MATH1115, Analysis Lecture 4

February 27, 2014

Least upper bounds revisited

Notation:
The set of real numbers is denoted by $\mathbb{R}$.
The set of rational numbers is $\mathbb{Q}$.
The integers are denoted by $\mathbb{Z}$.
The natural numbers $1, 2, 3, \ldots$ are denoted by $\mathbb{N}$.

I want to return briefly to a property of least upper bounds which we used implicitly last lecture. This is something we will use repeatedly and it sheds some light on the meaning of least upper bounds, I want to state it precisely here:

Proposition

Let A be a nonempty subset of $\mathbb{R}$ which is bounded above. Then for any $\varepsilon > 0$ there exists a number $a \in A$ such that $a > \sup A - \varepsilon$.

Proof. Suppose this is not true. Then there exists $\varepsilon > 0$ such that $a \leq \sup A - \varepsilon$ for every $a \in A$. That is, $\sup A - \varepsilon$ is an upper bound for A. This contradicts the fact that $\sup A$ is the least upper bound. $\square$

The Archimedean property

Theorem (Archimedean property)

For every real number x there exists a natural number n such that $x < n$.

This does not seem at all surprising, but it does require the completeness axiom to prove this: There are examples of ordered fields which do not satisfy the Archimedean property (see the discussion in Hutchinson's notes about the field of hyper-real numbers — and there are also examples of fields of rational functions which do not satisfy the Archimedean property).
Note that the Archimedean property is simply the statement that the natural numbers are not bounded above: A number x is an upper bound for the natural numbers $\mathbb{N}$ (that is, great than n for every natural number n) precisely when it is a counterexample to the Archimedean property.

Proof. Suppose there exists an upper bound for $\mathbb{N}$. Then by the completeness axiom there is a least upper bound x. Since x is an upper bound, $x \geq n$ for every $n \in \mathbb{N}$. In particular, for every $n \in \mathbb{N}$, $n+1$ is a natural number, so $x \geq n+1$. Rearranging this gives that $x - 1 \geq n$ for every natural number n. But this says that $x - 1$ is an upper bound for $\mathbb{N}$, contradicting the fact that x is the least upper bound. □

Density of the rationals and irrationals

A simple consequence of the Archimedean property is the following:

Corollary

For any positive real number ε there is a natural number n such that $\frac{1}{n} < \varepsilon$.

Proof. By the Archimedean property there is a natural number n such that $n \geq \frac{1}{\varepsilon}$. It follows that $\frac{1}{n} < \varepsilon$ (by yesterday's theorem on properties of inequalities). $\square$

An application of this is the following:

Proposition

Between every distinct two real numbers x and y there exists a rational number and an irrational number.

Proof. Since x and y are not equal, assume that $x < y$. By the corollary above there exists $n \in \mathbb{N}$ such that $1/n < y - x$. Then let m be the smallest integer such that $\frac{m}{n} > a$. I claim that $a < \frac{m}{n} < b$: We have $(m-1)/n \leq a$ by the choice of m, so that $m/n \leq a + \frac{1}{n} < a + (b-a) = b$. Alternatively, if we let k be the smallest integer such that $\sqrt{2} + k/n > a$, then we have $a < \sqrt{2} + k/n < b$ by the same argument. $\square$

Sequences

In the next several lectures we will be looking at the topic of *sequences* of real numbers. The relevant references in the texts are: Adams Chapter 9 (see also Appendix 3 — but this only has a very basic introduction and does not do much of what we will cover) and Hutchinson Chapter 2.
A sequence is an infinite ordered list of numbers with a first, but no last, element. For example:

$$1, 2, 3, 4, \ldots$$
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$
$$1, -1, 1, -1, \ldots$$

We will often write a sequence in the form

$$a_1, a_2, \ldots, a_n, \ldots$$

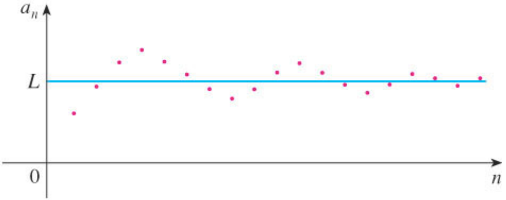
More formally, a sequence is a function a from the natural numbers $\mathbb{N}$ to the real numbers $\mathbb{R}$

$$n \in \mathbb{N} \mapsto a_n \in \mathbb{R}$$

We will often write (a_n) or $(a_n)_{n \geq 1}$ to represent a sequence.

Convergence of sequences

The most fundamental concept in the study of sequences is the notion of convergence. Informally, we say that a sequence (a_n) converges to a real number L if the terms of the sequence are ultimately within any prescribed distance of L. Here 'ultimately' means 'if we wait long enough' in the sense that we only look at terms after the Nth for some (usually large) natural number N depending on how close to a we require the terms to be.



