Ordinary Differential Equations

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1 Introduction to Laplace transforms

1.1 Laplace transforms

Introduction

In all previous sections, we went through several different methods to solve ordinary differential equations. For example, we have seen how to solve a second order linear differential equation of the type

$$y'' + ay' + by = r(x)$$

where a and b are constant and $r(x)$ is a continuous function, by using the undetermined coefficients or variation of parameters methods.

Unfortunately, these methods do not work if $r(x)$ is not continuous. This is a serious shortcoming since in many real life cases, such as in many mechanical and electrical systems,we may have as input a non-continuous driving function $r(x)$. I'll show you next that the use of Laplace transforms gives a much nicer and less tedious method to solve DEs which also overcomes the problem of having non-continuous $r(x)$.

The strategy at the core of the Laplace transform method is to transform differential equations into algebraic problems where solutions can be obtained quite easily. One then applies the Inverse Laplace transform to find the solutions to the original problem. This can be illustrated in Figure 1

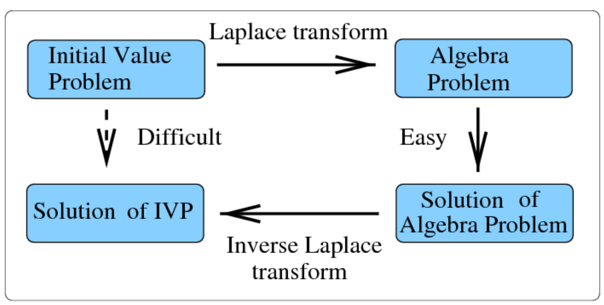

Figure 1: Strategy behind the Laplace transformation.

Heaviside

Oliver Heaviside (1850-1925) shown in Figure 2 is responsible for the early development of this method. Heaviside was an English electrical engineer who wanted to solve very practical engineering problems that could not be solved using standard techniques. Since he was an engineer and lacked mathematical rigour, the mathematicians of the time were not too impressed by his work. His response to the critics was “Shall I refuse my dinner because I do not fully understand ... digestion?”. Anyway, his methods worked and he managed to solve many outstanding problems related to currents and voltages propagation along transmission lines. This encouraged many mathematicians to search for a solid mathematical foundation for Heaviside's methods and their research recognised that the theoretical foundation to Heaviside's work had been developed about 100 years earlier by the French mathematician Pierre Simon de Laplace (1749-1827).

Heaviside was also responsible for the introduction of complex numbers to the study of electrical circuits and introduced the concepts of inductance, capacitance and impedance. He also developed the field of Vector Algebra and Analysis, known as the Gibbs-Heaviside approach and was the first to write Maxwell's equations in the modern vector form (although Hertz got the credit for this, since his formulation was clearer than Heaviside's). At roughly the same time as Kennelly, Heaviside proposed that Marconi was able to transmit trans-Atlantic radio waves because the electrified layer in the upper atmosphere (the Kennelly-Heaviside Zone) bounce off radio signal back to Earth.

Figure 2: Oliver Heaviside (1850-1925).

Plant

The method of Laplace transforms is very powerful and can solve a wide variety of initial-value problems in many different fields of science and engineering, particularly in the field of signals and linear systems analysis. A system (sometimes called "the plant") can be represented by a box, which is the mathematical model describing how the system deals with a certain input $u(t)$ to generate the output $x(t)$. When the input and output depend on the time t, one refers to them as *signals*. An engineer's task is to figure out what the output $x(t)$ is when the system is subjected to an input $u(t)$ applied at some time t (i.e. $t = 0$). See Figure 3.

1.1.1 Forward Laplace transforms

Laplace transforms

Let's see now what Laplace transforms are.

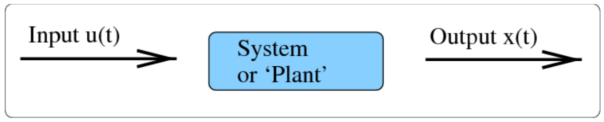

Figure 3: The plant

Definition 1 (Laplace transform)**.** *Let $f(t)$ be a function that is defined for all $t \geq 0$, then the function*

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

is said to be the Laplace transform of $f(t)$, provided the limit

$$\lim_{b \to \infty} \int_0^{b} e^{-st} f(t)\, dt$$

exists.

