$

$$= \frac{x}{r}\bar{i} + \frac{y}{r}\bar{j} + \frac{z}{r}\bar{k}$$

$$= \frac{x\bar{i} + y\bar{j} + z\bar{k}}{r} = \frac{\bar{r}}{r}$$

2. $\text{Grad}(1/r) = \nabla(1/r)$

$$= \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\bar{i} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\bar{j} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\bar{k}$$

$$= -\frac{1}{r^2}\cdot\frac{\partial r}{\partial x}\bar{i} - \frac{1}{r^2}\cdot\frac{\partial r}{\partial y}\bar{j} - \frac{1}{r^2}\cdot\frac{\partial r}{\partial z}\bar{k}$$

$$= -\frac{1}{r^2}\cdot\frac{x}{r}\bar{i} - \frac{1}{r^2}\cdot\frac{y}{r}\bar{j} - \frac{1}{r^2}\cdot\frac{z}{r}\cdot\bar{k} = -\frac{\bar{r}}{r^3}$$

3. $\text{Grad}(r^2) = \nabla r^2$

$$= \frac{\partial}{\partial x}(r^2)\bar{i} + \frac{\partial}{\partial y}(r^2)\bar{j} + \frac{\partial}{\partial z}(r^2)\bar{k}$$

$$= 2r.\frac{\partial r}{\partial x}\bar{i} + 2r.\frac{\partial r}{\partial y}\bar{j} + 2r.\frac{\partial r}{\partial z}\bar{k}$$

$$= 2r.\frac{x}{r}\bar{i} + 2.r.\frac{x}{r}\bar{j} + 2r.\frac{x}{r}\bar{k} = 2\bar{r}$$

4. $\text{Grad}(\log r) = \nabla(\log r)$

$$= \frac{\partial}{\partial x}(\log r)\bar{i} + \frac{\partial}{\partial y}(\log r)\bar{j} + \frac{\partial}{\partial z}(\log r)\bar{k}$$

$$= \frac{1}{r}\cdot\frac{\partial r}{\partial x}\bar{i} + \frac{1}{r}\cdot\frac{\partial r}{\partial y}\bar{j} + \frac{1}{r}\cdot\frac{\partial r}{\partial z}\bar{k}$$

$$= \frac{1}{r}\cdot\frac{x}{r}\bar{i} + \frac{1}{r}\cdot\frac{y}{r}\bar{j} + \frac{1}{r}\cdot\frac{z}{r}\cdot\bar{k} = \frac{\bar{r}}{r^2}$$

5. $\text{Grad}(r^n) = \nabla(r^n)$

$$= \frac{\partial}{\partial x}(r^n)\bar{i} + \frac{\partial}{\partial y}(r^n)\bar{j} + \frac{\partial}{\partial z}(r^n)\bar{k}$$

$$= nr^{n-1}.\frac{\partial r}{\partial x}\bar{i} + nr^{n-1}.\frac{\partial r}{\partial y}\bar{j} + nr^{n-1}.\frac{\partial r}{\partial z}\bar{k}$$

$$= nr^{n-1}.\frac{x}{r}\bar{i} + nr^{n-1}.\frac{y}{r}\bar{j} + nr^{n-1}.\frac{z}{r}\bar{k} = nr^{n-2}.\bar{r}$$

6. $\text{Grad}(e^{r^2}) = \nabla(e^{r^2})$

$$= e^{r^2}.2r\left[\frac{\partial r}{\partial x}\bar{i} + \frac{\partial r}{\partial y}\bar{j} + \frac{\partial r}{\partial z}\bar{k}\right]$$

$$= e^{r^2}.2r\left[\frac{x}{r}\bar{i} + \frac{y}{r}\bar{j} + \frac{z}{r}\bar{k}\right] = 2e^{r^2}\bar{r}$$

③

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Directional derivative: D.D of the surface $\phi$ in the direction of $\bar{r}$ is given by

$$D.D = \nabla\phi.\hat{n} \quad \text{where } \hat{n} = \frac{\bar{r}}{|\bar{r}|}$$

→ DD is maximum in the direction of $\nabla\phi$ (normal)

→ The maximum value of DD is $|\nabla\phi|$

prob: Find the DD of $\phi = 2xy + z^2$ in the direction of $\bar{r} = \bar{i} + 2\bar{j} + 2\bar{k}$ at the point $(1,-1,3)$

