Ordinary Differential Equations

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1 Laplace transforms, derivatives and integrals

1.1 Laplace transforms of derivatives

Laplace transforms of derivatives

Theorem 1 (Laplace transform of derivatives)**.** *Let $f(t)$ be a continuous function on $[0, \infty)$ and $f'(t)$ a piecewise continuous function on $[0, \infty)$, with both $f(t)$ and $f'(t)$ of exponential order k, then for $s > k$:*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

This is so because (by integration by parts):

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)\, dt \\
&= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt \\
&= -f(0) + s\mathcal{L}\{f(t)\}.
\end{aligned}$$

Note that $e^{-st} f(t) \to 0$ as $t \to \infty$ since $f(t)$ is of exponential order k.

This result tells us that the Laplace transform takes the derivative in the t domain to multiplication by s in the s domain (apart from the subtraction of $f(0)$). This is what makes Laplace transform very useful when we try to solve an initial value problem!

Let's take now the second derivative:

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0) = s\left[s\mathcal{L}\{f(t)\} - f(0)\right] - f'(0)$$

thus:

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

Similarly:

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0).$$

Therefore:

Theorem 2 (Laplace transforms and higher order derivatives)**.** *Let $f(t)$ and its derivatives $f'(t), f''(t), f'''(t), \cdots, f^{(n-1)}(t)$ be continuous on $[0, \infty)$ and let $f^{(n)}(t)$ be piecewise continuous on $[0, \infty)$ with all these functions of exponential order k, then for $s > k$:*

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0).$$

1.2 Examples: Laplace transforms of derivatives

Example 1 (Laplace transform of derivative)**.** *Use the theorem on the derivatives of the Laplace transform and*

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}$$

to find $\mathcal{L}\{\cos(at)\}$*.*

Solution

Let $f(t) = \sin(at)$. Then $f(0) = 0$ and $f'(t) = a\cos(at)$. Substitute in $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$:

$$\begin{aligned}
\mathcal{L}\{a\cos(at)\} &= s\mathcal{L}\{\sin(at)\} - 0 \\
\Rightarrow a\mathcal{L}\{\cos(at)\} &= \frac{sa}{s^2 + a^2} \\
\Rightarrow \mathcal{L}\{\cos(at)\} &= \frac{s}{s^2 + a^2}.
\end{aligned}$$

Example 2 (Laplace transform of derivative)**.** *Use the theorem on the derivatives of the Laplace transform to find* $\mathcal{L}\{\sin^2 t\}$*.*

Solution

We know that $f(0) = 0$ and

$$f'(t) = 2\sin t\cos t = \sin(2t).$$

Since

$$\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2 + 2^2}$$

then by using $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ we get:

$$\begin{aligned}
\mathcal{L}\{\sin(2t)\} &= s\mathcal{L}\{\sin^2 t\} - 0 \\
\Rightarrow \frac{2}{s^2 + 4} &= s\mathcal{L}\{\sin^2 t\} \\
\Rightarrow \mathcal{L}\{\sin^2 t\} &= \frac{2}{s(s^2 + 4)}.
\end{aligned}$$

1.3 Derivative of a transform

Derivative of a transform

Let $f(t)$ be a function with Laplace transform:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt, \qquad s > a.$$

Then the functions $t^n f(t), \quad (n = 1, 2, 3, \cdots)$ have the Laplace transform:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}, \qquad s > a$$

because by definition:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Thus:

$$\begin{aligned} \frac{d^n F(s)}{ds^n} &= \frac{d^n}{ds^n} \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_0^\infty \frac{d^n}{ds^n} \left[e^{-st} f(t) \right] dt \\ &= (-1)^n \int_0^\infty e^{-st} t^n f(t)\, dt \\ &= (-1)^n \mathcal{L}\{t^n f(t)\}. \end{aligned}$$

This means that differentiating the transform $F(s)$ of a function $f(t)$ with respect to s is the same as multiplying the function $f(t)$ by t.

1.3.1 Examples: Derivative of Laplace transforms

Example 3 (Derivative of Laplace transform)**.** *Find the transform of* $t^2 e^t$.

