MATH1115 ANALYSIS LECTURE 2
Thursday, Feb 20

Administrative details

Extra tutorial:

We will schedule an extra tutorial on Tuesdays at 3-5 PM. Please consider changing to this tutorial if you can, since otherwise some of the other tutorials will be overcrowded.

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AXIOMS OF THE REAL NUMBERS

In this section of the course we take an abstract point of view of the real numbers: We will ignore the question of what the real numbers are, and use as our starting point a list of their properties.

That is, we will *assume* that the real numbers satisfy certain axioms, and we will prove things about the real numbers using these axioms as hypotheses.

This approach has some advantages: It means that anything we prove about the real numbers is also true for any other set which satisfies the same axioms.

On the other hand, we run a certain risk: Since we don't actually construct the real numbers, it is possible that there is no such thing as a set which satisfies the axioms we assume! Then all out theorems would be theorems about the empty set....

For the purposes of this course there are two reasons for taking this approach: First, although there are many constructions of the real numbers, each with its own appeal and interest, all are quite complicated to write down in detail (PhB students may be interested in looking at this further as an ASC add-on to this course).

Second, the axiomatic approach to the real numbers gives us a very nice framework to practice constructing formal proofs: The axioms will provide a list of starting hypotheses from which we can in principle build up the entire edifice of calculus (and beyond) by pure deduction.

The Axioms

Our starting point is that the real numbers $\mathbb{R}$ is a set equipped with two algebraic operations (addition, which takes any two real numbers a and b and produces a new real number $a+b$, and multiplication which also takes any two real numbers a and b and produces a real number ab), two special real numbers called 0 and 1, and an *order* relation, which is a logical formula '$a < b$' which for any two real numbers a and b is either true or false. These are assumed to satisfy a collection of properties:

First, we assume a collection of axioms called the *algebraic* axioms of the real numbers. The first four concern addition:

(P1) For any real numbers a, b and c,

$$a + (b + c) = (a + b) + c.$$

Associative law for addition

It follows that the sum of several terms $a + b + c$ is well-defined.

(P2) If a is any real number, then

$$a + 0 = 0 + a = 0.$$

Existence of an additive identity

(P3) For any real number a, there exists a real number $-a$ such that

$$a + (-a) = (-a) + a = 0.$$

Existence of additive inverses

(P4) For any real numbers a and b,

$$a + b = b + a.$$

Commutative law for addition

Defining axioms of a *group*

From this we can deduce that 0 is unique:
Claim: If x satisfies $a + x = a$ for some real number a, then $x = 0$.
Proof: Suppose $a + x = a$.
Then $(-a) + (a + x) = (-a) + a = 0$ by (P3).
By (P1) the left-hand side equals $((-a) + a) + x$.
By (P3) this equals $0 + x$; By (P2) this equals x.
Therefore $x = 0$. QED

There are many other examples of groups: The integers $\mathbb{Z}$; complex numbers $\mathbb{C}$; the space of rotation matrices (under matrix multiplication); the space of invertible $n \times n$ matrices; the group of symmetries of a square, etc.

Note that at this point we can *define* the operation of subtraction by defining $a - b$ to be $a + (-b)$.

The next four axioms concern multiplication:

(P5) If a, b and c are any real numbers, then

$$a(bc) = (ab)c.$$

Associative law for multiplication

(P6) $1 \neq 0$, and if a is any real number, then

$$a.1 = 1.a = a.$$

Existence of multiplicative identity

(P7) For every real number $a \neq 0$, there exists a real number a^{-1} such that

$$a.a^{-1} = a^{-1}.a = 1.$$

Existence of multiplicative inverses

(P8) If a and b are real numbers, then

$$ab = ba.$$

Commutative law for multiplication

Some consequences:

Claim: If a, b and c are real numbers with $ab = ac$, and $a \neq 0$, then $b = c$.

Proof: If $ab = ac$ and $a \neq 0$, then by (P7) a has a multiplicative inverse a^{-1}, and

$$a^{-1}(ab) = a^{-1}(ac).$$

By (P5), the left hand side equals $(a^{-1}a)b = 1.b = b$ (by (P7) and (P6)).
Similarly, the right hand side equals c.
Therefore $b = c$. QED.