Sol:
$$\nabla\phi = \bar{i}(2y) + \bar{j}(2x) + \bar{k}(2z)$$
$$\nabla\phi|_{(1,-1,3)} = -2\bar{i} + 2\bar{j} + 6\bar{k}$$
$$\bar{r} = \bar{i} + 2\bar{j} + 2\bar{k} \Rightarrow \hat{n} = \frac{\bar{r}}{|\bar{r}|} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{9}} = \frac{\bar{i}+2\bar{j}+2\bar{k}}{3}$$
$$DD = \nabla\phi.\hat{n} = \frac{-2+4+12}{3} = 14/3$$

prob: Find D.D of $\phi = xyz^2 + xz$ at the point $(1,1,1)$ in the direction of normal to the surface $3xy^2 + y = z$ at $(0,1,1)$

Sol:
$$\nabla\phi = (yz^2+z)\bar{i} + (xz^2)\bar{j} + (2xyz+x)\bar{k}$$
$$\nabla\phi|_{(1,1,1)} = 2\bar{i} + \bar{j} + 3\bar{k}$$
$$\nabla S = (3y^2)\bar{i} + (6xy+1)\bar{j} + (-1)\bar{k}$$
$$\nabla S|_{(0,1,1)} = 3\bar{i} + \bar{j} - \bar{k}$$
$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{3\bar{i}+\bar{j}-\bar{k}}{\sqrt{11}}$$
$$DD = \nabla\phi.\hat{n} = \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

Divergence of vector: Let $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$ be a vector point function then

$$\text{div}(\bar{v}) = \nabla.\bar{v}$$
$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).\left(v_1\bar{i} + v_2\bar{j} + v_3\bar{k}\right)$$
$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

* Divergence of vector point function is a scalar point function.

* Divergence gives rate at which fluid flows out of a unit volume
* If divergence of $\bar{v} = 0$ then $\bar{v}$ is called solenoidal vector.

Curl of vector: Let $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$ be a vector point function then
$$\text{curl}(\bar{v}) = \nabla\times\bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

* curl of vector point function is again a vector point function
* curl gives rotation of the fluid
* If $\text{curl}(\bar{v}) = \bar{0}$ then $\bar{v}$ is called irrotational vector.
* If $\bar{v}$ is linear velocity then angular velocity $\bar{\omega} = \frac{1}{2}[\text{curl}\,\bar{v}]$

prob: If $\bar{v} = x^2yz\,\bar{i} + xy^2z\,\bar{j} + xyz^2\bar{k}$. Find div $\bar{v}$ & curl $\bar{v}$ at $(1,-1,1)$

Sol: $\text{div}\,\bar{v} = \nabla\cdot\bar{v} = 2xyz + 2xyz + 2xyz = 6xyz\Big|_{(1,-1,1)} = -6$

$$\text{curl}\,\bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix} = \bar{i}\,[xz^2 - xy^2] - \bar{j}\,[yz^2 - x^2y] + \bar{k}\,[y^2z - x^2z]$$
$$= \bar{0}$$

properties: (vector Identity)

GATE 96 1. $\text{curl}\,(\text{grad}\,\phi) = \bar{0}$

2. $\text{div}\,(\text{curl}\,\bar{v}) = 0$

3. $\text{div}\,(\nabla\phi) = \nabla^2\phi$

4. $\text{div}\,[\phi\bar{v}] = \phi\,\text{div}(\bar{v}) + \nabla\phi\cdot\bar{v}$

GATE-95 5. $\text{curl}\,[\phi\bar{v}] = \phi\,\text{curl}(\bar{v}) + \nabla\phi\times\bar{v}$

GATE-05 6. $\text{curl}\,[\text{curl}\,\bar{v}] = \nabla(\text{div}\,\bar{v}) - \nabla^2\bar{v}$

Prob: If $\bar{F} = \nabla(2x^3y^2z^4)$; Find div $\bar{F}$; curl $\bar{F}$

Sol: $\text{div}\,\bar{F} = \text{div}\,[\nabla(2x^3y^2z^4)]$
$$= \nabla^2(2x^3y^2z^4) \quad \text{from } (3)$$
$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

$\text{curl}\,\bar{F} = \text{curl}\,[\nabla(2x^3y^2z^4)]$
$$= \bar{0} \quad \text{from } (1)$$

prob: Find a, b, c of $\bar{F} = (x+2y+az)\bar{i} + (bx-3y+z)\bar{j} + (4x+cy+2z)\bar{k}$

If $\bar{F}$ is irrotational.

Sol: $$\text{curl}(\bar{F}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y+z & 4x+cy+2z \end{vmatrix}$$

$$= \bar{i}[c-1] - \bar{j}(4-a) + \bar{k}(b-2)$$

If $\bar{F}$ is irrotational then $\text{curl}(\bar{F}) = \bar{0} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

$$\therefore a = 4, \; b = 2, \; c = 1$$

prob: Find 'a' if $\bar{F} = 2xy\,\bar{i} + 3x^2y\,\bar{j} - 3ayz\,\bar{k}$ is solenoidal at (1,2,3).

