Lecture # 9

Counting Sort

Theorem 8.1

- *Any comparison-based sorting algorithm has worst-case running time* $\Omega(n \log n)$
  - Proof: (Book Page No 167, 2nd Edition)

- The lower bound implies that if we hope to sort numbers faster than O(n log n), we cannot do it by making comparisons alone.

- Is it possible to sort without making comparisons?
- The answer is yes, but only under very restrictive circumstances.

Counting Sort

- We will consider Counting Sort algorithm that is faster and work by not making comparisons.
- Counting sort assumes that the numbers to be sorted are in the range 1 to k where k is small. The basic idea is to determine the rank of each number in final sorted array.
- The rank of an item is the *number of elements that are less than or equal to it*.
- Once we know the ranks, we simply copy numbers to their final position in an output array.

Counting Sort

- The algorithm uses three arrays. As usual,
- A[1..n] holds the initial input,
- B[1..n] holds the sorted output and
- C[1..k] is an array of integers. C[x] is the rank of x in A, where $x \in [1..k]$.

Example

Input

	1	2	3	4	5	6	7	8	9	10	11
$A[1..n]$	7	1	3	1	2	4	5	7	2	4	3

$k = 7$

	1	2	3	4	5	6	7
$C[1..k]$	0	0	0	0	0	0	0

Input

	1	2	3	4	5	6	7	8	9	10	11
$A[1..n]$	7	1	3	1	2	4	5	7	2	4	3

k

	1	2	3	4	5	6	7
$C[1..k]$	0	0	0	0	0	0	1

Input

	1	2	3	4	5	6	7	8	9	10	11
$A[1..n]$	7	1	3	1	2	4	5	7	2	4	3

k

	1	2	3	4	5	6	7
$C[1..k]$	1	0	0	0	0	0	1

Input	1	2	3	4	5	6	7	8	9	10	11
$A[1..n]$	7	1	3	1	2	4	5	7	2	4	3

k

	1	2	3	4	5	6	7
$C[1..k]$	1	0	1	0	0	0	1

Input	1	2	3	4	5	6	7	8	9	10	11
$A[1..n]$	7	1	3	1	2	4	5	7	2	4	3

k

	1	2	3	4	5	6	7
$C[1..k]$	2	0	1	0	0	0	1

Input

	1	2	3	4	5	6	7	8	9	10	11
$A[1..n]$	**7**	**1**	**3**	**1**	**2**	**4**	**5**	**7**	**2**	**4**	**3**

k

	1	2	3	4	5	6	7
$C[1..k]$	**2**	**1**	**1**	**0**	**0**	**0**	**1**

Input

	1	2	3	4	5	6	7	8	9	10	11
$A[1..n]$	**7**	**1**	**3**	**1**	**2**	**4**	**5**	**7**	**2**	**4**	**3**

finally

	1	2	3	4	5	6	7
$C[1..k]$	**2**	**2**	**2**	**2**	**1**	**0**	**2**

Input

	1	2	3	4	5	6	7	8	9	10	11
$A[1..n]$	7	1	3	1	2	4	5	7	2	4	3

	1	2	3	4	5	6	7
$C[1..k]$	2	2	2	2	1	0	2

for i = 2 to 7
  do $C[i] = C[i] + C[i\text{-}1]$

	1	2	3	4	5	6	7
C	2	4	6	8	9	9	11

6 elements <= 3


Input
A[1..n]
7 1 3 1 2 4 5 7 2 4 3
1 2 3 4 5 6 7 8 9 10 11
Output
B[1..n]
3
B[6] = B[C[3]] = B[C[A[11]]] = A[11] = 3
1 2 3 4 5 6 7
C
2 4 6 8 9 9 11
C[A[11]] = C[A[11]] - 1
C
2 4 5 8 9 9 11

Input $A[1..n]$	7	1	3	1	2	4	5	7	2	4	3
	1	2	3	4	5	6	7	8	9	10	11
Output $B[1..n]$						3		4			

$$B[8] = B[C[4]] = B[C[A[10]]] = A[10] = 4$$

	1	2	3	4	5	6	7
C	2	4	5	8	9	9	11

$$C[A[10]] = C[A[10]] - 1$$

C	2	4	5	7	9	9	11


Input
A[1..n]
7 1 3 1 2 4 5 7 2 4 3
1 2 3 4 5 6 7 8 9 10 11
Output
B[1..n]
2 3 4
B[4] = B[C[2]] = B[C[A[9]]] = A[9] = 2
1 2 3 4 5 6 7
C
2 4 5 7 9 9 11
C[A[9]] = C[A[9]] - 1
C
2 3 5 7 9 9 11

