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Normal nonmetrizable Moore space from continuum hypothesis or nonexistence of inner models with measurable cardinals

(general topology/large cardinals/relative consistency results)

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ABSTRACT Assuming the continuum hypothesis, a normal nonmetrizable Moore space is constructed. This answers a question raised by F. B. Jones in 1931, using an axiom well known at that time. For the construction, a consequence of the continuum hypothesis that also follows from the nonexistence of an inner model with a measurable cardinal is used. Hence, it is shown that to prove the consistency of the statement that all normal Moore spaces are metrizable one must assume the consistency of the statement that measurable cardinals exist.

In 1931, Jones (1) asked whether all normal Moore spaces were metrizable. Towards this end, he proved that, assuming $2^{\aleph_0} < 2^{\aleph_1}$, separable Moore spaces are metrizable. Bing (2) showed that a Moore space is metrizable if and only if it is collectionwise normal. The most important results after Bing are: (*i*) assuming Martin's axiom plus not continuum hypothesis, there is a normal nonmetrizable Moore space (3); (*ii*) assuming Gödel's axiom of constructibility, normal Moore spaces are collectionwise Hausdorff (4); (*iii*) assuming the product measure extension axiom, all normal Moore spaces are metrizable (5). Nyikos (5) called this last result a provisional solution to the normal Moore space problem because a strongly compact cardinal was used in the consistency proof.

In this paper, a normal nonmetrizable Moore space is constructed, not from a new axiom, but from the familiar continuum hypothesis. Moreover, the solution of the normal Moore space problem is completed by showing that a large cardinal assumption is necessary to show the consistency of all normal Moore spaces being metrizable. These two things are done simultaneously by constructing a normal nonmetrizable Moore space from an axiom that is implied by the continuum hypothesis and whose failure implies the existence of inner models with measurable (and larger) cardinals.

I begin by setting out in detail the hypotheses assumed beyond the Zermelo–Frankel axioms for set theory with choice. Let *Hyp* be the axiom that asserts that for some κ:

there is an increasing sequence κ_n, $n \in \omega$, of cardinals

$$\text{cofinal in } \kappa \text{ such that for all } n \in \omega,\ 2^{\kappa_n} < \kappa, \qquad [1]$$

$$2^{\kappa} = \kappa^+, \qquad [2]$$

there is stationary subset, E, of κ^+ such that for all $\delta \in E$, there is an increasing sequence δ_i, $i \in \omega$, cofinal in δ, [3a]

and for all $\beta < \kappa^+$, $E \cap \beta$ is not stationary in β. [3b]

The continuum hypothesis implies *Hyp*; let $\kappa = \omega$ and let E be the set of limit ordinals less than ω_1. There are cardinals greater than ω that satisfy [1]; for example $\kappa = \sup\{\omega, 2^\omega, 2^{2^\omega}, \ldots\}$. If [2] or [3] fails for such a cardinal, then there is an inner model with a measurable cardinal, because the covering lemma would fail with respect to the core model; see ref. 6.

We now construct the space. Let F be the set of functions from ω to E. For $n \in \omega$, set $\Sigma_n = \{f|n: f \in F\}$ and set $\Sigma = \bigcup_{n\in\omega} \Sigma_n$. For $\sigma \in \Sigma$, set $[\sigma] = \{f \in F: \sigma \subset f\}$.

Let $\mathbf{Z}$ be the family of subsets, Z of Σ satisfying card $Z \leq \kappa$ and for some $n \in \omega$, $Z \subset \Sigma_n$. By [2], we can list $\mathbf{Z}$ as $\{Z_\alpha: \alpha < \kappa^+\}$. For $\sigma \in \Sigma$, define $\sigma*$ to be the greatest ordinal in range σ. For $\rho \in \Sigma$ set $\Sigma^\rho = \{\sigma: \sigma* < \rho*\}$ and $\mathbf{Z}(\rho) = \{Z_\alpha: \alpha < \rho*\}$. By induction on $n \in \omega$, we define, for $\sigma \in \Sigma_n$ and $m \in \omega$, the sets $A(\sigma,m)$ to satisfy

$$\text{card } A(\sigma,m) = \kappa_m \qquad [4]$$

$$\bigcup_{m\in\omega} A(\sigma,m) = \mathbf{Z}(\sigma) \cap \left(\bigcup_{k<n} \Sigma_k\right) \qquad [5]$$

$$\text{if } \sigma \subset \sigma' \text{ and } m \leq m', \text{ then } A(\sigma,m) \subset A(\sigma',m'). \qquad [6]$$

$$\text{Let } Q_k \text{ be the set of triples } \langle g,\rho,\tau\rangle \text{ satisfying } \rho,\ \tau \in \Sigma_k \qquad [7]$$

$$\text{for all } i < k - 1,\ \rho(i) < \tau(i) < \rho(i + 1) < \tau(i + 1) \qquad [8]$$

$$g \text{ is a function from } \mathbf{Z}(\rho*) \text{ to } \{0,1\}. \qquad [9]$$

Set $Q = \bigcup_{k\in\omega} Q_k$. For $\sigma \in \Sigma_n$, set $B(\sigma) = [\sigma] \cup G(\sigma)$, where $G(\sigma)$ is the set of triples $\langle g,\rho,\tau\rangle \in Q$ satisfying

$$\text{either } \sigma \subset \rho \text{ or } \sigma \subset \tau \qquad [10]$$

$$\text{for all } i < n,\ \rho(0)_i = \tau(0)_i \qquad [11]$$

if $Z \in A(\sigma,n)$ and $Z \subset \Sigma_m$,

$$\text{then } g(Z \cap \Sigma^{\rho|m}) = 1 \text{ iff } \sigma|m \in Z. \qquad [12]$$