The quantity e^{-st} is called the *kernel* of the transformation. The Laplace transform of $f(t)$ is denoted by $\mathcal{L}\{f(t)\}$. Thus:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt.$$

Thus the symbol $\mathcal{L}$ denotes the *Laplace transform operator* since when it operates on a function $f(t)$ in the t domain (also called *time domain*) it transforms it into a function $F(s)$ in the s domain (also called *frequency domain*). The functions $f(t)$ and $F(s)$ form a *Laplace transform pair* $\{f(t), F(s)\}$.

1.1.2 Inverse transform

Inverse transform

Since the Laplace transform of $f(t)$ is

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$$

the function $f(t)$ is called the *inverse transform* of $F(s)$ and is denoted $\mathcal{L}^{-1}\{F(s)\}$, that is:

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

The inverse Laplace transform takes $F(s)$ back to $f(t)$. It is a bit like the idea of derivatives versus antiderivatives.

1.1.3 Examples: Laplace transform

Example 1 (Laplace transform example 1)**.** *Find the Laplace transform of $f(t) = a$, where a is a constant.*

Solution

$$\begin{aligned}
\mathcal{L}\{f(t)\} = \mathcal{L}\{a\} &= \int_0^\infty a\, e^{-st}\, dt \\
&= \left[\frac{a\, e^{-st}}{-s}\right]_0^\infty \\
&= -\frac{a}{s}\left[e^{-st}\right]_0^\infty \\
&= -\frac{a}{s}\,[0-1] = \frac{a}{s}.
\end{aligned}$$

Thus:

$$\boxed{\mathcal{L}\{a\} = \tfrac{a}{s} \qquad\qquad (s > 0).}$$

Note that the integral above is an *improper integral*, since the interval of integration is infinite. You should not forget this! However, in order to simplify somewhat the notation we shall skip from now on (as we did above) the following intermediate step:

$$\lim_{b\to\infty}\left[\frac{a\, e^{-st}}{-s}\right]_0^b = \lim_{b\to\infty}\left[\frac{a\, e^{-sb}}{-s} - \frac{a}{-s}\right].$$

Example 2 (Laplace transform example 2)**.** *Find the Laplace transform of $f(t) = e^{at}$, where a is a constant.*

Solution

$$\begin{aligned}
\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{at}\} &= \int_0^\infty e^{at}\, e^{-st}\, dt \\
&= \int_0^\infty e^{(a-s)t}\, dt \\
&= \left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty \\
&= 0 - \frac{1}{a-s} \\
&= \frac{1}{s-a}.
\end{aligned}$$

Thus, provided that $a - s < 0$ (otherwise the integral diverges) :

$$\boxed{\mathcal{L}\{e^{at}\} = \tfrac{1}{s-a} \qquad\qquad (s > a).}$$

Example 3 (Laplace transform example 3). *Find the Laplace transform of $f(t) = \sin(at)$, where a is a constant.*

Solution

We can proceed in two different ways. One way, is to calculate:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(at)\} = \int_0^\infty \sin(at)\, e^{-st}\, dt$$

and then integrate by parts.

The other way (much better and more elegant) is to use Euler's formula

$$e^{it} = \cos t + i \sin t$$

so that the cos and sin functions are the real and imaginary parts of e^{it}, respectively. Thus $\sin(at) = \Im\left(e^{iat}\right)$. ($\Im$ is the imaginary part.)

Therefore:

$$\begin{aligned}
\mathcal{L}\{\sin(at)\} = \mathcal{L}\{\Im\left(e^{iat}\right)\} &= \Im \int_0^\infty e^{iat}\, e^{-st}\, dt \\
&= \Im \int_0^\infty e^{(ia-s)t}\, dt \\
&= \Im \left(\left[\frac{e^{(ia-s)t}}{ia-s} \right]_0^\infty \right) \\
&= \Im \left(\frac{1}{s-ia} \right) \\
&= \Im \left(\frac{s+ia}{s^2+a^2} \right) \\
&= \frac{a}{s^2+a^2}.
\end{aligned}$$

Thus, provide $s > 0$ (otherwise the integral diverges):

$$\boxed{\mathcal{L}\{\sin(at)\} = \tfrac{a}{s^2+a^2} \qquad (s > 0)}$$

We can obtain the Laplace transform of $f(t) = \cos(at)$ by taking the real part of the above quantity:

$$\begin{aligned}
\mathcal{L}\{\cos(at)\} &= \Re \left(\frac{s+ia}{s^2+a^2} \right) \\
&= \frac{s}{s^2+a^2}.
\end{aligned}$$