Definition

A sequence (a_n) converges to a real number L if for every positive number ε there exists natural number $N(\varepsilon)$ such that

$$n \geq N(\varepsilon) \implies |a_n - L| < \varepsilon.$$

Convergence and limits

Example

Show that the sequence given by $a_n = 1/n$ converges to zero.

Solution: Let $\varepsilon > 0$. We must show that there exists $N(\varepsilon) \in \mathbb{N}$ such that $|a_n| < \varepsilon$ for all $n \geq N(\varepsilon)$. By the Archimedean property we can choose $N(\varepsilon) \in \mathbb{N}$ such that $\frac{1}{N(\varepsilon)} < \varepsilon$. But then for any $n \geq N(\varepsilon)$ we have $a_n = \frac{1}{n} \leq \frac{1}{N(\varepsilon)} < \varepsilon$, as required. $\square$

Notation: If a sequence (a_n) converges to a number L, we call L the *limit* of the sequence, and write $\lim_{n\to\infty} a_n = L$. It is also common to write $a_n \to L$.
To justify this notation we should observe the following:

Proposition

A sequence can have at most one limit.

Proof. Suppose (a_n) converges to two limits a and b. Let $\varepsilon > 0$. Then by assumption, there exists $N_1(\varepsilon) \in \mathbb{N}$ such that $n \geq N_1(\varepsilon) \implies |a_n - a| < \varepsilon$. Also there exists $N_2(\varepsilon) \in \mathbb{N}$ such that $n \geq N_2(\varepsilon) \implies |a_n - b| < \varepsilon$. So for $n \geq \max\{N_1(\varepsilon), N_2(\varepsilon)\}$,

$$|a - b| \leq |a - a_n| + |a_n - b| < 2\varepsilon.$$

But now $|a - b|$ is non-negative, and less than 2ε for every $\varepsilon > 0$, so $|a - b| = 0$ and $a = b$. $\square$

Convergence and divergence

Definition

A sequence which does not converge is called *divergent*.
We say that a sequence (a_n) *diverges to infinity* if the terms a_n are ultimately greater than any specified real number: That is, for any $M \in \mathbb{R}$ there exists $N(M) \in \mathbb{N}$ such that $n \geq N(M) \implies a_n > M$.
We say that (a_n) *diverges to* $-\infty$ if for any $M \in \mathbb{R}$ there exists $N(M) \in \mathbb{N}$ such that $n \geq N(M)$ implies $a_n < M$.
In these case we write $\lim_{n\to\infty} a_n = \infty$ or $\lim_{n\to\infty} a_n = -\infty$ respectively.

Example

(1). The sequence (a_n) defined by $a_n = n$ diverges to ∞: Choose $N(M)$ to be any natural number greater than M (such a number exists by the Archimedean property). Then $n \geq N(M)$ means $n > M$, and so $a_n = n > M$ as required.

(2). Similarly the sequence $a_n = -n$ diverges to $-\infty$.

(3). The sequence $a_n = (-1)^n$ is divergent, but does not diverge to infinity or to minus infinity.

Theorem on limit laws

Here is a result which helps to find limits of sequences:

Theorem

Suppose that $a_n \to L$ and $b_n \to M$. Then

(i). $a_n + b_n \to L + M$;

(ii). *If* $c \in \mathbb{R}$, *then* $ca_n \to cL$;

(iii). $a_n b_n \to LM$;

(iv). *If* $M \neq 0$, *then* $b_n \neq 0$ *for sufficiently large* n, *and* $\frac{a_n}{b_n} \to \frac{L}{M}$.

Example

Let $a_n = \frac{\pi + \frac{1}{n}}{2+2^{-n}} + \frac{1}{n^2}$.
We know that $\lim_{n\to\infty} \frac{1}{n} = 0$.
Therefore (by the Theorem) $\lim_{n\to\infty}(\pi + \frac{1}{n}) = \pi$.
Also $\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{n} = 0$.

Lemma

$\lim_{n\to\infty} 2^{-n} = 0$.

Proof. We prove by induction that $2^n > n$ for $n \geq 1$. This is true for $n = 1$. Suppose for some $k \in \mathbb{N}$ we know that $2^k > k$. Then $2^{k+1} = 2 \cdot 2^k > 2k > k+1$. This completes the induction, and proves that $2^n > n$ for every $n \geq 1$. It follows that for $n \geq N(\varepsilon) > \frac{1}{\varepsilon}$, we have $|2^{-n}| < \frac{1}{n} < \varepsilon$, so $\lim_{n\to\infty} 2^{-n} = 0$. □

Putting all of these together using the theorem, we have $\lim_{n\to\infty} a_n = \frac{\pi}{2}$.

