Problems

Contents

2 Some Elementary Logic

Problem 2.1

1. Show that $p \Rightarrow q$, $\neg q \Rightarrow \neg p$, $\neg p \vee q$ and $\neg(p \wedge \neg q)$ have the same meaning, by showing that they have the same truth tables.

2. Do the same for $p \vee q$ and $\neg(\neg p \wedge \neg q)$.

3. Do the same for $\neg(p \wedge q)$ and $(\neg p) \vee (\neg q)$.

4. Do the same for $p \Leftrightarrow q$ and $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

5. Do the same for $\neg(p \Rightarrow q)$ and $p \wedge \neg q$.

Problem 2.2 Prove there is an infinite number of primes by assuming that there is a greatest prime p and deducing a contradiction. Set your proof out carefully.

Hint: Consider $q + 1$ where q is the product of all primes less than or equal to p. You may assume that any integer greater than one is either prime or is divisible by a prime.

Problem 2.3 Express each of the following in terms of $\forall$, $\exists$, $\neg$, $\vee$, $\wedge$, $\Rightarrow$ and $\Leftrightarrow$, as appropriate. Do the same for a sentence equivalent to the *negation* (do not just put a $\neg$ in front, you are supposed to find a more "natural" version of the negation). *Finally*, translate this version of the negation back into English.

1. If a real number is rational, so is its square.

2. No elephant can stand the sight of a mouse.

Problem 2.4 A *triangular number* is a number of the form $\frac{k(k+1)}{2}$ where k is a natural number. Use a proof by cases to show that every triangular number has remainder 0 or 1 when divided by 3.[1] (Can you see why such a number is called *triangular*?)

Problem 2.5

1. Express the following definition in terms of $\forall$, $\exists$, $\neg$, $\vee$, $\wedge$, $\Rightarrow$ and $\Leftrightarrow$, as appropriate.

   Definition A function $f : A\,(\subset \mathbb{R}) \to \mathbb{R}$ is *uniformly continuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever x and y are in A and $|x - y| < \delta$.

[1] This is an exercise in setting out a proof carefully. Be precise and to the point.

2. Express a "natural" version of what it means for a function to be *not* uniformly continuous:
   (a) in a form analogous to the previous definition;
   (b) in terms of $\forall$, $\exists$, $\neg$, $\vee$, $\wedge$, $\Rightarrow$ and $\Leftrightarrow$, as appropriate.
3. Give a simple example of a continuous but not uniformly continuous function in case $A = (0, 1)$. Explain.

Problem 2.6 1. Express the following definition in terms of $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\iff$, $\forall$, $\exists$ as appropriate.

Definition Suppose $f_1, f_2, \ldots, f_n, \ldots$ is a sequence of functions such that $f_n : [0, 1] \to \mathbb{R}$ for all n. Suppose that $f : [0, 1] \to \mathbb{R}$. Then *the sequence* $(f_n)_{n=1}^{\infty}$ *converges to* f *uniformly* if

for every $\epsilon > 0$ there exists N such that
$n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in [0, 1]$.

Note that when the displayed expression is rewritten in symbols, the quantifier for x should *precede* $|f_n(x) - f(x)| < \epsilon$.

2. Suppose $f_1, f_2, \ldots, f_n, \ldots$ is a sequence of functions such that $f_n : [0, 1] \to \mathbb{R}$ for all n. Suppose that $f : [0, 1] \to \mathbb{R}$. We say the sequence $(f_n)_{n=1}^{\infty}$ converges to f *pointwise* if $f_n(x) \to f(x)$ for every $x \in [0, 1]$.

   Let $f(x) = 0$ if $0 \leq x < 1/2$ and $f(x) = 1$ if $1/2 \leq x \leq 1$. Give an example of a sequence of functions $f_n : [0, 1] \to \mathbb{R}$ such that $(f_n)_{n=1}^{\infty}$ converges to f pointwise but not uniformly (see Section 12.1).
3. (a) Write out a definition of pointwise convergence analogous to that given for uniform convergence.
   (b) Write out a definition analogous to your answer to (2). Note that the important point will be where the quantifier $\forall x$ is placed.

3 The Real Number System

Problem 3.1 Prove that if A and B are two sets of real numbers, and $C = \{a + b : a \in A,\ b \in B\}$, then $\sup C = \sup A + \sup B$.

Problem 3.2 Suppose that $\sup_{x\in[a,b]} f(x)$ and $\sup_{x\in[a,b]} g(x)$ both exist.[2] Show[3] that

$$\sup_{x\in[a,b]} \Big(f(x) + g(x)\Big) \le \sup_{x\in[a,b]} f(x) + \sup_{x\in[a,b]} g(x).$$

Give a counterexample to equality.[4]

Problem 3.3 Prove the following theorems from the axioms. Set your proofs out carefully, using only *one* axiom for each line of your argument. Explicitly indicate which axiom is being used for each step.

(a) **Theorem** *Suppose a and b are real numbers and $a \neq 0$. Then there exists one, and only one, number x such that*

$$ax = b.$$

Moreover, $x = ba^{-1}$.

(b) **Theorem** *If a is a real number then*

$$a \cdot 0 = 0.$$

Problem 3.4 Prove that if A and B are two sets of *strictly positive* numbers that are bounded above and

$$C = \{a/b : a \in A,\ b \in B\},$$

then

$$\sup C = \frac{\sup A}{\inf B}.$$

You should realise that $\sup C$ may be $+\infty$.

Problem 3.5 From Problem 3.3 above there is a *unique* solution of the equation $a + x = b$, and also of the equation $ax = b$ if $a \neq 0$. In particular, given $a \in \mathbb{R}$ there is a *unique* $x \in \mathbb{R}$, which is denoted $-a$, such that $a + x = 0$. Similarly for the multiplicative inverse.

