12 Uniform Convergence of Functions

Problem 12.1 Let $f_n(x) = x^n$ for $x \in [0,1]$. Let $f(x) = 0$ if $x \in [0,1)$ and let $f(1) = 1$. Then clearly $f_n \to f$ pointwise in $[0,1]$. Prove *directly from the definition of uniform convergence* that f_n does not converge uniformly to f.

Note: It follows that f_n does not converge uniformly to any function g, since uniform convergence to g clearly implies pointwise convergence to g.

Problem 12.2 Let $f(x) = \sum_{k=1}^{\infty} (\sin kx)/k^2$. Use Theorem 12.3.1 to prove that f is continuous on $\mathbb{R}$.

Problem 12.3 Let $(a_{mn})_{m\geq 1,\ n\geq 1}$ be a *doubly infinite* sequence of real numbers as shown below:

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Assume that

$$\lim_{m\to\infty} a_{mn} = b_n$$

for $n = 1, 2, \ldots$ and that

$$\lim_{n\to\infty} a_{mn} = c_m$$

for $m = 1, 2, \ldots$.

1. Give an example where

   $$\lim_{m\to\infty} c_m, \quad \lim_{n\to\infty} b_n$$

   both exist but are *not* equal.

2. We say $a_{mn} \to b_n$ as $m \to \infty$ *uniformly in n* if:

   for each $\epsilon > 0$ there exists an M such that

   $$m \geq M \text{ implies } |a_{mn} - b_n| < \epsilon$$

   for all n. In this case

   (a) Prove that $(c_m)_{m=1}^{\infty}$ is Cauchy, and hence that $\lim_{m\to\infty} c_m$ exists. Denote the limit by c.

   (b) Deduce that $\lim_{n\to\infty} b_n$ exists, and that moreover

   $$\lim_{n\to\infty} b_n = c.$$

13 First Order Systems of Differential Equations

Problem 13.1 Convert the integral equation

$$x(t) = 1 + \int_0^t \big(x(s)\big)^2 \, ds,$$

where $t \in [0,1]$, into an initial value problem. What is $x(0)$?

Problem 13.2 Consider the system of differential equations

$$\begin{aligned} x''(t) + x'(t) + y(t) &= 0 \\ y'(t) + y(t) + x(t) &= 0 \end{aligned}$$

where

$$x(0) = 1, \ x'(0) = 0, \ y(0) = 1.$$

Convert this to an equivalent system of first-order differential equations. Carefully state the interpretation of the new variables, and their initial values.

Problem 13.3 Assume $K : [a,b] \times [a,b] \to \mathbb{R}$ and K is continuous. Recall that this implies K is uniformly continuous on $[a,b] \times [a,b]$.

Let $x : [a,b] \to \mathbb{R}$ be continuous and define

$$f(t) = \int_a^b K(s,t)\, x(s)\, ds.$$

Prove that f is continuous on $[a,b]$.

Problem 13.4 Consider the integral equation

$$x(t) = e^t + \frac{1}{2} \int_0^1 t \cos(ts) x(s)\, ds,$$

for $x \in \mathcal{C}[0,1]$ (i.e. x is a continuous function defined on $[0,1]$).

Show the integral equation has a solution in $\mathcal{C}[0,1]$.

Problem 13.5 Suppose that $y(t) \in C^1[0,+\infty)$ (meaning that y: $[0,+\infty) \to \mathbb{R}$ is continuously differentiable) satisfies

$$\begin{aligned} y'(t) &= 2\sqrt{|y(t)|} \quad \text{for}\, t > 0 \\ y(0) &= 0 \end{aligned}$$

Give a detailed proof that then there is $a \in [0,+\infty]$ such that

$$y(t) = \begin{cases} 0 & \text{if} \quad 0 \le t \le a \\ (t-a)^2 & \text{if}\, t \ge a. \end{cases}$$

(Hint: How to determine a?)

Problem 13.6 1. Let g: $[0,+\infty) \to [0,+\infty)$ be continuous and suppose that there are constants $A, B \geq 0$ such that

$$g(t) \leq A + B\int_0^t g(s)ds \quad \text{for every} \quad t \geq 0. \tag{1}$$

Show that $g(t) \leq Ae^{Bt}$ for every $t \geq 0$.
(Hint: Introduce $G(t) = \int_0^t g(s)ds$, multiply both sides of (1) by e^{-Bt} and integrate.)