Sol: $$\text{div}(\bar{F}) = (2y + 3x^2 - 3ay)\Big|_{(1,2,3)} = 0$$

$$4 + 3 - 6a = 0$$

$$a = 7/6$$

(GATE-93): If the linear velocity $\bar{V}$ is given by $\bar{V} = x^2y\bar{i} + xyz\bar{j} - yz^2\bar{k}$ then the angular velocity $\bar{W}$ at the point (1,1,−1) is

Sol: $$\bar{W} = \frac{1}{2}\text{curl}\,\bar{V} = \frac{1}{2}\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xyz & -yz^2 \end{vmatrix}$$

$$= \frac{1}{2}[\bar{i}(-z^2-xy) - \bar{j}(0-0) + \bar{k}(yz - x^2)]$$

At (1,1,−1) $= \frac{1}{2}[-2\bar{i} - 2\bar{k}] = -(\bar{i} + \bar{k})$

(GATE-94): The directional derivative of $f(x,y) = 2x^2 + 3y^2 + z^2$ at point P(2,1,3) in the direction of the vector $a = \bar{i} - 2\bar{k}$ is

Sol: $$\nabla f = 4x\bar{i} + 6y\bar{j} + 2z\bar{k}$$

$$\nabla f\Big|_{(2,1,3)} = 8\bar{i} + 6\bar{j} + 6\bar{k}$$

$$\hat{n} = \frac{\bar{a}}{|\bar{a}|} = \frac{1}{\sqrt{5}}(\bar{i} - 2\bar{k})$$

$$D.D = \nabla f \cdot \hat{n} = \frac{8-12}{\sqrt{5}} = \frac{-4}{\sqrt{5}}$$

(GATE-95): The derivative of $f(x,y)$ at point $(1,2)$ in the direction of vector $\bar{i}+\bar{j}$ is $2\sqrt{2}$ and in the direction of the vector $-2\bar{j}$ is $-3$. Then the derivative of $f(x,y)$ in the direction $-\bar{i}-2\bar{j}$ is

Sol: Let $\bar{a} = \bar{i}+\bar{j}$

$$(\nabla f)\cdot\frac{\bar{a}}{|\bar{a}|} = 2\sqrt{2} \quad \Rightarrow \left(\frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}\right)\cdot\left(\frac{\bar{i}+\bar{j}}{\sqrt{2}}\right) = 2\sqrt{2}$$

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 4 \longrightarrow ①$$

Let $\bar{b} = -2\bar{j}$

$$(\nabla f)\cdot\frac{\bar{b}}{|\bar{b}|} = -3 \quad \Rightarrow \frac{-2\,\partial f/\partial y}{2} = -3 \quad \Rightarrow \frac{\partial f}{\partial y} = 3 \rightarrow ②$$

from ① $\frac{\partial f}{\partial x} = 1$

$\therefore$ D.D in the direction $-\bar{i}-2\bar{j}$ is $\left(\frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}\right)\cdot\left(\frac{-\bar{i}-2\bar{j}}{\sqrt{5}}\right)$

$$= (\bar{i}+3\bar{j})\cdot\left(\frac{-\bar{i}-2\bar{j}}{\sqrt{5}}\right) = \frac{-7}{\sqrt{5}}$$

(GATE 96): The directional derivative of the function $f(x,y,z) = x+y$ at the point $P(1,1,0)$ along the direction $\bar{i}+\bar{j}$ is

Sol: $$\text{D.D} = \left(\frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}\right)_{(1,1,0)}\cdot\left(\frac{\bar{i}+\bar{j}}{\sqrt{2}}\right)$$

$$= \frac{1+1}{\sqrt{2}} = \sqrt{2}$$

(GATE 99): For the function $\phi = ax^2y - y^3$ to represent the velocity potential of an ideal fluid, $\nabla^2\phi$ should be equal to zero. In that case, the value of 'a' has to be

Sol: $\phi = ax^2y - y^3$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

$$ay - 3y = 0 \quad \Rightarrow a = 3$$

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(GATE-02): The directional derivative of the following function at (1,2) in the direction of $(4\bar{i}+3\bar{j})$ is : $f(x)=x^2+y^2$

Sol: $DD = (\nabla f)_{(1,2)} \cdot \hat{n}$

$$= (2x\bar{i} + 2y\bar{j})_{(1,2)} \cdot \frac{4\bar{i}+3\bar{j}}{\sqrt{25}} = \frac{8+12}{5} = 4$$

(GATE-03): The vector feild $F = x\bar{i} - y\bar{j}$ is

(a) divergence free, but not irrotational

(b) irrotational, but not divergence free

✓(c) divergence free and irrotational

(d) neither divergence free nor irrotational

Sol: $\text{div}\,\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 1 - 1 = 0$ (divergent free)