Examples of using the limit laws

Example

Show that the sequence n^{-k} converges to zero for any $k \in \mathbb{N}$.

Solution: We will prove this by induction on k. We proved the case $k = 1$ before. Now suppose we know that $n^{-k} \to 0$. Then we can write

$$n^{-(k+1)} = \left(n^{-k}\right)\left(n^{-1}\right).$$

That is $n^{-(k+1)}$ is the product of n^{-k} (which has limit zero by the inductive hypothesis) and n^{-1} (which we proved has limit zero). Therefore by the Theorem (part (iii)) we have $n^{-(k+1)} \to 0$. This completes the induction and proves that $n^k \to 0$ for every $k \in \mathbb{N}$.

Examples of using the limit laws

Example

Show that the sequence $a_n = \frac{c+dn+en^2}{f+gn+hn^2}$ converges to $\frac{e}{h}$ if $h \neq 0$.

Solution: Rewrite a_n by dividing through by n^2:

$$a_n = \frac{e + dn^{-1} + cn^{-2}}{h + gn^{-2} + fn^{-2}}.$$

Using the limit laws:

$$\begin{aligned}
n^{-1} \to 0 &\Longrightarrow^{(ii)} dn^{-1} \to 0; \\
n^{-2} \to 0 &\Longrightarrow^{(ii)} cn^{-2} \to 0; \\
&\Longrightarrow^{(i)} e + dn^{-1} + cn^{-2} \to e + 0 + 0 = e.
\end{aligned}$$

Similarly, on the denominator, $h + gn^{-2} + fn^{-2} \to h$.
Therefore, by part (iv) of the Theorem, $a_n \to e/h$.

Proof of the Theorem

The result is intuitively clear: For example in (i) we are saying that if the terms of (a_n) get close to L and the terms of (b_n) get close to M, then $a_n + b_n$ must get close to $L + M$.
To prove this formally:
Suppose that $a_n \to L$ and $b_n \to M$. That is:
For any $\varepsilon > 0$ there exists $N_1(\varepsilon) \in \mathbb{N}$ such that $n \geq N_1(\varepsilon) \implies |a_n - L| < \varepsilon$;
and
For any $\varepsilon > 0$ there exists $N_2(\varepsilon) \in \mathbb{N}$ such that $n \geq N_2(\varepsilon) \implies |b_n - M| < \varepsilon$;
Note that $|(a_n + b_n) - (L + M)| = |(a_n - L) + (b_n - M)| \leq |a_n - L| + |b_n - M|$ (triangle inequality).
So for $n \geq \max\{N_1(\varepsilon/2), N_2(\varepsilon/2)\}$ we have

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves that $a_n + b_n \to L + M$ as required.

The proof of (ii) is also very easy (Exercise!).

Proof of the limit laws Theorem, continued

The proof of (iii) is slightly more involved: We must show that if $a_n \to L$ and $b_n \to M$ then $a_n b_n \to LM$. For this we need to bound $|a_n b_n - LM|$ in terms of $|a_n - L|$ and $|b_n - M|$. We can write

$$\begin{aligned} |a_n b_n - LM| &= |(a_n - L)b_n + L(b_n - M)| \\ &\leq |(a_n - L)b_n| + |L(b_n - M)| \\ &\leq |a_n - L||b_n| + |L||b_n - M|. \end{aligned}$$

We need the following result to control the $|b_n|$ factor:

Lemma

Any convergent sequence is bounded. Precisely, if (b_n) is a sequence which converges to a limit M, then there exists $K > 0$ such that $|b_n| \leq K$ for every $n \in \mathbb{N}$.

Proof. For $n \geq N_1(1)$ we have $|b_n - M| < 1$, and hence $|b_n| \leq |M| + 1$. Therefore for any $n \in \mathbb{N}$ we have $|b_n| \leq K = \max\{|b_1|, |b_2|, \ldots, |b_{N_1(1)-1}|, |M| + 1\}$. $\square$

Then for $n \geq \max\{N_1(\varepsilon), N_2(\varepsilon)\}$ we have $|a_n b_n - LM| < (K + |L|)\varepsilon$. So for $n \geq \max\{N_1(\varepsilon/(K + |L|)), N_2(\varepsilon/(K + |L|))\}$, $|a_n b_n - LM| < \varepsilon$. This proves that $a_n b_n \to LM$. Exercise: Prove (iv).

Remarks

Note that in the lemma we can't do any better than 'there exists $K > 0$...' That is, there is in general no relation between K and the size of the limit M.

Example

Let

$$a_n = \begin{cases} n, & n \leq 10^{10^{10}}; \\ 0, & \text{otherwise.} \end{cases}$$

The sequence converges to zero, but the bound K in the Lemma is very large!