Solution

We know that

$$\mathcal{L}\{e^t\} = F(s) = \frac{1}{s-1} \qquad s > 1.$$

Thus:

$$\begin{aligned} \mathcal{L}\{t^2 e^t\} &= (-1)^2 \frac{d^2 F(s)}{ds^2} \\ &= (-1)^2 \frac{d^2}{ds^2} \left[\frac{1}{s-1} \right] \\ &= (-1) \frac{d}{ds} \left[\frac{1}{(s-1)^2} \right] \\ &= \frac{2}{(s-1)^3}, \qquad s > 1. \end{aligned}$$

Example 4 (Derivative of Laplace transform)**.** *Find the transform of $t\sin(4t)$.*

Solution
We know that

$$\mathcal{L}\{\sin(4t)\} = F(s) = \frac{4}{s^2+16}, \qquad s > 0.$$

Thus:

$$\begin{aligned}
\mathcal{L}\{t\sin(4t)\} &= (-1)\frac{dF(s)}{ds} \\
&= (-1)\frac{d}{ds}\left[\frac{4}{s^2+16}\right] \\
&= \frac{8s}{(s^2+16)^2}, \qquad s > 0.
\end{aligned}$$

1.3.2 Computing Laplace transforms with Matlab

How to calculate the Laplace transform

MATLAB can calculate the Laplace transform of simple functions. For example, if you want to find the Laplace transform of $f(t) = t^2 + \sin(t)$, you have to type:

```
syms t;
f = t^2+sin(t);
laplace(f)
```

where t is the variable in $f(t) = t^2 + \sin(t)$.

MATLAB can also help with the more difficult problem of inverting a Laplace transform (that is, computing $f(t) = \mathcal{L}^{-1}\{F(s)\}$). For example, if you want to find the inverse transform of $(s-1)^{-1} + \left(s^2 - 2^2\right)^{-1} + 1$ the commands are:

```
syms s;
F = 1/(s-1)+1/(s^2-2^2)+1;
ilaplace(F)
```

1.4 Differential equations and Initial Value Problems

Initial value problems

We'll see now (at last!) how the Laplace transforms can be used to solve differential equations.

Consider the initial value problem:

$$y'' + ay' + by = r(t), \qquad y(0) = K_0, \qquad y'(0) = K_1,$$

where a and b are constant, $r(t)$ is the input (driving force) and $y(t)$ is the output.

We can solve this IVP with Laplace method. Remember that

$$\begin{aligned} \mathcal{L}\{y''(t)\} &= s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0), \\ \mathcal{L}\{y'(t)\} &= s\mathcal{L}\{y(t)\} - y(0). \end{aligned}$$

Step 1

Set $Y = \mathcal{L}\{y\}$ and $R = \mathcal{L}\{r\}$. **Thus remember: the original functions are in lowercase letters and their transforms in uppercase letters.** Let's transform the DE:

$$\mathcal{L}\{y'' + ay' + by\} = \mathcal{L}\{r\}$$

$$\mathcal{L}\{y''\} + a\mathcal{L}\{y'\} + b\mathcal{L}\{y\} = \mathcal{L}\{r\}$$

$$\left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right] + a\left[s\mathcal{L}\{y\} - y(0)\right] + b\mathcal{L}\{y\} = \mathcal{L}\{r\}$$

$$s^2Y - sy(0) - y'(0) + asY - ay(0) + bY = R(s).$$

Now collect the Y terms and get the *subsidiary equation*:

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s).$$

Step 2

Now we solve the subsidiary equation algebraically for Y.

Let's introduce the so-called *transfer function* $Q(s)$:

$$Q(s) = \frac{1}{s^2 + as + b}$$

and let's multiply the subsidiary equation by $Q(s)$:

$$Y = \left[(s + a)y(0) + y'(0)\right]Q(s) + R(s)Q(s).$$

Note!! If $y(0) = 0$ and $y'(0) = 0$, then $Y = R(s)Q(s)$ and the transfer function is the quotient:

$$Q(s) = \frac{Y(s)}{R(s)} = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{r(t)\}} = \frac{\mathcal{L}\{\text{output}\}}{\mathcal{L}\{\text{input}\}}.$$

This is why the function $Q(s)$ is called the transfer function!!

Step 3

The solution is

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

I'll show you in a moment how we can find the inverse transform.

