Twist fields, the elliptic genus, and hidden symmetry

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We combine infinite dimensional analysis (in particular *a priori* estimates and twist positivity) with classical geometric structures, supersymmetry, and noncommutative geometry. We establish the existence of a family of examples of two-dimensional, twist quantum fields. We evaluate the elliptic genus in these examples. We demonstrate a hidden SL(2, $\mathbb{Z}$) symmetry of the elliptic genus, as suggested by Witten.

Deep and surprising relations between physics and geometry emerged from the attempt to formulate an appropriate mathematical framework to describe physics. Supersymmetry and quantized loops are basic ingredients of proposed theories combining quantum mechanics with general relativity. String theory was born in an effort to regularize the otherwise non-renormalizable aspects of gravitation as a quantum field.

The present paper focuses on the mathematics underlying a formulation of the fundamental laws of physics, but within a restricted context. We investigate infrared and ultraviolet convergence in certain nonlinear quantum field theories. Our nonperturbative analysis establishes the existence of solutions to the field equations, without being able to express the solutions in closed form. It also allows us to evaluate certain geometric invariants in the resulting field theory. We describe the results contained in a series of related papers (1–6).

Time-zero bosonic fields $\varphi(x)$ on a circle are called *loops*. The fields we consider are (multi-valued) maps from a circle (of period ℓ) into a target $\mathbb{C}^n$. Transporting x once around the circle yields the original value of φ multiplied by a phase $e^{i\chi}$, namely $\varphi(x+\ell) = e^{i\chi}\,\varphi(x)$. The angle χ characterizes the field φ. In what follows we study a family of fields with $\chi = \Omega\phi$, where Ω is a strictly positive, diagonal matrix, and where ϕ is a real parameter. We call $\varphi(x)$ a "twist field" depending on ϕ; it reduces to a periodic field as $\phi \to 0$. Twist fields have an intrinsic interest. In addition, the parameter ϕ provides an *infrared* regularization; in the next section we also introduce an *ultraviolet* regularization.

A space of functionals of fields φ is called a "loop space." Our loop space will be a space of quantum mechanical states, so it will also be a Hilbert space. To study analysis and geometry on loop space, we desire to have an exterior derivative D that acts on loop space. Some time ago, Witten suggested that natural candidates for D arise in supersymmetric physics. We study some such examples, where D has the structure $D = D_0 + D_I$, with D_0 an infinite-dimensional generalization of the de Rham derivative, and with D_I a connection determined by a holomorphic potential function V. In this context, Dirac fields are one-forms over loop space. The Fock space $\mathcal{H}$ is the exterior algebra over loop space defined by D_0, and it inherits a natural Hilbert space structure.

In our examples, D arises as a densely defined quadratic form. After regularization, D and D^* also determine densely defined operators, so for each D we introduce a "supercharge" operator Q on $\mathcal{H}$ as the (self-adjoint) closure of $Q = D + D^*$. This supercharge Q is a Dirac operator on loop space, and is a generator of supersymmetry. The theories we study possess *partial* supersymmetry. The nontrivial twist χ has the effect that we obtain only half the number of invariant charges Q from what one expects when $\chi = 0$.

Each operator Q that we consider is also invariant under a $U(1) \times U(1)$ group of translations and twists. We use this Lie symmetry to evaluate a fundamental geometric invariant for the dynamics on loop space, namely the elliptic genus. We justify a formula for the genus as a ratio of theta functions. The elliptic genus fits into the framework of noncommutative geometry, and it has the interpretation as an equivariant index of the supercharge Q. The elliptic genus is constant on each universality class of potentials V defined in **1.13**. Furthermore, as a function of ϕ and the two parameters of the symmetry group, the elliptic genus displays an $SL(2,\mathbb{Z})$ symmetry. Because this symmetry appears, even though the underlying quantum field theory is not conformal, we call the $SL(2,\mathbb{Z})$ symmetry "hidden."

1. Twist Constructive Quantum Field Theory

In constructive quantum field theory, the time-zero fields are operator-valued distributions on a Fock–Hilbert space $\mathcal{H}$. We study scalar fields $\varphi(x)$ with n components $\varphi_i(x)$, where $1 \le i \le n$, and a Dirac field $\psi(x)$ with $2n$ components $\psi_{\alpha,i}(x)$, where $\alpha = 1, 2$. The space $\mathcal{H} = \mathcal{H}^b \otimes \mathcal{H}^f$ is a tensor product of a bosonic Fock space $\mathcal{H}^b = \exp_{\otimes_s}(\mathcal{K})$ with a fermionic Fock space $\mathcal{H}^f = \exp_{\wedge}(\mathcal{K})$. The one-particle space $\mathcal{K}$ is $\mathcal{K} = \bigoplus_{i=1}^n L^2(S^1, dx) \oplus L^2(S^1, dx)$.