In the following use only the axioms or previously proved results. Use at most one axiom per line of argument.

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[2] We define

$$\sup_{x\in[a,b]} f(x) = \sup\{y : y = f(x) \text{ for some } x \in [a,b]\}.$$

[3] *Show* always means *prove*.

[4] *Counterexamples* should always be as simple as possible, in order to better illustrate the relevant features.

1. Prove $-(-a) = a$.
2. Prove $(-1)x = -x$.
3. Prove $a(-b) = -(ab) = (-a)b$.

Problem 3.6 Suppose that $A \subset \mathbb{R}$ is bounded above. Let $\sup A = \alpha$.

1. A has a maximum element iff $\alpha \in A$.
2. If $\alpha \notin A$ then for any $\varepsilon > 0$ there are infinitely many elements of A greater that $\alpha - \varepsilon$.

4 Set Theory

Problem 4.1 What is the cardinality of

$$S = \{(x, y) : x, y \text{ are rational }\}?$$

Problem 4.2 Find a one-one map from the set $\mathcal{P}[a, b]$ (the set of all subsets of $[a, b]$) into the set $F[a, b]$ (the set of all real-valued functions defined on $[a, b]$). Deduce that the cardinality of $F[a, b]$ is $\geq$ the cardinality of $\mathcal{P}[a, b]$, which as we saw in Theorem 4.10.1 is $> c$.

Problem 4.3

1. If A and B are disjoint denumerable sets, show[5] by means of an explicit enumeration that $A \cup B$ is denumerable.

2. What if they are not necessarily disjoint?

Problem 4.4 Prove that if A is denumerable then the set of all *finite* subsets of A is denumerable. (HINT: First show that the set of all subsets of cardinality one is denumerable, similarly for the set of all subsets of cardinality two, etc.)

Problem 4.5 Prove that the set of *all* subsets of a denumerable set has cardinality c.

Problem 4.6

1. If A has cardinality c and $B \subset A$ has cardinality d, prove that $A \setminus B$ has cardinality c. (HINT: Write $A_1 = A \setminus B$. Let B' be a denumerable subset of A_1. Then $A = (A_1 \setminus B') \cup (B \cup B')$ and $A_1 = (A_1 \setminus B') \cup B'$. Now construct a one-one correspondence.)

2. Deduce that the set of irrationals is uncountable.

Problem 4.7 A real number is *algebraic* if it is the solution of an equation of the form $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ for some natural number n and integers $a_0, a_1, \ldots, a_n$. Note that any rational number is algebraic, and that $\sqrt{2}$ is algebraic. Prove that the set of algebraic numbers is denumerable.

Problem 4.8

1. Prove, by giving an enumeration, that the set of all integer multiples of 5 is denumerable.

2. Prove, by giving an enumeration, that if A is denumerable and B is finite and disjoint from A, then $A \cup B$ is denumerable.

3. What if A and B are not necessarily disjoint? Explain.

4. Prove that the set of all complex numbers of the form $a + bi$, where a and b are rational, is denumerable.

[5] "show" always means "prove".

Problem 4.9 Suppose there is a function $f_1 : A \to B$ which is *one-one*, and a function $f_2 : A \to B$ which is *onto*. Prove that $\overline{\overline{A}} = \overline{\overline{B}}$.

Problem 4.10

1. Carefully prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (see Proposition 4.1). The proof should be in two parts: first show that if $x \in A \cup (B \cap C)$ then $x \in (A \cup B) \cap (A \cup C)$, then show the converse. Your proof should essentially just rely on the definitions of $\cap$ and $\cup$, and the meaning of the logical words *and* and *or*.
2. Carefully prove the two claims in (4.36).
3. Carefully prove the two claims in (4.34).
4. Give a *simple* counterexample to equality, instead of $\subset$, holding in the first claim of (4.34).

Problem 4.11

1. Suppose that the function $f : [0,1] \to \mathbb{R}$ is *increasing*, i.e. $x < y$ implies $f(x) \leq f(y)$.
   (a) Prove that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist for all $a \in [0,1]$.[6]
   (b) Prove that for each $\epsilon > 0$ there exist only finitely many numbers a such that $\lim_{x \to a^+} f(x) - \lim_{x \to a^+} f(x) > \epsilon$.
   (c) *Deduce* that the set of points at which f is discontinuous is countable.
2. Give a simple example of a function $f : [0,1] \to \mathbb{R}$ which is discontinuous everywhere.

Problem 4.12

1. Prove that $f\left[f^{-1}[A]\right] \subset A$.
2. Give a *simple* example to show "$\subset$" cannot be replaced by "$=$".
3. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f(x,y) = \left((x^2 + y^2)^{1/2}, x + y\right).$$

   Let $A = \{(x,y) : (x^2 + y^2)^{1/2} \leq a\}$, where $a > 0$ is a given real number. Find (i) $f[A]$, (ii) $f^{-1}[A]$.

Problem 4.13 Suppose $I_n = [a_n, b_n]$ is a sequence of intervals from $\mathbb{R}$ such that $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ and such that length$I_n \to 0$ as $n \to \infty$ (i.e. if $\epsilon > 0$ then there is an N such that $b_n - a_n < \epsilon$ for all $n \geq N$).

1. Prove that there exists a unique $x \in \mathbb{R}$ such that $x \in I_n$ for every n. *NOTE*: First look at the next two parts.

[6] We say $\lim_{x \to a^-} f(x)$ exists and equals c iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - c| < \epsilon$ whenever $a - \delta < x < a$. A similar definition applies to $\lim_{x \to a^+} f(x)$.