The point set of our space, X, is $F \cup Q$. A basis for the topology on X is $\{B(\sigma): \sigma \in \Sigma\} \cup \{\{q\}: q \in Q\}$. It is easy to see that X is T_1 and regular.

$$\text{Set } \mathbf{G}_n = \{B(\sigma): \sigma \in \Sigma_n\} \cup \{\{q\}: q \in Q_k \text{ where } k > n\}. \qquad [13]$$

For every infinite subset $\mathbf{U}$ of $\bigcup_{n\in\omega} \mathbf{G}_n$, if $x \in \mathbf{U}$, then $\mathbf{U}$ is a basis at x. Hence X is a metacompact Moore space. To show that X is normal, it suffices to separate, for each $n \in \omega$ and each $Z \subset \Sigma_n$, the sets $H_Z = \bigcup \{[\sigma]: \sigma \in Z\}$ and $K_Z = F - H_Z$. Define

$$C = \{\gamma \in \kappa^+: \text{if } \rho* < \gamma, \text{ then } Z \cap \Sigma^\rho = Z_\alpha \text{ for some } \alpha < \gamma\}. \qquad [14]$$

For $\beta \in \kappa^+$, define $\gamma(\beta)$ to be the least element of C greater than β. For $\sigma \in \Sigma_{n+3}$, define $j(\sigma) \geq n + 3$ to satisfy

$$\text{if } \gamma(\sigma(n)) < \sigma(n + 2), \text{ then } Z \cap \Sigma^{\sigma(n)} \in A(\sigma,j(\sigma)) \qquad [15]$$

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$$\text{if } \gamma(\sigma(n+1)) = \gamma(\nu(n+1)), \qquad [16]$$
$$\text{then for some } i < \max\{j(\sigma), j(\nu)\},\ \sigma(o)_i \neq \nu(o)_i$$

(We use [3b] to obtain [16]). Set $W_\sigma = \cup \{B(\rho): \sigma \subset \rho \in \Sigma_{j(\sigma)}\}$. To separate H_Z and K_Z, it will suffice to show that if σ, $\nu \in \Sigma_{n+3}$, $\sigma|n \in Z$, and $\nu|n \notin Z$, then $W(\sigma) \cap W(\nu) = \emptyset$. Aiming for a contradiction, assume that $\langle g,\rho,\tau\rangle \in W(\sigma) \cap W(\nu)$, where $\sigma(0) < \nu(0)$. If $\gamma(\nu(n)) < \sigma(n+2)$, we get a contradiction from [12]. Otherwise $\gamma(\sigma(n+1)) = \gamma(\nu(n+1))$, and we get a contradiction from [8], [16], and [11].

The following notion breaks the demonstration that X is not metrizable into several short combinatorial arguments.

Definition: A subset, S, of Σ_n is *stafull* if for all $\sigma \in S$ and $j < n$, $\{\tau(j): \sigma|j\ \tau \in S\}$ is stationary.

For each $\delta \in E$, set $Y_\delta = \{f \in F: f(0) = \delta\}$. Because $\{Y_\delta: \delta \in E\}$ is a discrete family of closed sets, if X were metrizable, there would be a disjoint family $\{U_\delta: \delta \in E\}$ of open sets with $Y_\delta \subset U_\delta$. Aiming for a contradiction, assume that $\{U_\delta: \delta \in E\}$ is such a family. Let $T = \{\sigma \in \Sigma: B(\sigma)\ U_{\sigma(0)}\}$. For some $n \in \omega$, $T \cap \Sigma_n$ contains a stafull set, S. (Assume not, then inductively define $y|i$ such that $y \notin U_{y(0)}$). Refine S to a stafull subset S' such that for all σ, $\nu \in S$ and $i < n$, $\sigma(0)_i = \nu(0)_i$. Next, inductively choose $\{\sigma_\alpha: \alpha < \kappa\} \subset S'$ such that $\sigma_\alpha(i) < \sigma_\beta(j)$ iff $i < j$ or $i = j$ and $\alpha < \beta$. Enumerate $A(\sigma_\alpha,n)$ as $\{Z(\alpha,\delta): \delta < \kappa_n\}$.

For $\alpha < \beta < \kappa$ there is g such that $\langle g,\sigma_\alpha,\sigma_\beta\rangle \in B(\sigma_\alpha) \cap B(\sigma_\beta)$ unless [12] fails. Thus for each pair α,β there are δ, η, and m such that $Z(\alpha,\delta) = Z(\beta,\eta) \cap \Sigma^{\sigma\beta|m}$ and either $\sigma_\alpha|m \in Z(\alpha,\delta)$ and $\sigma_\beta|m \notin Z(\beta\eta)$ or $\sigma_\alpha|m \notin Z(\alpha,\delta)$ and $\sigma_\beta|m \in Z(\beta,\eta)$. Because $\kappa > (2^{\kappa_n})^+$, we can apply the Erdös–Rado theorem, $\kappa \rightarrow (3)^2_{\kappa_n}$ to quickly get a contradiction.

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