2. Let $f = f(t,x)$: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and (globally) Lipschitz with respect to x, i.e. there is $L \geq 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x-y| \quad \text{for all} \quad t,x,y \in \mathbb{R}.$$

Let $x_0 \in \mathbb{R}$. Use the first part to show that the initial value problem

$$\begin{aligned} x'(t) &= f(t,x(t)) \quad \text{for}\, t > 0 \\ x(0) &= x_0 \end{aligned}$$

has at most one solution $x \in C^1[0,+\infty)$.
(Hint: Suppose that $x(t), y(t) \in C^1[0,+\infty)$ solve the IVP and consider $g(t) = |x(t) - y(t)|$ or $g(t) = |x(t) - y(t)|^2$.)

3. Under the assumptions of 2 show that the IVP has a solution $x(t) \in C^1[0,+\infty)$.
(Hint: Study the proof of Local Existence and Uniqueness Theorem 13.9.2. Check that in our situation h can be chosen independently of t_0 and x_0 and iterate.)

Problem 13.7 1. Let $T > 0$ and $L \geq 0$. Consider $C[0,T]$ (the space of all continuous functions on $[0,T]$) and for $x(t), y(t) \in C[0,T]$ define

$$\rho(x,y) = \sup_{0 \leq t \leq T} e^{-Lt}|x(t) - y(t)|.$$

Check that $(C[0,T], \rho)$ is a complete metric space.

2. Let $T > 0$. Under the assumptions of 2 consider an appropriate integral operator as in the proof of Theorem 13.9.2. Verify that this operator is a contraction on $(C[0,T], \rho)$. Deduce that (IVP) has a solution $x(t) \in C^1[0,T]$. Deduce from this that (IVP) has a (unique) solution $x \in C^1[0,+\infty)$.

Problem 13.8 1. Let (X,d) be a metric space, $A \subset X$ be compact and f: $A \to \mathbb{R}$ be L-Lipschitz, i.e. $L \geq 0$ and

$$|f(x) - f(y)| \leq L\, d(x,y) \quad \text{for all} \quad x,y \in A.$$

Define $\tilde{f}$: $X \to \mathbb{R}$ according to

$$\tilde{f}(x) = \sup\{f(a) - L\, d(a,x): \quad a \in A\} \quad \text{for} \quad x \in X.$$

Show that $\tilde{f}$ is well-defined, L-Lipschitz on X and $\tilde{f}_{|A} = f$ (that is, $\tilde{f}(a) = f(a)$ for $a \in A$.)

2. Let $A \subset \mathbb{R}^n$ be closed and bounded and $f\colon A \to \mathbb{R}^k$ be Lipschitz. Show that there exists $\tilde{f}\colon \mathbb{R}^n \to \mathbb{R}^k$ which is Lipschitz on $\mathbb{R}^n$ and satisfies $\tilde{f}_{|A} = f$.

3. Let $U \subset \mathbb{R}$ be open, $f = f(x)\colon U \to \mathbb{R}$ be locally Lipschitz and $x_0 \in U$. Consider the following autonomous initial value problem

$$\begin{aligned} x'(t) &= f(x(t)) \quad \text{for}\, t > 0 \\ x(0) &= x_0. \end{aligned}$$

Use 1 and 3 (or 2) show that there is a local solution (meaning that there is $h > 0$ and $x(t) \in C^1[0,h]$ satisfying $x(0) = x_0$ and $x'(t) = f(x(t))$ for $0 < t < h$.

4. ** A generalisation of 1 can be used to deduce the full (i.e., non-autonomous) Local Existence Theorem 13.9.2 from 3 (or 2). State and prove a required result and carry out the deduction mentioned above.

Problem 13.9 Suppose that $f(t,x)\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and $\frac{\partial f}{\partial x}$ exists everywhere and is bounded:

$$\exists\, K \geq 0\ \forall\, t, x \in \mathbb{R} \quad \left|\frac{\partial f}{\partial x}(t,x)\right| \leq K.$$

Show that for every $x_0 \in \mathbb{R}$ the initial value problem

$$\begin{aligned} x'(t) &= f(t, x(t)) \quad \text{for}\, t \in (-\infty, \infty) \\ x(0) &= x_0 \end{aligned}$$

has a unique solution $x \in C^1(-\infty, +\infty)$.

Problem 13.10 Consider the initial value problem:

$$\begin{aligned} x'(t) &= (x(t))^2, \\ x(0) &= 1 \end{aligned}$$

Solve this IVP. Where is the solution defined? Is there a unique solution? Does Local Existence and Uniqueness Theorem apply? If so, what value of h does Theorem 13.9.2 give? Discuss.

