PROOFS

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On writing proofs. A proof is a logical argument, beginning with the given information, using results already established, and proceeding to the conclusion.

Ideally, although you may not be able to prove a result, you should never give a false proof. A proof is a way of checking your own arguments. If you are not sure about the validity of one of your steps, then it is not a satisfactory proof. In this case you need to provide, or be able to provide, further details in the proof to "fill the gap".

It is often difficult to know how much detail to provide. This will depend on your level of experience. Early in the course you should provide more detail than later. An important aspect of writing a "good" proof is knowing what are the key/difficult parts and what are the easy/routine parts. The former should be explained clearly, the latter should be treated more briefly. Giving a lot of detail for the easy parts of a proof and little or no detail for the difficult parts, indicates you do not understand the material.

Proofs should be written in clear English, with properly structured and grammatically correct "sentences". Of course, your sentences will contain mathematical symbols.

Do not just write down a string of symbols. State clearly the hypotheses you are using, state clearly what theorems or previous results you are using, indicate if you are assuming something in order to obtain a contradiction, include words such as "assume, suppose, because, if, then, implies, and, or, there exists, for all" etc..

On quantifiers. Make sure that quantifiers, whether in words or symbols, are in the correct position. For example, working in the set of real numbers, consider the following:

$$\text{for all } x \text{ and } y, \text{ if } x < y \text{ there is a } z \text{ such that } x < z < y \ ,$$
$$\text{for all } x \text{ and } y \text{ there is a } z \text{ such that if } x < y \text{ then } x < z < y \ .$$

Both are true and have the same meaning, but the first is clearer. In symbols they are

$$\forall x, y \big(x < y \Longrightarrow \exists z (x < z < y)\big),$$
$$\forall x, y\ \exists z (x < y \Longrightarrow x < z < y).$$

Concerning the second version, if $x = 1$ and $y = 0$, what are the allowable values of z?[1]

It is probably OK to just write

$$\text{if } x < y \text{ there is a } z \text{ such that } x < z < y, \text{ or}$$
$$x < y \Longrightarrow \exists z (x < z < y).$$

In such a case it is understood that universal quantifiers are intended to occur for x and y at the *beginning* of each statement.

Next consider the statement

$$\text{there is a } z \text{ such that if } x < y \text{ then } x < z < y.$$

In symbols:

$$\exists z \text{ s.t. } (x < y \Longrightarrow x < z < y),$$

The variables x and y are free here, i.e. are not bounded, i.e. *are not within the scope of a quantifier.* It is not clear which of the following is intended:

$$\forall x, y\ \exists z \text{ s.t. } (x < y \Longrightarrow x < z < y),$$
$$\exists z \text{ s.t. } \forall x, y\ (x < y \Longrightarrow x < z < y).$$

The first is true, the second is false.

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[1] z can be any real number. Why?

A statement such as

$$\text{there is a } z \text{ such that } x < z < y \text{ for all } x \text{ and } y \text{ with } x < y,$$

is garbage. In symbols:

$$\exists z \text{ s.t. } (x < z < y \ \forall x, y \text{ with } x < y).$$

Does it mean

$$\exists z \text{ s.t. } \forall x, y (x < z < y \text{ if } x < y), \quad \text{i.e. } \exists z \text{ s.t. } \forall x, y \ (x < y \Longrightarrow x < z < y), \quad \text{or}$$
$$\forall x, y \ \exists z \text{ s.t. } (x < z < y \text{ if } x < y), \quad \text{i.e. } \forall x, y \ \exists z \text{ s.t. } (x < y \Longrightarrow x < z < y)?$$

It is not at all clear.

Sometimes one might write something like

$$2n \leq 1 + n^2 \quad (\text{for all } n).$$

This is probably okay, provided that it is clear the quantifier is really intended to occur at the beginning of the statement

The *guiding principle* is that it should always be clear what quantifiers apply (universal or existential), where they apply and in what order they apply.