Ordinary Differential Equations

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Contents

1 Laplace transforms and discontinuous functions

1.1 Heaviside (or unit step) function

Heaviside function

In many engineering applications, the driving force (or forcing function) is not continuous. The Heaviside function (or unit step function) is a typical engineering function that determines the "on" or "off" state of a mechanical or electrical driving force.

The Heaviside unit step function is defined by:

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} .$$

See Figure 1

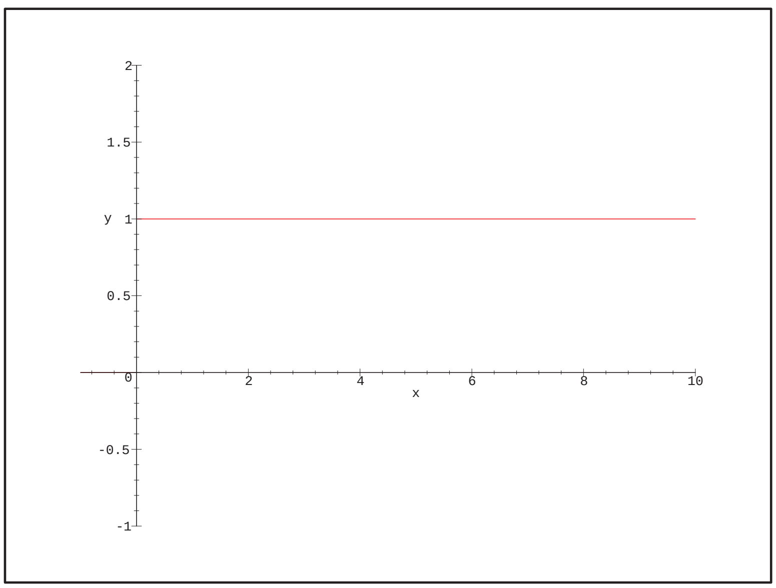

Figure 1: Heaviside function $H(t)$. The jump is at $t = 0$.

A function representing a unit step at $t = a$ can be obtained by a horizontal translation of duration a. In this case we'll have:

$$H(t-a) = \begin{cases} 0 & 0 \leq t < a \\ 1 & t \geq a \end{cases} .$$

See Figure 2.

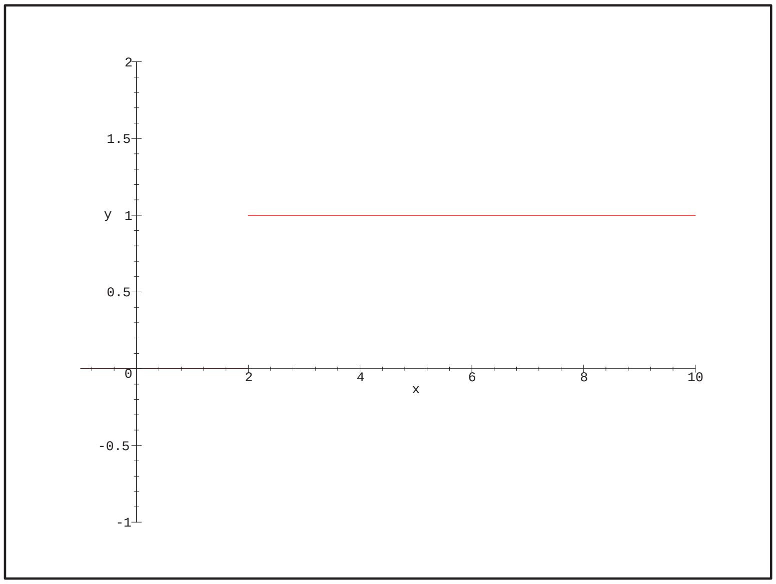

Figure 2: Heaviside function $H(t-2)$. The jump is at $t = 2$.

So, the function $H(t-a)$ can be interpreted as a device for switching on a certain function $f(t)$ at $t = a$:

$$f(t)H(t-a) = \begin{cases} 0 & t < a \\ f(t) & t \geq a \end{cases}.$$

See Figure 3.

1.1.1 Second shifting theorem

Second shifting theorem

The first shifting theorem (s-shifting) says:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad (s > a).$$

We are now going to see the *second shifting theorem* (t-shifting). Sometimes this is also called the *Heaviside* or *delay* theorem.

Theorem 1 (Second shifting theorem)**.** *This theorem states that if*

$$\mathcal{L}\{f(t)\} = F(s)$$

then:

$$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$$

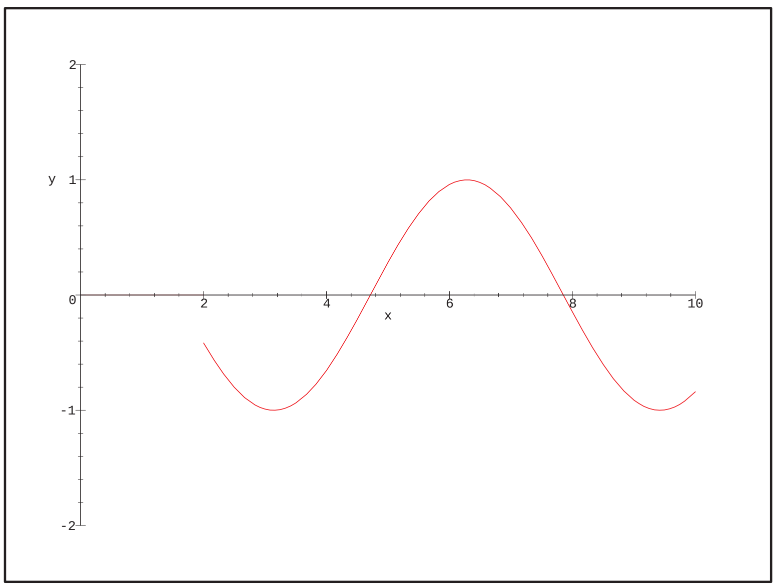

Figure 3: Heaviside function $cos(t)H(t-2)$. The cosine function is switched on at $t = 2$.

or, if we take the inverse on both sides:

$$f(t-a)H(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}.$$

Basically, if we know the Laplace transform $F(s)$ of $f(t)$, then we can easily calculate the Laplace transform of $f(t-a)H(t-a)$ by multiplying $F(s)$ by e^{-as}.

This theorem can be proven as follows.

By definition,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

thus:

$$\begin{aligned}
\mathcal{L}\{f(t-a)H(t-a)\} &= \int_0^\infty e^{-st} f(t-a)H(t-a)\, dt \\
&= \int_a^\infty e^{-st} f(t-a)\, dt.
\end{aligned}$$

Note that the integral is now from a to ∞, since the integrand is equal to zero for $t < a$, because of the heaviside function.

Now we make the substitution $T = t - a$, because we want to integrate from 0 (as required for a Laplace transform). Therefore:

$$\begin{aligned}
\mathcal{L}\{f(t-a)H(t-a)\} &= \int_0^\infty e^{-s(T+a)} f(T)\, dT \\
&= e^{-sa} \int_0^\infty e^{-sT} f(T)\, dT.
\end{aligned}$$

But

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-sT} f(T)\, dT$$

therefore:

$$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-sa}F(s).$$

1.1.2 How can we use the Heaviside (unit step) function?

Example uses of the Heaviside function

Consider the piecewise continuous function:

$$f(t) = \begin{cases} f_1(t) & 0 \le t < t_1 \\ f_2(t) & t_1 \le t < t_2 \\ f_3(t) & t \ge t_2 \end{cases}.$$

To construct this function, all we need to do is to

(a) At $t = 0$ switch on the function $f_1(t)$.