1.4.1 Example: Initial value problem

Example 5 (Initial value problem example 1)**.** *Solve the following IVP:*

$$y'' + 5y' + 6y = 2e^{-t}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution

Step 1.
Set $Y = \mathcal{L}\{y\}$ and $R = \mathcal{L}\{e^{-t}\}$ and remember that:

$$\begin{aligned} \mathcal{L}\{y''(t)\} &= s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0), \\ \mathcal{L}\{y'(t)\} &= s\mathcal{L}\{y(t)\} - y(0). \end{aligned}$$

Now transform the DE:

$$\begin{aligned} \mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} &= 2\mathcal{L}\{e^{-t}\} \\ \left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right] + 5\left[s\mathcal{L}\{y\} - y(0)\right] + 6\mathcal{L}\{y\} &= 2\mathcal{L}\{e^{-t}\} \\ s^2Y - sy(0) - y'(0) + 5sY - 5y(0) + 6Y &= \frac{2}{s+1} \\ s^2Y - s + 5sY - 5 + 6Y &= \frac{2}{s+1}. \end{aligned}$$

Now collect the Y terms and get the *subsidiary equation*:

$$(s^2 + 5s + 6)Y = (s + 5) + \frac{2}{s+1}.$$

Step 2.
Now we solve the subsidiary equation algebraically for Y. The transfer function $Q(s)$ is:

$$Q(s) = \frac{1}{s^2 + 5s + 6}.$$

Let's multiply the subsidiary equation by $Q(s)$:

$$Y = \frac{s+5}{s^2 + 5s + 6} + \frac{2}{(s^2 + 5s + 6)(s + 1)}.$$

Now note that $s^2 + 5s + 6 = (s + 2)(s + 3)$, thus:

$$Y = \frac{s+5}{(s+2)(s+3)} + \frac{2}{(s+2)(s+3)(s+1)}.$$

Now we need to reduce the above to a sum of terms whose inverse can be found in the Laplace tables. We can do this by using the partial fraction method

$$\begin{aligned}\frac{s+5}{(s+2)(s+3)}+\frac{2}{(s+2)(s+3)(s+1)} &= \frac{(s+5)(s+1)+2}{(s+2)(s+3)(s+1)} \\ &= \frac{A}{(s+2)}+\frac{B}{(s+3)}+\frac{C}{(s+1)}.\end{aligned}$$

Let's now use the "cover-up" method (see my notes on partial fraction decomposition). To find A, cancel $s+2$ from the left hand side and evaluate the result at $s=-2$. Obtain B and C similarly:

$$A = \left.\frac{(s+5)(s+1)+2}{(s+3)(s+1)}\right|_{s=-2} = \frac{(-2+5)(-2+1)+2}{(-2+3)(-2+1)} = 1,$$

$$B = \left.\frac{(s+5)(s+1)+2}{(s+2)(s+1)}\right|_{s=-3} = \frac{(-3+5)(-3+1)+2}{(-3+2)(-3+1)} = -1,$$

$$C = \left.\frac{(s+5)(s+1)+2}{(s+3)(s+2)}\right|_{s=-1} = \frac{(-1+5)(-1+1)+2}{(-1+3)(-1+2)} = 1.$$

So that:

$$Y(s) = \frac{(s+5)(s+1)+2}{(s+2)(s+3)(s+1)} = \frac{1}{(s+2)} - \frac{1}{(s+3)} + \frac{1}{(s+1)}.$$

Step 3.
The solution is

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

To find the inverse transform $\mathcal{L}^{-1}\{Y(s)\}$ we need to look at the Laplace tables:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)} - \frac{1}{(s+3)} + \frac{1}{(s+1)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+3)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} \\ &= e^{-2t} - e^{-3t} + e^{-t}.\end{aligned}$$

Example 6 (Initial value problem example 2)**.** *Solve the following IVP:*

$$y'' - 3y' + 2y = 12e^{4t}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution

Step 1.

