ÿ.

F

BR7802010

INIS-mf--4708

Zorn Algebra in General Relativity

Colber G. Oliveira
Departamento de Física
Universidade de Brasília

Marcos D. Maia
Departamento de Matemática
Universidade de Brasília
70.000 Brasília-DF - Brazil

Abstract

The covariant differential properties of the split Cayley subalgebra of local real quaternion tetrads is considered. Referred to this local quaternion tetrad several geometrical objects are given in terms of Zorn-Weyl matrices. Associated to a pair of real null vectors we define two-component spinor fields over the curved space and the associated Zorn-Weyl matrices which satisfy the Dirac equation written in terms of the Zorn algebra.

The formalism is generalized by considering a field of complex tetrads defining a Hermitian second rank tensor. The real part of this tensor describes the gravitational potentials and the imaginary part the electromagnetic potentials in the Lorentz gauge. The motion of a charged spin zero test body is considered. The Zorn-Weyl algebra associated to this generalized formalism has elements belonging to the full octonion algebra.

Introduction

In most applications of Cayley algebras to relativity theory a modification of the algebraic structure is required so as to make it compatible with the indefinite metric structure of the spacetime. Such modification can be obtained by using a suitable redefinition of the algebra over the complex field. In this case the algebra loses its division property and becomes a split algebra. A well known example is given by the associative subalgebra of complex quaternions (or biquaternions, or split quaternions) or the Cayley algebra in special relativity, where a Weyl representation by 2 x 2 complex matrices is used[1].

Recently the split quaternion subalgebra of the Zorn algebra has been applied to the Maxwell and Yang-Mills fields in special relativity[2]. Further applications to particle physics are also known in the literature[3][4][5]. The purpose of this paper is to investigate the application of the Zorn algebra to the study of the relativistic wave equations in curved space. It is found that by using the conventional tetrad formalism, which connects the tangent space to the pseudo-Riemannian spacetime, it is possible to construct an algebraic tetrad-structure belonging to a split quaternion subalgebra of the octonion algebra, where the algebra of octonions is here represented by a modified version of the Zorn matrices[6].

It follows that the role played by the second-rank Hermitian matrices of the two-component spinor formalism are taken over by four Zorn Weyl matrices which are associated to each local Weyl representation in the curved space. Since the "internal" symmetry group is the local $SL_2(C)$ we have two different Weyl representations, which define two sets $\overset{(a)}{\mathcal{H}}_\mu$ (a = 1,2) of the above matrices. The analogy between the $\overset{(a)}{\mathcal{H}}_\mu$ and the second-rank Hermitian matrices again indicates that the present formalism is equivalent to a local quaternion subalgebra of the full Cayley algebra. This property follows from the fact that the $\overset{(a)}{\mathcal{H}}_{\mu}$ are really 4×4 matrices as compared with the 2×2 matrices σ_{μ}, τ_{μ}. We also show that this local quaternion tetrad written as Zorn-Weyl matrices acts as projection operators which associate to each geometrical object (tensor or spinor) a well defined Zorn-Weyl matrix. The flat spacetime limit is then easily obtained and coincides with known results.

Our present Zorn-Weyl representation of octonions (and of quaternions) may also be thought as a 4×4 matrix similar a γ-matrix. Such matrices, and their non-associative law of product were already considered in the literature[5]. However, we mention that for our covariant treatment involving the "internal" group $SL_2(C)$ such analogy is not relevant. Indeed, it is not possible to associate to each $\overset{(a)}{\mathcal{H}}_{\mu}$ a γ-matrix since for each value of a we have only a type of Weyl basis, and as is known a γ-matrix contains the two Weyl basis. Due to this we interpret the $\overset{(a)}{\mathcal{H}}_{\mu}$ as Zorn matrices referred to a Weyl basis, the non-associative product being defined locally by introducing "scalar" and "wedge" products of the quaternion basis.

In the following sections we consider the definition of the differential operator in flat spaces and determine the Maxwell equations in the Zorn-Weyl formalism for the Lorentz gauge. Then we determine the Zorn-Weyl covariant derivative and apply the formalism for the relativistic spin $\frac{1}{2}$ wave equation in curved space.

Finally the Zorn-Weyl formalism is extended for the case of complex tetrads which generate a Hermitian second rank tensor field that plays the role of a generalized "metric". The symmetric (or real) part of this tensor describes gravitation according to general relativity and the antisymmetric (or imaginary) part describes the electromagnetic potentials in the Lorentz gauge. It is shown that the algebraic structure of the complex tetrad contains elements belonging to the split octonion algebra. The covariant differential properties are extended to this formalism, and as an application the problem of the motion of a charged spin zero test body in this generalized geometry is considered.

The conventions and notations which will be used throughout this paper are the following: the four-dimensional space of general relativity is assumed to have metric signature +2. Greek indices running from 1 to 4 denote tensor degrees of freedom. Latin indices are used with several different purposes: small Latin indices indicate spacelike degrees of freedom and run from 1 to 3. Bracketted indices are used to indicate tetrad indices. Capital dotted, or undotted, Latin indices are reserved for two component spinor degrees of freedom and run from 1 to 2. Finally, boldface Latin indices are used for algebraic elements, running from 1 to 7 for the capital indices and from 1 to 3 for the other indices. Summation convention is used throughout and applies to all kinds of indices.

Note to the printer: Due to difficulties in typing, boldface indices are represented in this preprint by a Latin indice with a bar on the top.

2 - The split Cayley algebra in the Zorn representation

Let $\{e_{\bar A}\}$ be a basis in a seven dimensional real vector space with an inner product. The real Cayley algebra, or octonion algebra, $\mathcal{O}$ is the linear algebra constructed in the above space, with the product operation defined by

$$e_{\bar A}\, e_{\bar B} = \varepsilon_{\bar A\bar B\bar C}\, e_{\bar C} - \delta_{\bar A\bar B}\, e_{\bar o} \qquad (2.1)$$

where $\varepsilon_{\bar A\bar B\bar C}$ is totally antisymmetric and satisfies $\varepsilon_{\bar A\bar B\bar C} = 1$ when $\bar A, \bar B, \bar C$ assume the values $(\bar 1,\bar 2,\bar 3)$, $(\bar 5,\bar 1,\bar 6)$, $(\bar 6,\bar 2,\bar 4)$, $(\bar 4,\bar 3,\bar 5)$, $(\bar 6,\bar 7,\bar 3)$, $(\bar 4,\bar 7,\bar 1)$ and $(\bar 5,\bar 7,\bar 2)$. For all other cases $\varepsilon_{\bar A\bar B\bar C}$ vanishes. The identity element of the algebra is $e_{\bar o}$.

It follows immediately from (2.1) that if the indices vary on each one of the seven above triads $(\bar A,\bar B,\bar C)$ a quaternion subalgebra is obtained. Thus, the real Cayley algebra contains seven quaternion subalgebras. In the basis $\{e_{\bar o}, e_{\bar A}\}$ a general real Cayley number is expressed by

$A = x_{\bar o}\, e_{\bar o} + x_{\bar A}\, e_{\bar A}$, $x_{\bar o}$, $x_{\bar A} \in \mathbb{R}$. The multiplication table implies that the product operation is in general nonassociative. Furthermore it follows that a real Cayley algebra is a division algebra.

Now we consider the algebra of complex Cayley numbers $\mathcal{O}/\mathbb{C}$ which may be taken as the set of elements of the form

$$A = z_{\bar o} e_{\bar o} + z_{\bar A} e_{\bar A}\ ;\ z_{\bar o},\ z_{\bar A} \in \mathbb{C} \qquad (2.4)$$

The complex conjugation applied to the components $z_{\bar o}$, $z_{\bar A}$ gives a new Cayley number: $A^* = z^*_{\bar o}\, e_{\bar o} + z^*_{\bar A}\, e_{\bar A}$

and this operation commutes with the Cayley number conjugation defined previously: $(\bar{A})^* = (\overline{A^*})$. In particular consider the following complex Cayley numbers

$$u_{\bar{o}} = 1/2(e_{\bar{o}} + ie_{\bar{7}})\,; \quad u_{\bar{i}} = 1/2(e_{\bar{i}} + ie_{\bar{i}+3})$$

and their complex conjugates. From the multiplication table (2.1) it follows that the above set of complex Cayley numbers together with their complex conjugates form a basis for O/C. The product between these basis elements are given by the equations

$$u_{\bar{i}}u_{\bar{j}} = \varepsilon_{\bar{i}\,\bar{j}\,\bar{k}}u^*_{\bar{k}}, \quad u_{\bar{i}}u^*_{\bar{j}} = -\delta_{\bar{i}\bar{j}}u_{\bar{o}},$$

$$u_{\bar{i}}u_{\bar{o}} = 0, \quad u_{\bar{i}}u^*_{\bar{o}} = u_{\bar{i}}, \quad u_{\bar{o}}u_{\bar{i}} = u_{\bar{i}},$$

$$u^*_{\bar{o}}u_{\bar{i}} = 0, \quad u_{\bar{o}}u_{\bar{o}} = u_{\bar{o}}, \quad u_{\bar{o}}u^*_{\bar{o}} = 0,$$

and their complex conjugates.