A unitary group e^{itH} on $\mathcal{H}$ generated by a self-adjoint Hamiltonian H determines the time evolution,

$$\varphi(x,t) = e^{itH}\,\varphi(x)\,e^{-itH}, \qquad \textbf{[1.1]}$$

and

$$\psi(x,t) = e^{itH}\,\psi(x)\,e^{-itH}. \qquad \textbf{[1.2]}$$

We also denote the bosonic conjugate time-zero field by $\pi(x)$,

$$\pi(x) = [iH, \varphi(x)^*]. \qquad \textbf{[1.3]}$$

The self-adjoint momentum operator P commutes with H and implements spatial translations. For example,

$$e^{-i\sigma P}\varphi(x)e^{i\sigma P} = \varphi(x+\sigma), \qquad \textbf{[1.4]}$$

with similar action on $\pi(x)$ and $\psi(x)$. Furthermore, the unitary twist group $e^{i\theta J}$ has a generator J commuting with H and P. For all $x \in S^1$

$$e^{i\theta J}\,\varphi_i(x)e^{-i\theta J} = e^{i\theta\Omega_i^b}\,\varphi_i(x), \qquad \textbf{[1.5]}$$

so consequently

$$e^{i\theta J}\,\pi_i(x)e^{-i\theta J} = e^{-i\theta\Omega_i^b}\,\pi_i(x), \qquad \textbf{[1.6]}$$

and the fermionic field satisfies

$$e^{i\theta J}\,\psi_{\alpha,i}(x)e^{-i\theta J} = e^{i\theta\Omega_{\alpha,i}^f}\,\psi_{\alpha,i}(x). \qquad \textbf{[1.7]}$$

The twisting angles $\Omega = \{\Omega_i^b, \Omega_{\alpha,i}^f\}$ are given constants that characterize the twist generator J, up to an additive constant. We choose this additive constant $\hat{c}/2$ so that $\pm J$ have the same spectrum. Then the zero-particle vector $\Omega_0 \in \mathcal{H}$ satisfies

$$J\,\Omega_0 = \frac{1}{2}\hat{c}\,\Omega_0, \ \text{ with } \ \hat{c} = \sum_{i=1}^n \left(\Omega_{2,i}^f - \Omega_{1,i}^f\right). \qquad \textbf{[1.8]}$$

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Definition 1.1: Twist quantum fields on a circle S^1 of length ℓ are quantum fields of the type above, such that the initial data $\{\varphi_i(x),\ \pi_i(x),\ \psi_{\alpha,i}(x)\}$ satisfy

$$\varphi_i(x+\ell) = e^{i\chi_i^b}\,\varphi_i(x), \qquad \pi_i(x+\ell) = e^{-i\chi_i^b}\,\pi_i(x), \quad \textbf{[1.9]}$$

and

$$\psi_{\alpha,i}(x+\ell) = e^{i\chi_{\alpha,i}^f}\,\psi_{\alpha,i}(x), \quad \textbf{[1.10]}$$

for all $x \in S^1$. The set of twisting angles $\chi = \{\chi_i^b,\ \chi_{\alpha,i}^f\}$ is taken so that no twisting phase equals one,

$$e^{i\chi_i^b} \neq 1, \qquad \textit{and} \qquad e^{i\chi_{\alpha,i}^f} \neq 1, \ \textit{for all}\ i,\ \alpha. \quad \textbf{[1.11]}$$

We study twist fields from two complementary points of view: as canonical quantum theories or via probability theories. Such interplay is standard in constructive quantum field theory without twists. The canonical quantum theory involves the direct study of linear transformations on Hilbert space; it is the traditional approach to quantum theory. It leads to harmonic analysis and nonlinear hyperbolic equations. The quantum fields satisfy systems of nonlinear equations with canonical constraints on their initial data. On the other hand, the probability approach relies on expectations over a classical configuration space, and a fundamental representation of expectations of the heat kernel of the Hamiltonian as a functional integral. This approach for the bosonic fields involves the definition of non-Gaussian, quasi-invariant measures on the space of functionals $\mathcal{S}'(\mathbb{T})$ of classical fields on a torus. The inclusion of fermionic fields requires the extension of the classical space to include a tensor product with an infinite dimensional Grassmann algebra equipped with a Gaussian functional that defines a Gaussian integral. These two points of view are unified through the "Feynman–Kac representation," which shows the equality of moments of the functional integral as expectations of time-ordered products of fields.

The nonlinearity of the systems we study is determined by a holomorphic polynomial $V\colon \mathbb{C}^n \mapsto \mathbb{C}$, called the superpotential. Denote the degree of this polynomial by

$$\tilde{n} = \text{degree}(V), \ \text{ and we assume } \ \tilde{n} \geq 2. \quad \textbf{[1.12]}$$

We have shown elsewhere the existence of solutions to these equations under certain assumptions on $V(z)$ and on the twisting angles that we detail below. These assumptions include the fact that $V(z)$ is a holomorphic, quasihomogeneous polynomial of degree at least two. In other words, there exist n rational numbers Ω_i called *weights*, with $\Omega_i \in (0, \frac{1}{2}]$, and such that

$$V(z) = \sum_{i=1}^{n} \Omega_i\, z_i\, V_i(z), \quad \textbf{[1.13]}$$

where $V_i(z) = \partial V(z)/\partial z_i$. Each set of weights $\{\Omega_i\}$ determines a universality class of potentials V. In these examples, the Hamiltonian $H = H(V)$ takes the form

$$H = H_0 + \int_0^{\ell} H_I(x)\, dx, \quad \textbf{[1.14]}$$

where $H_0 = H(0)$ denotes the free Hamiltonian, and

$$\begin{aligned} H_I(x) = {} & \sum_{j=1}^{n} \left|V_j(\varphi(x))\right|^2 \\ & + \sum_{i,\,j=1}^{n} \psi_{i,1}(x)\psi_{j,2}(x)^* V_{ij}(\varphi(x)) \\ & + \sum_{i,\,j=1}^{n} \psi_{i,2}(x)\psi_{j,1}(x)^* V_{ij}(\varphi(x))^*. \end{aligned} \quad \textbf{[1.15]}$$