Set $Y = \mathcal{L}\{y\}$ and $R = 12\mathcal{L}\{e^{4t}\}$ and remember that:

$$\begin{aligned}
\mathcal{L}\{y''(t)\} &= s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0), \\
\mathcal{L}\{y'(t)\} &= s\mathcal{L}\{y(t)\} - y(0).
\end{aligned}$$

Now transform the DE:

$$\mathcal{L}\{y'' - 3y' + 2y\} = 12\mathcal{L}\{e^{4t}\}$$

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 12\mathcal{L}\{e^{4t}\}$$

$$\left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right] - 3\left[s\mathcal{L}\{y\} - y(0)\right] + 2\mathcal{L}\{y\} = 12\mathcal{L}\{e^{4t}\}$$

$$s^2Y - sy(0) - y'(0) - 3sY + 3y(0) + 2Y = \frac{12}{s-4}$$

$$s^2Y - s - 3sY + 3 + 2Y = \frac{12}{s-4}.$$

Now collect the Y terms and get the *subsidiary equation*:

$$(s^2 - 3s + 2)Y = (s-3) + \frac{12}{s-4}.$$

Step 2.

Now we solve the subsidiary equation algebraically for Y. The transfer function $Q(s)$ is:

$$Q(s) = \frac{1}{s^2 - 3s + 2} = \frac{1}{(s-1)(s-2)}.$$

Let's multiply the subsidiary equation by $Q(s)$:

$$\begin{aligned}
Y &= \frac{s-3}{(s-1)(s-2)} + \frac{12}{(s-1)(s-2)(s-4)} \\
&= \frac{(s-3)(s-4)+12}{(s-1)(s-2)(s-4)}.
\end{aligned}$$

Now we need to reduce the above to a sum of terms whose inverse can be found in the Laplace tables:

$$\frac{(s-3)(s-4)+12}{(s-1)(s-2)(s-4)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-4)}.$$

To find A, B and C we use again the "cover-up" method. To find A, cancel $s-1$ from the left hand side and evaluate the result at $s=1$. Obtain B and C similarly:

$$\begin{aligned}
A &= \left.\frac{(s-3)(s-4)+12}{(s-2)(s-4)}\right|_{s=1} = \frac{(1-3)(1-4)+12}{(1-2)(1-4)} = 6,\\
B &= \left.\frac{(s-3)(s-4)+12}{(s-1)(s-4)}\right|_{s=2} = \frac{(2-3)(2-4)+12}{(2-1)(2-4)} = -7,\\
C &= \left.\frac{(s-3)(s-4)+12}{(s-1)(s-2)}\right|_{s=4} = \frac{12}{(4-1)(4-2)} = 2.
\end{aligned}$$

So that:

$$Y(s) = \frac{6}{(s-1)} - \frac{7}{(s-2)} + \frac{2}{(s-4)}.$$

Step 3.

The solution is

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

To find the inverse transform $\mathcal{L}^{-1}\{Y(s)\}$ we need to look at the Laplace tables.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{6}{(s-1)} - \frac{7}{(s-2)} + \frac{2}{(s-4)}\right\}\\
&= 6\mathcal{L}^{-1}\left\{\frac{1}{(s-1)}\right\} - 7\mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s-4)}\right\}\\
&= 6e^{t} - 7e^{2t} + 2e^{4t}.
\end{aligned}$$

Example 7 (Initial value problem example 3)**.** *Solve the following IVP:*

$$y'' - 4y' + 4y = t^2, \qquad y(0) = 0, \qquad y'(0) = 1.$$

Solution

Step 1.

Set $Y = \mathcal{L}\{y\}$ and $R = \mathcal{L}\{t^2\}$ and remember that:

$$\begin{aligned}
\mathcal{L}\{y''(t)\} &= s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0),\\
\mathcal{L}\{y'(t)\} &= s\mathcal{L}\{y(t)\} - y(0).
\end{aligned}$$

Now transform the DE:

$$\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\{t^2\},$$

$$
\begin{aligned}
\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} &= \mathcal{L}\{t^2\}, \\
\left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right] - 4\left[s\mathcal{L}\{y\} - y(0)\right] + 4\mathcal{L}\{y\} &= \mathcal{L}\{t^2\}, \\
s^2 Y - sy(0) - y'(0) - 4sY + 4y(0) + 4Y &= \frac{2}{s^3}, \\
s^2 Y - 1 - 4sY + 4Y &= \frac{2}{s^3}.
\end{aligned}
$$

Collect the Y terms and get the *subsidiary equation*:

$$
(s^2 - 4s + 4)Y = 1 + \frac{2}{s^3} = \frac{s^3 + 2}{s^3}.
$$

Step 2.