Thus:

$$\boxed{\mathcal{L}\{\cos(at)\} = \tfrac{s}{s^2+a^2} \qquad (s > 0)}$$

Example 4 (Laplace transform example 4). *Find the Laplace transform of $f(t) = t^n$, where n is a positive integer.*

Solution

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^n\} = \int_0^\infty t^n\, e^{-st}\, dt.$$

We integrate by parts:

$$\begin{aligned}
\mathcal{L}\{t^n\} &= \int_0^\infty t^n\, e^{-st}\, dt \\
&= \left[\frac{t^n\, e^{-st}}{-s}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}\, e^{-st}\, dt \\
&= 0 + \frac{n}{s}\int_0^\infty t^{n-1}\, e^{-st}\, dt.
\end{aligned}$$

What we found above, is simply the integral we started with, but with $n-1$ instead of n!

So, we can say that if

$$I_n = \int_0^\infty t^n\, e^{-st}\, dt,$$

then

$$I_{n-1} = \int_0^\infty t^{n-1}\, e^{-st}\, dt.$$

Thus:

$$I_n = \frac{n}{s} I_{n-1}, \qquad I_{n-1} = \frac{n-1}{s} I_{n-2}, \qquad I_{n-2} = \frac{n-2}{s} I_{n-3}, \cdots$$

And finally:

$$I_n = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \frac{n-3}{s} \cdots \frac{n-(n-1)}{s} I_0.$$

Since $I_0 = \mathcal{L}\{t^0\} = \mathcal{L}\{1\} = \frac{1}{s}$, then

$$\begin{aligned}
I_n &= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \frac{n-3}{s} \cdots \frac{n-(n-1)}{s}\frac{1}{s} \\
&= \frac{n(n-1)(n-2)(n-3)(n-4)\cdots 3\cdot 2\cdot 1}{s^{n+1}} \\
&= \frac{n!}{s^{n+1}}.
\end{aligned}$$

Thus:

$$\boxed{\mathcal{L}\{t^n\} = \tfrac{n!}{s^{n+1}} \qquad n = 1, 2, 3\cdots, \qquad (s > 0).}$$

Example 5 (Laplace transform example 5). *Find the Laplace transform of $f(t) = t^a$, where a is a positive real number.*

Solution
We have:

$$\mathcal{L}\{t^a\} = \int_0^\infty t^a\, e^{-st}\, dt.$$

Set $x = st$:

$$\begin{aligned}\mathcal{L}\{t^a\} = \int_0^\infty t^a\, e^{-st}\, dt &= \int_0^\infty \left(\frac{x}{s}\right)^a e^{-x}\,\frac{dx}{s} \\ &= \frac{1}{s^{a+1}} \int_0^\infty e^{-x}\, x^a\, dx.\end{aligned}$$

with $s > 0$ (otherwise the integral diverges).

The last integral is the so-called *gamma function*:

$$\Gamma(a+1) = \int_0^\infty x^a\, e^{-x}\, dx.$$

Thus, the Laplace transform of t^a is:

$$\boxed{\mathcal{L}\{t^a\} = \tfrac{\Gamma(a+1)}{s^{a+1}} \qquad a > 0, \qquad (s > 0).}$$

Let's see what happens if a is a positive integer. We should find the same result we obtained earlier, that is $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ $(s > 0)$. Let's see whether this is true.

Set $a = 0$ to find $\Gamma(1)$:

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = \left[-e^{-t}\right]_0^\infty = 1.$$

Now, since $\Gamma(a+1) = a\Gamma(a)$ (check by integrating by parts!), then

$$\Gamma(2) = \Gamma(1) = 1!, \qquad \Gamma(3) = 2\Gamma(2) = 2!, \qquad \Gamma(4) = 3\Gamma(3) = 3!, \cdots$$

Therefore:

$$\Gamma(n+1) = n! \qquad (n = 0, 1, 2, 3, \cdots).$$

Thus, the gamma function generalises the factorial function! Therefore the Laplace transform found for n positive integer really follows from the more general transform of t^a.

1.2 Linearity of the Laplace transform

Some properties

The Laplace transform is a linear operation, that is, consider two functions $f(t)$ and $g(t)$ whose Laplace transforms $F(s)$ and $G(s)$ exist for $s > a_1$ and $s > a_2$ respectively, then for s greater than the maximum of a_1 and a_2:

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} = aF(s) + bG(s)$$

where a and b are arbitrary constants.