Problem 13.11 Let A be an $n \times n$ matrix, and consider the linear system

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

Show that a solution is

$$\mathbf{x}(t) = e^{tA}\mathbf{x}(0),$$

where given an $n \times n$ matrix B,

$$e^B = \sum_{n=0}^{\infty} \frac{B^n}{n!}.$$

What is the interval of existence? Is this solution unique?

Problem 13.12 Let $K\colon [0,1] \times [0,1] \to (-1,1)$ and $\varphi\colon [0,1] \to \mathbb{R}$ both be continuous. Prove that there is a unique continuous function $f\colon [0,1] \to \mathbb{R}$ such that

$$f(x) = \varphi(x) + \int_0^1 K(x,y)f(y)\,dy \quad \text{for all } x \in [0,1].$$

(Hint: show that an appropriate integral operator is a contraction on $(C[0,1], d_u)$.)

14 Fractals

Problem 14.1 Show that there is no non-empty interval I with $I \subset C$, where C is the Cantor set.

Problem 14.2 If $f : A\,(\subset \mathbb{R}^n) \to \mathbb{R}$, the graph of f is defined by

$$G(f) = \{(x, f(x)) : x \in A\} \subset \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}.$$

Suppose that also $f_k : A \to \mathbb{R}$ for $k = 1, 2, \ldots$. Assume that A is compact and f is continuous.

1. Prove that $G(f)$ is compact.
2. Prove that $f_k \to f$ uniformly implies $G(f_k) \to G(f)$ in the Hausdorff metric sense.

15 Compactness

Problem 15.1 Prove that a subset of a metric space is totally bounded iff its closure is totally bounded.

Problem 15.2 Let $f\colon[0,1]\times[0,1]\to\mathbb{R}$ be continuous. For each $y\in[0,1]$, define the function $f_y\colon[0,1]\to\mathbb{R}$ by $f_y(x)=f(x,y)$.

Prove that the family of functions $\mathcal{F}=\{f_y : y\in[0,1]\}$ is equicontinuous.

Problem 15.3

1. Give an example of a function $f\colon(0,1)\to\mathbb{R}$ which is continuous, but such that there is no continuous function $g\colon[0,1]\to\mathbb{R}$ which agrees with f on $(0,1)$.
2. Suppose $f : A\,(\subset\mathbb{R}^n)\to\mathbb{R}$. Prove that if f is uniformly continuous then there is a unique continuous function[16] $g : \overline{A}\to\mathbb{R}$ which agrees with f on A.
3. Generalise.

Problem 15.4 Let X be a compact metric space. Suppose that $(F_i)_{i=1}^{\infty}$ is a nested (that is, $F_{i+1}\subseteq F_i$) sequence of nonempty closed subsets of X such that diameter$(F_i)\to 0$ as $i\to\infty$. Show that there is exactly one point in $\bigcap_{i=1}^{\infty}F_i$. (By definition, diameter$(F_i)=\sup\{d(x,y)\colon\ x,y\in F_i\}$.)

** The assumption that X be compact in 15.4 can be somewhat relaxed. State and prove an appropriate result.

Problem 15.5

1. Let $\emptyset\neq A,B\subset X$ with A closed, B compact and $A\cap B=\emptyset$. Show that there is $\epsilon>0$ such that $d(a,b)>\epsilon$ for all $a\in A$ and $b\in B$.
2. Is 1 true if A,B are merely closed?

Problem 15.6 Let X be compact and $T\colon X\to X$ be an isometry (meaning that $d(Tx,Ty)=d(x,y)$ for all $x,y\in X$). Show that T is a surjection.
(Hint: If T is not surjective, select $y\notin T[X]$ and consider the sequence $y,Ty,T(Ty),\ldots$. Use 1.)

Problem 15.7 Let X be the set of all sequences $(x_n)_{n=1}^{\infty}$ such that $x_n\in[0,1]$ for all n. For $x=(x_n)_{n=1}^{\infty}$, $y=(y_n)_{n=1}^{\infty}\in X$ put

$$d(x,y)=\sum_{n=1}^{\infty}\frac{1}{2^n}|x_n-y_n|.$$

Show that d makes X into a metric space. Prove that (X,d) is compact.
(Hint: Show that X is sequentially compact. Given a sequence in X, extract a

[16] In other words, there is *exactly one* such continuous function.

subsequence whose first components converge, from this subsequence extract a subsequence whose second components converge etc. Then use a diagonal process as in the proof of Theorem 15.5.2.)