A general complex Cayley number in this basis assumes the form

$$A = au^*_{\bar{o}} + bu_{\bar{o}} + x_{\bar{i}}u^*_{\bar{i}} + y_{\bar{i}}u_{\bar{i}} \tag{2.5}$$

where the coefficientes $a, b, x_{\bar{i}}$ and $y_{\bar{i}}$ are in general complex numbers. In particular they can be real, with A still a complex Cayley number.

In order to introduce a representation of the complex Cayley algebra we consider an application Z from O/C assuming values on the set $M_{2\times 2}/H$ of 2×2 matrices defined over the quaternion field H. Such application is defined by

$$Z(u^*_{\bar{o}}) = \begin{pmatrix} e_{\bar{o}} & 0 \\ 0 & 0 \end{pmatrix}, \quad Z(u_{\bar{o}}) = \begin{pmatrix} 0 & 0 \\ 0 & e_{\bar{o}} \end{pmatrix}, \quad Z(u_{\bar{i}}) = \begin{pmatrix} 0 & 0 \\ e_{\bar{i}} & 0 \end{pmatrix}, \quad Z(u^*_{\bar{i}}) = \begin{pmatrix} 0 & -e_{\bar{i}} \\ 0 & 0 \end{pmatrix}. \tag{2.6}$$

Defining the sum of two such matrices and multiplication by $\mathbb{C}$ in the usual way, it follows that the application Z is linear in $\mathcal{O}/C$ and from (2.5) and (2.6) we have

$$Z(A) = \begin{pmatrix} ae_{\bar{o}} & -x_{\bar{i}}e_{\bar{i}} \\ y_{\bar{i}}e_{\bar{i}} & be_{\bar{o}} \end{pmatrix} = \begin{pmatrix} a & -x \\ y & b \end{pmatrix} \qquad (2.7)$$

where we have denoted $x_{\bar{i}}e_{\bar{i}}$ by x, the same for y.

The set of matrices of the form (2.7) may define a representation of $\mathcal{O}/C$ in $M_{2\times2}/_H$ provided a product between such matrices is defined in such a way that the application Z is an homomorphism. In this case the matrices (2.7) are called Zorn matrices[6].

In order to introduce the definition of the Zorn product for matrices of the form (2.7) we define the scalar and wedge product of quaternions as

$$e_{\bar{i}} * e_{\bar{j}} = -\frac{1}{2}(e_{\bar{i}}e_{\bar{j}} + e_{\bar{j}}e_{\bar{i}}) = \delta_{\bar{i}\bar{j}}e_{\bar{o}} \qquad (2.8)$$

$$e_{\bar{i}} \wedge e_{\bar{j}} = \frac{1}{2}(e_{\bar{i}}e_{\bar{j}} - e_{\bar{j}}e_{\bar{i}}) = \varepsilon_{\bar{i}\bar{j}\bar{k}}e_{\bar{k}} \qquad (2.9)$$

The Zorn product between Zorn matrices is now defined in such way that it reproduces the multiplication table of the complex Cayley basis:

$$Z(AB) = Z(A) \odot Z(B) = \begin{pmatrix} ac - x*w & -az - dx - y\wedge w \\ cy + bw + x\wedge z & bd - y*z \end{pmatrix} \qquad (2.10)$$

for

$$Z(A) = \begin{pmatrix} a & -x \\ y & b \end{pmatrix}, \qquad Z(B) = \begin{pmatrix} c & -z \\ w & d \end{pmatrix}.$$

The unit element of the resulting Zorn matrix algebra is

$$1 = \begin{pmatrix} e_{\bar{o}} & 0 \\ 0 & e_{\bar{o}} \end{pmatrix} = Z(u_{\bar{o}}^{*}) + Z(u_{\bar{o}}). \qquad (2.11)$$

As it can be seen a complex octonion like (2.5) reduces, in general, to a complex quaternion when $a = b$, $x = y$. This quaternion belongs to the quaternion subalgebra (1,2,3) of the octonion algebra. The Zorn matrix associated to this quaternion is

$$Z(A) = \begin{pmatrix} a & -x \\ x & a \end{pmatrix}. \qquad (2.12)$$

The conjugation operation defined by (2.3) induces on the complex basis the transformation $u_{\bar{o}} \to u_{\bar{o}}^{*}$, $u_{\bar{i}} \to -u_{\bar{i}}$. Thus, a general complex Cayley number B such as (2.5) which in the Zorn representation reads as (2.7), transforms under conjugation to

$$Z(\bar{B}) = \begin{pmatrix} b & x \\ -y & a \end{pmatrix}.$$

The norm of this octonion is given by

$$Z(B\bar{B}) = Z(B) \odot Z(\bar{B}) = (ab + x_{\bar{i}}y_{\bar{i}})\mathbb{1}$$

Here we are interested in the situation where the octonion B reduces to a quaternion of the form (2.12). In this case the norm of this quaternion in the Zorn representation will be $(a^2 + x_{\bar{i}}x_{\bar{i}})\mathbb{1}$. If we take $\underline{a}$ as a imaginary number, $a = ix_{\bar{o}}$, and $x_{\bar{o}}$, $x_{\bar{i}}$ reals, it follows that

$$Z(A) \odot Z(\bar{A}) = Q(A)\cdot \mathbb{1} = -x_{\bar{o}}^2 + x_{\bar{i}}x_{\bar{i}} \qquad (2.13)$$

Therefore, the Zorn matrix (2.12) with $a = ix_{\bar{o}}$ may be thought as representing a four-vector in Minkowski spacetime (in this case the algebraic indices become world indices). Likewise given an general octonion like (2.7) with $a \neq b$, $x \neq y$ but with a,b imaginary numbers, $a = ix_{\bar{o}}$, $b = iy_{\bar{o}}$, and all $x_{\bar{o}}$, $x_{\bar{i}}$, $y_{\bar{o}}$, $y_{\bar{i}}$ reals, we can associate to this octonion a pair of four-vectors in Minkowski spacetime. From now on we will consider only this particular type of octonions and quaternions and their Zorn matrices. The corresponding quaternion subalgebra of the complex Cayley algebra given by the set of matrices of the form (2.12), with the law of product given by (2.10), and with norm given by (2.13) is a split algebra. This corresponds to the property that the Minkowski spacetime contains isotropic vectors. Similarly we have a split octonion algebra which corresponds to the property that the Minkowski spacetime contains pairs of orthogonal four-vectors.