(b) At $t = t_1$ switch off the function $f_1(t)$ and switch on the function $f_2(t)$.

(c) At $t = t_2$ switch off the function $f_2(t)$ and switch on the function $f_3(t)$.

How do we accomplish this?? We can express the function $f(t)$ in terms of the Heaviside function:

$$\begin{aligned} f(t) &= f_1(t)H(t) + \left[f_2(t) - f_1(t)\right] H(t - t_1) \\ &\quad \left[f_3(t) - f_2(t)\right] H(t - t_2). \end{aligned}$$

We can check quite easily whether the function above is indeed the function $f(t)$:

(a) For $0 \leq t < t_1$ $H(t) = 1$, $H(t - t_1) = 0$ and $H(t - t_2) = 0$, yielding $f(t) = f(t_1)$.

(b) For $t_1 \leq t < t_2$ $H(t) = 1$, $H(t - t_1) = 1$ and $H(t - t_2) = 0$, yielding $f(t) = f_1(t) + [f_2(t) - f_1(t)] = f(t_2)$.

(c) For $t > t_2$ $H(t) = 1$, $H(t - t_1) = 1$ and $H(t - t_2) = 1$, yielding $f(t) = f_1(t) + [f_2(t) - f_1(t)] + [f_3(t) - f_2(t)] = f(t_3)$.

1.1.3 Piecewise continuous functions

Piecewise continuous function and Heaviside functions

Consider the piecewise continuous function (shown in Figure 4):

$$f(t) = \begin{cases} t^2 & 0 \leq t < 3 \\ t + 2 & 3 \leq t < 6 \\ 4t & t \geq 6 \end{cases}.$$

To construct this function, all we need to do is:

(a) At $t = 0$ switch on the function $f_1(t) = t^2$.

(b) At $t = 3$ switch off the function $f_1(t) = t^2$ and switch on the function $f_2(t) = t + 2$.

(c) At $t = 6$ switch off the function $f_2(t) = t + 2$ and switch on the function $f_3(t) = 4t$.

In terms of the Heaviside functions:

$$\begin{aligned} f(t) &= t^2 H(t) + \left[(t + 2) - t^2\right] H(t - 3) \\ &\quad + \left[4t - (t + 2)\right] H(t - 6). \end{aligned}$$

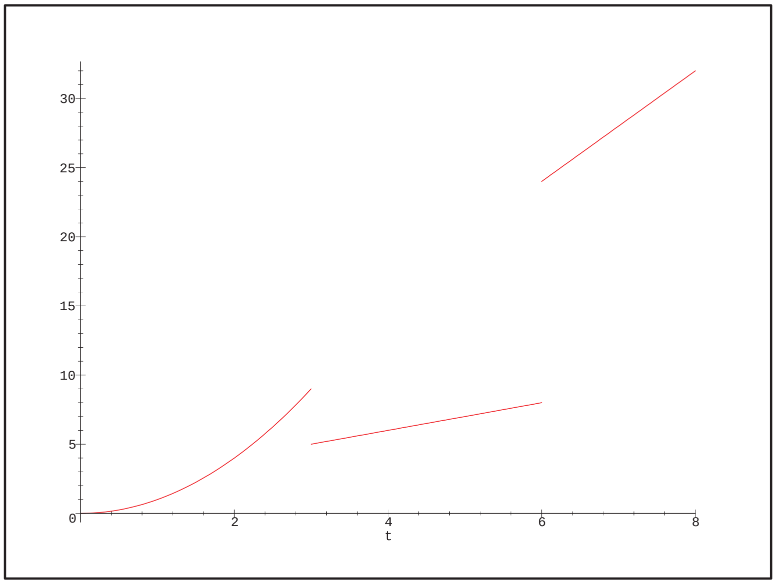

Figure 4: Piecewise continuous function.

Another piecewise continuous function

Consider the piecewise continuous function (shown in Figure 5):

$$f(t) = \begin{cases} 0 & 0 \le t < 2 \\ 1 & 2 \le t < 4 \\ 2 & 4 \le t < 6 \\ 3 & 6 \le t < 8 \\ 2 & 8 \le t < 10 \\ 1 & 10 \le t < 12 \\ 0 & t \ge 12 \end{cases} .$$

In terms of the Heaviside functions:

$$\begin{aligned} f(t) &= (1-0)H(t-2) + (2-1)H(t-4) + (3-2)H(t-6) \\ &\quad +(2-3)H(t-8) + (1-2)H(t-10) + (0-1)H(t-12) \\ &= H(t-2) + H(t-4) + H(t-6) - H(t-8) \\ &\quad -H(t-10) - H(t-12). \end{aligned}$$

Yet another piecewise continuous function

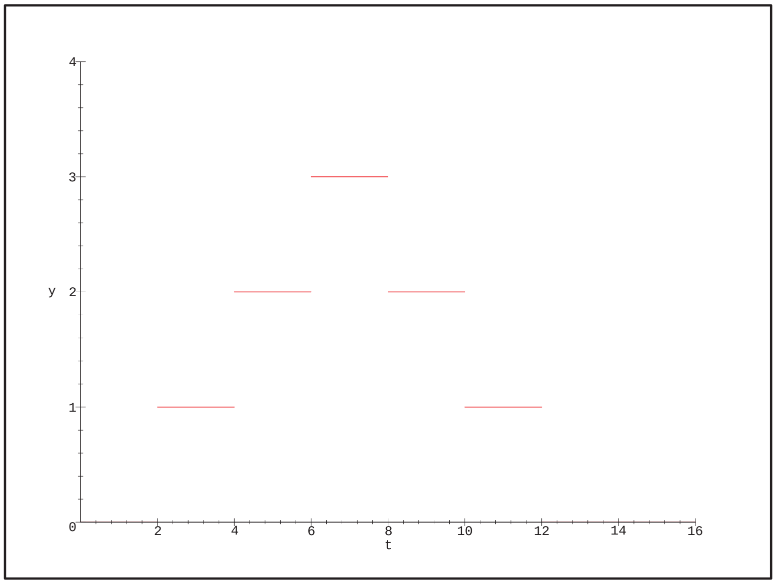

Figure 5: Another piecewise continuous function.