Problem 15.8 Let (X, d) be a compact metric space. Let $\{F_s\}_{s\in S}$ be a family of closed subsets of X, $U \subset X$ be open and suppose that $\bigcap_{s\in S} F_s \subset U$. Show that there is a finite set $\{s_1, s_2, \ldots, s_k\} \subset S$ such that $\bigcap_{i=1}^{k} F_{s_i} \subset U$.

Problem 15.9 Let (X, d) be a compact metric space, let $\mathcal{F}$ be an equicontinuous family of functions from X to X and let $g\colon X \to \mathbb{R}$ be continuous. Show that the family $\mathcal{K} = \{g \circ f\colon\ f \in \mathcal{F}\} \subset C(X; \mathbb{R})$ is equicontinuous and that $\overline{\mathcal{K}}$ is compact.

Problem 15.10 Let (X, d) and (Y, ρ) be metric spaces and let $f\colon X \to Y$ be uniformly continuous and onto. Show that (Y, ρ) is totally bounded if (X, d) is totally bounded.

Problem 15.11 Let (X, d) be a compact metric space and let $T\colon X \to X$ satisfy $d(T(x), T(y)) < d(x, y)$ for all $x, y \in X$, $x \neq y$. Show that T has a unique fixed point, that is, there is precisely one $x \in X$ such that $T(x) = x$.

16 Connectedness

Problem 16.1 Let $C_1, C_2, \ldots$ be a sequence of connected subspaces of X such that $C_i \cap C_{i+1} \neq \emptyset$ for $i = 1, 2, \ldots$. Show that the union $\bigcup_{i=1}^{\infty} C_i$ is connected.

Problem 16.2 Let X be connected. Show that for every pair $x, y \in X$ and $\epsilon > 0$ there exists a finite sequence $x_1, x_2, \ldots, x_k$ of points of X such that $x_1 = x$, $x_k = y$ and $d(x_i, x_{i+1}) < \epsilon$ for $i = 1, 2, \ldots, k-1$.

Problem 16.3 Prove that every compact space X satisfying the condition in 16.2 is connected. Is the assumption of compactness essential?
(Hint: Use 1.)

Problem 16.4 Consider the following two possible properties for a subset X of $\mathbb{R}^n$:

1. **i** There is a point $x_0 \in X$ such that every other point $x \in X$ can be joined to x_0 by a straight line in X.
2. **ii** There is a point $x_0 \in X$ such that every other point $x \in X$ can be joined to x_0 by a differentiable path in X.

Give examples of each kind of set that are not convex. Show that either of these conditions implies connectedness of X. Show that if X satisfies either of these conditions and $f\colon X \to \mathbb{R}$ is a differentiable function with zero derivative, then f is constant. Show that if X is an open subset of $\mathbb{R}^n$ then the following are equivalent: condition **ii** above, path connectedness of X, connectedness of X.

Problem 16.5 Let X be the space of all continuous functions from $[0, 1]$ to $[0, 1]$ equipped with the sup metric. Let X_i be the set of injective and X_s be the set of surjective elements of A and let $X_{is} = X_i \cap X_s$. Prove or disprove: i) X_i is closed, ii) X_s is closed, iii) X_{is} is closed, iv) X is connected, v) X is compact.

Problem 16.6 If A and B are closed subsets of a metric space X, whose union and intersection are connected, show that A and B themselves are connected. Give an example showing that the assumption of closedness is essential.

Problem 16.7 Show that a metric space X is not connected if and only if there exists a continuous surjection $f\colon X \to \{0, 1\}$.

Problem 16.8 Let X and Y be compact metric spaces and let $f\colon X \to Y$ be a continuous onto map with the property that $f^{-1}[\{y\}]$ is connected for every $y \in Y$. Show that if Y is connected then so is X.

17 Differentiation of Real-valued Functions

Problem 17.1 1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \frac{x^2y^2}{\sqrt{x^2+y^2}} & \text{for} \quad (x,y) \neq (0,0) \\ 0 & \text{for} \quad (x,y) = (0,0). \end{cases}$$

Is f differentiable at $(0,0)$?

2. Find a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ that is differentiable at each point but whose partials are not continuous at $(0,0)$.

Problem 17.2 Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} e^{-\frac{x^2}{y^2}-\frac{y^2}{x^2}} & \text{for}\, xy \neq 0 \\ 0 & \text{for}\, xy = 0. \end{cases}$$

Is f continuous? Is f differentiable? Do $\frac{\partial^{m+n} f}{\partial x^n \partial y^m}$ exist?