This is because:

$$\begin{aligned}
\mathcal{L}\{af(t)+bg(t)\} &= \int_0^\infty (af(t)+bg(t))\ e^{-st}\,dt \\
&= a\int_0^\infty f(t)e^{-st}\,dt + b\int_0^\infty g(t)e^{-st}\,dt \\
&= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} \\
&= aF(s)+bG(s).
\end{aligned}$$

The inverse Laplace transform $\mathcal{L}^{-1}$ is also a linear operation, that is:

$$\mathcal{L}^{-1}\{aF(s)+bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\} = af(t)+bg(t).$$

1.2.1 Hyperbolic functions

Example: hyperbolic functions

As you know:

$$\cosh(at) = \frac{e^{at}+e^{-at}}{2}, \qquad \sinh(at) = \frac{e^{at}-e^{-at}}{2}.$$

So, if we want to find the Laplace transform of $\cosh(at)$ and $\sinh(at)$, we can use the property of linearity and $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}\ \ (s>a)$ and $\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}\ \ (s>-a)$:

$$\begin{aligned}
\mathcal{L}\{\cosh(at)\} &= \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\} \\
&= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] \\
&= \frac{s}{s^2-a^2}.
\end{aligned}$$

Similarly:

$$\begin{aligned}
\mathcal{L}\{\sinh(at)\} &= \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\} \\
&= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] \\
&= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] \\
&= \frac{a}{s^2-a^2}.
\end{aligned}$$

Thus:

$$\boxed{\mathcal{L}\{\cosh(at)\} = \tfrac{s}{s^2-a^2}, \qquad (s > |a|),}$$

and:

$$\boxed{\mathcal{L}\{\sinh(at)\} = \tfrac{a}{s^2-a^2}, \qquad (s > |a|).}$$

1.2.2 Example: Linearity of Laplace transforms

Example 6 (Linearity of Laplace transform)**.** *Find the Laplace transform of* $f(t) = 2t + 4e^{5t}$*.*

Solution

To find the Laplace transform of $f(t) = 2t + 4e^{5t}$ we can use again the property of linearity and the following transforms:

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \ \ (s > 0), \qquad \mathcal{L}\{e^{5t}\} = \frac{1}{s-5} \ \ (s > 5).$$

Thus:

$$\begin{aligned} \mathcal{L}\{2t + 4e^{5t}\} &= 2\mathcal{L}\{t\} + 4\mathcal{L}\{e^{5t}\} \\ &= \frac{2}{s^2} + \frac{4}{s-5} \qquad (s > max\{0, 5\}). \end{aligned}$$

Therefore:

$$\mathcal{L}\{2t + 4e^{5t}\} = \frac{2}{s^2} + \frac{4}{s-5} \qquad (s > 5).$$

1.3 First shifting theorem

Shifting

Theorem 1 (First shifting theorem)**.** *The first shifting theorem states that if* $f(t)$ *is a function with Laplace transform* $F(s)$*:*

$$F(s) = \mathcal{L}\{f(t)\} \qquad\qquad (s > b)$$

then

$$F(s-a) = \mathcal{L}\{e^{at}f(t)\} \qquad\qquad (s - a > b)$$

or, by taking the inverse on both sides:

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t).$$

This is because:

$$\begin{aligned} F(s-a) &= \int_0^\infty e^{-(s-a)t} f(t)\, dt \\ &= \int_0^\infty e^{-st} \left(e^{at}f(t)\right)\, dt \\ &= \mathcal{L}\{e^{at}f(t)\}. \end{aligned}$$

Thus, the transform $\mathcal{L}\{e^{at}f(t)\}$ is the same as $\mathcal{L}\{f(t)\}$ except that s is replaced everywhere by $(s-a)$.