The phrase "Wess–Zumino equations" or sometimes "Landau–Ginzburg equations" identifies these examples in the literature. For cubic V, the equations reduce to the coupling of a nonlinear boson field to the Dirac field by a Yukawa interaction, so occasionally these equations are also called "generalized Yukawa" equations.

Denote the fermion number operator on $\mathcal{H}^f$ by N^f, and let $\Gamma = (-I)^{N^f}$ denote a $\mathbb{Z}_2$-grading on $\mathcal{H}$. As Γ commutes with H_0, it follows from **1.15** that Γ commutes with $H(V)$. We require above that the Hamiltonian is invariant under both the translation group $e^{-i\sigma P}$ and the twist group $e^{i\theta J}$. The free Hamiltonian H_0 and the first term in **1.15** have this property as long as the bosonic twisting angles χ^b and the bosonic twisting parameters Ω^b in **1.5** are both proportional to the weights $\{\Omega_i\}$ in **1.13**. We obtain all possible twisting phases by considering χ^b and Ω^b modulo 2π. We choose a normalization for θ such that the bosonic parameters exactly equal the weights. Thus we choose

$$\Omega_i^b = \Omega_i \qquad \text{and} \qquad \chi_i^b = \Omega_i \phi, \quad \textbf{[1.16]}$$

and for convenience we restrict the parameter ϕ to lie in the interval $\phi \in (0, \pi]$.

The boson–fermion interaction occurs in the two last terms in **1.15**. We want the Hamiltonian to be invariant both under the J-twist group and the translation group. If we have $e^{i\theta J} H_I(x) e^{-i\theta J} = H_I(x)$ (for all θ) and $H_I(x+\ell) = H_I(x)$ (for all x), then it follows that $[J, H_I] = [P, H_I] = 0$. Inserting **1.5–1.7** and **1.16** into **1.15** shows that J-twist invariance requires

$$\Omega_{1,\,i}^f - \Omega_{2,\,j}^f + 1 - \Omega_i - \Omega_j \in 2\pi\mathbb{Z}, \quad \textbf{[1.17]}$$

for all $1 \leq i,\ j \leq n$. Similarly substituting **1.9** and **1.10** and **1.16** into **1.15**, translation invariance requires that

$$\chi_{1,\,i}^f - \chi_{2,\,j}^f + \phi - \Omega_i\phi - \Omega_j\phi \in 2\pi\mathbb{Z}, \quad \textbf{[1.18]}$$

for all $1 \leq i,\ j \leq n$. Because we may also reduce χ^f and Ω^f modulo 2π, the right sides of the constraints **1.17** and **1.18** may be taken to equal zero. The solution to these constraints is

$$\Omega_{1,\,i}^f = \Omega_i - \frac{1}{2} + \epsilon, \qquad \Omega_{2,\,i}^f = -\Omega_i + \frac{1}{2} + \epsilon, \quad \textbf{[1.19]}$$

and

$$\chi_{\alpha,\,i}^f = \Omega_{\alpha,\,i}^f \phi + \mu, \quad \textbf{[1.20]}$$

where ϵ and μ are real parameters that may depend on ϕ, but are independent of i and α. The normalization constant $\hat{c}$ defined in **1.8** is independent of ϵ and μ, and it equals

$$\hat{c} = \sum_{i=1}^{n} (1 - 2\Omega_i)\,. \quad \textbf{[1.21]}$$

The choices above yield a translation-invariant Hamiltonian, as well as one invariant under J-twists. A further restriction on the possible values of μ and ϵ allows us also to define a translation-invariant supercharge Q. The two possibilities are $\epsilon = \pm\frac{1}{2}$, both with $\mu = 0$, leading to self-adjoint supercharge operators Q_+ or Q_- that anticommute with Γ. These restrictions are necessary even in case $V = 0$. They correspond to the occurrence of $\psi_{1,i}\pi_i$ or $\psi_{2,i}\pi_i^*$ in $Q_\pm$, respectively. The supersymmetry relations are

$$Q_+^2 = H + P \ \text{ or } \ Q_-^2 = H - P. \quad \textbf{[1.22]}$$

No single choice of ϵ leads to both relations (**1.22**), so the χ-twist cuts in half the number of translation-invariant supercharges. It turns out that each charge $Q_\pm$ also commutes with a respective

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J-twist $J_\pm$. We study the case $\epsilon = \frac{1}{2}$, and call this supercharge $Q = Q_0 + Q_I(V)$, with Q_0 independent of V and with $Q_I(V)$ linear in V. The J-twist parameters and the χ-angles are in this case[†]

$$\{\Omega_i^b,\, \Omega_{1,i}^f,\, \Omega_{2,i}^f\} = \{\Omega_i,\, \Omega_i,\, 1-\Omega_i\}, \text{ and}$$
$$\{\chi_i^b,\, \chi_{1,i}^f,\, \chi_{2,i}^f\} = \{\Omega_i\phi,\, \Omega_i\phi,\, (1-\Omega_i)\phi\}. \qquad \textbf{[1.23]}$$