The last algebraic axiom involves both addition and multiplication:

(P9) If a, b and c are real numbers, then

$$a(b + c) = ab + ac.$$

Distributive law

The algebraic axioms (P1)–(P9) are the defining axioms of a *field*.

Other examples (apart from the real numbers) include:

- The rational numbers $\mathbb{Q}$;
- The complex numbers $\mathbb{C}$;
- The set of real numbers of the form $p + q\sqrt{2}$, where p and q are rational

Claim: If a is any real number, then $a.0 = 0$.

Proof: Let a be any real number. Then by the distributive law,

$$a.0 + a.0 = a(0 + 0).$$

By the definition of 0, $0 + 0 = 0$, so we have

$$a.0 + a.0 = a.0$$

By the existence of additive inverses, we can add $-(a.0)$ to both sides:

$$(a.0 + a.0) - (a.0) = a.0 - a.0 = 0.$$

By the associative law for addition, the left hand side equals

$$a.0 + (a.0 - a.0) = a.0 + 0 = a.0$$

Therefore $a.0 = 0$ as required. QED

Claim: If a and b are real numbers with $ab = 0$, then either $a = 0$ or $b = 0$.

Proof: Suppose a and b are real numbers with $ab = 0$, and suppose $a \neq 0$.

Then $ab = 0 = a.0$ by the previous claim.

Therefore $b = 0$ (by the Claim on the previous slide).

This completes the proof: If $ab = 0$ then either $a = 0$ or $a \neq 0$, and in the latter case we have proved $b = 0$. QED

Example:
Show that two real numbers a and b satisfy $a - b = b - a$ if and only $a = b$.

Proof:
If $a = b$ then $a - b = a - a = 0$ and $b - a = a - a = 0$, so $a - b = b - a$.
Conversely, if $a - b = b - a$, then we can add b to both sides:

$$(a - b) + b = (b - a) + b.$$

The left hand side equals $a + (-b + b) = a + 0 = a$. The right hand side equals $b + (-a + b) = b + (b - a) = (b + b) - a$.
Therefore $a = (b + b) - a$.
Adding a to both sides gives

$$a + a = ((b + b) - a) + a = (b + b) + (-a + a) = (b + b) + 0 = b + b.$$

We can write this as $(1 + 1)a = (1 + 1)b$.
By our previous result, $a = b$. QED

Definition: $2 = 1 + 1$.

CAUTION: THIS ASSUMES $2 \neq 0$.
In fact we don't know this — at least, it does NOT follow from the algebraic axioms! A counterexample is given by the field of integers modulo 2, which has only two elements 0 and 1. In this case $1 + 1 = 0$.

We need to assume more: There are three axioms concerning the order relation.

(P10) For every real number a, one and only one of the following holds:

(i) $a = 0$,
(ii) $a > 0$, or
(iii) $-a > 0$.

Trichotomy law

(P11) If a and b are real numbers with $a > 0$ and $b > 0$, then $a + b > 0$. Closure of positivity under addition

(P12) If a and b are real numbers with $a > 0$ and $b > 0$, then $ab > 0$. Closure of positivity under multiplication

Claim: $1 > 0$.

Proof: We have by assumption $1 \neq 0$. Therefore by trichotomy, either $1 > 0$ or $-1 > 0$. We argue by contradiction: Suppose $-1 > 0$. Then by closure under multiplication $(-1).(-1) > 0$. But we have

$$(-1).(-1) + (-1) = (-1).(-1) + 1.(-1) = (-1+1).(-1) = 0.(-1) = 0.$$

Adding 1 to both sides we get

$$(-1).(-1) = 1.$$

But then $1 > 0$ and $-1 > 0$, which contradicts the trichotomy law. Therefore the only possibility is $1 > 0$. QED

Corollary: $1 + 1 \neq 0$.

Proof: Since $1 > 0$, $1+1 > 0$ by closure of positivity under addition. Therefore $1 + 1 \neq 0$ by the trichotomy law. QED