Consider the piecewise continuous function (shown in Figure 6):

$$f(t) = \begin{cases} 1 & 0 \leq t < \pi \\ 0 & \pi \leq t < 2\pi \\ \cos t & t > 2\pi \end{cases} .$$

In terms of the Heaviside functions:

$$\begin{aligned} f(t) &= H(t) + (0-1)H(t-\pi) + (\cos t - 0)H(t-2\pi) \\ &= H(t) - H(t-\pi) + H(t-2\pi)\cos t. \end{aligned}$$

1.1.4 Top hat function

Top hat function

We can construct piecewise continuous functions with the *top hat function* $H(t-a) - H(t-b)$ (shown in Figure 7):

$$H(t-a) - H(t-b) = \begin{cases} 1 & a \leq t < b \\ 0 & \text{otherwise} \end{cases} .$$

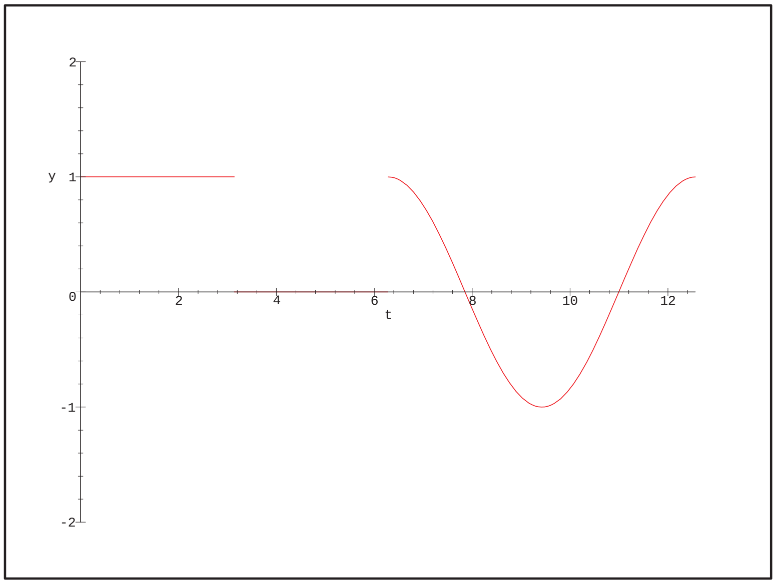

Figure 6: Yet another piecewise continuous function.

Consider again the piecewise continuous function:

$$f(t) = \begin{cases} f_1(t) & 0 \le t < t_1 \\ f_2(t) & t_1 \le t < t_2 \\ f_3(t) & t \ge t_2 \end{cases} .$$

and let's construct it in terms of the top hat function:

$$\begin{aligned} f(t) &= f_1(t)\,[H(t) - H(t-t_1)] + f_2(t)\,[H(t-t_1) - H(t-t_2)] \\ &\quad + f_3(t)\,[H(t-t_2)]\,. \end{aligned}$$

1.1.5 Laplace transform of the Heaviside function

Laplace transform of Heaviside function

The Heaviside unit step function is defined by:

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \ge 0 \end{cases} .$$

We can find the Laplace transform from the definition:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

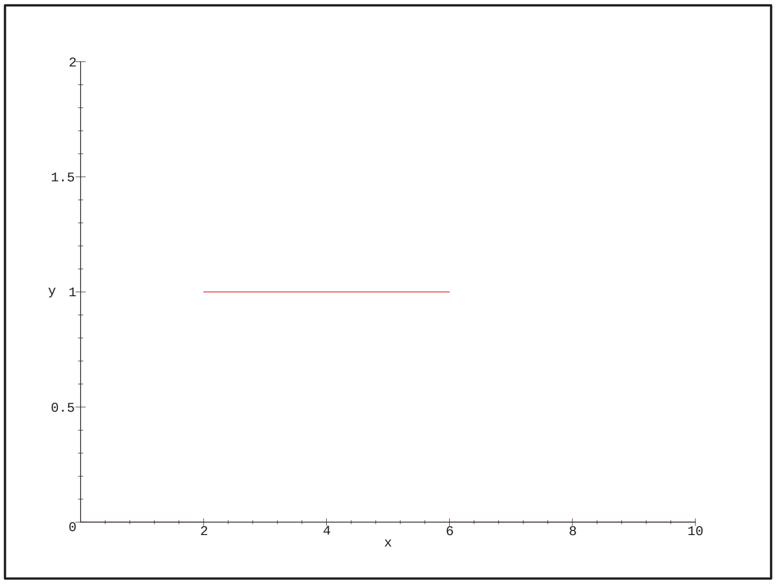

Figure 7: Top hat function $H(t-a)-H(t-b)$. The Jump is at $t=a$ and $t=b$.

that is:

$$\begin{aligned}
\mathcal{L}\{H(t-a)\} &= \int_0^\infty e^{-st} H(t-a)\,dt \\
&= \int_0^a e^{-st} 0\,dt + \int_a^\infty e^{-st} 1\,dt \\
&= \left[-\frac{1}{s}e^{-st}\right]_a^\infty \\
&= \frac{1}{s}e^{-as}
\end{aligned}$$

with $s>0$ otherwise the integral diverges.

Thus we have worked out another important Laplace transform:

$$\boxed{\mathcal{L}\{H(t-a)\} = \tfrac{1}{s}e^{-as}, \qquad (s>0).}$$

1.1.6 Example: Laplace transform of Heaviside functions

Example 1 (Laplace transform of Heaviside functions). *Find the Laplace transform of the following function:*

$$f(t) = \begin{cases} 3 & 0 < t \leq \pi \\ 0 & \pi < t \leq 2\pi \\ \sin t & t > 2\pi \end{cases}.$$

Solution

Let's write this function in terms of Heaviside functions:

$$\begin{aligned} f(t) &= 3H(t) - 3H(t-\pi) + H(t-2\pi)\sin t \\ &= 3H(t) - 3H(t-\pi) + H(t-2\pi)\sin(t-2\pi). \end{aligned}$$

Here, I have replaced the last term $\sin t$ with $\sin(t - 2\pi)$ so that I can use the second shifting theorem $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$. Also remember that the Laplace transform of the Heaviside function is $\mathcal{L}\{H(t-a)\} = \frac{1}{s}e^{-as}$. Thus:

$$\mathcal{L}\{f(t)\} = \frac{3}{s} - \frac{3e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}.$$

1.1.7 Application: RC circuit

RC-circuit

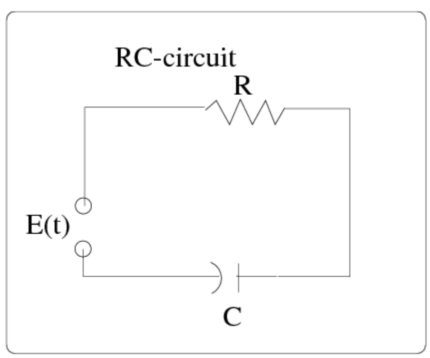

Figure 8: Example RC circuit

Consider a RC-circuit, as shown in Figure 8, and let's see what the resulting current $i(t)$ is if we apply a single square pulse of voltage V_0.

By Kirchoff's voltage law and $i = dq/dt$, we get:

$$\frac{1}{C}\int i(t)\,dt + Ri = v(t)$$

where $v(t)$ is given by:

$$v(t) = V_0 \left[H(t-a) - H(t-b)\right].$$

Thus:

$$\frac{1}{C}\int i(t)\,dt + Ri = V_0 \left[H(t-a) - H(t-b)\right].$$

Let's take the Laplace transforms. We can use the Laplace transform of the integral of a function:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$$

and we get:

$$RI(s) + \frac{I(s)}{Cs} = \frac{V_0}{s}\left[e^{-as} - e^{-bs}\right].$$

Thus:

$$I(s) = \frac{\frac{V_0}{R}}{s + \frac{1}{RC}}\left(e^{-as} - e^{-bs}\right) = F(s)\left(e^{-as} - e^{-bs}\right)$$

where we have set $F(s) = \frac{\frac{V_0}{R}}{s+\frac{1}{RC}}$.