We require an analytic condition on the polynomial $V(z)$, to ensure that the spectrum of the Hamiltonian **1.14** is discrete and that the eigenvalues increase sufficiently rapidly. We call this an *elliptic bound*, and assume that given $0 < \epsilon$, there exists $M < \infty$ such that the function V satisfies

$$|\partial^\alpha V| \le \epsilon\, |\partial V|^2 + M, \text{ and } |z|^2 + |V| \le M\left(|\partial V|^2 + 1\right). \qquad \textbf{[1.24]}$$

Here $\partial^\alpha V$ denotes any multi-derivative of V, while $|z|$ denotes the magnitude of z, and $|\partial V|^2 = \sum_{i=1}^n |\partial V/\partial z_i|^2$ is the squared magnitude of the gradient of V. We begin by stating our standard hypotheses and the fundamental existence result.

Definition 1.2: The Standard Hypotheses (SH). The potential V is a holomorphic, quasihomogeneous polynomial **1.12–1.13**, *satisfying the elliptic bound* **1.24**. *The relations* **1.23** *for J-twists and for χ-twists hold, yielding* **1.21**.

THEOREM 1.3. *Assume* SH. *There is a self-adjoint $Q = Q(V)$ commuting with the two-parameter unitary group $e^{-i\sigma P - i\theta J}$, anticommuting with Γ, and such that $H = Q^2 - P$. The Hamiltonian $H(V)$ is bounded from below, and the heat kernel $e^{-\beta H}$ is trace class for all $\beta > 0$.*

2. The Elliptic Genus and Noncommutative Geometry

Definition 2.1: The elliptic genus is defined as the partition function

$$\mathfrak{Z}^V = \mathrm{Tr}_{\mathcal{H}}\left(\Gamma e^{-i\theta J - i\sigma P - \beta H}\right). \qquad \textbf{[2.1]}$$

We investigate the elliptic genus by representing it as a functional integral, as embodied in the following:

THEOREM 2.2. *Assume* SH. *Then there exists a positive, non-Gaussian, and countably additive Borel measure $d\mu$ on the space $\mathcal{S}' = \mathcal{S}'(\mathbb{T}^2)$ of distributions on the 2-torus, and an integrable, regularized Fredholm determinant* $\det_3$ *arising from the boson–fermion interaction, such that*

$$\mathfrak{Z}^V = \int_{\mathcal{S}'} \det{}_3\, d\mu. \qquad \textbf{[2.2]}$$

The remarkable positivity of the measure $d\mu$ is a feature of the bosonic theory; we call it *twist positivity* (3). We recognized this property and established it for untwisted fields in ref. 3, we generalized this to twist fields in ref. 5, and we abstract this property in a forthcoming joint work with O. Grandjean and J. Tyson. The positivity of $d\mu$ arises from the fact that the measure $d\mu$ has the structure $d\mu = e^{-S} d\mu_0$, where S is a measurable real action functional, and where $d\mu_0$ is a Gaussian measure. In fact $d\mu_0$ is a Gaussian with mean zero, with covariance $(-\Delta_{\mathcal{T}})^{-1}$, and with an appropriate normalization. Here $\Delta_{\mathcal{T}}$ is the twisted Laplacian on the torus $\mathbb{T}$ with periods ℓ and β; the Laplacian acts on the vector bundle $\bigoplus_{i=1}^n L^2(\mathbb{T}^2)$, being uniquely determined by its action on the domain of smooth, n-component functions $f(x, t)$ satisfying the twist relations,

$$f_i(x, t+\beta) = e^{-i\Omega_i\theta} f_i(x+\sigma, t), \qquad \textbf{[2.3]}$$

and

$$f_i(x+\ell, t) = e^{-i\Omega_i\phi} f_i(x, t). \qquad \textbf{[2.4]}$$

The functions $e_i^{\{k,E,j\}}(x,t) = \delta_{ij} e^{ikx + iEt + ik\sigma t/\beta}$, in the case that $\ell k \in 2\pi\mathbb{Z} - \Omega_j\phi$, that $\beta E \in 2\pi\mathbb{Z} - \Omega_j\theta$, and that $1 \le j \le n$, form an orthogonal basis of eigenfunctions for Δ_τ.

Another aspect of our work involves understanding the interaction introduced by non-zero potentials V, yielding the action S in the measure $d\mu$ and the regularized Fredholm determinant $\det_3$ in the representation **2.2**. We need to identify and study the cancellations that occur in this representation and in the derivative of this representation with respect to some parameter. Some of these cancellations occur directly in the heat kernel itself: these are renormalization cancellations and can be handled by extensions of known methods. Other cancellations are more delicate and only occur in estimates on differences of partition functions. We begin with some basic operator estimates. Let N denote the total number operator on $\mathcal{H}$.