Now we solve the subsidiary equation algebraically for Y. The transfer function $Q(s)$ is:

$$
Q(s) = \frac{1}{s^2 - 4s + 4} = \frac{1}{(s-2)^2}.
$$

Let's multiply the subsidiary equation by $Q(s)$:

$$
Y = \frac{s^3 + 2}{s^3(s-2)^2}.
$$

The denominator now consists of repeated factors $(s - 0)$ (three times) and $(s - 2)$ (twice). The partial fraction decomposition becomes:

$$
\begin{aligned}
\frac{s^3+2}{s^3(s-2)^2} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{(s-2)} + \frac{E}{(s-2)^2} \\
\Rightarrow \frac{s^3+2}{s^3(s-2)^2} &= \frac{As^2(s-2)^2}{s^3(s-2)^2} + \frac{Bs(s-2)^2}{s^3(s-2)^2} + \frac{C(s-2)^2}{s^3(s-2)^2} \\
&\quad + \frac{Ds^3(s-2)}{s^3(s-2)^2} + \frac{Es^3}{s^3(s-2)^2}.
\end{aligned}
$$

Now we have to find A, B, C, D and E. Set $s = 2$ and we get immediately:

$$
8 + 2 = 8E \qquad \Rightarrow \qquad E = \frac{5}{4}.
$$

Now set $s = 0$ and we get

$$
2 = 4C \qquad \Rightarrow \qquad C = \frac{1}{2}.
$$

Now set $s = 1$, then $s = 3$ and $s = -1$ and use the values just found for C and E and get the following system:

$$\begin{aligned} 3 &= A + B + \frac{1}{2} - D + \frac{5}{4} \\ 29 &= 9A + 3B + \frac{1}{2} - 27D + \frac{135}{4} \\ 1 &= 9A - 9B + \frac{9}{2} + 4D - \frac{5}{4} \end{aligned}$$

Which gives $A = \frac{3}{8}$, $B = \frac{1}{2}$ and $D = -\frac{3}{8}$ (check!). So:

$$Y(s) = \frac{3}{8s} + \frac{1}{2s^2} + \frac{1}{2s^3} - \frac{3}{8(s-2)} + \frac{5}{4(s-2)^2}.$$

Step 3.

The solution is

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

To find the inverse transform $\mathcal{L}^{-1}\{Y(s)\}$ we need to look at the Laplace tables.

$$y(t) = \frac{3}{8} + \frac{t}{2} + \frac{t^2}{4} - \frac{3}{8}e^{2t} + \frac{5}{4}te^{2t}.$$

1.4.2 Application: RLC-circuit

RCL-circuit

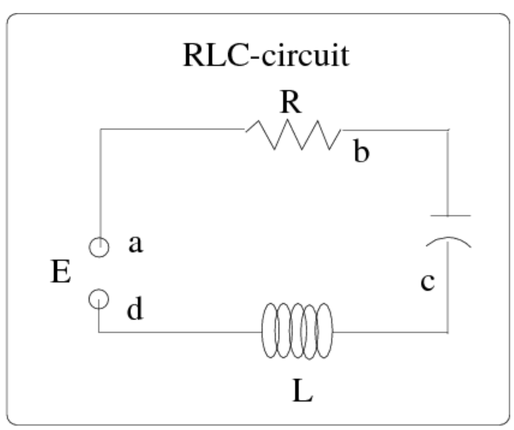

Figure 1: Example RCL circuit

Let's look again at the RCL circuit shown in Figure 1.

$$E(t) = (V_d - V_a) = (V_b - V_a) + (V_c - V_b) + (V_d - V_c).$$

The above identity can be written as:

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i(\tau)\,d\tau = E(t).$$

Since $i = \frac{dq}{dt}$ we get:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t).$$

Now set $R = 200$ ohm, $L = 1$ henry, $C = 10^{-4}$ farad and $E(t) = 50$ volt. Let's see what the charge $q(t)$ on the capacitor and the resulting current $i(t)$ are if prior to closing the switch (at time $t = 0$) both the charge on the capacitor and the current are zero.