From the Laplace tables we get:

$$\mathcal{L}^{-1}\{F(s)\} = \frac{V_0}{R}e^{-t/RC}.$$

Now we use the second shifting theorem $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$ to find:

$$\begin{aligned} i(t) &= \mathcal{L}^{-1}\{I(s)\} \\ &= \mathcal{L}^{-1}\{e^{-as}F(s)\} - \mathcal{L}^{-1}\{e^{-bs}F(s)\} \\ &= \frac{V_0}{R}\left[e^{-(t-a)/RC}H(t-a) - e^{-(t-b)/RC}H(t-b)\right]. \end{aligned}$$

Therefore, if $i(t) = 0$ when $t < a$:

$$i(t) = \begin{cases} K_1 e^{-t/RC} & a < t \leq b \\ (K_1 - K_2)e^{-t/RC} & t > b \end{cases}$$

where $K_1 = \frac{V_0}{R}e^{a/RC}$ and $K_2 = \frac{V_0}{R}e^{b/RC}$. See Figure 9.

1.1.8 The impulse function

Why use the impulse function

In many applications, particularly in engineering, one may want to see what the response of a system is if the forcing function is applied suddenly and for a very short time.

Situations of this type can arise if we hit something with a hammer, if an aeroplane suffers from a hard landing, if a ship is hit by huge single wave, etc.

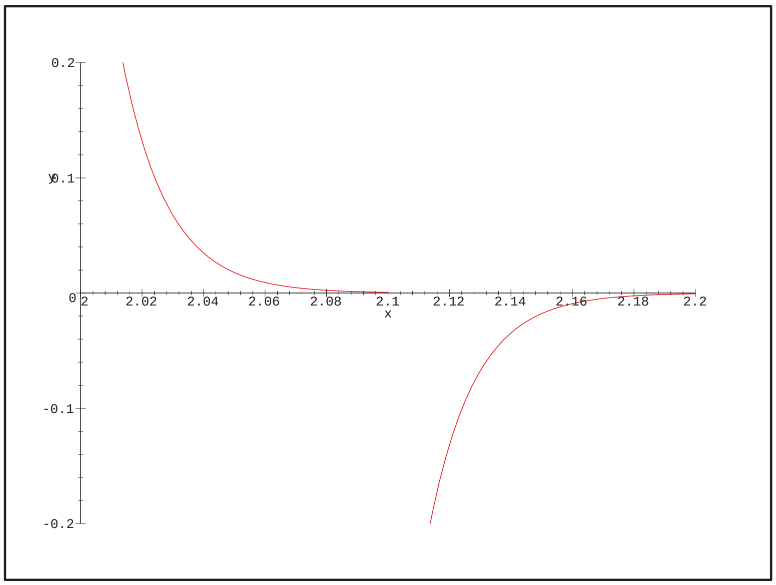

Figure 9: Current in a RC-circuit subjected to a single square wave with voltage V_0.

Mathematically, these situations can be described by functions known as *impulsive forces* which are idealised by the *impulse function*.

In mechanics, the impulse of a force $f(t)$ over a certain time interval is the integral of $f(t)$ over that interval. But what happens if the impulse of a force acts only for an instant? To deal with this problem, we consider the function:

$$f_k(t-a) = \begin{cases} 0 & 0 \le t < a-k \\ \frac{1}{2k} & a-k \le t < a+k \\ 0 & t \ge a+k \end{cases} .$$

Now take the integral:

$$I_k = \int_0^\infty f_k(t-a)\,dt = \int_{a-k}^{a+k} \frac{1}{2k}\,dt = 1$$

independently of k. Thus as $k \to 0$ the sequence of functions tends to a function that is zero everywhere except at $t = a$ where it tends to ∞. And all this while the area under the curve remains equal to 1! See Figure 10.

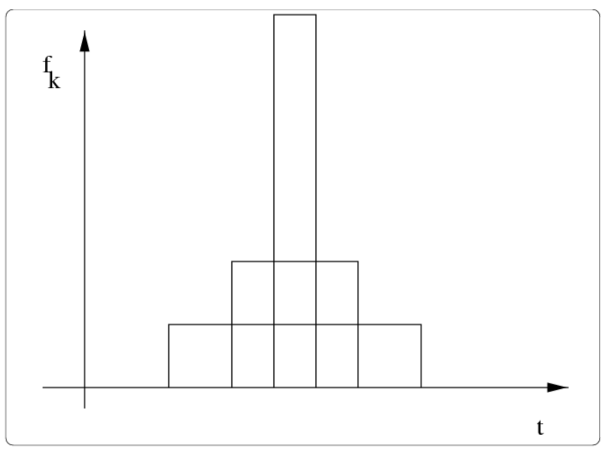

Figure 10: The area under the curve is always 1.

We can write the function $f_k(t-a)$ in terms of two Heaviside functions, that is:

$$f_k(t-a) = \frac{1}{2k}\left[H(t-(a-k)) - H(t-(a+k))\right].$$

Thus, the Laplace transform of the function $f_k(t-a)$ is:

$$\begin{aligned} \mathcal{L}\{f_k(t-a)\} &= \frac{1}{2ks}\left[e^{-(a-k)s} - e^{-(a+k)s}\right] \\ &= e^{-as}\frac{e^{ks} - e^{-ks}}{2ks}. \end{aligned}$$

1.1.9 Dirac delta function

Dirac delta function

We now define a new function, called the *Dirac delta function* or also the *unit impulse function* by taking the limit $k \to 0$ of $f_k(t-a)$:

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a).$$

The Laplace transform of this function can also be obtained by taking the limit $k \to 0$ of the Laplace transform of $f_k(t-a)$:

$$\mathcal{L}\{\delta(t-a)\} = \lim_{k\to 0} e^{-as}\frac{e^{ks}-e^{-ks}}{2ks} = e^{-as}.$$

And now we have another important Laplace transform to remember:

$$\boxed{\mathcal{L}\{\delta(t-a)\} = e^{-as}, \qquad (s>0).}$$

The Dirac delta function is not a function in the standard sense, since according to our definitions,

$$\delta(t-a) = \begin{cases} \infty & t=a \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int_0^\infty \delta(t-a)\,dt = 1.$$

But a function that is equal to zero everywhere except at a single point should really have the integral equal to zero! However, there is a branch of mathematics known as *the Theory of Distribution* that deals with *generalised functions*, such as the Dirac delta function.

1.1.10 Application: Mass-spring system

Mass-spring system

Let's see what the response of a damped mass-spring system is if we hit it with a hammer.

The differential equation of the forced mechanical oscillator is:

$$my'' + cy' + ky = r(t),$$

where $r(t)$ is the force applied to the mass. If this force is due to a hammer blow at $t=1$, we can take $r(t)$ to be the Dirac delta function.