THEOREM 2.3. *Assume* SH. *Then there exist constants $M_1 = M_1(V,\phi)$, $M_2 = M_2(V,\phi)$, and $M = M(\beta, V, \phi)$ that are independent of j, independent of $\lambda \in (0, 1]$, and such that*

$$N + H_0^{1/2} \le M_1 H_j(\lambda V) + M_2, \qquad \textbf{[2.5]}$$

and

$$\mathrm{Tr}_{\mathcal{H}}\left(e^{-\beta H_j(\lambda V)}\right) \le M. \qquad \textbf{[2.6]}$$

Also, for fixed V, fixed $\lambda \in [0, 1]$, fixed $\tau \in \mathbb{H}$, fixed $\theta \in \mathbb{R}$, and fixed $\phi \in (0, \pi]$,

$$\lim_{j\to\infty} \left|\, \mathfrak{Z}_j^{\lambda V} - \mathfrak{Z}^{\lambda V} \,\right| = 0. \qquad \textbf{[2.7]}$$

In ref. 6, we show that this estimate results in *a priori* estimates on the operators P and J. For a fixed V and fixed $\phi \in (0, \pi]$, there exist constants M_1 and M_2 depending on ϕ, such that $\pm P \le M_1 H_j + M_2$ and $\pm J \le M_1 H_j + M_2$. It follows that Q_j, defined as the regularized Q, can be estimated in terms of H_j, and $Q_j\, e^{-\beta H_j}$ is trace class, uniformly in j, for each $\beta > 0$. The bounds satisfied by Q_j and the convergence of $\mathfrak{Z}_j^{\lambda V}$ as $j \to \infty$, lead to the analyticity of the elliptic genus. Let $\mathbb{H}$ denote the interior of the upper-half plane. Define τ in terms of the space-time parameters σ and β to be

$$\tau = \frac{\sigma + i\beta}{\ell} \in \mathbb{H}. \qquad \textbf{[2.8]}$$

THEOREM 2.4 (6). *Assume* SH. *Then, for fixed real θ and ϕ, the elliptic genus $\mathfrak{Z}^V(\tau, \theta, \phi)$ defined in* (**2.1**) is a holomorphic function of $\tau \in \mathbb{H}$.

The elliptic genus is one member in a family of K-theory invariants arising from non-commutative geometry (entire cyclic cohomology), as explained in §IX of ref. 1 and based on ref. 10. It is the equivariant index of a Dirac operator Q on loop space. The elliptic genus, however, is only one invariant from a whole family of invariants, resulting from pairing the JLO-cocycle (11). Therefore, it may be possible, within the framework of the Wess–Zumino examples that we study here, to find closed form expressions for some other invariants given in ref. 1. We formulated various representations for such invariants in refs. 2 and 4, and these might be useful in computation.

As a computational tool, we wish to know under what hypotheses the partition function $\mathfrak{Z}^{V_\lambda}$ is independent of a parameter λ in the potential function $V_\lambda(z)$. The answer to this question depends both on analytic as well as on geometric data. The geometric requirement is satisfied if one symmetry group $e^{i\theta J}$ (with fixed Ω) is applicable to the family V_λ of potentials that

[†]While earlier methods required a massive bosonic free Hamiltonian, see for example (7–9), twist fields allow a massless one. This provides a big advantage in preserving other symmetries. Even though twists partially break supersymmetry, our computation of invariants requires only one charge, but not both. It is a remarkable feature of these examples that they do not require infinite "renormalization," as long as the twist relations **1.23** hold. Natural cancellations of divergences occur between the bosonic and the fermionic degrees of freedom.

we consider. Complementary to that, the analytic input involves whether the function

$$\lambda \mapsto \mathfrak{Z}^{V_\lambda}, \qquad \textbf{[2.9]}$$

is *a priori* continuous—or even differentiable—in λ. We would like precise conditions under which $\mathfrak{Z}^{V_\lambda}$ is differentiable, and for which the derivative equals the expression obtained by exchanging the order of differentiation and taking traces.

In this paper, we study the case $V_\lambda = \lambda V$, so variation of λ does not change the quasihomogeneous weights of V. As a consequence, if $\Gamma e^{-i\sigma P - i\theta J}$ commutes with $H(\lambda V)$ for one value of $\lambda \in (0, 1]$, it commutes for all $\lambda \in [0, 1]$. Thus we study

$$\mathfrak{Z}^{\lambda V} = \mathrm{Tr}_{\mathcal{H}}\left(\Gamma e^{-i\theta J - i\sigma P - \beta H(\lambda V)}\right). \qquad \textbf{[2.10]}$$

For $V = 0$, the heat kernel $e^{-\beta H_0}$ is also trace class, on account of the non-zero twisting parameter ϕ, and we choose in J the twisting parameters associated with V. We let the partition function $\mathfrak{Z}^0$ have an implicit dependence on V, brought about through the choice that J be appropriate for the associated family $H(\lambda V)$.

We approach the study of the properties of $\mathfrak{Z}^{\lambda V}$ as a function of λ by introducing an approximating family of regularized partition functions $\mathfrak{Z}_j^{\lambda V}$. We obtain this family by replacing $Q_I(V)$ by $Q_{I,j}(V)$, and $Q(V)$ by $Q_j(V) = Q_0 + Q_{I,j}(V)$, leading to a regularized Hamiltonian $H_j(V) = Q_j(V)^2 - P$. The approximating operators satisfy certain basic *a priori* estimates, some of which are uniform in j, while others are not. At a technical level, the family of approximations we use involves mollifiers with "slow decrease at infinity," as we introduced in ref. 12 to study a related problem, and as implemented in ref. 6 for this problem.