We have:

$$\frac{d^2q}{dt^2} + 200\frac{dq}{dt} + 10^4 q = 50.$$

Now remember that

$$\begin{aligned} \mathcal{L}\{q''(t)\} &= s^2\mathcal{L}\{q(t)\} - sq(0) - q'(0), \\ \mathcal{L}\{q'(t)\} &= s\mathcal{L}\{q(t)\} - q(0). \end{aligned}$$

Transform the DE and use the initial values $q(0) = 0$ and $q'(0) = 0$:

$$\mathcal{L}\{q'' + 200q' + 10^4 q\} = \mathcal{L}\{50\}$$

$$\mathcal{L}\{q''\} + 200\mathcal{L}\{q'\} + 10^4\mathcal{L}\{q\} = \mathcal{L}\{50\}$$

$$\left[s^2\mathcal{L}\{q\} - sq(0) - q'(0)\right] + 200\left[s\mathcal{L}\{q\} - q(0)\right] + 10^4\mathcal{L}\{q\} = \mathcal{L}\{50\}$$

$$s^2Q + 200sQ + 10^4Q = \frac{50}{s}.$$

Now collect the Q terms and get the *subsidiary equation*:

$$(s^2 + 200s + 10^4)Q = \frac{50}{s}$$

that is:

$$Q = \frac{50}{s(s^2 + 200s + 10^4)} = \frac{50}{s(s+100)^2}.$$

Decompose into partial fractions:

$$Q = \frac{50}{s(s+100)^2} = \frac{A}{s} + \frac{B}{s+100} + \frac{C}{(s+100)^2}.$$

Let's evaluate A, B and C:

$$\frac{50}{s(s+100)^2} = \frac{A(s+100)^2 + Bs(s+100) + Cs}{s(s+100)^2}$$

thus:

$$\begin{array}{llll} (s^2 \text{ terms}) & A+B & = 0 & \Rightarrow A = -B \\ (s \text{ terms}) & 200A + 100B + C & = 0 & \Rightarrow C = -100A \\ (\text{constant terms}) & 10^4 A & = 50 & \Rightarrow A = \frac{50}{10^4} = \frac{1}{200} \end{array}$$

so that $A = \frac{1}{200}$, $B = -\frac{1}{200}$ and $C = -\frac{1}{2}$.
So:

$$Q = \frac{1}{200s} - \frac{1}{200(s+100)} - \frac{1}{2(s+100)^2}.$$

The charge on the capacitor is obtained by taking the inverse Laplace transform of Q:

$$q(t) = \mathcal{L}^{-1}\{Q(s)\} = \frac{1}{200} - \frac{1}{200}e^{-100t} - \frac{t}{2}e^{-100t}.$$

(Since $\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = 1/s^2$, we have used the first shifting theorem which says that $F(s-a) = \mathcal{L}\{e^{at}f(t)\}$ to find the inverse transform of $Q(s)$).

The current $i(t)$ can be obtained by taking the derivative of $q(t)$:

$$i(t) = \frac{dq}{dt} = \frac{1}{2}e^{-100t} - \frac{1}{2}e^{-100t} + 50te^{-100t} = 50te^{-100t}.$$

See Figure 2.

1.4.3 Application: automatic pilot

Automatic pilot

We are now going to model the servo-mechanism of an automatic pilot (auto-pilot), as it is used on boats or aeroplanes, such as those shown in figures 3 and 4. This mechanism allows an aircraft to maintain a set course and level flight without human control.

The auto-pilot works as follows. You set the direction $f(t)$ of where you want to go with $y(t)$ the actual direction (angle) of motion. The deviation between where you want to go and where you are actually going is given, at time t, by:

$$d(t) = y(t) - f(t).$$

The auto-pilot can detect the deviation $d(t)$ and impress to the steering mechanism a torque proportional to the deviation but of opposite sign. The torque τ is given by:

$$\tau = I\alpha$$

where I is the moment of inertia and α is the angular acceleration.