Take $m=1$, $c=3$ and $k=2$ and consider the IVP:

$$y'' + 3y' + 2y = \delta(t-1), \qquad y(0)=0, \qquad y'(0)=0.$$

Take the Laplace transforms:

$$\begin{aligned}
\mathcal{L}\{y'' + 3y' + 2y\} &= \mathcal{L}\{\delta(t-1)\} \\
\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= \mathcal{L}\{\delta(t-1)\} \\
s^2Y - sy'(0) - y(0) + 3sY - 3y(0) + 2Y &= e^{-s} \\
s^2Y + 3sY + 2Y &= e^{-s}.
\end{aligned}$$

Thus:

$$Y = \frac{e^{-s}}{(s+1)(s+2)} = F(s)e^{-s}.$$

To find the inverse Laplace transform, we can use the second shifting theorem: $f(t-a)H(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}$ where

$$F(s) = \frac{1}{(s+1)(s+2)}.$$

Let's decompose $F(s)$ into partial fractions:

$$\begin{aligned}
F(s) &= \frac{1}{(s+1)(s+2)} \\
&= \frac{A}{(s+1)} + \frac{B}{(s+2)} \\
&= \frac{1}{(s+1)} - \frac{1}{(s+2)}.
\end{aligned}$$

Take the inverse transform:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{-t} - e^{-2t}.$$

Use now the second shifting theorem $f(t-a)H(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}$ and obtain:

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\{e^{-s}F(s)\} \\
&= f(t-1)H(t-1) \\
&= \left[e^{-(t-1)} - e^{-2(t-1)}\right] H(t-1)
\end{aligned}$$

which can be written as follows:

$$f(t-1)H(t-1) = \begin{cases} 0 & 0 \le t \le 1 \\ e^{-(t-1)} - e^{-2(t-1)} & t > 1 \end{cases}.$$

See Figure 11.

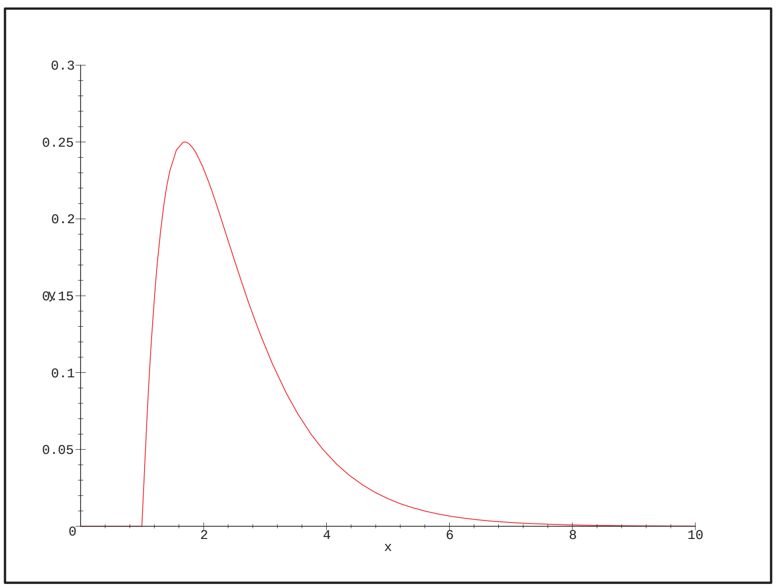

Figure 11: Response function to a hammer blow on a damped mass-spring system.

1.2 Convolution

Introduction to convolutions

As you may already have noticed, we find ourselves frequently in the situation where we know what the inverse Laplace transforms of $F(s)$ and $G(s)$ are, but we don't know what the inverse transform of their product $H(s) = F(s)G(s)$ is. But we are lucky, since there is a theorem that can help us find the inverse $h(t)$ of the product of transforms $H(s)$ if we know $f(t)$ and $g(t)$.

This theorem states that if the functions $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order k, then the inverse $h(t)$ of the product $H(s) = F(s)G(s)$ is given by the *convolution of* $f(t)$ *and* $g(t)$ which is denoted by $f(t) * g(t)$:

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)G(s)\} = f * g = \int_0^t f(\tau)g(t-\tau)\, d\tau.$$

Let's see why this works.

By definition we have:

$$\begin{aligned} F(s)G(s) &= \left[\int_0^\infty e^{-st_1} f(t_1)\, dt_1\right]\left[\int_0^\infty e^{-st_2} g(t_2)\, dt_2\right] \\ &= \int_0^\infty \int_0^\infty e^{-s(t_1+t_2)} f(t_1)g(t_2)\, dt_1\, dt_2. \end{aligned}$$

Now we make a change of variables. Set $t = t_1 + t_2$ and $\tau = t_2$. This change of variables will give new upper and lower limits of integration. If before we were integrating over the first quadrant in the t_1, t_2 plane, now we are integrating over a region in the t, τ plane that is bounded by the lines $\tau = 0$ and $\tau = t$ (see Figure 12).

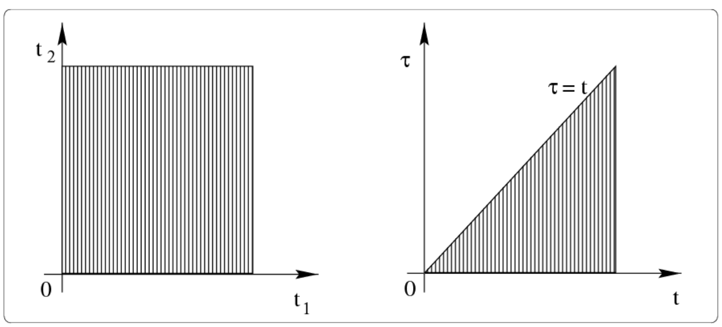

Figure 12: Left: old region of integration. Right: new region of integration.

Thus, the double integral becomes:

$$\begin{aligned}
F(s)G(s) &= \int_0^\infty \int_0^t e^{-st} f(t-\tau)g(\tau)\,d\tau\,dt \\
&= \int_0^\infty e^{-st}\left[\int_0^t f(t-\tau)g(\tau)\,d\tau\right] dt \\
&= \int_0^\infty e^{-st} g * f\,dt \\
&= \int_0^\infty e^{-st} f * g\,dt.
\end{aligned}$$

In the last step I have used the commutative law of convolution:

$$\int_0^\infty f(\tau)g(t-\tau)\,d\tau = \int_0^\infty f(t-\tau)g(\tau)\,d\tau.$$

Properties

The properties of convolution are summarised below:

- Commutative law: $f * g = g * f$
- Distributive law: $f * (g + h) = (f * g) + (f * h)$
- Associative law: $(f * g) * h = f * (g * h)$
- Zero: $f * 0 = 0$

But note that in general $f * 1 \neq f$!!