3 - Extension of the method to curved spacetimes in the tetrad representation

In this section we apply the previous algebraic methods to a curved fourdimensional spacetime. Such type of formalism is an extension of previous works which apply these algebraic methods to special relativity[2]. The formalism which will be developed in this section corresponds to the use of only a part of the Cayley algebra, namely the quaternion subalgebra of the complex Cayley algebra. As in the previous section we will use this algebra referred to the Zorn matrices defined in a quaternion basis. In applications to relativity it is of interest to use the quaternion basis in terms of the three Pauli matrices and the 2×2 identity matrix, that is, in terms of a Weyl representation of the quaternion algebra[7]. From the algebraic point of view this Weyl representation is obtained by the application $W: H/C \to M_{2\times 2/C}$ which may be defined either by

$$W(e_{\bar{o}}) = \overset{o}{\sigma}_{\bar{o}}, \quad W(e_{\bar{i}}) = \overset{o}{\sigma}_{\bar{i}}, \tag{3.1}$$

or by,

$$W(e_{\bar{o}}) = \overset{o}{\sigma}_{\bar{o}}, \quad W(e_{\bar{i}}) = \left(\frac{1}{i}\right)\overset{o}{\sigma}_{\bar{i}}. \tag{3.2}$$

The symbols $\overset{o}{\sigma}_{\bar{i}}$, $\overset{o}{\sigma}_{\bar{o}}$ denote the Pauli matrices and the 2×2 identity matrix respectively. In the first case they satisfy the usual law of multiplication of the Pauli matrices together with the 2×2 identity matrix, and in the second case they satisfy the same law of product as the $e_{\bar{o}}$, $e_{\bar{i}}$, namely

$$\begin{aligned} W(e_{\bar{i}})\, W(e_{\bar{j}}) &= -\delta_{\bar{i}\bar{j}}\, W(e_{\bar{o}}) + \varepsilon_{\bar{i}\bar{j}\bar{k}}\, W(e_{\bar{k}}) \\ W(e_{\bar{i}})\, W(e_{\bar{o}}) &= W(e_{\bar{o}})\, W(e_{\bar{i}}) = W(e_{\bar{i}}) \end{aligned} \tag{3.3}$$

We will use the second alternative. It should be observed that the Zorn representation of the quaternion algebra treated on the last section is distinct from the usual Weyl representation of quaternions. However, since the Zorn matrices associated to quaternions are defined over the quaternion field, a combined Zorn-Weyl representation of quaternions can be obtained by considering the Weyl representation of elements of the Zorn matrices. Denoting the resulting composition by ZW, we have for a Zorn matrix like (2.7) for $a = ix_{\bar{o}}$ (Which gives the Zorn representation of the split quaternion subalgebra of the complex Cayley algebra)

$$ZW(A) = \begin{pmatrix} W(a) & -W(x) \\ W(x) & W(a) \end{pmatrix} = \begin{pmatrix} ix_{\bar{o}}W(e_{\bar{o}}) & -x_{\bar{i}}W(e_{\bar{i}}) \\ x_{\bar{i}}W(e_{\bar{i}}) & ix_{\bar{o}}W(e_{\bar{o}}) \end{pmatrix}$$

which from (3.2) takes the form:

$$ZW(A) = \begin{pmatrix} ix_{\bar{o}}\overset{o}{\sigma}_{\bar{o}} & -\frac{1}{i}x_{\bar{i}}\overset{o}{\sigma}_{\bar{i}} \\ \frac{1}{i}x_{\bar{i}}\overset{o}{\sigma}_{\bar{i}} & ix_{\bar{o}}\overset{o}{\sigma}_{\bar{o}} \end{pmatrix} \tag{3.4}$$

Presently we have to adapt this notation to our problem of a curved spacetime with a Riemannian structure. For this purpose we consider only the local properties of this spacetime translated in terms of the Zorn-Weyl algebra. With this in mind we consider the local tangent space at each point of the Riemannian spacetime, and the set of four local tetrad vectors $h_\alpha = (h_{(\lambda)\alpha})$. All algebraic quantities, with indices $\bar{o}$, $\bar{i}$, $\bar{j}$ etc. now become quantities defined on the local tangent space with indices (o), (i), (j) etc. The metric $g_{\mu\nu}$ is related to the Minkowskian metric $\eta_{\alpha\beta}$ by means of the local tetrad field

$$g_{\mu\nu} = h_\mu^{(\alpha)} h_\nu^{(\beta)} \eta_{\alpha\beta}.$$

The matrices $\overset{o}{\sigma}_{(o)}$, $\overset{o}{\sigma}_{(i)}$, σ_i, σ_j have degrees of freedom given by the indices K and $\overset{o}{M}$ which run from 1 to 2, both indices taken as contravariant indices. The matrices denoted by τ have both indices as covariant indices, also of the type $K\dot{M}$. The matrix denoted by $\underline{I}$ is the 2×2 identity matrix of the form (δ_s^k). In the local tangent space which here is taken with signature +2 we can define three types of identity matrices: $\overset{\circ}{\sigma}{}^{(o)}$, $\overset{\circ}{\tau}_{(o)}$ and $\underline{I}$. The matrix $\overset{\circ}{\tau}_{(o)}$ is defined by $\overset{\circ}{\tau}_{(o)} = \varepsilon\overset{\circ}{\sigma}{}^{*}{}_{(o)}\varepsilon = I$ and is numerically identical to the identity 2×2 matrix, but has covariant indices of the type $K\dot{M}$. Here $\underline{\varepsilon}$ is the matrix

$$\varepsilon = (\varepsilon^{KM}) = (\varepsilon_{KM}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since local indices are raised and lowered respectively by $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$, we have $\overset{\circ}{\sigma}_{(o)} = -I$. Covariant matrix product is defined only between matrices of the type σ and τ. This implies that the algebraic formulas defining the quaternion structure like (3.3), and the formulas defining the ZW representation of quaternions have to be translated in this covariant notation. With this in mind we define the following Weyl representations on the local tangent space, which are associated to $SL_2(C)$:

$$\begin{aligned} W_1(e_{(i)}) &= \frac{1}{i}\overset{\circ}{\sigma}_{(i)}, \qquad W_1(e_{(o)}) = \overset{\circ}{\sigma}{}^{(o)} \\ W_2(e_{(i)}) &= \frac{1}{i}\overset{\circ}{\tau}_{(i)}, \qquad W_2(e_{(o)}) = \overset{\circ}{\tau}_{(o)} \end{aligned} \tag{3.5}$$

In the limit where we consider only the action of $SU_2(C)$ on the spinor degrees of freedom they degenerate in the representation given by (3.2) for the spacelike degrees of freedom: $W_1(e_{(i)}) \to W_2(e_{(i)})$. Besides this we also define

$$W_3(e_{(o)}) = I = (\delta^k_s)$$

It can be shown that the covariant law of product for the Weyl representations (3.5) has the form

$$W_1(e_{(i)})W_2(e_{(j)}) = -\delta_{(i)(j)}W_3(e_{(o)})+\varepsilon_{(i)(j)(k)}W_1(e_{(k)})W_2(e_{(o)}) \tag{3.6}$$

$$W_1(e_{(o)})W_2(e_{(o)}) = W_3(e_{(o)}). \tag{3.7}$$

These formulas presently substitute the formulas (3.3).

Given the Zorn matrix associated to a quaternion we can write it in the Weyl representation of the type W_1 as

$$ZW_1(A) = \begin{bmatrix} ia_{(o)}W_1(e_{(o)}) & -a_{(i)}W_1(e_{(i)}) \\ a_{(i)}W_1(e_{(i)}) & ia_{(o)}W_1(e_{(o)}) \end{bmatrix} \tag{3.8}$$

For the same quaternion, or in general for any other quaternion, we can write the ZW matrix of the type W_2 by replacing the subscripts 1 by 2 .

On the remaining of this section we will use these matrices in place of the matrices (3.4).

The product of these matrices is defined similarly as before (see Eq. (2.10)):

$$ZW_1(A) \odot ZW_2(B) = ZW_3(C) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \tag{3.9}$$

where,

$$\alpha = \delta = -a_{(o)}b_{(o)}W_3(e_{(o)}) - a_{(i)}b_{(j)}W_1(e_{(i)})*W_2(e_{(j)})$$

$$\beta = -\gamma = -(ia_{(o)}b_{(i)}+ia_{(i)}b_{(o)})W_1(e_{(i)})W_2(e_{(o)}) -$$

$$- a_{(i)}b_{(j)}W_1(e_{(i)}) \wedge W_2(e_{(j)}).$$

From (3.6) we define in analogy with (2.8) and (2.9)

$$W_1(e_{(i)}) * W_2(e_{(j)}) = -\frac{1}{2} \underset{(i,j)}{S} \left(W_1(e_{(i)}) W_2(e_{(j)})\right) = \delta_{(i)(j)} W_3(e_{(o)}), \tag{3.10}$$

$$\begin{aligned} W_1(e_{(i)}) \wedge W_2(e_{(j)}) &= \frac{1}{2} \underset{(i,j)}{E} \left(W_1(e_{(i)}) W_2(e_{(j)})\right) = \\ &= \frac{1}{2} \varepsilon_{(i)(j)(k)} \left[W_1(e_{(k)}) W_2(e_{(o)}) + W_1(e_{(o)}) W_2(e_{(k)})\right] = \\ &= \varepsilon_{(i)(j)(k)} W_1(e_{(o)}) W_2(e_{(k)}) = \varepsilon_{(i)(j)(k)} W_1(e_{(k)}) W_2(e_{(o)}) \end{aligned} \tag{3.11}$$

where $\underset{(i,j)}{S} T_{(i)(j)}$ and $\underset{(i,j)}{E} T_{(i)(j)}$, for any $T_{(i)(j)}$ mean

$$\underset{(i,j)}{S} T_{(i)(j)} = T_{(i)(j)} + T_{(j)(i)}, \quad \underset{(i,j)}{E} T_{(i)(j)} = T_{(i)(j)} - T_{(j)(i)}.$$