2. Give an example to show that this is not true if $\mathbb{R}$ is replaced by $\mathbb{Q}$.

3. Give an example to show that the result is not true if the I_n are of the form (a_n, b_n).

Give a new proof that an interval $[a, b]$ (where $a < b$) is uncountable by beginning as follows:

> Suppose (in order to obtain a contradiction) that $[a, b]$ is countable. Let $x_1, x_2, x_3, \ldots, x_n, \ldots$ be a sequence which enumerates $[a, b]$. Divide $[a, b]$ into 3 intervals $[a, a + (b - a)/3]$, $[a + (b - a)/3, a + 2(b - a)/3]$ and $[a + 2(b - a)/3, b]$. Then for at least one of these intervals, which we call I_1, we have $x_1 \notin I_1$ (why do we need to divide $[a, b]$ into 3, and not 2, for this to be true?). Now divide I_1 into 3 intervals ....

Problem 4.14 Use Proposition 4.8.4 and the fact $\mathbb{N} \times \mathbb{N}$ is countable to prove Theorem 4.9.1-3 .

Problem 4.15 1. Prove that if A is infinite and B has cardinality d, then $\overline{\overline{A \cup B}} = \overline{\overline{A}}$. *Hint*: use the argument, but *not* the result, of Problem 4.6.

2. *Hence* deduce that the set of irrationals has cardinality c.

Problem 4.16 1. Let S be the set of all sequences of the form

$$(a_1, a_2, a_3, \ldots, a_i, \ldots),$$

where each $a_i = 0$ or 1. Show that S has cardinality c. *Hint*: use binary expansions of real numbers in the interval $[0, 1]$.

2. Let S_0 be the set of all *finite* sequences of the form

$$(a_1, a_2, a_3, \ldots, a_n),$$

where n can be *any* (positive) integer and where each $a_i = 0$ or 1. Show that S_0 has cardinality d (thus S_0 is the set of *all* finite sequences whose terms are 0 or 1).

3. Deduce that the set of all subsets of $\mathbb{N}$ has cardinality c and that the set of all finite subsets of $\mathbb{N}$ has cardinality d.

Problem 4.17 Suppose $\epsilon > 0$ (think of ϵ as small).

1. Show there exists a set $A \subset [0, 1]$ of the form

$$A = \bigcup_{i=1}^{\infty} (a_i, b_i)$$

where the intervals (a_i, b_i) are mutually disjoint[7], and such that

[7] That is, $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ if $i \neq j$.

   (a) $\mathbb{Q} \cap (0,1) \subset A$,
   (b) $\sum_{i=1}^{\infty}(b_i - a_i) \leq \epsilon$.

2. Show that if $x \in A^c$ then every open interval containing x [8] meets A [9].

3. Show A^c has cardinality c.

[8]That is, every open interval of the form $(a - \delta_1, a + \delta_2)$ for some $\delta_1, \delta_2 > 0$.
[9]That is, has non-empty intersection with A.

5 Vector Space Properties of $\mathbb{R}^n$

Problem 5.1 Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$, and let

$$C = \left\{ \mathbf{x} : \mathbf{x} = \sum_{i=1,n}^{n} t^i \mathbf{v} v_i,\ 0 \le t^i \le 1 \text{ for } i = 1, \ldots, n \right\}.$$

The set C is called an *n-cube*. If each $t^i = 0$ or 1, $\mathbf{x}$ is called a *vertex*. What are the various possible distances between the vertices of C? *HINT:* First think about the cases $n = 1, 2, 3$.

Problem 5.2 Let V be a subspace of $\mathbb{R}^n$ of dimension k. Consider the *orthogonal complement*

$$V^{\perp} = \{\mathbf{y} : \mathbf{y} \cdot \mathbf{x} = 0\ \forall \mathbf{x} \in V\}.$$

(a) Find an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for $\mathbb{R}^n$, such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthonormal basis for V and $\{\mathbf{v}_{\mathbf{k+1}}, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for $V^{\perp}$. *Hint:* Apply the Gram-Schmidt process to a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ for $\mathbb{R}^n$ where $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a basis for V.

(b) Show that each $\mathbf{x} \in \mathbb{R}^n$ can be written in one and only one way as $\mathbf{x} = \mathbf{y} + \mathbf{z}$ where $\mathbf{y} \in V$ and $\mathbf{z} \in V^{\perp}$.

Problem 5.3

1. Prove the following identities hold in any inner product space:

$$\begin{aligned}
(x, y) &= \frac{1}{4}\left[||x+y||^2 - ||x-y||^2\right], \\
(x, y) &= \frac{1}{2}\left[||x+y||^2 - ||x||^2 - ||y||^2\right], && (1) \\
||x+y||^2 + ||x-y||^2 &= 2\left[||x||^2 + ||y||^2\right]. && (2)
\end{aligned}$$

2. **Prove that if (2) is true in a normed space, then (1) defines an inner product on the space.

   Thus a normed space has its norm induced from an inner product iff (2) is true (and the inner product is then determined from the norm via (1)).

6 Metric Spaces

Problem 6.1 Find int$A$, ∂A and $\overline{A}$ where A is

1. $\{\mathbf{x} : 0 < |\mathbf{x} - \mathbf{x}_0| \le \delta\},\ \delta > 0.$
2. $\{(r\cos\theta,\ r\sin\theta) : 0 < r < 1,\ 0 < \theta < 2\pi\}.$
3. $\{(x, y) :$ at least one of x or y is irrational$\}$.

Problem 6.2 In the previous question, which sets are open and which are closed?

Problem 6.3 Let c be a real number and suppose $\mathbf{z} \in \mathbb{R}^n$. Show that the *half space* $\{\mathbf{x} : \mathbf{z} \cdot \mathbf{x} < c\}$ is an open set. *HINT:* $|\mathbf{z} \cdot \mathbf{y} - \mathbf{z} \cdot \mathbf{x}| \le |\mathbf{z}|\,|\mathbf{y} - \mathbf{x}|$.