1.2.1 Examples: Convolutions

Example 2 (Convolution example)**.** *Use the convolution theorem to find the inverse Laplace transform of*

$$H(s) = \frac{2}{s^2(s^2+4)}.$$

Solution

We can see that

$$H(s) = F(s)G(s) = \left[\frac{1}{s^2}\right]\left[\frac{2}{(s^2+4)}\right].$$

We know what the inverse Laplace transforms of $F(s)$ and $G(s)$ are:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t$$

and

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \sin(2t).$$

Thus:

$$\begin{aligned}
h(t) &= \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)G(s)\} \\
&= \int_0^t f(t-\tau)g(\tau)\,d\tau \\
&= \int_0^t (t-\tau)\sin(2\tau)\,d\tau \\
&= t\int_0^t \sin(2\tau)\,d\tau - \int_0^t \tau\sin(2\tau)\,d\tau \\
&= t\left[-\frac{1}{2}\cos(2\tau)\right]_0^t - \left[\frac{1}{4}\sin(2\tau) - \frac{\tau}{2}\cos(2\tau)\right]_0^t \\
&= -\frac{t}{2}\cos(2t) + \frac{t}{2} - \frac{1}{4}\sin(2t) + \frac{t}{2}\cos(2t) \\
&= \frac{t}{2} - \frac{1}{4}\sin(2t).
\end{aligned}$$

Example 3 (Convolution example)**.** *Use the convolution theorem to find the inverse Laplace transform of*

$$H(s) = \frac{1}{s^2(s+3)}.$$

Solution
We can see that

$$H(s) = F(s)G(s) = \left[\frac{1}{s^2}\right]\left[\frac{1}{(s+3)}\right].$$

We know what the inverse Laplace transforms of $F(s)$ and $G(s)$ are:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t$$

and

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{-3t}.$$

Thus:

$$\begin{aligned}
h(t) &= \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)G(s)\} \\
&= \int_0^t f(t-\tau)g(\tau)\,d\tau \\
&= \int_0^t (t-\tau)e^{-3\tau}\,d\tau \\
&= t\int_0^t e^{-3\tau}\,d\tau - \int_0^t \tau e^{-3\tau}\,d\tau \\
&= t\left[-\frac{e^{-3\tau}}{3}\right]_0^t - \left[-\frac{\tau e^{-3\tau}}{3} - \frac{e^{-3\tau}}{9}\right]_0^t \\
&= -\frac{te^{-3t}}{3} + \frac{t}{3} + \frac{te^{-3t}}{3} + \frac{e^{-3t}}{9} - \frac{1}{9} \\
&= \frac{t}{3} + \frac{e^{-3t}}{9} - \frac{1}{9}.
\end{aligned}$$

Example 4 (Convolution example)**.** *Using the convolution theorem, verify that*

$$\mathcal{L}^{-1}\{\frac{1}{s^3}\} = \frac{t^2}{2}.$$

Solution
Write

$$H(s) = F(s)G(s) = \left[\frac{1}{s^2}\right]\left[\frac{1}{s}\right].$$

We know what the inverse Laplace transforms of $F(s)$ and $G(s)$ are:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t$$

and

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = 1.$$

Thus:

$$
\begin{aligned}
h(t) &= \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)G(s)\} \\
&= \int_0^t f(t-\tau)g(\tau)\,d\tau \\
&= \int_0^t (t-\tau)1\,d\tau \\
&= \left[t\tau - \frac{\tau^2}{2}\right]_0^t \\
&= t^2 - \frac{t^2}{2} \\
&= \frac{t^2}{2}.
\end{aligned}
$$

Note: $f * 1 \neq f$!!

1.2.2 The convolution theorem applied to DEs

Differential equations

Consider the initial value problem:

$$y'' + ay' + by = r(t), \qquad y(0) = K_0, \qquad y'(0) = K_1$$

where a and b are constant, $r(t)$ is the input (driving force) and $y(t)$ is the output.

Then

$$
\begin{aligned}
\mathcal{L}\{y''(t)\} &= s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0) \\
\mathcal{L}\{y'(t)\} &= s\mathcal{L}\{y(t)\} - y(0).
\end{aligned}
$$

So we have

$$
\begin{aligned}
\mathcal{L}\{y'' + ay' + by\} &= \mathcal{L}\{r\} \\
\mathcal{L}\{y''\} + a\mathcal{L}\{y'\} + b\mathcal{L}\{y\} &= \mathcal{L}\{r\} \\
\left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right] + a\left[s\mathcal{L}\{y\} - y(0)\right] + b\mathcal{L}\{y\} &= \mathcal{L}\{r\} \\
s^2Y - sy(0) - y'(0) + asY - ay(0) + bY &= R(s).
\end{aligned}
$$

Now collect the Y terms and get the subsidiary equation:

$$(s^2 + as + b)Y = (s+a)y(0) + y'(0) + R(s).$$

The transfer function $Q(s)$ is:

$$Q(s) = \frac{1}{s^2 + as + b}$$

thus:

$$Y = [(s+a)y(0) + y'(0)]\, Q(s) + Q(s)R(s).$$

If $y(0) = 0$ and $y'(0) = 0$, then $Y = Q(s)R(s)$, therefore, we can use the convolution theorem to find $y(t)$!

$$y(t) = \int_0^t q(t-\tau)r(\tau)\, d\tau$$

where $q(t) = \mathcal{L}^{-1}\{Q(s)\}$.

1.2.3 Application: Response of a damped mass-spring system to a single square wave input

Response of a damped mass-spring system to a single square wave input

The differential equation of the forced mechanical oscillator is:

$$my'' + cy' + ky = r(t)$$

where $r(t)$ is the force applied to the mass due to a single square wave:

$$r(t) = \begin{cases} 2 & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}.$$

In terms of Heaviside functions:

$$r(t) = 2(H(t-1) - H(t-2)).$$

Take $m = 1$, $c = 3$ and $k = 2$ and consider the IVP:

$$y'' + 3y' + 2y = 2(H(t-1) - H(t-2)), \qquad y(0) = 0, \qquad y'(0) = 0$$

Thus,

$$\mathcal{L}\{y'' + 3y' + 2y\} = R$$

(where $R = 2\mathcal{L}\{(H(t-1) - H(t-2))\}$)

$$s^2Y - sy'(0) - y(0) + 3sY - 3y(0) + 2Y = R\}$$

$$s^2Y + 3sY + 2Y = R\}$$

$$Y(s^2 + 3s + 2) = R\}.$$

So

$$Q(s) = \frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}.$$

The inverse Laplace transform of $Q(s)$ is:

$$q(t) = \mathcal{L}^{-1}\{Q(s)\} = e^{-t} - e^{-2t}.$$

We can now use the convolution theorem to find $y(t)$:

$$\begin{aligned} y(t) &= \int_0^t q(t-\tau) r(\tau)\, d\tau \\ &= \int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) 2(H(\tau-1) - H(\tau-2))\, d\tau. \end{aligned}$$

Now, if $0 \le t < 1$, then $y(t) = 0$, since $2(H(\tau-1) - H(\tau-2)) = 0$.
If $1 \le t \le 2$, then $2(H(\tau-1) - H(\tau-2)) = 2$:

$$\begin{aligned} y(t) &= 2\int_1^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau \\ &= 2\left[e^{-(t-\tau)} - \frac{e^{-2(t-\tau)}}{2}\right]_1^t \\ &= 2 - 1 - 2e^{-(t-1)} + e^{-2(t-1)} \\ &= 1 - 2e^{-(t-1)} + e^{-2(t-1)}. \end{aligned}$$

If $t > 2$, the integral becomes:

$$\begin{aligned} y(t) &= 2\int_1^2 \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) d\tau \\ &= 2\left[e^{-(t-\tau)} - \frac{e^{-2(t-\tau)}}{2}\right]_1^2 \\ &= 2e^{-(t-2)} - e^{-2(t-2)} - 2e^{-(t-1)} + e^{-2(t-1)}. \end{aligned}$$

Figure 13 shows the solution for the different regions.