Of fundamental importance are the ZW matrices associated to the tetrad field, which are defined by

$$\overset{(1)}{\mathcal{H}}_{\mu} = ZW_1(H_\mu) = \begin{bmatrix} i h_{\mu(o)} W_1(e_{(o)}) & -h_{\mu(i)} W_1(e_{(i)}) \\ h_{\mu(i)} W_1(e_{(i)}) & i h_{\mu(o)} W_1(e_{(o)}) \end{bmatrix} \tag{3.12}$$

$$\overset{(2)}{\mathcal{H}}_{\mu} = ZW_2(H_\mu) = \begin{bmatrix} i h_{\mu(o)} W_2(e_{(o)}) & -h_{\mu(i)} W_2(e_{(i)}) \\ h_{\mu(i)} W_2(e_{(i)}) & i h_{\mu(o)} W_2(e_{(o)}) \end{bmatrix} \tag{3.13}$$

where $H_\mu = h_{\mu(o)}e_{(o)} + h_{\mu(i)}e_{(i)}$.

From these definitions it follows that the metric of the Riemannian spacetime is given in terms of the ZW matrices (3.12) and (3.13) as

$$2g_{\mu\nu}ZW_3(e_{(o)}) = ZW_1(H_\mu) \odot ZW_2(\bar{H}_\nu) + ZW_1(H_\nu) \odot ZW_2(\bar{H}_\mu) \qquad (3.14)$$

where

$$ZW_3(e_{(o)}) = \begin{pmatrix} W_3(e_{(o)}) & 0 \\ 0 & W_3(e_{(o)}) \end{pmatrix} = I$$

The world indices labeling the several elements of the algebraic quantities given by (3.12) and (3.13) are raised by the metric field $g^{\mu\nu}$. The process of raising (lowering) world indices is presently equivalent to a sum of terms representing the multiplication of Zorn-Weyl scalars, the metric components, by Zorn-Weyl matrices which display free world indices: $\overset{(1)}{\mathcal{H}}_\mu = g^{\mu\nu}\overset{(1)}{\mathcal{H}}_\nu$, $\overset{(2)}{\mathcal{H}}{}^\mu = g^{\mu\nu}\overset{(2)}{\mathcal{H}}_\nu$. This process is extended to any other ZW matrix possessing free world indices.

4 - The Zorn-Weyl differential operator in flat spacetime

We define a flat spacetime Zorn-Weyl differential operator in the quaternion representation as

$$D_a = \begin{pmatrix} iW_a(e_o)\partial_o & -W_a(e_j)\partial_j \\ W_a(e_j)\partial_j & -iW_a(e_o)\partial_o \end{pmatrix} = ZW_a(\partial_\mu), \qquad (4.1)$$

where $\underline{a}$ takes the values 1 or 2. Here ∂_o, ∂_j denote the usual partial derivatives. The operator (4.1) acts on a Zorn-Weyl matrix as $D_a(A) = D_a \odot A$. It follows that

$$D_1 \odot \bar{D}_2 = \bar{D}_1 \odot D_2 = D_2 \odot \bar{D}_1 = \bar{D}_2 \odot D_1 = \Box \cdot \mathbb{I} \qquad (4.2)$$

where $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$, $\eta^{\alpha\beta} = \mathrm{diag}(-1,+1,+1,+1)$.

If A_μ is the electromagnetic potential its associated Zorn-Weyl matrix is given by $A_a = ZW_a(A_\mu)$. Using (4.1) it is possible to show that the Maxwell equations in the Lorentz gauge assume the form[2]

$$D_2 \odot (\bar{D}_1 \odot A_2) = -\bar{J}_2 \qquad (4.3)$$

where $J_a = ZW_a(j_\mu)$ are the two Zorn-Weyl representations of the current four vector.

5 - Zorn Weyl matrices associated to geometrical objects.

The Zorn-Weyl matrices associated to the four vectors of the tetrad $H_\mu = (h_{\mu(0)}, h_{\mu(1)}, h_{\mu(2)}, h_{\mu(3)})$ are given by $\overset{(a)}{\mathcal{H}}_\mu = ZW_a(H_\mu)$. Then the Zorn-Weyl matrices associated to a vector B_μ are defined by

$$\overset{(a)}{B} = \overset{(a)}{\mathcal{H}}{}^\mu B_\mu = \overset{(a)}{\mathcal{H}}{}^\mu \odot \mathbb{I} B_\mu. \qquad (5.1)$$

Algebraic Zorn-Weyl objects may also be associated to tensors, spinors and mixed geometrical objects. To $B_{\sigma\mu}$, a tensor of rank two, we can associate the Zorn-Weyl matrices

$$\overset{(a)}{C}_\lambda = \overset{(a)}{\mathcal{H}}{}^\mu B_{(\lambda)\mu} \qquad a = 1,2 \qquad (5.2)$$

where $B_{(\lambda)\mu} = h^\sigma_{(\lambda)} B_{\sigma\mu}$.

It is also possible to associate to $B_{\sigma\mu}$ a further Zorn-Weyl matrix given by

$$C = (\overset{(1)}{\mathcal{H}}{}^{\mu} \odot \overline{\overset{(2)}{\mathcal{H}}}{}^{\nu}) B_{\mu\nu}. \qquad (5.3)$$

If $B_{\mu\nu}$ is symmetric the expression (5.4) becomes $C = g^{\mu\nu}B_{\mu\nu}\cdot\mathbb{I}$. If $B_{\mu\nu}$ is antisymmetric C contains only non diagonal "matrix elements".

Now we consider the problem of associating ZW matrices to spinor fields in curved spaces. This correspondence is obtained by recalling that two-component spinors are related to tensors through well known formulas. Here we are mainly interested in two-component spinor fields of the type $\chi_A, \omega^{\dot{A}}$ since we want to obtain the Dirac equation for a massive spin $\frac{1}{2}$ particle in terms of the Zorn algebra. With this in mind we consider a pair of real null vectors $V_\mu(x)$ and $W_\mu(x)$. Then

$$V_\mu(x) = \frac{1}{2}\,\sigma_\mu{}^{A\dot{B}}(x)\chi_A(x)\chi_{\dot{B}}(x), \qquad (5.4)$$

$$W_\mu(x) = \frac{1}{2}\,\sigma_\mu{}^{A\dot{B}}(x)\omega_A(x)\omega_{\dot{B}}(x) = \frac{1}{2}\,\sigma_{\mu A\dot{B}}(x)\omega^{A}(x)\omega^{\dot{B}}(x), \qquad (5.5)$$

where $\sigma_{\mu(x)}^{A\dot{B}} = h_{\mu(\alpha)}(x)\overset{\circ}{\sigma}{}^{(\alpha)A\dot{B}}$. In matrix notation we have

$$\chi = (\chi_A) = \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix},\ \chi^\dagger = (\chi_{\dot{A}}) = (\chi_{\dot{1}}\ \chi_{\dot{2}}),\ \sigma_\mu = (\sigma_\mu{}^{A\dot{B}}) = \sigma_\mu{}^{\dagger}.$$

Similarly we denote

$$\tau_\mu = \varepsilon\cdot\overset{*}{\sigma}_\mu\cdot\varepsilon = -(\overset{*}{\alpha}_{\mu A\dot{B}}) = \tau_\mu^\dagger,\ \Omega = (\omega^{A}) = \begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix},\Omega^\dagger = (\omega^{\dot{A}}) = (\omega^{\dot{1}},\omega^{\dot{2}}).$$