1.3.1 Examples: First shifting theorem

Example 7 (First shifting theorem)**.** *Find the Laplace transform of* $f(t) = e^{at}\cos(\omega t)$*.*

Solution
We can use the first shifting theorem to find the Laplace transform of $f(t) = e^{at}\cos(\omega t)$. To do this, let's use the transform of $\cos(\omega t)$:

$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2} \qquad (s > 0)$$

Now replace s with $(s-a)$ to find the transform of $e^{at}\cos(\omega t)$:

$$\boxed{\mathcal{L}\{e^{at}\cos(\omega t)\} = \tfrac{s-a}{(s-a)^2+\omega^2} \qquad (s > a).}$$

Similarly,

$$\boxed{\mathcal{L}\{e^{at}\sin(\omega t)\} = \tfrac{\omega}{(s-a)^2+\omega^2} \qquad (s > a).}$$

Example 8 (First shifting theorem)**.** *Find the Laplace transform of* $f(t) = te^{-2t}$*.*

Solution
We can use the first shifting theorem to find the Laplace transform of $f(t) = te^{-2t}$. To do this, let's use the Laplace transform of t:

$$\mathcal{L}\{t\} = \frac{1}{s^2} \qquad (s > 0).$$

Now replace s with $(s-a) = (s+2)$:

$$\mathcal{L}\{te^{-2t}\} = \frac{1}{(s+2)^2} \qquad (s > -2).$$

1.4 Existence of Laplace transforms

1.4.1 Piecewise continuity

Piecewise continuity

Definition 2 (Piecewise continuity)**.** *A function $f(t)$ is said to be piecewise continuous on a finite interval $[a, b]$ if $f(t)$ is continuous at every point in $[a, b]$ except for a finite number of points where $f(t)$ has a jump discontinuity.*

A function $f(t)$ has a jump discontinuity at $t = t_0$ if $f(t)$ is not continuous at t_0, but the limits:

$$\lim_{t\to t_0^-} f(t), \qquad \text{and} \qquad \lim_{t\to t_0^+} f(t)$$

exist as a finite numbers. If the discontinuity is at one of the endpoints (a or b), a jump discontinuity exists if the one sided limit of $f(t)$ ($t \to a^+$ or $t \to b^-$) exists as a finite number.

An example is the following function (see Figure 4):

$$f(t) = \begin{cases} t, & 0 < t < 2 \\ 4, & 2 < t \leq 4 \\ (t-4)^2, & 4 < t \leq 5 \end{cases}$$

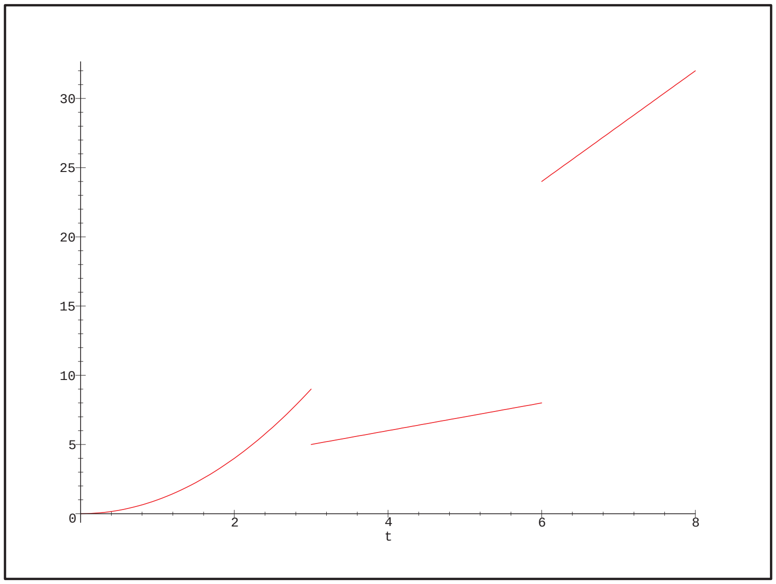

Figure 4: A piecewise continuous function (see text).

This function is continuous on the intervals $(0, 2), (2, 4)$ and $(4, 5]$. At the points of discontinuity ($t = 0, 2, 4$), $f(t)$ has jump discontinuities, since all one-sided limits exist as finite numbers (e.g. at $t = 2$ the left-hand limit is 2 and the right-hand limit is 4). Thus, $f(t)$ is piecewise continuous on the interval $[0, 5]$.