Change of wave input

Solve

$$y'' + 3y' + 2y = \sin t - \sin t\, H(t-\pi), \qquad y(0) = 0, \quad y'(0) = 0.$$

Rewrite the equation as $y''+3y'+2y = r(t)$ where $r(t) = \sin t - \sin t\, H(t-\pi)$. Then, following the working from the previous example

$$\begin{aligned} y(t) &= \int_0^t q(t-\tau) r(\tau)\, d\tau \\ &= \int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) [\sin t - \sin t\, H(t-\pi)]\, d\tau. \end{aligned}$$

Note:

$$\int \sin ax\, e^{bx}\, dx = \frac{e^{bx}}{a^2+b^2}\left(b \sin ax - a\cos ax\right).$$

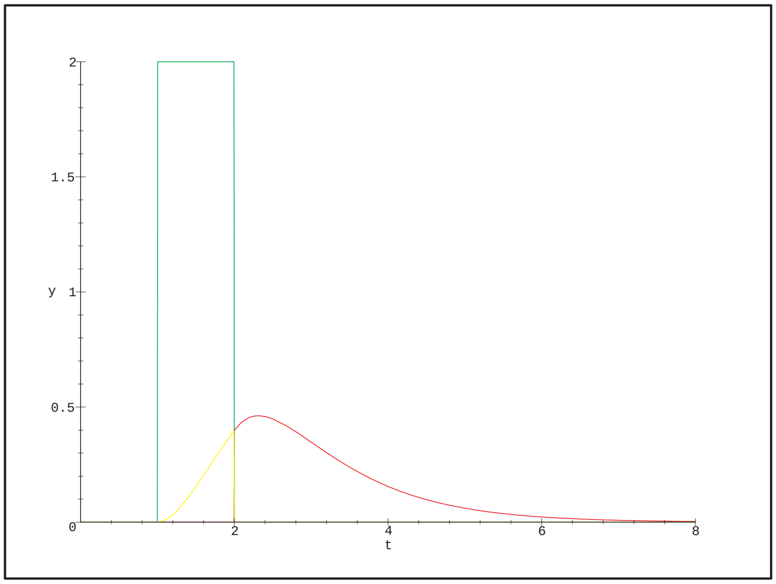

Figure 13: Square wave input and system response

If $0 \leq t < \pi$,

$$
\begin{aligned}
y(t) &= \int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) \sin\tau \, d\tau \\
&= e^{-t} \int_0^t e^{\tau} \sin\tau \, d\tau - e^{-2t} \int_0^t e^{2\tau} \sin\tau \, d\tau \\
&= e^{-t} \left[\frac{e^{\tau}}{2} (\sin\tau - \cos\tau)\right]_{\tau=0}^{\tau=t} - e^{-2t} \left[\frac{e^{2\tau}}{5} (2\sin\tau - \cos\tau)\right]_{\tau=0}^{\tau=t} \\
&= \frac{1}{10} \left(5e^{-t} - 2e^{-2t} + \sin t - 3\cos t\right).
\end{aligned}
$$

If $t \geq \pi$,

$$\begin{aligned}
y(t) &= \int_0^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) [\sin\tau - \sin\tau\, H(\tau-\pi)]\ d\tau \\
&= \int_0^\pi \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) \sin\tau\, d\tau \\
&\quad + \int_\pi^t \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) \times 0, d\tau \\
&= \int_0^\pi \left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right) \sin\tau\, d\tau \\
&= e^{-t}\left[\frac{e^\tau}{2}(\sin\tau - \cos\tau)\right]_{\tau=0}^{\tau=\pi} - e^{-2t}\left[\frac{e^{2\tau}}{5}(2\sin\tau - \cos\tau)\right]_{\tau=0}^{\tau=\pi} \\
&= \frac{1}{2}e^{-t}\left(e^\pi + 1\right) - \frac{1}{5}e^{-2t}\left(e^{2\pi}+1\right).
\end{aligned}$$

1.3 Laplace transforms of periodic functions

Periodic functions

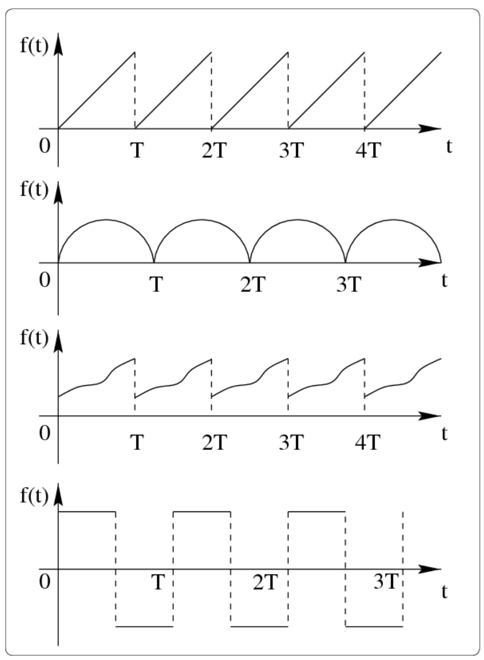

Figure 14: Some examples of periodic functions

In many applications, particularly those concerning mechanical vibrations and electrical oscillations, one may have to deal with period input functions.

Therefore, it is a good idea to find out what the Laplace transform of a periodic function is. The following theorem provides an explicit expression for the Laplace transform of periodic functions.

If $f(t)$ is a piecewise continuous function on $[0, \infty)$ with a period T, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt, \qquad s > 0.$$

Let's see how this works.

We can write the Laplace transform as a series of integrals in the following way:

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty f(t) e^{-st}\, dt \\
&= \int_0^T f(t) e^{-st}\, dt + \int_T^{2T} f(t) e^{-st}\, dt + \int_{2T}^{3T} f(t) e^{-st}\, dt \\
&\quad + \int_{3T}^{4T} f(t) e^{-st}\, dt + \cdots + \int_{(n-1)T}^{nT} f(t) e^{-st}\, dt + \cdots .
\end{aligned}$$

Now make the substitution

$$t = \tau + nT$$

where $n = 0, 1, 2, 3, 4, \cdots$.

Thus:

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \sum_{n=0}^\infty \int_0^T f(\tau + nT) e^{-s(\tau + nT)}\, d\tau \\
&= \sum_{n=0}^\infty \int_0^T f(\tau) e^{-s(\tau + nT)}\, d\tau \\
&= \sum_{n=0}^\infty \int_0^T f(\tau) e^{-s\tau} e^{-nTs}\, d\tau \\
&= \sum_{n=0}^\infty \left[e^{-nTs}\right] \int_0^T f(\tau) e^{-s\tau}\, d\tau.
\end{aligned}$$

Here, I used the fact that $f(t)$ is a periodic function of period T and thus $f(\tau + nT) = f(\tau)$.

Now note that:

$$\sum_{n=0}^\infty e^{-nTs} = 1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \cdots$$

This is an infinite geometric progression so we know what its sum is!!

$$\sum_{n=0}^\infty e^{-nTs} = \frac{1}{1 - e^{-Ts}}.$$

Therefore:

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T f(t)e^{-st}\,dt.$$

Note that if $f(t)$ is a periodic function with period T and

$$g(t) = f(t)\left(H(t) - H(t-T)\right)$$

then:

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}} \mathcal{L}\{g(t)\}.$$

This is because the function $f(t)$ is periodic and $g(t) = 0$ for $t > T$.