Thus:

$$Iy''(t) = -kd(t)$$

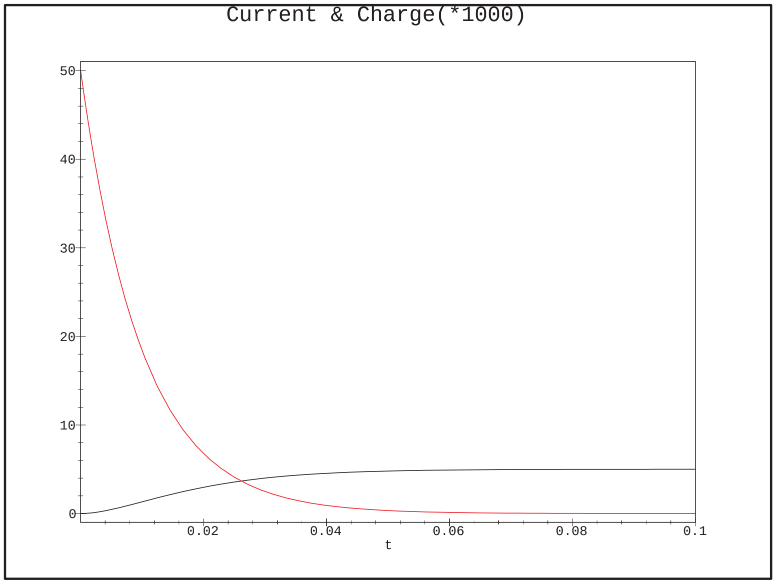

Figure 2: Current $i(t)$ (red curve) and charge $q(t)$ (black curve) as a function of time. Here, $q(t)$ has been multiplied by a factor 1000 to display both curves together.

Figure 3: ZPG-2N Blimp (Credit: “Larry’s U.S. Navy Airship Picture Book” `http://www.geocities.com/capecanaveral/1022/lakehurs.html`)

Figure 4: On auto-pilot! (Credit: "Larry's U.S. Navy Airship Picture Book" `http://www.geocities.com/capecanaveral/1022/lakehurs.html`)

where k is a positive constant.

Let's calculate the deviation $d(t)$ if the steering mechanism is initially at rest in the zero direction and $f(t) = at$, where a is a constant. Thus, we have to solve the IVP:

$$Iy''(t) = -kd(t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

Take the Laplace transforms and set $D(s) = \mathcal{L}\{d(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$:

$$\begin{aligned} I\mathcal{L}\{y''(t)\} &= -k\mathcal{L}\{d(t)\} \\ I\left[s^2Y(s) - sy(0) - y'(0)\right] &= -kD(s) \\ I\left[s^2Y(s)\right] &= -kD(s). \end{aligned}$$

Since $d(t) = y(t) - at$, then:

$$\begin{aligned} D(s) &= \mathcal{L}\{y(t) - at\} \\ &= Y(s) - a\mathcal{L}\{t\} \\ &= Y(s) - a\frac{1}{s^2} \\ \Rightarrow Y(s) &= D(s) + \frac{a}{s^2}. \end{aligned}$$

Therefore, since $Is^2Y(s) = -kD(s)$ (from above), then:

$$\begin{aligned} Is^2\left(D(s) + \frac{a}{s^2}\right) &= -kD(s) \\ \Rightarrow D(s) &= -\frac{aI}{Is^2 + k} \\ &= -\left[\frac{a}{\sqrt{\frac{k}{I}}}\right]\frac{\sqrt{\frac{k}{I}}}{s^2 + \frac{k}{I}}. \end{aligned}$$

Now we can take the inverse Laplace transform to find $d(t)$:

$$\mathcal{L}^{-1}\{D(s)\} = d(t) = -\frac{a}{\sqrt{\frac{k}{I}}}\sin\sqrt{\frac{k}{I}}t.$$

This equation says that the auto-pilot will oscillate back and forth about the desired direction, always over-steering by the factor $a/\sqrt{k/I}$. What we could do, is to make the deviation smaller by taking k large compared to the moment of inertia I. However, this would make the term $\sqrt{k/I}$ become large causing the deviation to oscillate more rapidly! The oscillations due to over-steering can be damped by introducing a damping torque proportional to $d'(t)$ but of opposite sign.