Raising (lowering) of spinor indices is obtained by the use of the spinors ε_{AB}, ε^{AB}, as usually. $\psi^{A} = \varepsilon^{AK}\psi_{K}$, $\psi_{A} = \psi^{K}\varepsilon_{KA}$. Now the equation (5.5) can be written as $W_{\mu} = \frac{1}{2}\,\Omega^{T}\rho_{\mu}\Omega$, where $\rho_{\mu} = (\sigma_{\mu A\dot{B}}) = \rho_{\mu}^{\dagger}$. Since W_{μ} is a real vector field

$$W_{\mu}^{*} = W_{\mu} = \frac{1}{2}\,\Omega^{T}\rho_{\mu}^{*}\Omega^{*} = -\frac{1}{2}\,\Omega^{T}\tau_{\mu}\Omega^{*}.$$

Similarly equation (5.4) gives

$$V_{\mu} = \frac{1}{2}\,\chi^{T}\sigma_{\mu}\chi.$$

Therefore we get the Zorn-Weyl matrices associated to the spinor χ and Ω:

$$\overset{(a)}{B} = \overset{(a)}{\mathcal{X}}{}^{\mu}V_{\mu} = \frac{1}{2}\,\overset{(a)}{\mathcal{X}}{}^{\mu} \odot \mathbb{I}\chi^{T}\sigma_{\mu}\chi, \tag{5.6}$$

$$\overset{(a)}{C} = \overset{(a)}{\mathcal{X}}W_{\mu} = -\frac{1}{2}\,\overset{(a)}{\mathcal{X}}{}^{\mu} \odot \mathbb{I}\Omega^{T}\tau_{\mu}\Omega^{*}. \tag{5.7}$$

Zorn-Weyl matrices associated to higher rank spinor fields may also be constructed. For example, if $\phi_{A\dot{B}C\dot{D}}$ is a spinor associated to a second rank tensor field

$$B_{\mu\nu}(x) = \frac{1}{4}\,\sigma^{A\dot{B}}(x)\sigma^{C\dot{D}}(x)\phi_{A\dot{B}\;C\dot{D}}(x).$$

Then Zorn-Weyl matrices of the type (5.2) or (5.3) may be obtained. Finally we may also construct Zorn-Weyl matrices associated to mixed spin-tensor objects. For example for the spin-tensor field $\psi_{\alpha}^{A}(x)$, the associated Zorn-Weyl matrices are

$$\overset{(a)}{F}{}^{A} = \overset{(a)}{\mathcal{X}}{}^{\mu}\psi_{\mu}^{A} = \overset{(a)}{\mathcal{X}}{}^{\mu} \odot \mathbb{I}\cdot\psi_{\mu}^{A}. \tag{5.8}$$

6 - The covariant Zorn-Weyl differential operator and Field equations.

Now we consider the problem of forming higher order tensors, spinors, or mixed objects by taking covariant derivatives in the Zorn-Weyl formulation of these objects. For that purpose we introduce an affine connection and define the differential operator

$$\mathcal{D}_\mu = 1\cdot\partial_\mu + \Gamma_\mu \tag{6.1}$$

and using (5.1) its associated Zorn-Weyl covariant differential operators are constructed as

$$\overset{(a)}{\mathbb{D}} = g_{\mu\nu} \overset{(a)}{\mathcal{Z}}{}^{\mu} \mathcal{D}^{\nu} = \overset{(a)}{\mathcal{Z}}{}_{\mu} \mathcal{D}^{\mu}. \tag{6.2}$$

The definition of Γ_μ will depend on the space where $\overset{(a)}{\mathbb{D}}$ operates. We can write $\Gamma_\mu = (\Gamma_\mu{}^{\bar\alpha}{}_{\bar\beta})$ where the indices $\bar\alpha$, $\bar\beta$ are to be taken as world indices, or spinor indices. Thus if $\overset{(a)}{\mathbb{D}}$ operates on a Zorn-Weyl matrix associated to a world vector then $\Gamma_\mu{}^{\bar\alpha}{}_{\bar\beta}$ is given by the Christoffel symbols $\{\mu{}^{\alpha}\beta\}$. On the other hand if $\overset{(a)}{\mathbb{D}}$ operates on a Zorn-Weyl matrix associated to a spinor field the $\Gamma_\mu{}^{\bar\alpha}{}_{\bar\beta}$ are the components of the spinor connection. We may also consider $\overset{(a)}{\mathbb{D}}$ acting on Zorn-Weyl matrices which are associated with mixed objects displaying vector and spinor indices, in this case $\Gamma_\mu{}^{\bar\alpha}{}_{\bar\beta}$ is a more complicated object where the indices $\bar\alpha$, $\bar\beta$ take on the values of spinor and tensor indices. In this case $(\Gamma_{\mu}{}^{\bar{\alpha}}{}_{\bar{\beta}})$ are represented by a sum of terms involving the Christoffel symbols and the spinor affinities. In the equation (6.1) $\underline{1}$ denotes the identity element with the same index structure as the term in Γ_{μ}. For example, considering the mixed object $\psi^{A}{}_{\mu}$ of (5.8) we have

$$\mathcal{D}_{\mu}\psi^{A}{}_{\nu} = \left\{\delta^{A}_{D}\delta^{\rho}_{\nu}\partial_{\mu} - \{{}^{\rho}_{\mu\nu}\}\delta^{A}_{D} + \Gamma_{\mu}{}^{A}{}_{D}\delta^{\rho}_{\nu}\right\}\psi_{\rho}{}^{D}.$$

Therefore in this case $\underline{1}$ is represented by $\delta^{A}_{D}\delta^{\rho}_{\nu}$ and

$$(\Gamma_{\mu}{}^{\bar{\alpha}}{}_{\bar{\beta}}) \to (-\{{}^{\rho}_{\mu\nu}\}\delta^{A}_{D} + \Gamma_{\mu}{}^{A}{}_{D}\delta^{\rho}_{\nu}).$$

Here $\Gamma_{\mu}{}^{A}{}_{D}$ represents the spinor affinity associated to local unimodular transformations of the spinor indices. In the flat space limit, in Cartesian coordinates, $h^{\mu}_{(\beta)} \to \delta^{\mu}_{\beta}$, $h_{\mu(\beta)} \to \eta_{\mu\beta}$ and $\Gamma_{\mu} \to 0$, so that $\overset{(a)}{\mathcal{D}} \to D_{a}$. Now using the covariant operator (6.2) we may construct covariant wave equations involving tensors spinors or mixed objects. As a first example consider the expression

$$\overset{\overline{(1)}}{\mathcal{D}} \odot \overset{(2)}{B} = \overset{\overline{(1)}}{\mathcal{H}}{}^{\mu}\mathcal{D}_{\mu} \odot \overset{(2)}{\mathcal{H}}{}^{\nu}B_{\nu} = \overset{\overline{(1)}}{\mathcal{H}}{}^{\mu} \odot \overset{(2)}{\mathcal{H}}{}^{\nu}\mathcal{D}_{\mu}B_{\nu}, \qquad (6.3)$$

where $B_{\nu}(x)$ is an arbitrary vector field. Using the properties

$$\mathcal{D}_{\mu}h^{\alpha}_{(\nu)} = 0, \quad \mathcal{D}_{\mu}\mathbb{I} = 0 \qquad (6.4)$$

and assuming that $B_{\mu} = A_{\nu}$ (the electromagnetic potential) in (6.3), then $\overset{\overline{(1)}}{D} \odot \overset{(2)}{B}$ gives the Zorn-Weyl matrix associated to the eletromagnetic field $F_{\mu\nu}$ in curved spacetime.
A straightforward calculation shows that in the flat limit

this expression reduces to the expression $\bar{D}_1 \odot A_2$ of (4.3). Now consider the Zorn covariant derivative of a real null vector associated to the spinor χ (see Eq. (5.6)). Denoting

$$\Lambda = \overset{(2)}{D} \odot \overset{\overline{(1)}}{B} \qquad (6.5)$$

and using unities such that $c = \hbar = 1$ and usual spinor connection condition $\mathcal{D}_\mu \sigma_\lambda = 0$, we have that the diagonal matrix elements in (6.5) are given by $\frac{1}{2}(\chi^\dagger_{;\alpha}\sigma^\alpha\chi + \chi^\dagger\sigma^\alpha\chi_{;\alpha})\mathbb{I}$