1.3.1 Example: Periodic functions

Example 5 (Periodic functions example 1)**.** *Find the Laplace transform of the repeated pulse wave function (shown in Figure 15):*

$$f(t) = \begin{cases} 5 & 0 \le t \le 2 \\ 0 & 2 < t < 4 \end{cases}.$$

Solution

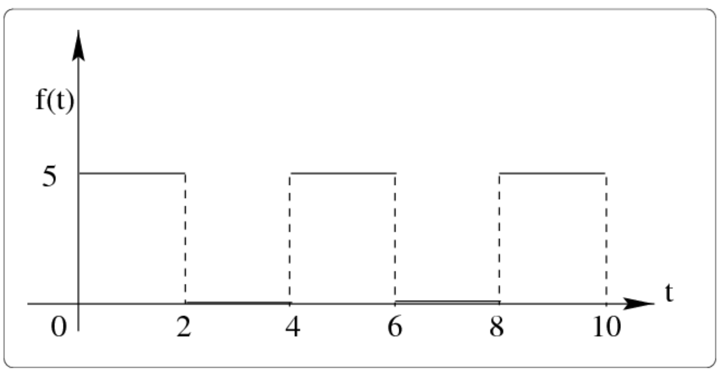

Figure 15: Repeated pulse wave function

The period of this function is $T = 4$. In terms of Heaviside functions, $g(t)$ can be written as

$$\begin{aligned} g(t) &= f(t)\left(H(t) - H(t-T)\right) = f(t)\left(H(t) - H(t-4)\right) \\ &= 5\left(H(t) - H(t-2)\right) + 0\left(H(t-2) - H(t-4)\right) \\ &= 5\left(H(t) - H(t-2)\right). \end{aligned}$$

Here I used the fact that $f(t) = 0$ for $2 < t < 4$.

Now we can find the Laplace transform using:

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}} \mathcal{L}\{g(t)\}.$$

So we need to calculate $\mathcal{L}\{g(t)\}$:

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \mathcal{L}\{5\,(H(t) - H(t-2))\} \\
&= \frac{5}{s} - \frac{5e^{-2s}}{s}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{5}{1-e^{-4s}}\left[\frac{1}{s} - \frac{e^{-2s}}{s}\right] \\
&= \frac{5}{1-e^{-4s}}\left[\frac{1-e^{-2s}}{s}\right] \\
&= \frac{5}{s(1+e^{-2s})}.
\end{aligned}$$

Example 6 (Periodic functions example 2)**.** *Find the Laplace transform of the sawtooth wave function (shown in Figure 16):*

$$f(t) = t, \qquad 0 < t < 3.$$

Solution

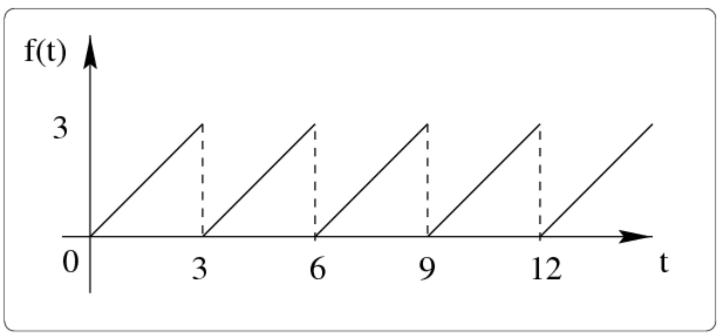

Figure 16: Sawtooth wave function

The period of this function is $T = 3$. Thus:

$$g(t) = f(t)\,(H(t) - H(t-T)) = t\,(H(t) - H(t-3))\,.$$

The Laplace transform of $f(t)$ is given by:

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}}\mathcal{L}\{g(t)\}$$

therefore:

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \mathcal{L}\{t\,(H(t) - H(t-3))\} \\
&= \frac{1}{s^2} - \frac{3se^{-3s} + e^{-3s}}{s^2}.
\end{aligned}$$

Note that I used the formula of differentiation of transform: $\mathcal{L}\{tk(t)\} = -K'(s)$, where, in our case, $k(t) = H(t-3)$.

So, finally we have:

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-3s}}\left[\frac{1}{s^2} - \frac{3se^{-3s}+e^{-3s}}{s^2}\right] \\ &= \frac{1-3se^{-3s}-e^{-3s}}{s^2\left(1-e^{-3s}\right)}. \end{aligned}$$

Example 7 (Periodic functions example 3)**.** *Find the Laplace transform of the half-wave rectifier function (shown in Figure 17):*

$$f(t) = \begin{cases} \sin(at) & 0 < t < \frac{\pi}{a} \\ 0 & \frac{\pi}{a} < t < 2\frac{\pi}{a} \end{cases}$$

Solution

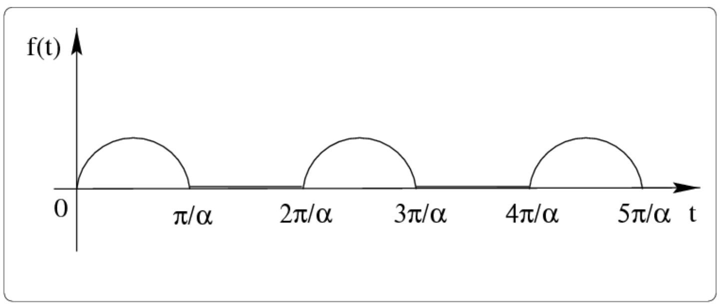

Figure 17: Half-wave rectifier. The period is $T = 2\pi/a$

The period of this function is $T = \frac{2\pi}{a}$. In terms of Heaviside functions, $g(t)$ can be written as

$$\begin{aligned} g(t) &= f(t)\left(H(t) - H(t-T)\right) \\ &= f(t)\left(H(t) - H(t-\frac{2\pi}{a})\right) \\ &= \sin(at)\left(H(t) - H(t-\frac{\pi}{a})\right) + 0\left(H(t-\frac{\pi}{a}) - H(t-\frac{2\pi}{a})\right) \\ &= \sin(at)\left(H(t) - H(t-\frac{\pi}{a})\right) \\ &= \sin(at)H(t) + \sin\left[a\left(t-\frac{\pi}{a}\right)\right]H\left(t-\frac{\pi}{a}\right). \end{aligned}$$

Here I used the fact that $f(t) = 0$ for $\frac{\pi}{a} < t < \frac{2\pi}{a}$ and $\sin(at) = -\sin\left[a\left(t-\frac{\pi}{a}\right)\right]$.

The Laplace transform of $f(t)$ is given by:

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}}\mathcal{L}\{g(t)\}.$$

Thus, the Laplace transform of $g(t)$ is given by:

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \mathcal{L}\left\{\sin(at)H(t) + \sin a\left(t - \frac{\pi}{a}\right) H\left(t - \frac{\pi}{a}\right)\right\} \\
&= \frac{a}{s^2 + a^2} + \frac{ae^{\frac{-s\pi}{a}}}{s^2 + a^2} \\
&= \frac{a(1 + e^{\frac{-s\pi}{a}})}{s^2 + a^2}.
\end{aligned}$$

Here I used the second shifting theorem $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$.
And we finally have:

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1}{1 - e^{\frac{-2s\pi}{a}}} \frac{a\left(1 + e^{\frac{-s\pi}{a}}\right)}{s^2 + a^2} \\
&= \frac{a}{\left(1 - e^{\frac{-s\pi}{a}}\right)(s^2 + a^2)}.
\end{aligned}$$