We write the elliptic genus $\mathfrak{Z}^{\lambda V}$ as a function of the invariant charge $Q(\lambda V)$, and likewise we write $\mathfrak{Z}_j^{\lambda V}$ as a function of $Q_j(V)$, which is also invariant. We establish certain properties of $\mathfrak{Z}_j^{\lambda V}$ depending on estimates that are *not* uniform in j.

THEOREM 2.5 (6). *Assume* SH. *Then, the map*

$$\lambda \mapsto \mathfrak{Z}_j^{\lambda V}(\tau, \theta, \phi) \qquad \textbf{[2.11]}$$

is differentiable in λ *for* $\lambda \in (0, 1]$, *and the order of* λ-*differentiation and the trace in* **2.1** *can be interchanged.*

Furthermore, with $\tilde{n}$ *given in* **1.12** *and with* $0 \le \alpha < 2/(\tilde{n} - 1)$, *there exists a constant* $M = M(\alpha, \beta, j, V, \phi)$ *such that for* $\lambda \in [0, 1]$,

$$\left|\mathfrak{Z}_j^{\lambda V} - \mathfrak{Z}^0\right| \le M\,\lambda^\alpha. \qquad \textbf{[2.12]}$$

These two analytic results lie at the very heart of evaluating the elliptic genus, as they establish the existence of a homotopy between the genus $\mathfrak{Z}_j^V$ and $\mathfrak{Z}^0$. Generically the genus $\mathfrak{Z}_j^{\lambda V}$ is piecewise constant in λ, but not globally constant. We show that $\mathfrak{Z}_j^{\lambda V}$ is differentiable on the open interval $0 < \lambda$, and we evaluate the derivative by showing the existence of and identifying the limit

$$\begin{aligned}\frac{\partial}{\partial\lambda}\mathfrak{Z}_j^{\lambda V} &= \lim_{\lambda'\to\lambda}\frac{\mathfrak{Z}_j^{\lambda V} - \mathfrak{Z}_j^{\lambda' V}}{\lambda - \lambda'} \\ &= -\mathrm{Tr}_{\mathcal{H}}\left(\Gamma Q_j(\lambda V)\, Q_{I,j}(V)\, e^{-i\theta J - i\sigma P - \beta H_j(V)}\right) \\ &\quad - \mathrm{Tr}_{\mathcal{H}}\left(\Gamma Q_{I,j}(V)\, Q_j(\lambda V) e^{-i\theta J - i\sigma P - \beta H_j(V)}\right). \qquad \textbf{[2.13]}\end{aligned}$$

We can then use *a priori* bounds on $Q_j(\lambda V)$, $Q_{I,j}(V)$, and $H_j(\lambda V)$ to show that $\partial\mathfrak{Z}_j^{\lambda V}/\partial\lambda = 0$, for $\lambda \in (0, 1]$. An important fact is that we can use cyclicity of the trace $\mathrm{Tr}_{\mathcal{H}}(AB) = \mathrm{Tr}_{\mathcal{H}}(BA)$, in the case that A is trace class, namely $\|A\|_1 < \infty$, and that B is bounded. The trace norm $\|A\|_1$ is the Schatten norm $\|A\|_p$ with $p = 1$, defined by $\|A\|_p^p = \mathrm{Tr}_{\mathcal{H}}((A^*A)^{p/2})$ for $p \ge 1$. In case $\|A\|_p < \infty$, then $\|A\| = \liminf_{p' \ge p}\|A\|_{p'}$. These norms satisfy Hölder's inequality with $\|A\|_\infty = \|A\|$ (see ref. 13).

The behavior of $\mathfrak{Z}_j^{\lambda V}$ at the endpoint $\lambda = 0$ is trickier. The second statement of the theorem claims that $\mathfrak{Z}_j^{\lambda V}$ is Hölder continuous at the origin, with an exponent that may be arbitrarily small for potentials V of large degree. In ref. 6, we prove the identity

$$\begin{aligned}\mathfrak{Z}_j^{\lambda V} - \mathfrak{Z}^0 = -\beta\lambda^2\int_0^1 \mathrm{Tr}_{\mathcal{H}}\big(\Gamma e^{-i\sigma P - i\theta J} e^{-s\beta H_j(\lambda V)/2} \\ \times Q_{I,j}(V) e^{-(1-s)H_0} Q_{I,j}(V) e^{-s\beta H_j(\lambda V)/2}\big)\, ds. \qquad \textbf{[2.14]}\end{aligned}$$

Choose α so that $0 < \alpha(\tilde{n} - 1)/2 < 1$. We also establish in ref. 6 the existence of $M_3 = M_3(\alpha, \beta, j, V, \phi)$ such that

$$\begin{aligned}&\left\| e^{-s\beta H_j(\lambda V)/4} Q_{I,j}(V) e^{-(1-s)H_0/4}\right\| \\ &\quad\le M_3\lambda^{-1+\alpha/2}s^{-1/2+\alpha/4}(1-s)^{-\alpha(\tilde{n}-1)/4}. \qquad \textbf{[2.15]}\end{aligned}$$