In contrast, the function $f(t) = 1/t$ is not piecewise continuous on any interval containing the origin, since (see Figure 5):

$$\lim_{t \to 0^-} \frac{1}{t} = -\infty, \qquad \text{and} \qquad \lim_{t \to 0^+} \frac{1}{t} = \infty.$$

1.4.2 Exponential order

Exponential order k

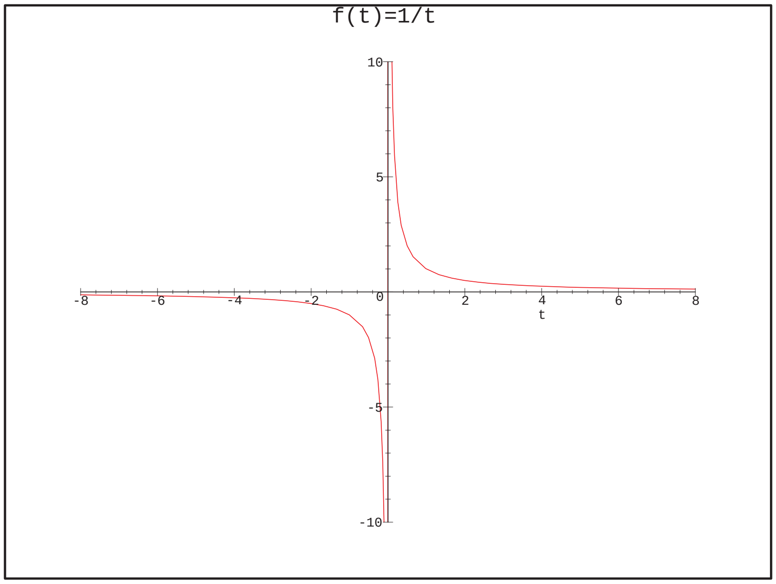

Figure 5: Infinite jump at the origin.

The integral involved in the determination of the Laplace transform of a function $f(t)$:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

exists only if $\lim_{t\to\infty} e^{-st} f(t) \to 0$ at least as fast as an exponential function with a negative exponent.

Thus, $f(t) = e^{t^2}$ increases too fast as $t \to \infty$. This function does not have a Laplace transform. Thus, a function $f(t)$ is said to be of *exponential order* k if there exists positive constants T and M such that:

$$|f(t)| \le Me^{kt} \qquad \text{for} \qquad t \ge T.$$

For example, the function:

$$f(t) = \cos(3t)e^{4t}$$

is of exponential order $k = 4$ since

$$\left|\cos(3t)e^{4t}\right| \le e^{4t}$$

where $M = 1$ and T is any positive constant.

The function $f(t) = e^{t^2}$ is not of exponential order since

$$\lim_{t\to\infty} \frac{e^{t^2}}{e^{kt}} = \lim_{t\to\infty} e^{t(t-k)} = \infty$$

for any value of k.

Now that we have introduced the concepts of piecewise continuity and exponential order k, we can at last give the following condition for the existence of the Laplace transform.

Theorem 2. *If a function $f(t)$ is piecewise continuous on the interval $[0, \infty)$ and of exponential order k, then the Laplace transform $\mathcal{L}\{f(t)\}$ exists for $s > k$.*

To prove this theorem, we must show that the integral

$$\int_0^\infty e^{-st} f(t)\, dt$$

converges for $s > k$.

So:

$$\begin{aligned} |\mathcal{L}\{f(t)\}| &= \left|\int_0^\infty e^{-st} f(t)\, dt\right| \\ &\le \left|\int_0^T e^{-st} f(t)\, dt\right| + \left|\int_T^\infty e^{-st} f(t)\, dt\right|. \end{aligned}$$

The first integral on the RHS exists since the integrand is piecewise continuous and thus the integral exists over any finite interval. So we need to look only at the second integral.

$$\begin{aligned} \left|\int_T^\infty e^{-st} f(t)\, dt\right| &\le \int_0^\infty e^{-st} |f(t)|\, dt \\ &\le \int_0^\infty e^{-st} M e^{kt}\, dt \\ &= \int_0^\infty M e^{(k-s)t}\, dt \\ &= \left[\frac{M e^{(k-s)t}}{k-s}\right]_0^\infty \\ &= \frac{M}{s-k} \end{aligned}$$

The condition $s > k$ is necessary for the existence of the last integral.

At this point, I must say that the conditions in this theorem are sufficient, rather than necessary. A typical example of this is given by the function $f(t) = 1/\sqrt{t}$ which although is infinite for $t = 0$ since it breaks the condition of piecewise continuity on $[0, \infty)$, nevertheless it does have a Laplace transform. In fact, by using the result $\mathcal{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}$ we have:

$$\mathcal{L}\left\{t^{-\frac{1}{2}}\right\} = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \sqrt{\frac{\pi}{s}}.$$

Here, I have used the result $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Finally, the function $f(t) = 1/t$ also breaks the condition of piecewise continuity on $[0, \infty)$, but in this case this function does not have a Laplace transform.