Problem 17.3 Prove or disprove: if $\{f_n\}_{n=1}^{\infty}$ is a sequence in $C^2(\mathbb{R};\mathbb{R})$, for some finite M'' and all n, $\sup_{\mathbb{R}} |f_n''| \leq M''$, and $f_n \to 0$ uniformly as $n \to \infty$, then for some M' and all n, $\sup_{\mathbb{R}} |f_n'| \leq M'$.

18 Differentiation of Vector-valued Functions

Problem 18.1 1. Let $\mathbf{f}\colon [0,1] \to \mathbb{R}^n$ be a path. For $t \in [0,1]$ define $s(t) = \int_0^t |\mathbf{f}'(\tau)| d\tau$, so that $s(t)$ represents the distance between $\mathbf{f}(0)$ and $\mathbf{f}(t)$ measured along the curve parametrised by $\mathbf{f}$. Let $l = s(1)$ denote the length of the path. Show that $s\colon [0,1] \to [0,l]$ is strictly increasing, $s \in C^1$, and that t can be regarded as a function of s: $t = t(s)$, where $t\colon [0,l] \to [0,1]$ is C^1. Let $\mathbf{T}$ denote the unit tangent vector to the curve, i.e.

$$\mathbf{T}(t) = \frac{\mathbf{f}'(t)}{|\mathbf{f}'(t)|}.$$

Check that $\mathbf{T} = \dfrac{d\mathbf{f}}{ds}$.

2. Now suppose also that $\mathbf{f} \in C^2$. The curvature κ of the curve parametrised by $\mathbf{f}$ is defined as

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|,$$

so that κ measures the rate of change of the unit tangent direction of the curve with respect to the arc length. Show that $\mathbf{T}$ and $\frac{d\mathbf{T}}{ds}$ are orthogonal vectors. Show that for $n = 3$

$$\kappa = \frac{|\mathbf{f}' \times \mathbf{f}''|}{|\mathbf{f}'|^3},$$

where "$\times$" denotes the vector product in $\mathbb{R}^3$.

Problem 18.2 Let $\mathbf{f}\colon [0,1] \to \mathbb{R}^3$ be given by $\mathbf{f}(t) = (t, t^2, \frac{2}{3}t^3)$. Compute the curvature of this curve in two ways.

Problem 18.3 Let $f\colon \mathbb{R} \to \mathbb{R}$ be C^2 and suppose that

$$M_0 = \sup |f|, \quad M_1 = \sup |f'|, \quad M_2 = \sup |f''|$$

are finite. Show that $M_1^2 \le 4M_0M_2$. Does this result extend to vector-valued functions?
(Hint: From Taylor's theorem $f'(x) = \frac{1}{2h}(f(x+2h) - f(x)) + hf''(c)$ and therefore $|f'| \le hM_2 + \frac{M_0}{h}$ for every $h > 0$.)

Problem 18.4 Suppose that $\mathbf{f}\colon [0,1] \to \mathbb{R}^n$ is continuous and differentiable in $(0,1)$. Prove that there exists $c \in (0,1)$ such that

$$|\mathbf{f}(1) - \mathbf{f}(0)| \le |\mathbf{f}'(c)|.$$

Can "$\le$" be replaced by "$=$"?
(Hint: Consider $\phi\colon [0,1] \to \mathbb{R}$ given by $\phi(t) = (\mathbf{f}(1) - \mathbf{f}(0)) \cdot \mathbf{f}(t)$.)

Problem 18.5 Suppose that $U \subset \mathbb{R}^n$ is open and $\mathbf{f} \in C^1(U; \mathbb{R}^m)$ satisfies

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \le |\mathbf{x} - \mathbf{y}| \quad \text{for all } \mathbf{x}, \mathbf{y} \in U.$$

Show that $\|\mathbf{f}'(\mathbf{x})\| \le 1$ for every $\mathbf{x} \in U$.

19 Inverse Function Theorem

Problem 19.1

1. Let $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\mathbf{f}(x, y) = (e^x \cos y, e^x \sin y)$. Show that $\mathbf{f}$ is locally invertible near every point, but is not invertible.

2. Investigate whether the system

$$\begin{aligned} u(x, y, z) &= x + yz \\ v(x, y, z) &= 2e^x \sin z + y^2 \\ w(x, y, z) &= xyz + y \end{aligned}$$

   can be solved for x, y, z in terms of u, v, w near $(0, 0, 0)$.