Although this bound is not uniform in j, it serves our purpose. We estimate **2.14** using $\left|\mathrm{Tr}_{\mathcal{H}}(A)\right| \le \|A\|_1$. We then apply a Hölder inequality in the Schatten norms. We conclude that there is a constant $M_4 = M_4(\alpha, \beta, j, V, \phi)$ such that

$$\begin{aligned}&\left|\mathfrak{Z}_j^{\lambda V} - \mathfrak{Z}^0\right| \\ &\le \beta M_3^2\lambda^\alpha\int_0^1 s^{-1+\alpha/2}(1-s)^{-\alpha(\tilde{n}-1)/2} \\ &\quad\times\left(\mathrm{Tr}_{\mathcal{H}}\left(e^{-\beta H_j(\lambda V)/4}\right)\right)^s\left(\mathrm{Tr}_{\mathcal{H}}\left(e^{-\beta H_0/4}\right)\right)^{1-s} ds \\ &\le M_4\lambda^\alpha, \qquad \textbf{[2.16]}\end{aligned}$$

establishing the Hölder continuity of $\mathfrak{Z}_j^{\lambda V}$ at $\lambda = 0$.

Combining the vanishing of $\partial\mathfrak{Z}_j^{\lambda V}/\partial\lambda$ for $\lambda > 0$, with continuity at $\lambda = 0$ (where $\mathfrak{Z}_j^{\lambda V}$ is independent of j), we infer the following from *Theorem 2.4*.

THEOREM 2.6. *Assume* SH. *Then the map* $\lambda \mapsto \mathfrak{Z}^{\lambda V}$ *is constant for* $\lambda \in [0, 1]$.

3. Hidden Symmetry

In a seminal paper (14), Witten suggested that one could calculate the elliptic genus of these examples in closed form. He gave a proposed formula (for $\phi = 0$) based on a free field computation and pointed out why one expects that answer. Kawai, Yamada, and Yang (15) elaborated on the algebraic aspects Witten's work and made contact with related proposals of Vafa (16). These insights require further elaboration, as the representation **2.1** is ill-defined if both $V = 0$ and $\phi = 0$. We prove here the representation for the elliptic genus $\mathfrak{Z}^V$ for $\phi \in (0, \pi]$, relying on the analysis above to reduce the problem to the case $V = 0$, and to a calculation carried out in ref. 6, similar to Witten's consideration for $\phi = 0$.

Define the variables

$$q = e^{2\pi i\tau}, \ \text{ so } |q| < 1, \qquad y = e^{i\theta}, \qquad \textbf{[3.1]}$$

so $|y| = 1$, and

$$z = e^{i\phi\tau}, \ \text{ so } |z| < 1. \qquad \textbf{[3.2]}$$

Consider partition functions as functions of τ, θ, and ϕ, related to q, y, and z as above. The Jacobi theta function of the first kind $\vartheta_1(\tau, \theta)$, defined for $\tau \in \mathbb{H}$ and for $\theta \in \mathbb{C}$, is given by

$$\begin{aligned}\vartheta_1(\tau, \theta) &= iq^{\frac{1}{8}}\left(y^{-\frac{1}{2}} - y^{\frac{1}{2}}\right) \\ &\quad\times\prod_{n=1}^\infty(1 - q^n)(1 - q^n y)(1 - q^n y^{-1}). \qquad \textbf{[3.3]}\end{aligned}$$

This function is odd in the second variable, namely $\vartheta_1(\tau, \theta) = -\vartheta_1(\tau, -\theta)$. We use the notation in §21.3 of Whittaker and Watson (17).

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THEOREM 3.1 (6). *Assume* SH. *Then the elliptic genus* $\mathfrak{Z}^V$ *depends on* V *only through its universality class, as determined by the weights* $\{\Omega_i\}$, *and it equals*

$$\mathfrak{Z}^V(\tau, \theta, \phi) = z^{\hat{c}/2} \prod_{i=i}^{n} \frac{\vartheta_1(\tau, (1-\Omega_i)(\theta - \phi\tau))}{\vartheta_1(\tau, \Omega_i(\theta - \phi\tau))}. \qquad \textbf{[3.4]}$$

Remark: Theorem 3.1 shows that $\mathfrak{Z}^V(\tau, \theta, \phi)$ extends to a holomorphic function for $\tau \in \mathbb{H}$, $\theta \in \mathbb{C}$, and $\phi \in \mathbb{C}$. If $a, b, c, d \in \mathbb{Z}$, and $ad - bc = 1$, then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$. Let

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad \theta' = \frac{\theta}{c\tau + d}, \qquad \textbf{[3.5]}$$

and

$$\phi' = \frac{\phi\tau}{a\tau + b}. \qquad \textbf{[3.6]}$$

The analytic continuation of the partition function $\mathfrak{Z}^V(\tau, \theta, \phi)$ obeys the transformation law

$$\mathfrak{Z}^V(\tau', \theta', \phi') = e^{2\pi i\left(\frac{\hat{c}}{8}\right)\left(\frac{c(\theta-\phi\tau)^2}{c\tau+d}\right)} \mathfrak{Z}^V(\tau, \theta, \phi). \qquad \textbf{[3.7]}$$

One obtains limiting values from the representation **3.4** as the parameters ϕ, θ, or q vanish; these limits are not uniform and do not commute. Define the integer-valued index of the self-adjoint operator Q (restricted to the subspace $H = 0$ and with respect to the grading Γ) as the difference in the dimension of the kernel and the dimension of the cokernel of Q as a map from the $+1$ eigenspace of Γ to the -1 eigenspace of Γ. Denote this integer by $\mathrm{Index}_\Gamma(Q)$.

