MATH1115, Analysis Lecture 1

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MATH1115 Analysis
Lecture 1, Wednesday Feb 19

LOGIC AND PROOF

Logic is the study of methods and principles used to distinguish good (correct) from bad (incorrect) reasoning.

Basic terminology:

Statement: A *statement* is a sentence expressed in words (or mathematical symbols) that is either true or false. For example

- "Two plus two equals four"
- "Two plus two equals five"
- "There is no largest integer"
- "Every differentiable function is continuous"

Statements are *simple* if they cannot be broken down into other statements — such as the first two examples above. A *composite* statement contains several simple statements connected by punctuation and/or words such as and, although, or, thus, then, therefore, because, for, moreover, however, and so on.

For example:

- "If n is an odd integer then n^2 is an odd integer"
- "If f is a continuous function then f is integrable"
- "An integer is either odd or even"

$P =$"n is an odd integer"

$Q =$"n^2 is an odd integer"

Composite statement: "If P then Q"

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For example:

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- "If f is a continuous function then f is integrable"
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Hypothesis: A *hypothesis* is a statement that it is assumed to be true, and from which some consequence follows. (For example, in the sentence "If we work on this problem, we will understand it better" the statement "we work on this problem" is the hypothesis.)

Note this use of 'hypothesis' in mathematics, which is a *deductive* enterprise, is different from that used in the *empirical* sciences (e.g. statistics, biology, psychology), where scientists discuss the need "to test the hypothesis."

Conclusion: A *conclusion* is a statement that follows as a consequence from previously assumed conditions (hypotheses). (For example, in the sentence "If we work on this problem, we will understand it better" the statement "we will understand it better" is the conclusion.)

Definition: A *definition* is an unequivocal statement of the precise meaning of a word or phrase, a mathematical symbol or concept, ending all possible confusion.

Proof: A *proof* is a logical argument that establishes the truth of a statement beyond any doubt. A proof consists of a finite chain of steps, each one of them a logical consequence of the previous one.

Theorem: A *theorem* is a mathematical statement for which the truth can be established using logical reasoning on the basis of certain assumptions that are explicitly given or implied in the statement (i.e. by constructing a proof)

Lemma: A *lemma* is an auxilliary theorem proved beforehand so it can be used in the proof of another theorem.

The proofs of some theorems are long and difficult to follow, so it useful to break them up into smaller, simpler Lemmas.

Often the results stated in lemmas are not very interesting by themselves, but serve as building blocks in the proof of more important results.

On the other hand, some lemmas end up being used in many places and are so important that they are named after famous mathematicians, such as "Zorn's Lemma", "Gauss's Lemma"

Corollary: A *corollary* is a theorem that follows with little or no proof from another theorem.

Conjecture: A statement thought to be true but unproved (sometimes also 'hypothesis' as in 'The Riemann Hypothesis'). e.g. Fermat's Conjecture (now Wiles's Theorem), the Poincaré Conjecture (which should be now called 'Perelman's Theorem', but since it has been a famous conjecture for a hundred years it is still usually called the Poincaré conjecture).

Proposition, Claim: Synonyms for Theorem.

How not to prove a theorem

1. Start from what you are trying to prove and end up with a tautology.

Example: Show that for any real numbers a, b and c, $a(b-c) = ab - ac$.
Proof:

$$
\begin{aligned}
a(b-c) &= ab - ac \\
a(b+(-c)) &= ab - ac \quad \textbf{definition of } b-c \\
ab + a(-c) &= ab - ac \quad \textbf{distributive law} \\
a(-c) &= -(ac) \\
ac + a(-c) &= ac - (ac) \\
ac + a(-c) &= 0 \quad \textbf{Definition of } -(ac) \\
a(c+(-c)) &= 0 \quad \textbf{Distributive Law; definition of } -(ac) \\
a.0 &= 0 \quad \textbf{Definition of } -c; \\
0 &= 0 \quad \textbf{property of } 0.
\end{aligned}
$$

QED

Include your assumptions.