Problem 6.4 Show that

$$\partial A = \partial(A^c) \text{ and } \overline{A} = \overline{\left(\overline{A}\right)}.$$

Problem 6.5 Give a simple example to show that the following is *not* necessarily true:

$$\mathrm{int}(\overline{A}) = \mathrm{int}A.$$

Problem 6.6 Let A be open and B be closed. Prove that $A \setminus B$ is open and that $B \setminus A$ is closed.

Problem 6.7 Prove that

$$\mathrm{int}(A \cap B) = (\mathrm{int}A) \cap (\mathrm{int}B).$$

Problem 6.8 Prove that

$$\mathrm{int}(A \cup B) \supset (\mathrm{int}A) \cup (\mathrm{int}B).$$

Problem 6.9 Let (X, d) be a metric space. define

$$\overline{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Prove that $\overline{d}$ is a metric. Also prove that the metrics d and $\overline{d}$ have the same open sets.

Note: The metric $\overline{d}$ has the occasional advantage that it is bounded, since $\overline{d}(x, y) < 1$ for all x, y.

Problem 6.10 Suppose $1 \le s \le n-1$. Regard $\mathbb{R}^n$ as the product $\mathbb{R}^s \times \mathbb{R}^{n-s}$ and write $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$, where $\mathbf{x}' = (x^1, \ldots, x^s)$, $\mathbf{x}'' = (x^{s+1}, \ldots, x^n)$. Let $\pi(\mathbf{x}) = \mathbf{x}'$ be the projection of $\mathbb{R}^n$ onto $\mathbb{R}^s$. Show that $\pi(A)$ is an open subset of $\mathbb{R}^s$ if A is open in $\mathbb{R}^n$. *Hint:* First consider the case $s = 1,\ n = 2$.

Problem 6.11 In the following, S has the metric induced from $\mathbb{R}$. In each case state whether A is open in S, closed in S, or neither. Justify your answers.

1. $S = [a, c) \cup (c, b]$, $A = [a, c)$, $a < c < b$.
2. $S = (0, 1]$ and $A = \{1, 1/2, 1/3, \ldots\}$.
3. $S = [0, 1]$ and $A = \{1, 1/2, 1/3, \ldots\}$.

Problem 6.12 Let S be an open (closed) subset of a metric space (X, d). Prove that a subset of S is open (closed) in S iff it is open (closed) in X.

Problem 6.13 Let X be any set. We define the *discrete* metric on X by

$$d(x, y) = \begin{cases} 1 & \text{if } x \ne y \\ 0 & \text{if } x = y \end{cases}$$

1. Prove d is a metric.
2. Describe the open balls $B_r(x)$ about $x \in X$ (you will need to consider different values of r).
3. Find $\mathrm{int}\{x\}$, $\mathrm{ext}\{x\}$, $\partial\{x\}$, $\overline{\{x\}}$.

Problem 6.14 Let (X, d) be any metric space. Define

$$\overline{d}(x, y) = \min\Big(1, d(x, y)\Big).$$

1. Prove $\overline{d}$ is a metric.
2. If (X, d) is $\mathbb{R}^2$ with the usual metric, describe the open balls $B_r(\mathbf{0})$. Draw a diagram.
3. Prove (in the general case) that d and $\overline{d}$ give the same open sets.

Problem 6.15 Let (X, d_1) and (X, d_2) be two metric spaces (with the same underlying set X).

We say that the metrics are *comparable* if there exist real numbers $\alpha > 0$ and $\beta > 0$ such that

$$\begin{aligned} d_1(x, y) &\le \alpha d_2(x, y) \\ d_2(x, y) &\le \beta d_1(x, y) \end{aligned}$$

for all $x, y \in X$.

1. Suppose B^1 and B^2 denote balls corresponding to two equivalent metrics d_1 and d_2. Prove that $B^2_r(x) \subset B^1_{\alpha r}(x)$ and $B^1_r(x) \subset B^2_{\beta r}(x)$ for all $x \in X$ and $r > 0$. Deduce that the open sets are the same for both metrics.

2. Write down an expression for the sup metric (induced from the sup norm) on $\mathbb{R}^n$. Prove it is equivalent to the (standard) Euclidean metric.[10]

3. Prove that the Euclidean metric on $\mathbb{R}^2$, and the metric induced from the Euclidean metric as in Problem 6.14, are not equivalent.

4. Write down an expression for the sup metric (induced from the sup norm) on $\mathcal{C}[a, b]$. The L_1 metric on $\mathcal{C}[a, b]$ is defined by

$$d_1(f, g) = \int_a^b |f - g|.$$

*Prove that the L_1 metric is bounded by a multiple of the sup metric, but not conversely.

5. *Give an example of a set open with respect to the sup metric on $\mathcal{C}[a, b]$, but not open with respect to the L_1 metric.

Problem 6.16 1. Prove Proposition 6.3.5

2. Carefully write out the proof of Theorem 6.4.8 for the case of arbitrary (not necessarily finite) intersections.

Problem 6.17 In the following, we are working with subsets of a fixed metric space (X, d). You should first think of $\mathbb{R}^2$ (or $\mathbb{R}$).

1. Prove that int$A$ is the *largest* open subset of A, in the sense that:
   (a) If $B \subset A$ and B is open, then $B \subset \text{int}A$;
   (b) $\text{int}A = \bigcup_{O \in \mathcal{F}} O$, where $\mathcal{F}$ is the family of all open subsets of A.

2. Prove that for any set A, $\text{int}A = \overline{A^c}^{\,c}$ and $\overline{A} = (\text{int}A^c)^c$.

3. (a) Formulate a result similar to (1) for the *closure* of a set.
   (b) Deduce this result from (1) and (2).