Defining for any Zorn-Weyl matrix

$$S(\overset{(a)}{N}) = \overset{(a)}{N} + \overset{\overline{(a)}}{N} = \mathrm{tr}\,\overset{(a)}{N}\begin{bmatrix} W_a(e_{(o)}) & 0 \\ 0 & W_a(e_{(o)}) \end{bmatrix},$$

we have for (6.5)

$$S(\overset{(2)}{D} \odot \overset{\overline{(1)}}{B}) = (\chi^\dagger_{;\alpha}\sigma^\alpha\chi + \chi^\dagger\sigma^\alpha\chi_{;\alpha})\mathbb{I}\,.. \qquad (6.6)$$

Similarly for the matrix (5.7) we have

$$S(\overset{(2)}{D} \odot \overset{\overline{(1)}}{C}) = -(\Omega^T_{;\alpha}\tau^\alpha\Omega^* + \Omega^T\tau^\alpha\Omega^*_{;\alpha})\mathbb{I} \qquad (6.7)$$

Introducing

$$L = (s_1\Omega^T\cdot\chi + s_2\chi^\dagger\cdot\Omega^*)\mathbb{I}\ ,\ K = (s_3\chi^\dagger\Omega^* + s_4\Omega^T\cdot\chi)\mathbb{I} \qquad (6.8)$$

where s_1, s_2, s_3 and s_4 are constant numbers to be determined, we find

$$S(\overset{(2)}{D} \odot \overset{\overline{(1)}}{B}) + L = \{[\chi^\dagger{}_{;\alpha}\sigma^\alpha + s_1\Omega^T]\chi + \chi^\dagger[\sigma^\alpha\chi_{;\alpha} + s_2\Omega^*]\}\mathbb{I}\ , \qquad (6.9)$$

$$S(\overset{(2)}{D} \odot \overset{\overline{(1)}}{C}) + K = \{\Omega^T[-\tau^\alpha\Omega^*_{;\alpha} + s_4\chi] + [-\Omega^T_{;\alpha}\tau^\alpha + s_3\chi^\dagger]\Omega^*\}\mathbb{I}. \qquad (6.10)$$

For the choice $s_1 = s_2 = -m$, $s_3 = s_4 = m$ the terms between brackets in the right hand side of (6.9) and (6.10) give the left hand side of the Dirac equation written in terms of two-component spinors[8]. Here $\underline{m}$ is the rest mass of the spin $\frac{1}{2}$ particle. According to our method we may present the Dirac equation in the Zorn algebra on a curved space as

$$\begin{cases} S(\overset{(2)}{D} \odot \overline{\overset{(1)}{B}}) - 2m\,\mathrm{Re}(\Omega^T \cdot \chi) ZW_3(e_{(o)}) = 0 \\ S(\overset{(2)}{D} \odot \overline{\overset{(1)}{C}}) + 2m\,\mathrm{Re}(\Omega^T \cdot \chi) ZW_3(e_{(o)}) = 0 \end{cases} \tag{6.11}$$

Note that from the right hand side of (6.9) and (6.10) the Dirac equation is writen as $\gamma^{\alpha}\psi_{,\alpha} - im\psi = 0$, for

$$\gamma^{\alpha} = \begin{pmatrix} 0 & -i\overset{*\alpha}{\sigma}_{\dot{B}A} \\ i\sigma^{\alpha\dot{B}A} & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\tau^{\alpha} \\ i\sigma^{\alpha} & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} \chi_{\dot{A}} \\ \omega^{\dot{A}} \end{pmatrix} = \begin{pmatrix} \chi \\ \Omega^* \end{pmatrix} \tag{6.12}$$

from our previous definitions it follows that

$$\gamma_{(\alpha}\gamma_{\beta)} = -2g_{\alpha\beta} \cdot 1 \ .$$

It is also possible to derive a direct analogue of the left hand side of the Dirac equation without the problem presented by (6.11) which is quadratic in the spinor χ, Ω. For obtaining such direct analogy we recall the definition (5.8) and rewrite the relations (5.6) and (5.7) as $\overset{(a)}{B} = \chi_{\dot{A}} \overset{(a)}{\mathcal{M}}{}^{\dot{A}}$, $\overset{(a)}{C} = \omega^{A} \overset{(a)}{\mathcal{N}}{}_{A}$, where

$$\overset{(a)}{\mathcal{M}}{}^{\dot{A}} = \frac{1}{2} \overset{(a)}{\mathcal{B}}{}^{\mu} \sigma_{\mu}{}^{\dot{A}B} \chi_{B} = \frac{1}{2} \overset{(a)}{\mathcal{B}}{}^{\mu} \xi_{\mu}^{\dot{A}}, \tag{6.13}$$

$$\overset{(a)}{\mathcal{N}}{}_{A} = \frac{1}{2} \overset{(a)}{\mathcal{B}}{}^{\mu} \overset{*}{\sigma}_{\mu B \dot{A}} \omega^{\dot{B}} = \frac{1}{2} \overset{(a)}{\mathcal{B}}{}^{\mu} \eta_{\mu A}. \tag{6.14}$$

These expressions are linear in the spinors χ, Ω and a straightforward calculation gives

$$S(\overset{(2)}{\mathcal{D}} \odot \overline{\overset{(1)}{M}}{}^{\dot{A}}) = \sigma^{\nu\dot{A}B}\chi_{B;\nu}\mathbb{I}. \qquad (6.15)$$

Similarly

$$S(\overset{(2)}{\mathcal{D}} \odot \overline{\overset{(1)}{N}}_{A}) = \overset{*}{\sigma}{}^{\mu}_{B\dot{A}}\omega^{\dot{B}}{}_{;\mu}\mathbb{I}. \qquad (6.16)$$

Define $P^{\dot{A}} = -m\,\omega^{\dot{A}}\mathbb{I}$, $F_{A} = m\chi_{A}\mathbb{I}$. Then, from (4.16) and (4.17) we have

$$S(\overset{(2)}{\mathcal{D}} \odot \overline{\overset{(1)}{M}}{}^{\dot{A}}) + P^{\dot{A}} = [\sigma^{\nu\dot{A}B}\chi_{B;\nu} - m\omega^{\dot{A}}], \qquad (6.17)$$

$$S(\overset{(2)}{\mathcal{D}} \odot \overline{\overset{(1)}{N}}_{A}) + F_{A} = [\overset{*}{\sigma}{}^{\mu}_{B\dot{A}}\,\omega^{\dot{B}}{}_{;\mu} + m\chi_{A}]. \qquad (6.18)$$

Therefore the Dirac equation in the Zorn algebra may also be directly obtained from (6.17) and (6.18) as

$$S(\overset{(2)}{\mathcal{D}} \odot \overline{\overset{(1)}{M}}{}^{\dot{A}}) + P^{\dot{A}} = 0, \quad S(\overset{(2)}{\mathcal{D}} \odot \overline{\overset{(1)}{N}}_{A}) + F_{A} = 0. \qquad (6.19)$$

Now we derive the Zorn-Weyl version of the Klein Gordon equation. Defining the spinor operators (or Cartan matrices associated to the covariant derivative) $\overset{*}{\sigma}_{\dot{C}A}\mathcal{D}_{\mu}$ and $\sigma^{\mu C\dot{B}}\mathcal{D}_{\mu}$, we can form the ZW elements

$$\hat{\mathcal{D}}^{(2)}_{\dot{C}A} = ZW_3(e_{(o)})\overset{*}{\sigma}_{\dot{C}A}\mathcal{D}_{\mu}, \quad \hat{\mathcal{D}}^{(1)C\dot{B}} = ZW_3(e_{(o)})\sigma^{\mu C\dot{B}}\mathcal{D}_{\mu}$$

From (6.15) we find

$$\hat{\mathcal{D}}^{(2)}_{\dot{C}A} \odot S(\overset{(2)}{\mathcal{D}} \odot \overline{\overset{(1)}{M}}{}^{\dot{A}}) = \mathbb{I}\cdot\overset{*}{\sigma}_{\dot{C}A}\sigma^{\nu\dot{A}B}\chi_{B;\nu;\mu}. \qquad (6.20)$$

$$\hat{\mathcal{D}}^{(1)A\dot{B}} \odot S(\overset{(2)}{\mathcal{D}} \odot \overline{\overset{(1)}{N}}_{A}) = \mathbb{I}\cdot\sigma^{\mu A\dot{B}}\overset{*}{\sigma}_{C\dot{A}}\omega^{\dot{C}}{}_{;\nu;\mu}. \qquad (6.21)$$

