MATH1014

Semester 1

Administrative Overview

Lecturers:

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Assessment

- Midsemester exam (date TBA) (25%)
- Final exam (45%)
- Web Assign quizzes (10%)
- Tutorial quizzes (10%)
- Tutorial participation (5%)
- Written assignment (5%)

Tips for success:

- Ask questions!
- Make use of the available resources!
- Don't fall behind!

Linear Algebra

- We will be covering most of the material in Stewart, Sections 10.1, 10.2, 10.3 and 10.4, and Lay Chapters 4 and 5, and Chapter 6, Sections 1 - 6.
- Vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$, dot products, cross products in $\mathbf{R}^3$, planes and lines in $\mathbf{R}^3$ (Stewart).
- Properties of Vector Spaces and Subspaces.
- Linear Independence, bases and dimension, change of basis.
- Applications to difference equations, Markov chains.
- Eigenvalues and eigenvectors.
- Orthogonality, Gram-Schmidt process. Least squares problem.

Coordinates, Vectors and Geometry in $\mathbb{R}^3$

From Stewart, §10.1, §10.2

Question: How do we describe 3-dimensional space?

1. Coordinates
2. Lines, planes, and spheres in $\mathbb{R}^3$
3. Vectors

Euclidean Space and Coordinate Systems

We identify points in the plane ($\mathbb{R}^2$) and in three-dimensional space ($\mathbb{R}^3$) using coordinates.

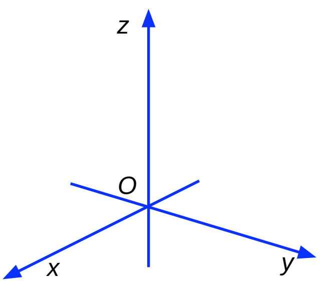
$$\mathbb{R}^3 = \{(x, y, z) \ : \ x, y, z \in \mathbb{R}\}$$

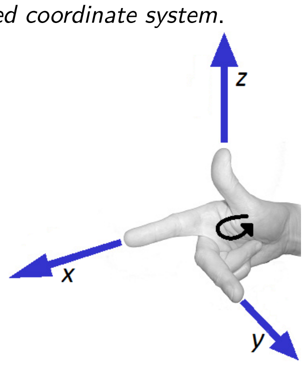
reads as "$\mathbb{R}^3$ is the set of ordered triples of real numbers".

We first choose a fixed point $\mathbf{O} = (0, 0, 0)$, called the *origin*, and three directed lines through O that are perpendicular to each other. We call these the *coordinate axes* and label them the x-axis, the y-axis and the z-axis.

Usually we think of the x- and y-axes as being horizontal and the z-axis as being vertical.

Together, $\{x, y, z\}$ form a *right-handed coordinate system*.





Compare this to the axes we use to describe $\mathbb{R}^2$, where the x-axis is horizontal and the y-axis is vertical.

The Distance Formula

Definition

The *distance* $\mid P_1P_2 \mid$ between the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$\mid P_1P_2 \mid = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Definition

The *distance* $\mid P_1P_2 \mid$ between the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is

$$\mid P_1P_2 \mid = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

1.1 Surfaces in $\mathbb{R}^3$

Lines, planes, and spheres are special sets of points in $\mathbb{R}^3$ which can be described using coordinates.

Example 1

The sphere of radius r with centre $C = (c_1, c_2, c_3)$ is the set of all points in $\mathbb{R}^3$ with distance r from C:

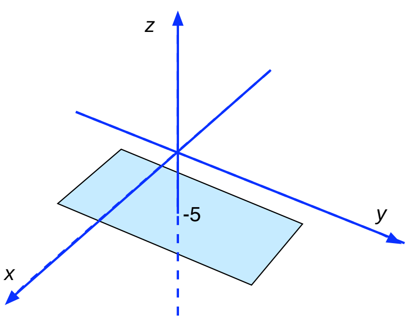
$$S = \{P \ : \ |PC| = r\}.$$

Equivalently, the sphere consists of all the solutions to this equation:

$$(x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 = r^2.$$

Example 2

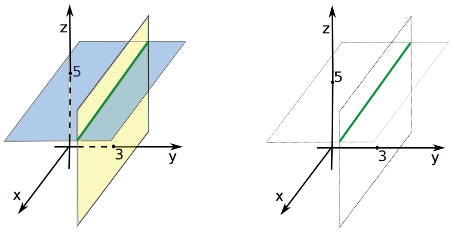
The equation $z = -5$ in $\mathbb{R}^3$ represents the set $\{(x, y, z) \mid z = -5\}$, which is the set of all points whose z-coordinate is -5. This is a horizontal plane that is parallel to the xy-plane and five units below it.



Example 3

What does the pair of equations $y = 3, z = 5$ represent? In other words, describe the set of points

$$\{(x, y, z)\ : y = 3 \text{ and } z = 5\} = \{(x, 3, 5)\}.$$



Connections with linear equations

Recall from 1013 that a system of linear equations defines a *solution set*. When we think about the unknowns as coordinate variables, we can ask what the solution set looks like.

- A single linear equation with 3 unknowns will **usually** have a solution set that's a plane. (e.g., Example 2 or $3x + 2y - 5z = 1$)
- Two linear equations with 3 unknowns will **usually** have a solution set that's a line. (e.g., Example 3 or $3x + 2y - 5z = 1$ and $x + z = 2$)
- Three linear equations with 3 unknowns will **usually** have a solution set that's a point (i.e., a unique solution).

Question

When do these heuristic guidelines fail?

Vectors

We'll study vectors both as formal mathematical objects and as tools for modelling the physical world.

Definition

A *vector* is an object that has both magnitude and direction.

Physical quantities such as velocity, force, momentum, torque, electromagnetic field strength are all "vector quantities" in that to specify them requires both a magnitude and a direction.

Vectors

Definition

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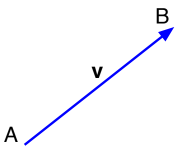
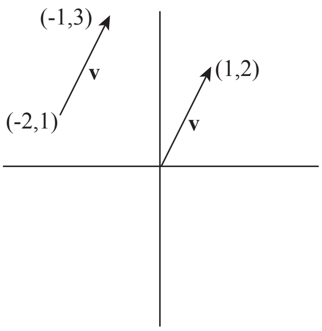


We represent vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ by arrows. For example, the vector $\mathbf{v}$ has initial point A and terminal point B and we write $\mathbf{v} = \vec{AB}$.
The *zero vector* $\mathbf{0}$ has length zero (and no direction).

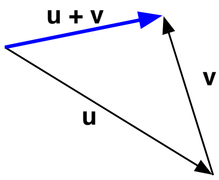
Since a vector doesn't have "location" as one of its properties, we can slide the arrow around as long as we don't rotate or stretch it.

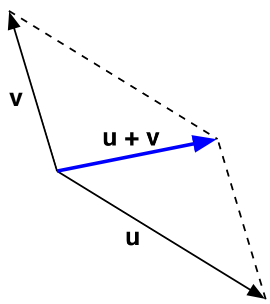


We can describe a vector using the coordinates of its head when its tail is at the origin, and we call these the *components* of the vector. Thus in this example $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and we say the components of $\mathbf{v}$ are 1 and 2.

Vector Addition

If an arrow representing $\mathbf{v}$ is placed with its tail at the head of an arrow representing $\mathbf{u}$, then an arrow from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$ represents the sum $\mathbf{u} + \mathbf{v}$.





Suppose that $\mathbf{u}$ has components a and b and that $\mathbf{v}$ has components x and y. Then $\mathbf{u} + \mathbf{v}$ has components $a + x$ and $b + y$:

$$\mathbf{u} + \mathbf{v} = (a, b) + (x, y) = (a + x, b + y)$$

Scalar Multiplication

If $\mathbf{v}$ is a vector, and t is a real number (*scalar*), then the *scalar multiple* of $\mathbf{v}$ is a vector with magnitude $|t|$ times that of $\mathbf{v}$, and direction the same as $\mathbf{v}$ if $t > 0$, or opposite to that of $\mathbf{v}$ if $t < 0$.