1.5 Laplace transforms of an integral

Differentiation v's Integration

We have seen that the differentiation of a function corresponds to the multiplication of its transform by s (well, roughly). So, now we expect that integration of a function will lead to division of its transform by s. That is

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s),$$

(Remembering that $\mathcal{L}\{f(\tau)\} = F(s)$) or, if we take the inverse Laplace transform on both sides:

$$g(t) = \int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}.$$

This is because:

$$\begin{aligned} g(t) &= \int_0^t f(\tau)\,d\tau \\ \Rightarrow \frac{dg}{dt} &= f(t). \end{aligned}$$

Initial value problem

Consider now the IVP:

$$\frac{dg}{dt} = f(t), \qquad g(0) = 0.$$

Solve using Laplace transforms method:

$$\begin{aligned} \mathcal{L}\left\{\tfrac{dg}{dt}\right\} &= \mathcal{L}\{f(t)\}, \\ sG(s) - g(0) &= F(s), \\ sG(s) &= F(s), \\ \Rightarrow \quad G(s) = \tfrac{F(s)}{s}. \end{aligned}$$

But $G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}$, then:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s).$$

Example application

The result we have just seen on the Laplace transform of integrals of functions is very useful, since in many applications one may have to solve integro-differential equations such as:

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\, d\tau = E(t).$$

Usually, in order to determine the current, we use $i = dq/dt$ to get rid of the integral. However this is not necessary. We can use the theorem on transform of integrals and proceed as follows.

Take, as before, $R = 200$ ohm, $L = 1$ henry, $C = 10^{-4}$ farad and $E = 50$ volt. Let's see what the current $i(t)$ is if prior to closing the switch both the charge on the capacitor and the current are zero ($i(0) = 0,\ q(0) = 0$).

$$200\mathcal{L}\{i\} + \mathcal{L}\{i'\} + 10^4\mathcal{L}\left\{\int_0^t i\, d\tau\right\} = \mathcal{L}\{50\}.$$

Thus:

$$\begin{aligned} 200I + sI(s) - i(0) + \frac{10^4}{s}I(s) &= \frac{50}{s} \\ 200I + sI(s) + \frac{10^4}{s}I(s) &= \frac{50}{s} \\ \left(200 + s + \frac{10^4}{s}\right)I(s) &= \frac{50}{s} \end{aligned}$$

$$\Rightarrow I(s) = \frac{50}{s^2 + 200s + 10^4} = \frac{50}{(s+100)^2}.$$

Since $\mathcal{L}\{t\} = 1/s^2$, then we can use the first shifting theorem ($F(s-a) = \mathcal{L}\{e^{at}f(t)\}$) to find the inverse transform of $I(s)$:

$$\mathcal{L}^{-1}\{I(s)\} = i(t) = 50te^{-100t}$$

which is the same result we obtained before. If you look at our previous result (example) on the Laplace transform of the charge, then you'll see that $I(s) = sQ(s)$. This is to be expected, since $i = dq/dt$ and $q(0) = 0$...

1.5.1 Examples: Laplace transformations of integrals

Example 8 (Laplace transformations of an integral example 1)**.** *If*

$$G(s) = \mathcal{L}\{g(t)\} = \frac{1}{s(s^2 + a^2)}$$

what is $g(t)$?

Solution

Here we can use the result of the theorem we have just seen. A look at the table will tell us immediately that

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin(at).$$

The theorem says that:

$$g(t) = \int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}.$$

In our case:

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} \\ &= \frac{1}{a}\int_0^t \sin(a\tau)\,d\tau \\ &= \frac{1}{a^2}\left[-\cos(a\tau)\right]_0^t \\ &= \frac{1}{a^2}(1-\cos(at)). \end{aligned}$$

Example 9 (Laplace transformation of an integral example 2)**.** *If*

$$G(s) = \mathcal{L}\{g(t)\} = \frac{1}{s^2(s^2+a^2)}$$

what is $g(t)$?

Solution

Well, this is easy! We can use the result from the previous example. That is:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \frac{1}{a^2}(1-\cos(at)).$$

All that is required, is to apply the theorem again!

The theorem says that:

$$g(t) = \int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}.$$

Thus, in our case:

$$\begin{aligned}
g(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} \\
&= \frac{1}{a^2}\int_0^t (1-\cos(a\tau))\,d\tau \\
&= \frac{1}{a^2}\left[\int_0^t d\tau - \int_0^t \cos(a\tau)\,d\tau\right] \\
&= \frac{1}{a^2}\left[t - \frac{\sin(at)}{a}\right].
\end{aligned}$$