Ordinary Differential Equations

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1 Laplace transforms: Systems of differential equations

System of equations

In many applications, it often happens that one may have to solve a set of simultaneous linear differential equations. But we are lucky, because the Laplace transform method can be applied to solve systems.

So, consider:

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + g_1(t), \\ y_2' &= a_{21}y_1 + a_{22}y_2 + g_2(t). \end{aligned}$$

Now if we indicate with $Y_1 = \mathcal{L}\{y_1\}$, $Y_2 = \mathcal{L}\{y_2\}$, $G_1 = \mathcal{L}\{g_1\}$ and $G_2 = \mathcal{L}\{g_2\}$, then, since $\mathcal{L}\{y'\} = sY - y(0)$, we have:

$$\begin{aligned} sY_1 - y_1(0) &= a_{11}Y_1 + a_{12}Y_2 + G_1, \\ sY_2 - y_2(0) &= a_{21}Y_1 + a_{22}Y_2 + G_2. \end{aligned}$$

Now collect the Y_1 and Y_2 terms:

$$\begin{aligned} (a_{11} - s)Y_1 + a_{12}Y_2 &= -y_1(0) - G_1 \\ a_{21}Y_1 + (a_{22} - s)Y_2 &= -y_2(0) - G_2. \end{aligned}$$

Then solve for Y_1 and Y_2. The solution is then obtained by taking the inverse Laplace transforms of Y_1 and Y_2.

1.1 Application: Electrical network

Electrical network

Find the currents $i_1(t)$ and $i_2(t)$ in the network shown in Figure 1 if $L = 1$ henry, $R = 50$ ohm, $C = 10^{-4}$ farad, $v(t) = 60$ volt, and the current is zero at $t = 0$.

As you can see, the current $i_1(t)$ divides into currents $i_2(t)$ and $i_3(t)$ at the branch point A. Thus, Kirchhoff's current law says:

$$i_1(t) = i_2(t) + i_3(t).$$

Furthermore, we have to apply Kirchhoff's voltage law to each loop:

$$\begin{aligned} L\frac{di_1}{dt} + Ri_2 &= v(t) \\ \frac{1}{C}\int_0^t i_3(\tau)\,d\tau - Ri_2 &= 0. \end{aligned}$$

Differentiate the equation for the second loop with respect to time and get:

$$\begin{aligned} L\frac{di_1}{dt} + Ri_2 &= v(t) \\ \frac{i_3}{C} - Ri_2' &= 0. \end{aligned}$$

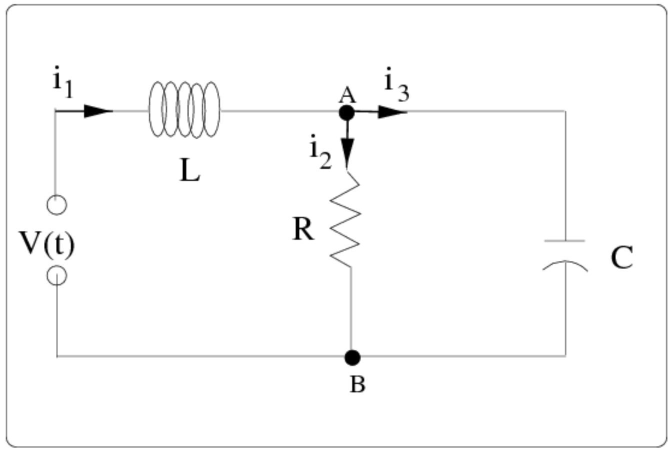

Figure 1: An electrical network

Since $i_3(t) = i_1(t) - i_2(t)$, then:

$$\begin{aligned} L\frac{di_1}{dt} + Ri_2 &= v(t) \\ \frac{i_1 - i_2}{C} - Ri_2' &= 0. \end{aligned}$$

By tidying things up we finally obtain:

$$\begin{aligned} Li_1' + Ri_2 &= v(t) \\ CRi_2' + i_2 - i_1 &= 0. \end{aligned}$$

Let's put the values in:

$$\begin{aligned} i_1' + 50i_2 &= 60 \\ 50i_2' + 10^4 i_2 - 10^4 i_1 &= 0. \end{aligned}$$

Now apply Laplace transform to each equation, remembering that $i_1(0) = 0$ and $i_2(0) = 0$:

$$\begin{aligned} sI_1 + 50I_2 &= \frac{60}{s} \\ 50sI_2 + 10^4 I_2 - 10^4 I_1 &= 0. \end{aligned}$$

Collect I_1 and I_2:

$$\begin{aligned} sI_1 + 50I_2 &= \frac{60}{s} \\ -200I_1 + (s + 200)I_2 &= 0. \end{aligned}$$

Now we have to solve for I_1 and I_2.