Problem 6.18 In this question we will establish a number of interesting and important inequalities. We will also discuss some very important normed spaces

[10] As usual, it is easier to begin with a simpler case. Try the case of $\mathbb{R}^2$.

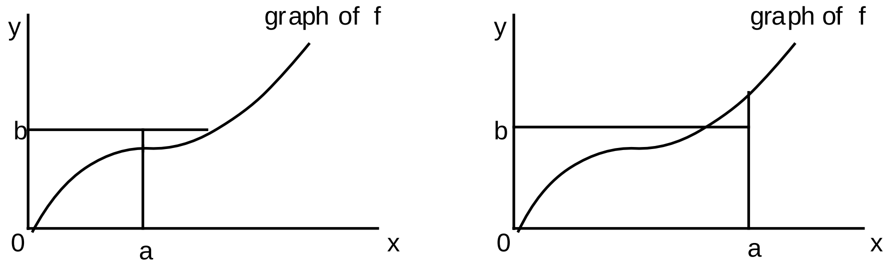
1. *Young's Inequality* Let $f:[0,\infty)\to[0,\infty)$ be strictly increasing with $f(0)=0$ and $\lim_{x\to\infty} f(x)=\infty$. Let $g:[0,\infty)\to[0,\infty)$ be the inverse function defined by

$$g(y)=x \quad \text{iff} \quad f(x)=y.$$

Argue informally, using the following diagrams according as $f(a)\le b$ or $f(a)>b$, to show that if $a,b\ge 0$ then

$$ab \le \int_0^a f + \int_0^b g,$$

and equality holds iff $f(a)=b$.



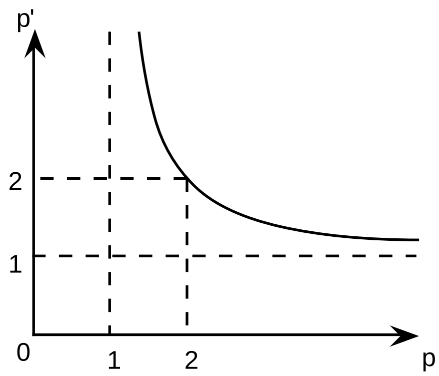
2. *Young's Inequality* If $p>1$ the *conjugate* p' of p is defined by

$$\frac{1}{p}+\frac{1}{p'}=1,$$

i.e.

$$p'=\frac{p}{p-1}.$$

Note that the graph of p' plotted against p looks like:



Deduce from 1. that if $p>1$ and $a,b\ge 0$ then

$$ab\le \frac{a^p}{p}+\frac{b^{p'}}{p'},$$

and equality holds iff $a^p=b^{p'}$.

3. *Hölder's Inequality* Suppose $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are real numbers. Suppose $p > 1$. Show that

$$\sum |a_i b_i| \le \left(\sum |a_i|^p\right)^{1/p} \left(\sum |b_i|^{p'}\right)^{1/p'}.$$

Hint: Use Young's Inequality to first prove the result in the case $\sum a_i{}^p = \sum b_i{}^{p'} = 1$. Then note that by dividing each a_i by some constant α we may assume $\sum a_i{}^p = 1$, and similarly by dividing each b_i by some constant β we may assume $\sum b_i{}^{p'} = 1$.

4. *Hölder's Inequality* Suppose $f, g \in \mathcal{C}[a, b]$. Suppose $p > 1$. Show that

$$\int_a^b |fg| \le \left(\int_a^b |f|^p\right)^{1/p} \left(\int_a^b |g|^{p'}\right)^{1/p'}.$$

Hint: First prove the result in the case $\int_a^b |f|^p = \int_a^b |g|^{p'} = 1$.

5. For $x \in \mathbb{R}^n$ and $p \ge 1$ define

$$\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}.$$

Prove this defines a norm. *Hint:* The main point is the *triangle inequality*, which is also called *Minkowski's Inequality*. For this, first assume that $\|x + y\|_p = 1$ and apply Hölder's inequality.

6. For $f \in \mathcal{C}[a, b]$ define

$$\|f\|_p = \left(\int_a^b |f|^p\right)^{1/p}.$$

Prove this defines a norm. Again, the main point is the *triangle inequality*, which is also called *Minkowski's Inequality*.

**Remark* The last result generalises with essentially the same proof to the Lebesgue integral over an arbitrary measure space. The penultimate result is also true for infinite sequences, again with almost exactly the same proof; it is in fact a particular case of the Lebesgue integral result.

Problem 6.19 1. The *unit circle* in $\mathbb{R}^2$ is defined by

$$S^1 = \{(\cos\theta, \sin\theta) : 0 \le \theta \le 2\pi\}.$$

If $p_i = (\cos\theta_i, \sin\theta_i) \in S^1$ for $i = 1, 2$ define

$$d(p_1, p_2) = |\theta_1 - \theta_2|.$$

Show that d defines a metric on S^1 (you may assume the usual properties of the trigonometric functions). Describe this metric geometrically in one sentence.

2. In any metric space, prove that $\partial A = \overline{A} \setminus \mathrm{int} A$ (your proof should only be a couple of lines).

3. Let $X = [0,1]$ with the standard metric from $\mathbb{R}$. Describe the (open) balls of radius 2 and of radius 1/2 about 0 (i.e. what are the members?).

4. What if $X = \mathbb{N}$?

5. Give an example of a set in $\mathbb{R}$ with exactly three limit points.

6. Let (x_n) be a sequence in$\mathbb{R}$. Let L be the set of all points $x \in \mathbb{R}$ for which there is a subsequence of (x_n) converging to x.

   (a) Give an example of a sequence for which the corresponding set L has exactly two members.

   (b) Give an example for which the corresponding set L has uncountably many members.