If $t = 0$, then $t\mathbf{v}$ is the zero vector $\mathbf{0}$.
If $\mathbf{u}$ has components a and b, then $t\mathbf{v}$ has components tx and ty:

$$t\mathbf{v} = t\langle x, y\rangle = \langle tx, ty\rangle.$$

Example

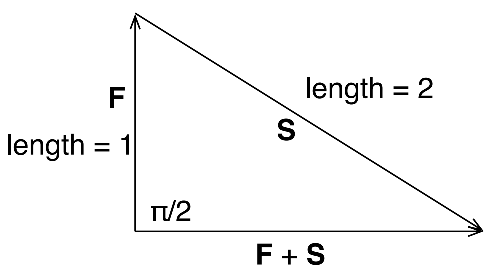
Example 4

A river flows north at 1km/hr, and a swimmer moves at 2km/hr relative to the water.

- At what angle to the bank must the swimmer move to swim east across the river?
- What is the speed of the swimmer relative to the land?

There are several velocities to be considered:
The velocity of the river, $\mathbf{F}$, with $\|\mathbf{F}\| = \mathbf{1}$;
The velocity of the swimmer relative to the water, $\mathbf{S}$, so that $\|\mathbf{S}\| = \mathbf{2}$;
The resultant velocity of the swimmer, $\mathbf{F} + \mathbf{S}$, which is to be perpendicular to $\mathbf{F}$.

The problem is to determine the *direction* of $\mathbf{S}$ and the *magnitude* of $\mathbf{F} + \mathbf{S}$.



From the figure it follows that the angle between $\mathbf{S}$ and $\mathbf{F}$ must be $2\pi/3$ and the resulting speed will be $\sqrt{3}$ km/hour. □

Standard basis vectors in $\mathbb{R}^2$

The vector $\mathbf{i}$ has components 1 and 0, and the vector $\mathbf{j}$ has components 0 and 1.

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The vector $\mathbf{r}$ from the origin to the point (x, y) has components x and y and can be expressed in the form

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{i} + y\mathbf{j}.$$

The length of of a vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ is given by

$$\|\mathbf{v}\| = \sqrt{x^2 + y^2}$$

Standard basis vectors in $\mathbb{R}^3$

In the Cartesian coordinate system in 3-space we define three **standard basis vectors** $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ represented by arrows from the origin to the points $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ respectively:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Any vector can be written as a sum of scalar multiples of the standard basis vectors:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}.$$

If $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, the *length* of $\mathbf{v}$ is defined as

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}\ .$$

This is just the distance from the origin (with coordinates $0, 0, 0$) of the point with coordinates a, b, c.

A vector with length 1 is called a *unit vector*.

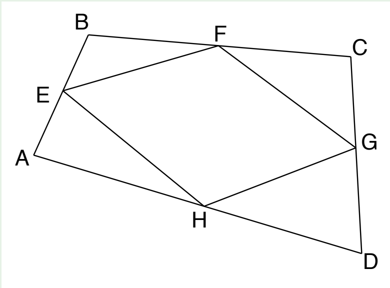
If $\mathbf{v}$ is not zero, then $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is the unit vector in the same direction as $\mathbf{v}$.

The zero vector is not given a direction.

Vectors and Shapes

Example 5

The midpoints of the four sides of any quadrilateral are the vertices of a parallelogram.

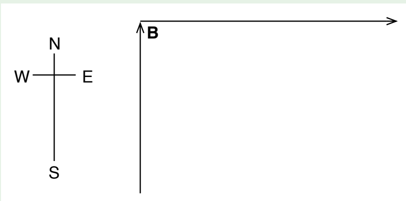


Can you prove this using vectors?