We have

$$\chi_{B;\nu;\mu} = \frac{1}{2}(\chi_{B;\nu;\mu}+\chi_{B;\mu;\nu}) + \frac{1}{2}(\chi_{B;\nu;\mu}-\chi_{B;\mu;\nu}) = \frac{1}{2}\,\chi_{B(;\nu;\mu)} + \frac{1}{2}\,P^{R}_{\nu\mu B}\chi_{R}, \tag{6.22}$$

where $P_{\nu\mu}$ is the curvature 2-spinor. Similarly

$$\omega^{\dot{C}}{}_{;\nu;\mu} = \frac{1}{2}\,\omega^{\dot{C}}{}_{(;\nu;\mu)} + \frac{1}{2}\,P^{\dot{C}}_{\nu\mu\dot{R}}\omega^{\dot{R}}. \tag{6.23}$$

Substitution of (6.22) into (6.20) and (6.23) into (6.21) gives

$$\hat{0}^{(2)}_{\dot{C}A} \odot S(\overset{(2)}{D} \odot \overline{\overset{(1)}{M}}{}^{\dot{A}}) = -ZW_3(e_{(o)})\chi_{C;\mu;\nu}g^{\mu\nu} + \frac{1}{4}\,ZW_3(e_{(o)})P^{R}_{\nu\mu B}\Sigma^{\mu\nu B}{}_{C}\chi_{R}, \tag{6.24}$$

$$\hat{0}^{(1)A\dot{B}} \odot S(\overset{(2)}{D} \odot \overline{\overset{(1)}{N}}_{A}) = -ZW_3(e_{(o)})\omega^{\dot{B}}{}_{;\mu;\nu}g^{\mu\nu} - \frac{1}{4}\,ZW_3(e_{(o)})P^{\dot{C}}_{\nu\mu\dot{R}}\Sigma^{\mu\nu\dot{B}}{}_{\dot{C}}\omega^{\dot{R}}, \tag{6.25}$$

where

$$\Sigma^{\mu\nu B}{}_{C} = \overset{*}{\sigma}{}^{\mu}_{\dot{C}A}\sigma^{\nu\dot{A}B} - \overset{*}{\sigma}{}^{\nu}_{\dot{C}A}\sigma^{\mu\dot{A}B}.$$

From (6.18) it follows that

$$\hat{0}^{(2)}_{\dot{C}A} \odot P^{\dot{A}} = -m\mathbb{I}\,\overset{*}{\sigma}{}^{\mu}_{\dot{C}A}\omega^{\dot{A}}{}_{;\mu} = m^{2}\mathbb{I}\chi_{C}\ .$$

Similarly from (6.17) and (6.19)

$$\hat{0}^{(1)A\dot{B}} \odot T_{A} = m\mathbb{I}\sigma^{\mu A\dot{B}}\chi_{A;\mu} = m^{2}\mathbb{I}\omega^{\dot{B}}.$$

Then, the Klein-Gordon equation for each component of X and Ω^{*} has the form

$$\hat{0}^{(2)}_{\dot{C}A} \odot \{S(\overset{(2)}{D} \odot \overline{\overset{(1)}{M}}{}^{\dot{A}}) + P^{\dot{A}}\} = \mathbb{I}\{-\Box\chi_{C} + \frac{1}{4}P^{R}_{\nu\mu B}\Sigma^{\mu\nu B}{}_{C}\chi_{R} + m^{2}\chi_{C}\} = 0, \tag{6.26}$$

$$\hat{0}^{(1)A\dot{B}} \odot \{S(\overset{(2)}{\mathbb{D}} \odot \overline{\overset{(1)}{N}_A}) + T_A\} = \mathbb{j}\,\{-\Box\omega^{\dot{B}} - \frac{1}{4}\,\Sigma^{\mu\nu\dot{B}}{}_{\dot{C}}P_{\nu\mu}{}^{\dot{C}}{}_{\dot{R}}\omega^{\dot{R}} + m^2\omega^{\dot{B}}\} = 0, \tag{6.27}$$

where $\Box$ represents the covariant D'Alembert operator. The equations (6.26) and (6.27) in the limit of flat space reduce to the correct Klein-Gordon equation, for the signature (+2), in special relativity.

7 - Symmetric - antisymmetric theory in a complex tetrad formalism

As was seen in the previous sections the geometry of the four-dimensional Riemannian space, described locally by the tetrad field, is algebraically described as a split quaternion subalgebra of the Cayley algebra. In this section we look for a generalization of this geometry in such way that part of its algebraic description is contained in the full Cayley algebra. With this in mind we consider a general second rank tensor field $G^{\mu\nu}(x^\alpha)$ given in terms of a complex tetrad as

$$G^{\mu\nu} = h^{\mu}{}_{(\alpha)}h^{*\nu}{}_{(\beta)}\eta^{\alpha\beta}. \tag{7.1}$$

Here $\eta^{\alpha\beta}$ indicates the Minkowski tensor with signature (+2). The matrix $G = (G^{\mu\nu})$ is Hermitian. $G^{*\nu\mu} = G^{\mu\nu}$. The symmetric and antissymmetric parts of this matrix are given by

$$\begin{aligned} G^{(\mu\nu)} &= \tfrac{1}{2}(G^{\mu\nu} + G^{\nu\mu}) = \mathrm{Re}(G^{\mu\nu}), \\ G^{[\mu\nu]} &= \tfrac{1}{2}(G^{\mu\nu} - G^{\nu\mu}) = i\,\mathrm{Im}(G^{\mu\nu}). \end{aligned} \tag{7.2}$$

Denoting the matrices associated to the symmetric and antissymmetric parts of $G^{\mu\nu}$ by $\underline{g}$ and $\underline{if}$ we have $G = g + if$. The matrices $\underline{g}$ and $\underline{f}$ are supposed to be non-singular, and the matrix g is used for raising for-dimensional indices (and g^{-1} for lowering these indices).

$$A_{\mu\ldots} = G_{(\mu\nu)}A^{\nu}\ldots,\quad A^{\mu\ldots} = G^{(\mu\nu)}A_{\nu}\ldots,\quad G_{(\mu\nu)}G^{(\nu\sigma)} = \delta^{\sigma}_{\mu}.$$

The use of complex tetrads is known in the literature[9], and our present formalism giving the Hermitian tensor $G^{\mu\nu}$ in terms of a complex tetrad is a condensed notation for a formalism due to P. Smith[10].

From (5.1) we have

$$G_{\mu\nu} = h_{\mu(\alpha)}h^{*}_{\nu(\beta)}\eta^{\alpha\beta},\quad h_{\mu(\alpha)} = G_{(\mu\rho)}h^{\rho}_{(\alpha)}. \tag{7.3}$$

In matrix notation this takes the form: $K = (G_{\mu\nu})$,
$K = g^{-1} + ig^{-1}\cdot f\cdot g^{-1} = K^{\dagger}$.

Associated to the field of complex tetrads we define in each Zorn-Weyl basis the set of four split octonion elements (for each of the two values of a)

$$\overset{(a)}{\mathcal{X}}{}^{\mu} = \begin{bmatrix} ih^{\mu}_{(o)}W_a(e_{(o)}) & -h^{*\mu}_{(k)}W_a(e_{(k)}) \\ h^{\mu}_{(k)}W_a(e_{(k)}) & ih^{*\mu}_{(o)}W_a(e_{(o)}) \end{bmatrix}, \tag{7.4}$$

which may be written as $\overset{(a)}{\mathcal{X}}{}^{\mu} = ZW_a(K^{\mu})$, where

$$K^{\mu} = ih^{\mu}_{(o)}u^{*}_{(o)} + ih^{*\mu}_{(o)}u_{(o)} + h^{\mu}_{(k)}u_{(k)} + h^{*\mu}_{(k)}u^{*}_{(k)}.$$

In the limit $\mathrm{Im}(h^{\mu}_{(\alpha)}) \to 0$ the Cayley numbers K^{μ} degenerate in elements of the split quaternion subalgebra of the octonion algebra. A straightforward calculation gives

$$ZW_1(K^{(\mu}) \odot ZW_2(\bar{K}^{\nu)}) = G^{(\mu\nu)}ZW_3(e_{(o)}), \tag{7.5}$$

where for any quantities A^μ, B^μ

$$A^{(\mu}\,B^{\nu)} = \tfrac{1}{2}(A^\mu B^\nu + A^\nu B^\mu).$$

Therefore the ZW elements $\overset{(a)}{\mathcal{Z}}{}^\mu$ are associated to the symmetric part of the Hermitian tensor $G^{\mu\nu}$.