Input $A[1..n]$

1	2	3	4	5	6	7	8	9	10	11
7	1	3	1	2	4	5	7	2	4	3

Output $B[1..n]$

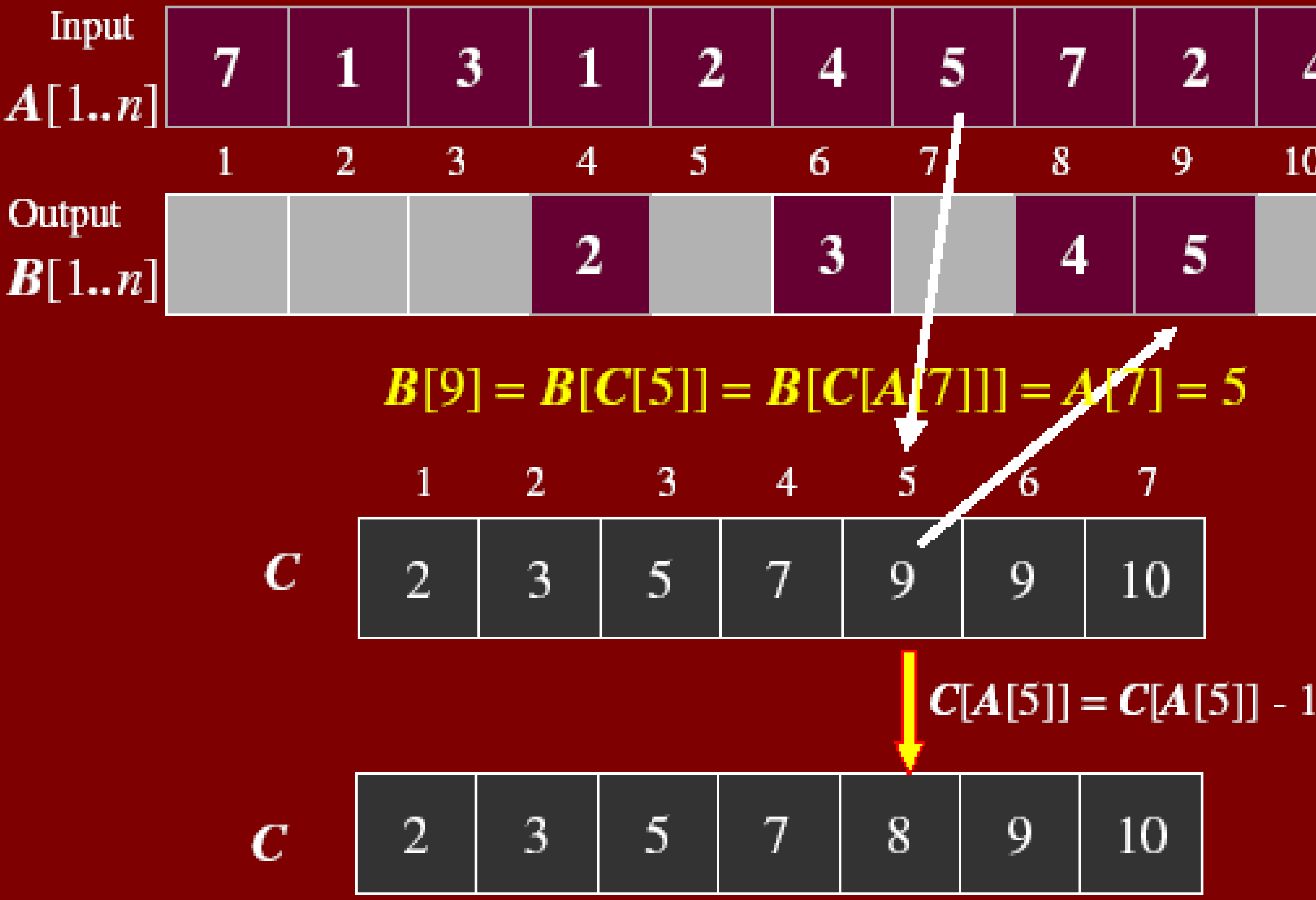
1	2	3	4	5	6	7	8	9	10	11
			2		3		4			7

$$B[11] = B[C[7]] = B[C[A[8]]] = A[8] = 7$$

	1	2	3	4	5	6	7
C	2	3	5	7	9	9	11

$$C[A[8]] = C[A[8]] - 1$$

C	2	3	5	7	9	9	10


Input
A[1..n]
7 1 3 1 2 4 5 7 2 4 3
1 2 3 4 5 6 7 8 9 10 11
Output
B[1..n]
2 3 4 5 7
B[9] = B[C[5]] = B[C[A[7]]] = A[7] = 5
1 2 3 4 5 6 7
C
2 3 5 7 9 9 10
C[A[5]] = C[A[5]] - 1
C
2 3 5 7 8 9 10