3. Let $\mathbf{g}\colon \mathbb{R}^n \to \mathbb{R}^n$ be given by $\mathbf{g}(\mathbf{x}) = L(\mathbf{x}) + \mathbf{f}(\mathbf{x})$, where $L\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism and $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and there is a constant M such that

$$|\mathbf{f}(\mathbf{x})| \leq M|\mathbf{x}|^2 \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

   Show that $\mathbf{g}$ is locally invertible near 0.

Problem 19.2 Let $D \subset \mathbb{R}^n$ be open and let $\mathbf{f}\colon D \to \mathbb{R}^n$ be a C^1 map with $\det(\mathbf{f}'(\mathbf{x})) \neq 0$ for every $\mathbf{x} \in U$. Show that $\mathbf{f}[D]$ is an open subset of $\mathbb{R}^n$.

Problem 19.3 Denote $B = B_1(0) \subset \mathbb{R}^n$ and let $\mathbf{f} \in C^1(B; \mathbb{R}^n)$. Show that there is $\delta > 0$ such that if $\sup_{\mathbf{x} \in B} \|\mathbf{f}'(\mathbf{x}) - \mathrm{id}\| < \delta$ then $\mathbf{f}$ is one-to-one on B.

Problem 19.4 Let $f \in C^2(\mathbb{R}^n; \mathbb{R})$ and assume that $f'(\mathbf{x}_0) = 0$ and $(f''(\mathbf{x}_0))^{-1}$ exists. Show that there is an open set U containing $\mathbf{x}_0$ such that $f'(\mathbf{y}) \neq 0$ for all $\mathbf{y} \in U \setminus \{\mathbf{x}_0\}$.

Problem 19.5 Let $\mathbf{f} \in C^1(\mathbb{R}^n; \mathbb{R}^m)$ and $\mathbf{x}_0 \in \mathbb{R}^n$.

1. Suppose that $\mathbf{f}'(\mathbf{x}_0)$ has rank m (i.e., $\mathbf{f}'(\mathbf{x}_0)$ as a linear map is onto). Show that there is a whole neighborhood of $\mathbf{f}(\mathbf{x}_0)$ lying in the image of $\mathbf{f}$.

2. Suppose that $\mathbf{f}'(\mathbf{x}_0)$ is one-to-one. Show that $\mathbf{f}$ is one-to-one on some neighborhood of $\mathbf{x}_0$.

Problem 19.6 Find (if it exists) the best linear approximation to the inverse function (if it exists) of the function $\mathbf{f}(x, y, z) = (x^2 + y^2, x^2 - y^2, z)$ near $(1, 2, 3)$.

Problem 19.7 Let $M = \{(x, y) \in \mathbb{R}^2\colon\ xy = 0\}$. Is M a manifold? If so, of what dimension? If not, explain why not. Same questions for $M = \{(x, y, z) \in \mathbb{R}^3\colon\ xy = yz = 0\}$,

$$M = \{(x, y, z) \in \mathbb{R}^3\colon\ ((x-1)^2 + y^2 + z^2 - 1) \cdot ((x+1)^2 + y^2 + z^2 - 1) = 0\}$$

and $M = \{(x, y, z) \in \mathbb{R}^3\colon\ x^2 + y^2 - 1 = 0 = x + y + z - 1\}$. In all cases determine tangent and normal spaces T_aM and N_aM.

Problem 19.8 Let $f \in C^1(\mathbb{R}^2; \mathbb{R})$ and assume that $f(x_0, u_0) = 0$ and $f_x(x_0, u_0) > 0$. Prove the implicit function theorem from the intermediate value theorem. (Hint: it is sufficient and easier to look at rectangular, rather than circular, neighborhoods).

Problem 19.9 Suppose that $F\colon \mathbb{R}^3 \to \mathbb{R}$ is C^1 and at $(0, 0, 0)$ all the partials of F are non-zero: $F_x(0, 0, 0) \neq 0$, $F_y(0, 0, 0) \neq 0$ and $F_z(0, 0, 0) \neq 0$. By the Implicit Function Theorem the equation $F(x, y, z) = 0$ can be solved near $(0, 0, 0)$ for each variable in terms of the remaining two: $x = f(y, z)$, $y = g(x, z)$ and $z = h(x, y)$. Show that

$$\frac{\partial f}{\partial y} \cdot \frac{\partial g}{\partial z} \cdot \frac{\partial h}{\partial x} = -1 \quad \Big(\text{equivalently, } \frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = -1\Big).$$