It is also possible to introduce ZW elements belonging to the split quaternion subalgebra associated to the complex tetrad:

$$\overset{(a)}{\mathcal{Z}}{}^\mu = \begin{bmatrix} ih^\mu{}_{(o)}W_a(e_{(o)}) & -h^\mu{}_{(s)}W_a(e_{(s)}) \\ h^\mu{}_{(s)}W_a(e_{(s)}) & ih^\mu{}_{(o)}W_a(e_{(o)}) \end{bmatrix}.$$

Since these objects are 4×4 matrices we may introduce their Hermitian conjugates

$$\overset{(a)}{\mathcal{Z}}{}^{\dagger\mu} = \begin{bmatrix} -ih^{*\mu}{}_{(o)}W_a^\dagger(e_{(o)}) & h^{*\mu}{}_{(s)}W_a^\dagger(e_{(s)}) \\ -h^{*\mu}{}_{(s)}W_a^\dagger(e_{(s)}) & -ih^{*\mu}{}_{(o)}W_a^\dagger(e_{(o)}) \end{bmatrix}$$

In this equation we have to use that $W_a^\dagger(e_{(o)}) = W_a(e_{(o)})$ and $W_a^\dagger(e_{(s)}) = -W_a(e_{(s)})$. Defining for any ZW element the operation $\mathcal{J}(\overset{(a)}{M}) = \overset{(a)}{M} + \overline{\overset{(a)}{M}}$, we find by a direct calculation

$$\mathcal{J}(\overset{(1)}{\mathcal{Z}}{}^{[\mu} \odot \overset{(2)}{\mathcal{Z}}{}^{\dagger\nu]}) = -2G^{[\mu\nu]}\,ZW_3(e_{(o)}).$$

This is a relation involving product in the ZW algebra which generates the antissymmetric part of $G^{\mu\nu}$.

In the formalism presently considered the real part of $G^{\mu\nu}$ plays the role of metric of a Riemannian geometry with affinity $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$ (the Christoffel Symbols). Thus, only one kind of covariant differentiation is used, namely the usual covariant differentiation used in general relativity. Therefore, all previous conventions regarding covariant differentiation in the ZW algebra apply here. The antisymmetric part of the Hermitian tensor $G^{\mu\nu}$ is related to the electromagnetic potentials by the definition[10]

$$A^{\mu} = \lambda G^{[\mu\nu]}_{\ \ \ ;\nu} \tag{7.6}$$

where $\underline{\lambda}$ is a constant. The potentials A^{μ} satisfy the covariant Lorentz condition $A^{\mu}_{\ ;\mu} = -\lambda R_{\mu\nu} G^{[\mu\nu]} = 0$, where $R_{\mu\nu} = R^{\alpha}_{\ \mu\nu\alpha}$ is the Ricci tensor of the Riemannian geometry.

The operator of covariant differentiation in the ZW algebra is here defined similarly as before by

$$\overset{(a)}{D} = \overset{(a)}{\chi}{}^{\mu} \partial_{\mu}.$$

As an application of the present formalism we consider the motion of a charged spin zero massive test body under the action of gravitation and electromagnetism, described by the corresponding covariant Klein-Gordon equation. We take unities such that $c = \hbar = 1$. The equation of motion takes the form

$$\left(G^{(\mu\nu)} \frac{D}{Dx^{\mu}} \frac{D}{Dx^{\nu}} + m^2\right)\Psi = 0, \tag{7.7}$$

$$\frac{D}{Dx^{\mu}} = \frac{1}{i} \partial_{\mu}\Psi - eA_{\mu}\Psi. \tag{7.8}$$

Using the notation $P_{\mu}\Psi = \frac{1}{i} \partial_{\mu}\Psi$, the equation (7.7) takes the form (in the Lorentz gauge)

$$G^{(\mu\nu)}P_{\mu}P_{\nu}\Psi - 2\lambda G^{[\mu\nu]}_{;\nu}P_{\mu}\Psi = -(m^2 + \lambda^2 e^2 G^{[\mu\nu]}_{;\nu} G^{;\alpha}_{[\mu\alpha]})\Psi. \qquad (7.9)$$

writing $\overset{(a)}{B} = \overset{(a)}{\mathcal{K}}{}^{\mu}P_{\mu}\Psi$ we have

$$\overline{\overset{(1)}{D}} \odot \overset{(2)}{B} = \overline{\overset{(1)}{\mathcal{K}}}{}^{\mu}\partial_{\mu} \odot \overset{(2)}{\mathcal{K}}{}^{\nu}P_{\nu}\Psi = i\overline{\overset{(1)}{\mathcal{K}}}{}^{\mu} \odot \overset{(2)}{\mathcal{K}}{}^{\nu}P_{\mu}P_{\nu}\Psi.$$

From (7.5) we get

$$\overline{\overset{(1)}{D}} \odot \overset{(2)}{B} = iG^{(\mu\nu)}P_{\mu}P_{\nu}\Psi = iG^{\mu\nu}P_{\mu}P_{\nu}\Psi. \qquad (7.10)$$

A similar operation may be extend for the vector field $\frac{D\Psi}{Dx^{\mu}} = P_{\mu}\Psi - \lambda e G_{[\mu\alpha]}^{;\alpha}\Psi$. Defining

$$\overset{(a)}{\Pi} = \overset{(a)}{\mathcal{K}}{}^{\mu}\frac{D}{Dx^{\mu}} = \frac{1}{i}\overset{(a)}{D} - e\overset{(a)}{A}, \quad \overset{(a)}{R} = \overset{(a)}{\mathcal{K}}{}^{\mu}\frac{D\Psi}{Dx^{\mu}},$$

we have

$$\overline{\overset{(1)}{\Pi}} \odot \overset{(2)}{R} = \overline{\overset{(1)}{\mathcal{K}}}{}^{\mu} \odot \overset{(2)}{\mathcal{K}}{}^{\nu}\frac{D}{Dx^{\mu}}\frac{D}{Dx^{\nu}}\Psi.$$

Then,

$$\mathcal{S}(\overline{\overset{(1)}{\Pi}} \odot \overset{(2)}{R}) = \mathcal{S}(\overline{\overset{(1)}{\mathcal{K}}}{}^{\mu} \odot \overset{(2)}{\mathcal{K}}{}^{\nu})\frac{D}{Dx^{\mu}}\frac{D}{Dx^{\nu}}\Psi = 2G^{(\mu\nu)}\cdot\mathbb{1}\cdot\frac{D}{Dx^{\mu}}\frac{D}{Dx^{\nu}}\Psi.$$

Therefore the Klein-Gordon equation takes the simple form

$$\mathcal{S}(\overline{\overset{(1)}{\Pi}} \odot \overset{(2)}{R}) + 2m^2\mathbb{1}\cdot\Psi = 0.$$

The first term on the left hand side of this equation is the Zorn-Weyl gauge invariant covariant "D'Alembertian" (divided by a factor $\frac{1}{2}$).

Notice that the effect of the introduction of the octonions (complex tetrads) is to absorb the electromagnetic interaction of the test particle.

REFERENCES

1 - P.A.M. Dirac - Proc. Royal Irish Academy 50A, 261 (1945).

2 - A.M. Buoncristiani - J. Math. Phys. 14, 849 (1973).

3 - M. Gnaydin, F. Gürsey - J. Math. Phys. 14, no. 11, 1651 (1973).

4 - R. Penney - N. Cim., 3B, 95 (1971) Am. J. Phys., 36, 871 (1968).

5 - M. Gnaydin - J. Math. Phys. 17, no. 10, 1875 (1976).

6 - M. Zorn - Abh. Math. Sem. Univ. Hamburg, 8, 123 (1931).
G.B. Seligman - Trans. Ann. Math. Soc., 97, 286 (1960).

7 - R. Brauer H. Weyl - Am. J. Maths, 57, 425 (1935).

8 - E.M. Corson - Introduction to tensors, spinors and relativistic wave equations, Blackie & Son Ltd (1954), page 10.

9 - D.W. Sciama - N. Cim. 8, 417 (1958).

10 - P.D.P. Smith - N. Cim. 24B, no. 2, 255 (1974).