Proof: Let a, b and c be arbitrary real numbers.
Then $a(c+(-c)) = a.0 = 0$ by the definition of $-c$.
By the distributive law, $ac + a(-c) = 0$.
Adding $-(ac)$ to both sides and using $-(ac)+(ac) = 0$, we find $a(-c) = -(ac)$.
Adding ab to both sides, we get $ab + a(-c) = ab - ac$.
Therefore by the distributive law, $a(b-c) = ab - ac$.
QED

The proof should read in sentences.
Don't leave out the words!

Logical arguments

To prove a theorem we have to start from the hypotheses, and deduce a chain of consequences leading to the desired conclusion. To be a valid argument, each statement must follow from the hypotheses and the preceding statements.

It is important to distinguish the *validity* of an argument from the truth or falsity of its conclusions. A valid argument produces a true conclusion if the hypotheses are true, but it is perfectly possible to have a valid argument starting from false hypotheses which leads to a false conclusion.

P

Q

For example:

> All birds are able to fly.
> Penguins are birds.
> Therefore, penguins are able to fly.

This is a valid argument, but one of the hypotheses is false.

Equally, it is possible to have an invalid argument which leads to a true conclusion:

> If I win the lottery I can afford to buy a new house.
> I didn't win the lottery.
> Therefore, I cannot afford to buy a new house.

If P then Q.
P is false.
Therefore Q is false.

The conclusion might be true, but the chain of reasoning is not valid.

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Logical operations

Statements can be combined in various ways using logical operations:

NOT: The statement 'NOT P' is true if and only if P is false.

We can capture what a logical operation does using a 'truth table', which gives the values of the result for each possible value ('true' or 'false') of its arguments.

Here is the truth table for 'NOT' which has only one argument: 'NOT' takes a statement P to its negation 'NOT P'

P	**NOT P**
T	F
F	T

$\neg P$

AND: If P and Q are two statements, then the statement 'P and Q' is true if both P and Q are true.

Here is the corresponding truth table, which gives the values of 'P AND Q' for all the possible values of P and of Q:

P	**Q**	**P AND Q**
T	T	T
T	F	F
F	T	F
F	F	F

$P \wedge Q$

OR: In mathematics 'P or Q' is always taken to mean 'P or Q or both', i.e. it means the *inclusive* OR.

The corresponding truth table is:

P	**Q**	**P OR Q**
T	T	T
T	F	T
F	T	T
F	F	F

$P \vee Q$

IMPLIES: A common operation is that of implication. If P and Q are statements, then the statement 'P IMPLIES Q' is true if either P is false or Q is true.

We often use the symbol '$P \Longrightarrow Q$' for this. We might also write 'If P then Q'.

The truth table is:

P	**Q**	**P $\Longrightarrow$ Q**
T	T	T
T	F	F
F	T	T
F	F	T

$P \Longrightarrow Q$
If P then Q
Q if P
P only if Q

For example, 'London is in Russia implies Paris is in China' is true, 'London is in Russia implies Paris is in France' is true, but 'London is in England implies Paris is in China' is false. This is because London *is* in England, yet Paris is not in China!

IF AND ONLY IF: The statement 'P IF AND ONLY IF Q' (also written '$P \iff Q$') is the same as '$(P \implies Q)$ AND $(Q \implies P)$'. The truth table is as follows:

P	Q	P $\iff$ Q
T	T	T
T	F	F
F	T	F
F	F	T

Equivalent statements

Two statements are *logically equivalent* if they have the same truth table.

Example: The statement '$P \implies Q$' is logically equivalent to '(NOT P) OR Q':

P	Q	NOT P	(NOT P) OR Q	P$\implies$Q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The statement '$P \implies Q$' has several related statements, only some of which are equivalent:

• The *converse* is the statement $Q \implies P$. This is not equivalent, since the truth tables are not the same:

P	Q	Q$\implies$ P	P$\implies$Q
T	T	T	T
T	F	T	F
F	T	F	T
F	F	T	T

• The *inverse* of '$P \implies Q$' is the statement '$(NOTP) \implies (NOTQ)$. This is again not equivalent to the original statement, but it is equivalent to the converse:

P	Q	NOT P	NOT Q	(NOT P)$\implies$(NOT Q)	Q$\implies$P	P$\implies$Q
T	T	F	F	T	T	T
T	F	F	T	T	T	F
F	T	T	F	F	F	T
F	F	T	T	T	T	T

•The *contrapositive* is the statement '$(NOTQ) \implies (NOTP)$'. This is equivalent to $P \implies Q$:

P	Q	NOT P	NOT Q	(NOT Q)$\implies$(NOT P)	P$\implies$Q
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Here are some other examples:

- 'NOT($P \implies Q$)' is equivalent to 'P AND (NOT Q)';
- 'NOT(NOT P)' is equivalent to P;
- 'NOT(P AND Q)' is equivalent to '(NOT P) OR (NOT Q)';
  $\neg(P \wedge Q) = \neg P \vee \neg Q$
- 'NOT(P OR Q)' is equivalent to '(NOT P) AND (NOT Q)'.
  $\neg(P \vee Q) = \neg P \wedge \neg Q$

The last two are called De Morgan's Laws.

Variables and quantifiers

In mathematical arguments, we often have an additional complication: We often work with statements about whole classes of objects, and the statement can then be true for some objects in the class and false for others.

For example "x is an even integer".

Here x is a 'free variable', which could be allowed to range over all real numbers, for example.

We might write this statement in the form '$P(x)$' to indicate that x is a free variable.

You can think of P as a function which takes the set of all possible values of x (e.g. the real numbers) to the two possible values 'true' or 'false'.

Other examples are $P(x, y) =$"$x < y$". In most of this course our variables will be real numbers, but in principle they could range over any kind of set — for example the variable could represent a function, a subset, a vector, a matrix, etc.

A statement with free variables can be converted into a statement (i.e. one which is either true or false) using a *quantifier*. These come in two flavours:

• Existential quantifier $\exists$. If $P(x)$ is a logical formula in which x is a free variable, then the statement

$$\exists x : P(x)$$

is the statement 'there exists a value of x for which $P(x)$ is true'. Most of the time I will avoid using symbols like this, but it is occasionally useful.

• Universal quantifier $\forall$. If $P(x)$ is a logical formula in which x is a free variable, then the statement

$$\forall x : P(x)$$

is the statement '$P(x)$ is true for every value of x'.

For example, if $P(x)$ is the formula $x = 1$, then

$$\exists x : P(x)$$

is true.
But the statement

$$\forall x : P(x)$$

is false.

Similarly, if $P(x, y)$ is the formula $x = y$, the statement

$$\forall x : \ (\exists y : \ P(x, y))$$

is true. What about the statement

$$\exists y : \ (\forall x : \ P(x, y)) \ ?$$

Truth sets

When dealing with statements involving variables, it no longer makes sense to talk about truth tables.

Instead we can talk about *truth sets*. The truth set of the statement $P(x)$ is the set $\{x : \ P(x)\}$ (i.e. the set of values of the variable x for which $P(x)$ is true).

For example:

- the truth set of the statement '$x = 2$' consists of the single point 2;
- the truth set of '$x > 0$' is the set of positive real numbers;
- the truth set of '$x^2 < 4$' is the open interval $(-2, 2)$;
- the truth set of 'x is rational' is the set of rational numbers

The statement '$\forall x : \ P(x)$' simply means that the truth set of $P(x)$ is $\mathbb{R}$.

$$\{x : \ P(x)\} = \mathbb{R}$$

More generally, if A is a subset of the real numbers, then by the statement '$\forall x \in A : \ P(x)$' we mean '$P(x)$ is true for every $x \in A$, which is the same thing as saying that A is contained in the truth set of $P(x)$.

$$\{x : \ P(x)\} \supseteq A$$

The statement '$\exists x : \ P(x)$' is the same as saying the truth set is not empty:

$$\{x : \ P(x)\} \neq \emptyset$$

More generally, if A is a subset of $\mathbb{R}$, then the statement '$\exists x \in A : \ P(x)$' is the same as saying the truth set of $P(x)$ has nonempty intersection with A.

$$\{x : \ P(x)\} \cap A \neq \emptyset.$$

Claim: The statements

$$\neg(\exists x : P(x))$$

and

$$\forall x \ \big(\neg P(x)\big)$$

are logically equivalent.