From the second equation we find $I_2 = 200I_1/(s+200)$. If we insert this in the first equation we get:

$$\begin{aligned} sI_1 + 50\frac{200I_1}{s+200} &= \frac{60}{s} \\ \Rightarrow I_1 &= \frac{60(s+200)}{s^2(s+200)+10^4 s} \\ &= \frac{60s+12,000}{s(s+100)^2}. \end{aligned}$$

Decompose into partial fractions:

$$\begin{aligned} I_1 &= \frac{60s+12,000}{s(s+100)^2} \\ &= \frac{A}{s} + \frac{B}{s+100} + \frac{C}{(s+100)^2} \end{aligned}$$

which gives $A = 6/5$, $B = -6/5$ and $C = -60$.

$$I_1 = \frac{6/5}{s} - \frac{6/5}{s+100} - \frac{60}{(s+100)^2}.$$

Thus:

$$i_1(t) = \mathcal{L}^{-1}\{I_1(s)\} = \frac{6}{5} - \frac{6}{5}e^{-100t} - 60te^{-100t}.$$

For the last term, I have used the formula of differentiation of transforms $\mathcal{L}\{tf(t)\} = -F'(s)$, where, in our case, $F(s) = -\frac{1}{s+100}$.

Let's now find I_2. Since $I_2 = 200I_1/(s+200)$, then

$$\begin{aligned} I_2 &= \frac{12,000}{s(s+100)^2} \\ &= \frac{A}{s} + \frac{B}{s+100} + \frac{C}{(s+100)^2}. \end{aligned}$$

which gives $A = 6/5$, $B = -6/5$ and $C = -120$. Therefore:

$$I_2 = \frac{6/5}{s} - \frac{6/5}{s+100} - \frac{120}{(s+100)^2}.$$

Thus:

$$i_2(t) = \mathcal{L}^{-1}\{I_2(s)\} = \frac{6}{5} - \frac{6}{5}e^{-100t} - 120te^{-100t}.$$

Here I have used again, for the last term, the formula of differentiation of transform.

The current $i_3(t)$ through the capacitor is given by $i_1(t) - i_2(t)$:

$$i_3(t) = 60te^{-100t}.$$

Note that

$$\lim_{t\to\infty} i_3(t) = 0.$$

The different current are shown in Figure 2.

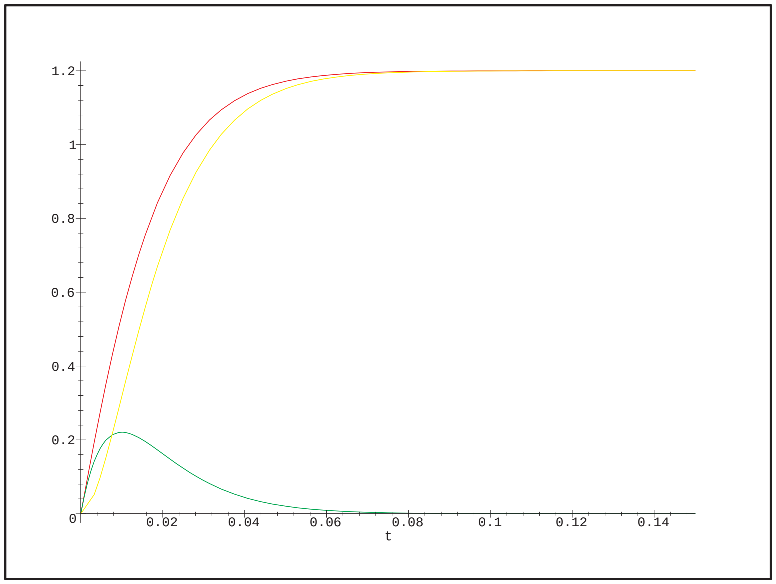

Figure 2: The currents $i_1(t)$ (red), $i_2(t)$ (yellow), $i_3(t)$ (green) in the electrical network discussed in the text.