Hint: how can you tell if two vectors are parallel? How can you tell if they have the same length?

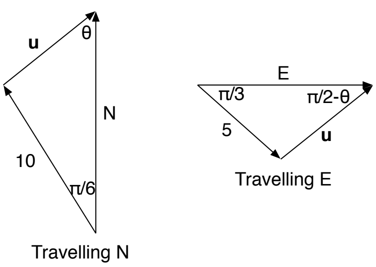
Example 6

A boat travels due north to a marker, then due east, as shown:



Travelling at a speed of 10 knots with respect to the water, the boat must head 30° west of north on the first leg because of the water current. After rounding the marker and reducing speed to 5 knots with respect to the water, the boat must be steered 60° south of east to allow for the current. Determine the velocity **u** of the water current (assumed constant).

A diagram is helpful. The vector **u** represents the velocity of the river current, and has the same magnitude and direction in both diagrams.



Applying the sine rule, we have

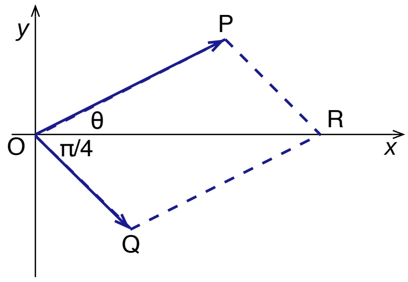
$$\frac{\sin\theta}{10} = \frac{\sin\frac{\pi}{6}}{\|\mathbf{u}\|} \qquad \frac{\cos\theta}{5} = \frac{\sin\frac{\pi}{3}}{\|\mathbf{u}\|}.$$

which are easily solvable for $\|\mathbf{u}\|$ and θ, and hence give $\mathbf{u}$. □

Example 7

An aircraft flies with an airspeed of 750 km/h. In what direction should it head in order to make progress in a true easterly direction if the wind is from the northwest at 100 km/h?

Solution The problem is 2-dimensional, so we can use plane vectors. Choose a coordinate system so that the x- and y-axes point east and north respectively.



$$
\begin{aligned}
\overrightarrow{OQ} &= \mathbf{v}_{air\ \ rel\ \ ground} \\
&= 100\cos(-\pi/4)\mathbf{i} + 100\sin(-\pi/4)\mathbf{j} \\
&= 50\sqrt{2}\mathbf{i} - 50\sqrt{2}\mathbf{j}
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{OP} &= \mathbf{v}_{aircraft\ \ rel\ \ air} \\
&= 750\cos\theta\mathbf{i} + 750\sin\theta\mathbf{j}
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{OR} &= \mathbf{v}_{aircraft\ \ rel\ \ ground} \\
&= \overrightarrow{OP} + \overrightarrow{OQ} \\
&= (750\cos\theta\mathbf{i} + 750\sin\theta\mathbf{j}) + (50\sqrt{2}\mathbf{i} - 50\sqrt{2}\mathbf{j}) \\
&= (750\cos\theta + 50\sqrt{2})\mathbf{i} + (750\sin\theta - 50\sqrt{2})\mathbf{j}
\end{aligned}
$$

We want $\mathbf{v}_{aircraft\ \ rel\ \ ground}$ to be in an easterly direction, that is, in the positive direction of the x-axis. So for ground speed of the aircraft v, we have

$$\overrightarrow{OR} = v\mathbf{i}.$$

Comparing the two expressions for $\overrightarrow{OR}$ we get

$$v\mathbf{i} = (750\cos\theta + 50\sqrt{2})\mathbf{i} + (750\sin\theta - 50\sqrt{2})\mathbf{j}.$$

This implies that

$$750\sin\theta - 50\sqrt{2} = 0 \quad \leftrightarrow \quad \sin\theta = \frac{\sqrt{2}}{15}.$$

This gives $\theta \approx 0.1$ radians $\approx 5.4^\circ$.
Using this information v can be calculated, as well as the time to travel a given distance.