"It is not true that the truth set of $P(x)$ is not empty"
i.e. $\{x : \ P(x)\} = \emptyset$

i.e. $\{x : \ \neg P(x)\} = \mathbb{R}$

$$\begin{aligned}\{x : \ P(x)\} &= \mathbb{R} \setminus \{x : \ \neg P(x)\} \\ &= \mathbb{R} \setminus \mathbb{R} \\ &= \emptyset\end{aligned}$$

In words the first statement means
"It is not true that there exists an x such that $P(x)$ holds"

while the second statement means
"For every x is is not true that $P(x)$ holds"

Similarly,

$$\mathbf{NOT}\big(\forall x\ P(x)\big)$$

The truth set of $P(x)$ is not $\mathbb{R}$

and

$$\exists x\ \big(\mathbf{NOT}P(x)\big)$$

The truth set of $\neg P(x)$ is not empty.

are logically equivalent.

Equivalent, since $\{x:\ \neg P(x)\} = \mathbb{R} \setminus \{x:\ P(x)\}$.

Examples:

The negation of the statement
'there exists a student in this room with green hair'
is logically equivalent to
'All students in this room do not have green hair'.
The negation of
'every student scored over 50% in the exam'
is
'there exists a student who did not score over 50% in the exam'.

We say two statements involving variables are *logically equivalent* if they have the same truth set.

Example:
'x is an even prime' is logically equivalent to '$x^3 = 8$'

Truth sets convert logical operations into set-theoretic ones:
AND corresponds to intersections:

$$\{x:\ P(x) \wedge Q(x)\} = \{x:\ P(x)\} \cap \{x:\ Q(x)\}$$

OR corresponds to unions:

$$\{x:\ P(x) \wedge Q(x)\} = \{x:\ P(x)\} \cup \{x:\ Q(x)\}$$

NOT corresponds to taking the complement:

$$\{x:\ \neg P(x)\} = \mathbb{R} \setminus \{x:\ P(x)\}$$

Similarly,

$$\textbf{NOT}(\forall x\ P(x))$$

and

$$\exists x\ (\textbf{NOT} P(x))$$

are logically equivalent.

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Methods of proof

Proving implications

To prove $P \implies Q$: Either

- Assume P, and deduce Q (Direct Proof); or
- Assume P is true, and Q is false, and arrive at a contradiction (Proof by contradiction);
- Assume Q is false, and deduce P is false (Proof by contrapositive).

Proving equivalence

To prove $P \iff Q$, you need to prove $P \implies Q$ **and** $Q \implies P$ (or $\neg P \implies \neg Q$)

Proofs involving quantifiers

- To prove $\forall x :\ P(x)$: Let x be arbitrary, and prove $P(x)$;
- To *disprove* $\forall x :\ P(x)$: Find an x such that $P(x)$ is false.
- To prove $\exists x :\ P(x)$: Find an x for which $P(x)$ is true;
- To *disprove* $\exists x :\ P(x)$: Let x be arbitrary, and disprove $P(x)$

In the next few lectures we will be looking in more detail at proofs, in the context of the real numbers. This material is not in either Adams or Stewart. Take a look at the notes by Terence Tao on the wattle site.

Some further recommendations: If you have difficulty in constructing proofs (and don't be dismayed if you do — this is probably the part of the course that more students struggle with than any other), there are some excellent books that will help you work through the process step by step. Here are three that are worth having a look at:

- "How to prove it: A structured approach" (Second Edition), Daniel J. Velleman (Cambridge University Press 2006)
- "The nuts and bolts of proof" (Third Edition), Antonella Cupillari (Elsevier Academic Press 2005)
- "Mathematical Proofs: A Transition to Advanced Mathematics" (Second Edition), Chartrand, Polimeni and Zhang, (Addison Wesley 2007).

For the material on real numbers, look at the lecture notes by John Hutchinson on the Wattle site, especially chapter 2, pp 1–8.