Input $A[1..n]$	7	1	3	1	2	4	5	7	2	4	3
	1	2	3	4	5	6	7	8	9	10	11
Output $B[1..n]$				2		3	4	4	5		7

$$B[7] = B[C[4]] = B[C[A[6]]] = A[6] = 4$$

	1	2	3	4	5	6	7
C	2	3	5	7	8	9	10

$$C[A[6]] = C[A[6]] - 1$$

C	2	3	5	6	8	9	10

	1	2	3	4	5	6	7	8	9	10	11
Input $A[1..n]$	7	1	3	1	2	4	5	7	2	4	3
Output $B[1..n]$			2	2		3	4	4	5		7

$$B[3] = B[C[2]] = B[C[A[5]]] = A[5] = 2$$

	1	2	3	4	5	6	7
C	2	3	5	7	8	9	10

$$C[A[5]] = C[A[5]] - 1$$

C	2	2	5	6	8	9	10

Input $A[1..n]$

1	2	3	4	5	6	7	8	9	10	11
7	**1**	**3**	**1**	**2**	**4**	**5**	**7**	**2**	**4**	**3**

Output $B[1..n]$

1	2	3	4	5	6	7	8	9	10	11
	1	**2**	**2**		**3**	**4**	**4**	**5**		**7**

$$B[2] = B[C[1]] = B[C[A[4]]] = A[4] = 1$$

C

1	2	3	4	5	6	7
2	2	5	7	8	9	10

$$C[A[4]] = C[A[4]] - 1$$

C

1	2	5	6	8	9	10

Input $A[1..n]$

1	2	3	4	5	6	7	8	9	10	11
7	1	3	1	2	4	5	7	2	4	3

Output $B[1..n]$

1	2	3	4	5	6	7	8	9	10	11
	1	2	2	3	3	4	4	5		7

$$B[5] = B[C[3]] = B[C[A[3]]] = A[3] = 3$$

C

1	2	3	4	5	6	7
1	2	5	7	8	9	10

$$C[A[3]] = C[A[3]] - 1$$

C

1	2	4	6	8	9	10

Input $A[1..n]$	7	1	3	1	2	4	5	7	2	4	3
	1	2	3	4	5	6	7	8	9	10	11
Output $B[1..n]$	1	1	2	2	3	3	4	4	5		7

$B[1] = B[C[1]] = B[C[A[2]]] = A[2] = 1$

	1	2	3	4	5	6	7
C	1	2	4	7	8	9	10

$C[A[3]] = C[A[3]] - 1$

C	0	2	4	6	8	9	10

Input $A[1..n]$

7	1	3	1	2	4	5	7	2	4	3
1	2	3	4	5	6	7	8	9	10	11

Output $B[1..n]$

1	1	2	2	3	3	4	4	5	7	7

$$B[10] = B[C[7]] = B[C[A[1]]] = A[1] = 7$$

C

1	2	3	4	5	6	7
0	2	4	7	8	9	10

$C[A[1]] = C[A[1]] - 1$

C

0	2	4	6	8	9	9

Counting Sort Algorithm

```
COUNTING-SORT( array A, array B, int k)
for i ← 1 to k
    do C[i] ← 0                                  k times
for j ← 1 to length[A]
    do C[A[j]] ← C[A[j]] + 1                     n times
// C[i] now contains the number of elements = i
for i ← 2 to k
    do C[i] ← C[i] + C[i − 1]                    k times
// C[i] now contains the number of elements ≤ i
for j ← length[A] downto 1
    do B[C[A[j]]] ← A[j]
       C[A[j]] ← C[A[j]] − 1                     n times
```

- There are four (unnested) loops, executed k times, n times, k - 1 times, and n times, respectively,

- so the total running time is **$\Theta(n + k)$** time.

- If $k = \Theta(n)$, then the total running time is **$\Theta(n)$**.