7 Sequences and Convergence

Problem 7.1 Use Theorem 7.5.1 to find the limit, if it exists, of the sequence in $\mathbb{R}^2$ given by $(x_n, y_n) = (1 - 2^{-n}, (n^2 + 3^n)/n!)$.

Problem 7.2 Use Corollary 7.6.2 to prove that if $A \subset \mathbb{R}^s$ and $B \subset \mathbb{R}^{n-s}$, and A and B are closed, then $A \times B$ is closed as a subset of $\mathbb{R}^n$.

Problem 7.3 Let $x_m \to x_0$ and $y_m \to y_0$ in $\mathbb{R}$, and assume $y_m \neq 0$ for $m = 0, 1, 2, \ldots$. Prove that $x_m/y_m \to x_0/y_0$. *HINT:* From (7.7) it is sufficient to show that $y_m^{-1} \to y_0^{-1}$.

Note that a similar result, with a similar proof, is true if (x_m) is a sequence in a normed space.

Problem 7.4 Prove that $x_0 = y_0$ in the Example in Section 7.4.

Problem 7.5 If $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^n$, use Corollary 7.6.2 to prove that A is closed. Your proof should work in any metric space. [HINT: $A = \{a_1\} \cup \cdots \cup \{a_n\}$, and so it is sufficient to show that any singleton $\{a\}$ is closed.]

Problem 7.6 If $A \subset \mathbb{R}^2$ is open, prove that A is a countable union of balls $B_r(\mathbf{x})$. [HINT: Let S be the set of all balls $B_r(\mathbf{x})$ where r is rational and the components of $\mathbf{x}$ are both rational. First prove S is countable]

Problem 7.7

1. If $x_n \to x$ in a normed space, prove $||x_n|| \to ||x||$.
2. In Corollary 7.6.2 we characterised closed subsets of a metric space in terms of convergent sequences. Prove the following analogous result for open sets:
   Let $A \subset X$ where (X, d) is a metric space. Then A is open iff:
   $$x \in A \text{ and } x_n \to x \text{ implies } x_n \in A \text{ for all sufficiently large } n.$$

Problem 7.8 In the following, we are working with subsets of a fixed metric space (X, d). You should first think of $\mathbb{R}^2$ (or $\mathbb{R}$).

1. If $A = B_1 \cup B_2$, use sequences to prove $\overline{A} = \overline{B_1} \cup \overline{B_2}$.
2. If $A = \bigcup_{i=1}^n B_i$, use sequences to prove $\overline{A} = \bigcup_{i=1}^n \overline{B_i}$.
3. If $A = \bigcup_{i=1}^\infty B_i$, use sequences to prove $\overline{A} \supset \bigcup_{i=1}^\infty \overline{B_i}$.
4. Give a simple counterexample in $\mathbb{R}$ to equality in (3).

Problem 7.9

1. Prove (7.7) of the Notes
2. (a) Show that $|\log(n+1) - \log n| \to 0$ as $n \to \infty$.
   (b) Is the sequence $(\log n)_{n=1}^\infty$ Cauchy? Explain.

8 Cauchy Sequences

Problem 8.1 Suppose $\mathbf{x} = (x^1, x^2, \ldots, x^n, \ldots)$ and $\mathbf{y} = (y^1, y^2, \ldots, y^n, \ldots)$ are infinite sequences of real numbers, and that c is a scalar (i.e. a real number). The *sum* and *scalar product* are defined by

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x^1 + y^1, x^2 + y^2, \ldots, x^n + y^n, \ldots), \\ c\mathbf{x} &= (cx^1, cx^2, \ldots, cx^n, \ldots). \end{aligned}$$

Let $\mathcal{V}$ (or more frequently ℓ_2) denote the set of all such sequences $\mathbf{x}$ for which

$$\sum_{n \geq 1} |x^n|^2 < \infty.$$

For $\mathbf{v} \in \mathcal{V}$ define

$$||\mathbf{x}|| = \left(\sum_{n \geq 1} |x^n|^2 \right)^{1/2}.$$

1. Prove that $(\mathcal{V}, ||\cdot||)$ is a normed vector space.
2. Prove that $(\mathcal{V}, ||\cdot||)$ is complete.
3. For $i = 1, 2, \ldots$ define $\mathbf{e_i} = (0, \ldots, 0, 1, 0, \ldots)$, where the 1 occurs in the ith position. Let $A = \{\mathbf{e}_1, \mathbf{e}_2, \ldots\}$. Prove that the set A is closed and bounded in $\mathcal{V}$.

Problem 8.2 An infinite series $\sum_{i=1}^{\infty} \mathbf{x}_i$ from $\mathbb{R}^k$ converges *absolutely* if the corresponding series (in $\mathbb{R}$) of absolute values $\sum_{i=1}^{\infty} |\mathbf{x}_i|$ converges. Prove that any absolutely convergent infinite series is convergent. (*Hint:* Prove that the sequence of partial sums is Cauchy.)

Give a simple counterexample in $\mathbb{R}$ to the converse.

Note: The same result and proof holds in any *complete* normed space.

Problem 8.3 Let $f : I \to \mathbb{R}$ where I is an interval from $\mathbb{R}$. Suppose f is differentiable and $|f'(x)| \leq \lambda$ for all $x \in I$.
(i) Show f is a contraction map if $\lambda < 1$ [HINT: Use the Mean Value Theorem].
(ii) If $f : I \to I$ and $\lambda < 1$ show the equation $f(x) = x$ has a unique solution.

Problem 8.4 Let $f : I \to I$ where $I = [0, \infty)$. Give an example where $|f(x) - f(y)| < |x - y|$ for all $x, y \in I$ and $x \neq y$, but f does not have a fixed point.

