MATH1115, Calculus Lecture 2

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Review

We are taking an axiomatic approach to the real numbers:

Assume: $\mathbb{R}$ is a set equipped with certain operations and relations satisfying a collection of axioms. Last time we introduced the algebraic axioms:

There are special elements 1 and 0 such that for all real numbers a, b and c,

(1). $a+b=b+a$ (commutative axiom for addition)

(2). $(a+b)+c=a+(b+c)$ (associative axiom for addition)

(3). $a+0=0+a=a$ (additive identity axiom)

(4). there is a real number $(-a)$ such that

$$a+(-a)=(-a)+a=0. \qquad \text{(additive inverse axiom)}$$

(5). $a\times b=b\times a$ (commutative axiom for multiplication)

(6). $(a\times b)\times c=a\times(b\times c)$ (associative axiom for multiplication)

(7). $a\times 1=1\times a=a$, and $0\neq 1$ (multiplicative identity axiom)

(8). If $a\neq 0$ then there is a real number a^{-1} such that

$$a\times a^{-1}=a^{-1}\times a=1 \qquad \text{(multiplicative inverse axiom)}$$

(9). $a\times(b+c)=a\times b+a\times c$ (distributive axiom)

Consequences of the algebraic axioms, continued

Last time we proved the cancellation law: If $a+b=a+c$ then $b=c$.

Theorem (Theorem 1.2.2 in Hutchinson)

If a, b, c and d are real numbers with $c \neq 0$ and $d \neq 0$, then

(1) $ac = bc$ implies $a = b$. *(Multiplicative version of the cancellation law)*

(2) $a0 = 0$. $a0 + a0 \overset{A9}{=} a(0+0) \overset{A3}{=} a0$, so $a0 = 0$ by cancellation

(3) $-(-a) = a$. $(-a) + (-(-a)) \overset{A4}{=} 0 \overset{A4}{=} (-a) + a$, so $-(-a) \overset{\text{cancellation}}{=} a$

(4) $\left(c^{-1}\right)^{-1} = c$. *(Multiplicative version of (3))*

(5) $(-1)a = (-a)$. $(-1)a + a \overset{A7}{=} (-1)a + 1a \overset{A9}{=} ((-1)+1)a \overset{A4}{=} 0a \overset{(2)}{=} 0$

(6) $a(-b) = -(ab) = (-a)b$. *Exercise*

(7) $(-a) + (-b) = -(a+b)$. *Exercise*

(8) $(-a)(-b) = ab$. *Exercise*

(9) $\frac{a}{c}\frac{b}{d} = \frac{ab}{cd}$. *Exercise*

(10) $\frac{a}{c} + \frac{b}{d} = \frac{ad+bc}{cd}$. *Exercise*

The axioms (1)-(9) define an algebraic object called a *field*. eg. $\mathbb{Q}$, $\mathbb{Z}_p$, $\mathbb{Z} + \sqrt{2}\mathbb{Z}, \ldots$

Order axioms

The next collection of axioms are about the order relation $x < y$. These together with the algebraic axioms define an *ordered field*. The order is a relation which is either true or false for any two real numbers a and b, and if true we write $a < b$.

We assume: For all real numbers a, b and c,

(10). exactly one of the following holds: $a < b$ or $a = b$ or $b < a$ (Trichotomy axiom)

(11). If $a < b$ and $b < c$ then $a < c$ (transitivity axiom)

(12). If $a < b$ then $a + c < b + c$ (addition order axiom)

(13). If $a < b$ and $0 < c$ then $a \times c < b \times c$ (multiplication order axiom)

Further definitions:

- If $0 < a$ we say a is positive, and if $0 < a$ we say a is negative.
- We say $a > b$ if and only if $b < a$, and $a \leq b$ if $a < b$ or $a = b$, etc.

Consequences of the order axioms

Example

Show that two real numbers a and b satisfy $a-b=b-a$ if and only if $a=b$.

Proof: We have to prove both implications. One is easy: If $a=b$ then $a-b=a-a=b-a$. Conversely, if $a-b=b-a$, then add b to both sides:

$$(a-b)+b=(b-a)+b.$$

The left-hand side equals $a+((-b)+b)=a+0=a$, using the definition of subtraction, the associative law of addition, the additive inverse axiom, and the additive identity axiom. The right-hand side equals

$$b+(b+(-a))=(b+b)+(-a)=(b+b)-a$$

using the commutative and the associative laws for addition. Therefore $a=(b-b)-a$. Adding a to both sides gives

$$a+a=((b+b)-a)+a\overset{A2}{=}(b+b)+((-a)+a)\overset{A4}{=}(b+b)+0\overset{A3}{=}b+b.$$

Since we know $a=1.a$ and $b=1.b$ by the additive identity law, we can write using the distributive law as

$$(1+1)a=(1+1)b,$$

so $a=b$ by the multiplicative cancellation law provided $1+1\neq 0$.

Consequences of the order axioms, continued

Proposition

$1+1 \neq 0$.

This cannot be proved using the algebraic axioms (cf. the field $\mathbb{Z}_2$)

Lemma

$0 < 1$.