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & 0 \end{vmatrix} = \bar{0} \quad \text{Irrotational}$$

(GATE-05 EE): For the scalar field $u = \frac{x^2}{2} + \frac{y^2}{3}$, the magnitude of the gradient at the point (1,3) is

Sol: $\text{grad}\,u = (\nabla u)_{(1,3)} = (x\bar{i} + \frac{2}{3}y\bar{j})_{(1,3)}$

$$= \bar{i} + 2\bar{j}$$

$$|\nabla u| = \sqrt{1+4} = \sqrt{5}$$

(GATE-05 IN): A scalar field is given by $f = x^{2/3} + y^{2/3}$, where x and y are cartesian co-ordinates. The derivative of 'f' along the line y=x directed away from the origin at point (8,8) is

Sol: <u>unit vector</u> along $y=x$ is

$$\hat{a} = \cos\frac{\pi}{4}\bar{i} + \sin\frac{\pi}{4}\bar{j} = \frac{1}{\sqrt{2}}\bar{i} + \frac{1}{\sqrt{2}}\bar{j}$$

$$(\nabla f)_{(8,8)} = \left(\frac{2}{3}x^{-1/3}\bar{i} + \frac{2}{3}y^{-1/3}\bar{j}\right)_{(8,8)}$$

$$= \frac{1}{3}\bar{i} + \frac{1}{3}\bar{j}$$

$$D.D = (\nabla f) \cdot \bar{a} = \frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$$

(GATE-06 CE): The D.D of $f(x,y,z) = 2x^2+3y^2+z^2$ at the point $P(1,2,3)$ in the direction of the vector $\bar{a} = \bar{i} - 2\bar{k}$ is

Sol: $$(\nabla f)_{(1,2,3)} = (4x\,\bar{i} + 6y\,\bar{j} + 2z\,\bar{k})_{(1,2,3)}$$
$$= 4\bar{i} + 12\bar{j} + 6\bar{k}$$
$$DD = (\nabla f)\cdot\frac{\bar{a}}{|\bar{a}|} = (4\bar{i} + 12\bar{j} + 6\bar{k})\cdot\left(\frac{\bar{i} - 2\bar{k}}{\sqrt{5}}\right)$$
$$= \frac{4-12}{\sqrt{5}} = -8/\sqrt{5}$$

(GATE-07 CE): The velocity vector is given as $\bar{v} = 5xy\,\bar{i} + 2y^2\,\bar{j} + 3yz^2\,\bar{k}$. The divergence of this velocity vector at $(1,1,1)$ is

Sol: $$\text{div}\,\bar{v} = 5y + 4y + 6yz$$
At $(1,1,1)$ $$\text{div}\,\bar{v} = 5+4+6 = 15$$

(GATE-07 PI): The angle between two planar vector $\bar{a} = \frac{\sqrt{3}}{2}\bar{i} + \frac{1}{2}\bar{j}$ and $\bar{b} = -\frac{\sqrt{3}}{2}\bar{i} + \frac{1}{2}\bar{j}$ is

Sol: $$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = \frac{-3/4 + 1/4}{\sqrt{\frac{3}{4}+\frac{1}{4}}\cdot\sqrt{\frac{3}{4}+1/4}} = -1/2$$
$$\theta = \cos^{-1}(-1/2) = 120^\circ$$

(GATE-07 EE): Divergence of the vector field $V(x,y,z) = -(x\cos xy + y)\bar{i} + (y\cos xy)\bar{j} + [(\sin z^2) + x^2 + y^2]\bar{k}$ is

Sol: $$\text{div}\,\bar{v} = \frac{\partial}{\partial x}[-(x\cos xy + y)] + \frac{\partial}{\partial y}(y\cos xy) + \frac{\partial}{\partial z}[(\sin z^2) + x^2 + y^2]$$
$$= 2z\cos z^2$$

(GATE-08 ME): The divergence of the vector field $(x-y)\bar{i} + (y-x)\bar{j} + (x+y+z)\bar{k}$ is

Sol: $$\text{div}\,\bar{v} = 1+1+1 = 3$$

(GATE-08 ME): The D.D of scalar function $f(x,y,z) = x^2 + 2y^2 + z$ at the point $P = (1,1,2)$ in the direction of vector $\bar{a} = 3\bar{i} - 4\bar{j}$ is

Sol: $$DD = (2x\,\bar{i} + 4y\,\bar{j} + \bar{k})_{(1,1,2)}\cdot\left(\frac{3\bar{i} - 4\bar{j}}{5}\right)$$
$$= \frac{6-16}{5} = -2$$