Why does this not contradict the Contraction Mapping Principle? Note that I is closed in $\mathbb{R}$ and so *is* complete with the metric induced from $\mathbb{R}$.

Problem 8.5 Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x,y) = (\frac{1}{3}\sin x - \frac{1}{3}\cos y + 2, \frac{1}{6}\cos x - \frac{1}{2}\sin y - 1).$$

Use the Contraction Mapping Principle to show that f has a fixed point.

Problem 8.6 1. Give a sequence (A_n) of closed non-empty subsets of $\mathbb{R}$ such that $A_1 \supset A_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

2. If $A \subset X$ then the *diameter* of A is defined by

$$\mathrm{diam} A = \sup\{d(x,y) : x \in A,\ y \in A\}.$$

Suppose (X,d) is complete, (A_n) is a sequence of closed non-empty subsets such that $A_1 \supset A_2 \supset \cdots$ and $\mathrm{diam} A \to 0$ as $n \to \infty$. Prove $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

HINT: Define an appropriate Cauchy sequence.

Problem 8.7 1. Let $F:\mathbb{R}^n \to \mathbb{R}^n$ be given by $F(x_1,\ldots,x_n) = (y_1,\ldots,y_n)$ where

$$y_i = \sum_{j=1}^{n} a_{ij}x_j + b_i \quad i = 1,\ldots,n.$$

(a) Show that F is a contraction map in the *sup* metric with contraction ratio λ if $\sum_j |a_{ij}| \leq \lambda < 1$ for each i.

(b) Show that F is a contraction map in the *standard* metric with contraction ratio $\lambda^{1/2}$ if $\sum_{i,j} {a_{i,j}}^2 \leq \lambda^2 < 1$.

(c) Deduce that $F(x) = x$ has a solution assuming the condition in either (a) or (b).

2. Suppose $F:X \to X$ where (X,d) is a complete metric space. Assume that F^n is a contraction map for some $n \geq 1$. Prove that F has a unique fixed point.

Problem 8.8 Suppose $F:\mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

1. Use *Hölder's inequality* to find a simple condition on $a_{11},\ldots,a_{22}$ such that F is a contraction map, and hence such that the Contraction Mapping Theorem applies.

2. What *is* the fixed point of F?

3. If

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

for which λ_1 and λ_2 is F a contraction map according to 1. ?

4. For which λ_1 and λ_2 is F actually a contraction map?

9 Sequences and Compactness

Problem 9.1 Prove that a subset of a compact metric space is compact iff it is closed.

Problem 9.2 Suppose $A \subset X$ where (X, d) is a metric space. For any $x \in X$ define $f(x) = d(x, A)$, where $d(x, A)$ is the distance from x to A defined in (9.1).

Prove that f is Lipschitz with Lipschitz constant 1. (Be careful: remember that it is not necessarily true that $d(x, A) = d(x, a)$ for some $a \in A$. When this *is* true the proof is easier.)

Problem 9.3

1. $A \subset \mathbb{R}^n$ is *convex* if whenever $x \in A$ and $y \in A$ then $\lambda x + (1 - \lambda)y \in A$ for all $0 < \lambda < 1$.

   Prove that if A is a closed bounded convex subset of $\mathbb{R}^n$ then for any $x \notin A$ there is a *unique* nearest point in A.

2. Suppose $x \in X$. Suppose that (x_n) is a sequence from X with the property that every subsequence contains a further subsequence which converges to x.

   Prove that the original sequence converges to x.

Problem 9.4

1. Use Definition 9.3.1 to prove that a closed subset of a compact set is compact.

2. Use Definition 9.3.1 to:

   (a) prove that the intersection of any (not necessarily finite) collection of compact sets is compact;

   (b) prove that the union of any *finite* collection of compact sets is compact.

3. Give a simple example in $\mathbb{R}$ to show that the union of a collection of compact sets need not be compact.

Problem 9.5 Let X be the collection of all sequences of the form

$$x = (x_1, x_2, \ldots)$$

for which there exists an integer N such that $x_i = 0$ if $i \geq N$ (of course, N will depend on x). Define

$$d(x, y) = \max_{1 \leq i < \infty} |x_i - y_i|.$$

1. Show (X, d) is a metric space.

2. Show it is not complete.

3. Find a subset which is closed and bounded but not compact (prove your claims).

10 Limits of Functions

Problem 10.1 Find the following limits, if they exist. Explain your reasoning.

$$
\begin{aligned}
&(1) \qquad \lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}, \\
&(2) \qquad \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}, \\
&(3) \qquad \lim_{|\mathbf{x}|\to\infty} \frac{|\mathbf{x}-\mathbf{x}_1|}{|\mathbf{x}-\mathbf{x}_2|}.
\end{aligned}
$$

Problem 10.2

1. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be given by

$$
f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}
$$

Let $a \in \mathbb{R}$ and define

$$
\begin{aligned}
S_1 &= \{(x,y) : y = ax\} \\
S_2 &= \{(x,y) : y = ax^2\} \\
S_3 &= \{(x,y) : x^2 = y^3\} \\
S_4 &= \mathbb{R}^2
\end{aligned}
$$

Evaluate each of the four limits (if they exist, and explain why not if they do not exist)

$$
\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in S_n}} f(x,y).
$$

Also evaluate the iterated limits (if they exist, and explain why not if they do not exist)

$$
\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right), \quad \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right).
$$

2. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be given by

$$
f(x,y) = \frac{x^2y}{x^6+y^2} \ (x,y) \neq (0,0).
$$

Show that f is not bounded in any open ball centred at (0,0), but the restriction of f to any straight line $L \subset \mathbb{R}^2$ which passes through the origin, is continuous on L.[11]

[11] We define continuity in the next Chapter. But you already know something about continuity from earlier courses.