Proof: The axioms give $1 \neq 0$, so by trichotomy either $0 < 1$ or $1 < 0$. Suppose the latter holds. We claim that $0 < -1$: By the addition order axiom we have $1+(-1) < 0+(-1) = -1$, so $0 < -1$. Therefore by the multiplication order axiom we have

$$1(-1) < 0(-1).$$

So $-1 < 0$ (using the multiplicative identity axiom on the left and result $a.0 = 0$ on the right). This contradicts the trichotomy axiom: We cannot have both $-1 < 0$ and $0 < -1$. Therefore we must have $0 < 1$, as claimed. □

Now we can prove the proposition: Since $0 < 1$, adding 1 to both sides gives $1 < 1+1$ (addition order axiom). But then $0 < 1 < 1+1$, so $0 < 1+1$ (transitivity axiom). By trichotomy we conclude that $1+1 \neq 0$.

Further consequences of order

Theorem (Theorem 1.2.3 in Hutchinson)

If a, b and c are real numbers, then

(1) $a < b$ *and* $c < 0$ *implies* $ac > bc$ *(we know* $0 < (-c)$*)*

(2) $0 < 1$ *and* $-1 < 0$ *(we just proved this)*

(3) $a > 0$ *implies* $a^{-1} > 0$ *(consequence of (2) and trichotomy)*

(4) $0 < a < b$ *implies* $0 < b^{-1} < a^{-1}$ *(multiply by* $a^{-1}b^{-1} > 0$*)*

(5) $|a+b| \leq |a| + |b|$ *(triangle inequality)* *(check various cases)*

(6) $||a| - |b|| \leq |a-b|$ *(consequence of the triangle inequality)*

Here we make the definition $|a| = a$ if $0 \leq a$, and $|a| = -a$ if $a < 0$.

The algebraic and order axioms are still not enough to characterise the real numbers: All of the axioms so far are also satisfied by the rational numbers $\mathbb{Q}$. The final axiom which distinguishes $\mathbb{R}$ and $\mathbb{Q}$ is called the *completeness axiom*, which captures the idea that the real numbers form a *continuum* without any ‘holes’.

Notation and definitions

Some definitions and notation:

• We use curly brackets to denote a set, for example $\mathbb{N} = \{1, 2, 3, \cdots\}$.
• The notation $a \in X$ means that a is an element of the set X (we also say 'a is in X').
• If a is not in the set X we can write $a \notin X$.
• We might also write $X = \{x \in Y : \ P(x)\}$ or $\{x \in Y \big| \ P(x)\}$ to denote 'the set of elements x of the set Y which also satisfy the property $P(x)$'. For example $\mathbb{R}_+ = \{x \in \mathbb{R} : \ 0 < x\}$ is the positive reals.
• The *union* $X \cup Y$ of two sets X and Y is defined by $\{x : \ x \in X \text{ or } x \in Y\}$. The *intersection* $X \cap Y$ is the set $\{x : \ x \in X \text{ and } x \in Y\}$.
• We say X is a subset of Y, and write $X \subset Y$, if every element of X is also an element of Y:

$$X \subset Y \iff a \in X \implies a \in Y.$$

Here $\iff$ is the equivalence symbol, read 'iff and only if' or 'is equivalent to', and $\implies$ is the implication symbol, read 'implies'.
Note: The notation $X \subset Y$ does not mean that X is a proper subset of Y — if you need to convey this you could use the symbol $X \subsetneq Y$.

Upper bounds and least upper bounds

Now we make some useful definitions:

Definition

Let $A \subset \mathbb{R}$. We say that a real number a is an *upper bound* for A if $x \in A \implies x \leq a$.

For example:
- If $X = [0,1] = \{x \in \mathbb{R} : \ 0 \leq x \leq 1\}$, then the numbers $1, 2, \ldots$ are all upper bounds for X.
- If $X = \emptyset$ is the empty set, then every real number is an upper bound for X.
- The subsets $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$ do not have upper bounds.

We say that a subset X of $\mathbb{R}$ is *bounded above* if there exists an upper bound for X.

Definition

Let $A \subset \mathbb{R}$. A number a is a *least upper bound* for A if

(i) a is an upper bound for A, and

(ii) If b is any upper bound for A, then $a \leq b$.

The completeness axiom

Now we are in a position to state the completeness axiom:

Axiom 14: Every non-empty subset X of $\mathbb{R}$ which is bounded above has a least upper bound.

To see the meaning of this, consider the following example:
$X = \{x \in \mathbb{R} : \ x > 0, \ x^2 < 2\}$.
X is non-empty: $1 \in X$.
X is bounded above:

Lemma

If $0 < x < y$ then $x^2 < y^2$ (the function $x \to x^2$ is an increasing function on positive numbers).

Proof: Since $0 < x$ and $x < y$ we have $x^2 < xy$. Since $0 < y$ (transitivity) and $x < y$ we have $xy < y^2$. Since $x^2 < xy$ and $xy < y^2$ we have $x^2 < y^2$ (transitivity). $\square$

Since $2^2 > 2$, we conclude that 2 is an upper bound for X: If any $x \in X$ satisfies $x > 2$ then we would have $x^2 > 2^2 > 2$, which is impossible since $x \in X \implies x^2 < 2$. So X is bounded above.