COROLLARY 3.2. *We have the following limits.*

(*i*) *As* ϕ *tends to zero, the partition function converges to* †

$$\lim_{\phi\to 0} \mathfrak{Z}^V = \prod_{i=i}^{n} \frac{\vartheta_1(\tau, (1-\Omega_i)\,\theta)}{\vartheta_1(\tau, \Omega_i\theta)}. \qquad \textbf{[3.8]}$$

As $\theta \to 0$, *the partition function converges to*

$$\lim_{\theta\to 0} \mathfrak{Z}^V = z^{\hat{c}/2} \prod_{i=1}^{n} \frac{\vartheta_1(\tau, (1-\Omega_i)\,\phi\tau)}{\vartheta_1(\tau, \Omega_i\phi\tau)}. \qquad \textbf{[3.9]}$$

(*ii*) *For* $\theta \in (0, \pi)$, *we may take the iterated limit as* $\phi \to 0$ *and then* $q \to 0$ *to obtain the equivariant, quantum-mechanical index studied in ref. 4,*

$$\lim_{q\to 0}\left(\lim_{\phi\to 0} \mathfrak{Z}^V\right) = \prod_{i=1}^{n} \frac{\sin((1-\Omega_i)\theta/2)}{\sin(\Omega_i\theta/2)}. \qquad \textbf{[3.10]}$$

(*iii*) *The integer-valued index* $\mathrm{Index}_\Gamma(Q)$ *can be obtained as*

$$\begin{aligned} \mathrm{Index}_\Gamma(Q) &= \lim_{\theta\to 0}\left(\lim_{\phi\to 0} \mathfrak{Z}^V\right) \\ &= \lim_{\phi\to 0}\left(\lim_{\theta\to 0} \mathfrak{Z}^V\right) \\ &= \lim_{\theta\to 0}\left(\lim_{q\to 0}\left(\lim_{\phi\to 0} \mathfrak{Z}^V\right)\right) \\ &= \prod_{i=1}^{n}\left(\frac{1}{\Omega_i} - 1\right). \end{aligned} \qquad \textbf{[3.11]}$$

(*iv*) *On the other hand,*

$$\lim_{\theta\to 0}\left(\lim_{q\to 0} \mathfrak{Z}^V\right) = \lim_{q\to 0}\left(\lim_{\theta\to 0} \mathfrak{Z}^V\right) = 1. \qquad \textbf{[3.12]}$$

Example 1: For any n, let $V(z) = \sum_{i=1}^{n} z_i^{k_i}$, with $2 \le k_i \in \mathbb{Z}$. Then

$$\Omega_i = \frac{1}{k_i}, \qquad \hat{c} = \sum_{i=1}^{n} \frac{k_i - 2}{k_i}, \qquad \textbf{[3.13]}$$

and

$$\mathrm{Index}_\Gamma(Q) = \prod_{i=1}^{n}(k_i - 1). \qquad \textbf{[3.14]}$$

Example 2: For $n = 2$, let $V(z) = z_1^{k_1} + z_1 z_2^{k_2}$. In this case,

$$\Omega_1 = \frac{1}{k_1}, \Omega_2 = \frac{k_1 - 1}{k_1 k_2}, \qquad \hat{c} = 2\frac{(k_1-1)(k_2-1)}{k_1 k_2}, \qquad \textbf{[3.15]}$$

and

$$\mathrm{Index}_\Gamma(Q) = k_1(k_2 - 1) + 1. \qquad \textbf{[3.16]}$$

Remark: The integer-valued index **3.11** is stable under a class of perturbations of V that are not necessarily quasi-homogeneous. Briefly, we require that $V = V_1 + V_2$, where V_1 satisfies the hypotheses 1.13–1.24 above. While V_2 is a holomorphic polynomial, it is not necessarily quasi-homogeneous. In place of this, we assume that the perturbation V_2 is small with respect to V_1 in the following sense: given $0 < \epsilon$, there exists a constant $M_2 < \infty$ such that for any multi-derivative ∂^α of total degree $|\alpha| \ge 0$,

$$|\partial^\alpha V_2| \le \epsilon|\partial V_1| + M_2. \qquad \textbf{[3.17]}$$

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†The existence of a field theory for $\phi = 0$ requires special analysis. For $\lambda \neq 0$, the existence of a $\phi = 0$ field theory (not just the partition function) is a consequence of the assumption **1.24** for V, and the $\phi = 0$ theory is also the $\phi \to 0$ limit of the twist field theory. The elliptic genus of the limiting theory is the limit **3.8**, and it agrees with the formula proposed in ref. 15. In the case $\lambda = 0$, the elliptic genus also has a $\phi \to 0$ limit as long as $0 < |\theta| < 2\pi$, but this limit is not the genus of a limiting theory.

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