11 Continuity

Problem 11.1 Use Theorem 11.4.1 to show that the following are closed

1. $\{x : -2 \le x \le 2,\ x^3 = x \ge 0\} \subset \mathbb{R}$.
2. $\{\mathbf{x} : \mathbf{y}_0 \cdot \mathbf{x} \le |\mathbf{x}|\} \subset \mathbb{R}^n$, where $\mathbf{y}_0$ is a given vector.

Problem 11.2 Let $D = \{\mathbf{x} : |\mathbf{x}| \le 1\}$ be the closed unit ball in $\mathbb{R}^n$. Let

$$f : D \to D$$

be a continuous function.

1. Use the Intermediate Value Theorem to prove that if $n = 1$ then f has a fixed point. [*Hint:* Draw a graph]
2. Assume (for arbitrary n) that f is Lipschitz with Lipschitz constant 1. Use the Contraction Mapping Principle to prove that f has a fixed point. [*Hint:* First consider the *contraction* maps $f_k = (1 - 1/k)f$]
3. Give an example where D is replaced by the annulus $A = \{\mathbf{x} : 1 \le |\mathbf{x}| \le 2\}$, f has Lipschitz constant 1, but f has no fixed point.

Remark: It is in fact true that *any* continuous $f : D \to D$ has a fixed point. This deep result is known as the Brouwer Fixed Point Theorem. You may prove it in a later course in topology.

Problem 11.3 1. Suppose that $a \in X$. Show that the function f defined by $f(x) = d(a, x)$ is continuous.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} x \sin \frac{1}{y} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

Is f continuous at $(0,0)$? Explain.

3. Use Corollary 11.4.2 to prove

$$\{(x,y) : x^2 \le y^3 \text{ and } \sin x \ge 3y\}$$

is closed.

Problem 11.4 1. Use Theorem 11.4.1 to prove that

$$\{(x,y) \in \mathbb{R}^2 : x^2 - 3xy < 7 \text{ or } \sin x \neq \frac{1}{2}\}$$

is open.

2. Give an example of a continuous function $f:\mathbb{R} \to [-1,1]$ such that f is not uniformly continuous.

3. Prove that $f(x) = x^3$ is uniformly continuous on $[-a,a]$ for each $a > 0$, but is *not* uniformly continuous on $\mathbb{R}$.

Problem 11.5 [12] If (X,d) is a metric space and A and B are non-empty disjoint closed subsets, prove that there is a continuous function $f:X \to [0,1]$ such that $f(x) = 0$ for $x \in A$, $f(x) = 1$ for $x \in B$, and $0 < f(x) < 1$ otherwise.

HINT: See Problem 9.2. Let

$$f(x) = \frac{d(x,A)}{d(x,A) + d(x,B)}.$$

Problem 11.6 [13] Suppose $K(x,y)$ is a Lipschitz function defined on $[a,b] \times \mathbb{R}$, with Lipschitz constant M. Suppose $c \in \mathbb{R}$.

Prove there is a unique continuous function u defined on $[a,a+h]$ such that

$$u(x) = c + \int_a^x K\Big(t, u(t)\Big)\, dt$$

for all $x \in [a,a+h]$, provided $h < \min\{b-a, 1/M\}$.

HINT:

1. Let $G:\mathcal{C}[a,b] \to \mathcal{C}[a,b]$ be given by

$$\Big(G(f)\Big)(x) = c + \int_a^x K\Big(t, f(t)\Big)\, dt.$$

   That is, if $f \in \mathcal{C}[a,b]$ then $G(f)$ is the function defined by the above equation. It is necessary to show that $G(f)$ is indeed continuous.

2. Prove that G is a contraction map on $\mathcal{C}[a,a+h]$.

3. Now consider the function which is the fixed point of G.

Remark The integral equation is essentially equivalent to the (initial value) differential equation problem

$$\begin{aligned} u'(x) &= K\Big(x, u(x)\Big) \\ u(a) &= c. \end{aligned}$$

Thus the preceding problem shows the existence and uniqueness of a solution to the differential equation problem on some interval $[a,a+h]$. The same proof easily generalises, apart from notational changes, to systems of differentail equations.

[12] This is an important Result.

[13] This is an important Result, and is at the centre of the work in the Chapter on Differential Equations. We will discuss these ideas in detail there.

Problem 11.7

1. Give an example of two functions $f:\mathbb{R} \to \mathbb{R}$ which are uniformly continuous and yet fg is not uniformly continuous.

2. Prove that if $f, g : X \to \mathbb{R}$ are uniformly continuous where (X, d) is a metric space, then $f + g$ is uniformly continuous.

Problem 11.8 Let (X_1, d_1), (X_2, d_2) and (X_3, d_3) be metric spaces. Suppose $f : X_1 \to X_2$ and $g : X_2 \to X_3$ are continuous. Prove $g \circ f : X_1 \to X_3$ is continuous

1. by using Theorem 11.1.2(3),

2. by using Theorem 11.4.1(2).

Problem 11.9 A subset D of a metric space (X, d) is *dense* if every member of X is a limit of a sequence of elements from D.

Suppose (X, d) and (Y, ρ) are metric spaces and D is a dense subset of X.[14]

1. Prove that if $f : D \to Y$ is uniformly continuous then there exists an extension[15] of f to a uniformly continuous function $\overline{f} : X \to Y$. *Hint*: if $d_n (\in D) \to x \in X$ define $\overline{f}(x) = \lim f(d_n)$.

2. Show the result is not true if "uniformly continuous" is everywhere replaced by "continuous".

[14] Think of the case $X = [0, 1]$, $D = (0, 1)$ and $Y = \mathbb{R}$.

[15] To say that $\overline{f}$ is an *extension* of f means that $f(d) = \overline{f}(d)$ for all $d \in D$.