Markov Chains

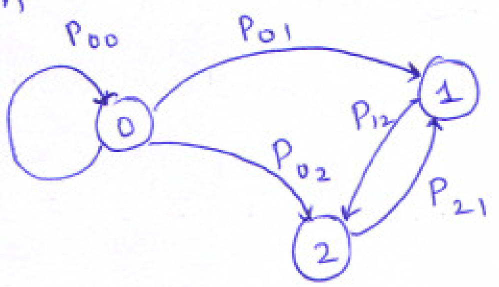
A sequence $\{X_n : n \geq 0\}$ of random variables taking values in a countable state space S is called a Markov chain if

$$P\{X_n = \alpha_n \mid X_{n-1} = \alpha_{n-1}, X_{n-2} = \alpha_{n-2}, \dots, X_0 = \alpha_0) = P\{X_n = \alpha_n \mid X_{n-1} = \alpha_{n-1}\}$$

ie. the probability that the chain will be in a certain state α_n at time n given all its past history depends only on its previous state ie state at time $n-1$.

Usually we also impose the condition of homogeinity $P\{X_{n+1} = j \mid X_n = i\} = P\{X_1 = j \mid X_0 = i\}$

A Markov chain can be represented by a graph.



Here P_{ij} represents the transition probability $P_{ij} = P\{X_n = j \mid X_{n-1} = i\}$

It is clear that P_{ij} satisfies

1) $P_{ij} \geq 0$ for all i, j

2) $\sum_j P_{ij} = 1$

A Markov chain is thus represented by a <u>transition matrix</u> P with entries

$$(P)_{ij} = P\{X_n = j \mid X_{n-1} = i\}$$

This matrix with properties $P_{ij} \geq 0$ and $\sum_j P_{ij} = 1$ is called a <u>Stochastic Matrix</u>

<u>Eg 1</u> <u>Random walk on a discrete lattice $\mathbb{Z}$</u>

A walker walks on the discrete lattice $\mathbb{Z}$ in the following way. At each time step a coin toss is made. The walker walks to the right with probability p and to the left with probability $1-p$, where p is the bias of coin. Let X_n be the position of the random walker at the n^{th} step. It is clear that $\{X_n : n \geq 0\}$ forms a Markov Chain since the probability that the walker will be at any given site j at time n depends only on the state at step $n-1$.

The transition matrix for the random walker is

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} & & -2 & -1 & 0 & 1 & 2 & & \end{array} \\
P = & \begin{array}{c} \\ -1 \\ \\ \\ \\ \end{array}
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & 1-p & 0 & p & 0 & \cdots & & \\
 & \cdots & 0 & 1-p & 0 & p & 0 & \cdots & \\
 & & \cdots & 0 & 1-p & 0 & p & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\end{array}
$$

ie.
$$
\begin{cases}
P_{i,i+1} = p \\
P_{i,i-1} = 1-p \\
P_{ij} = 0 \quad \text{if} \quad j \neq i+1 \text{ or } i-1
\end{cases}
$$

Eg 2

Branching Process

Let X_n be the number of individuals at the n^{th} generation of a family tree. Each member of the n^{th} generation gives birth to a family of, (possibly empty) of members of the $(n+1)^{st}$ generation. We make the following assumptions about the family sizes.

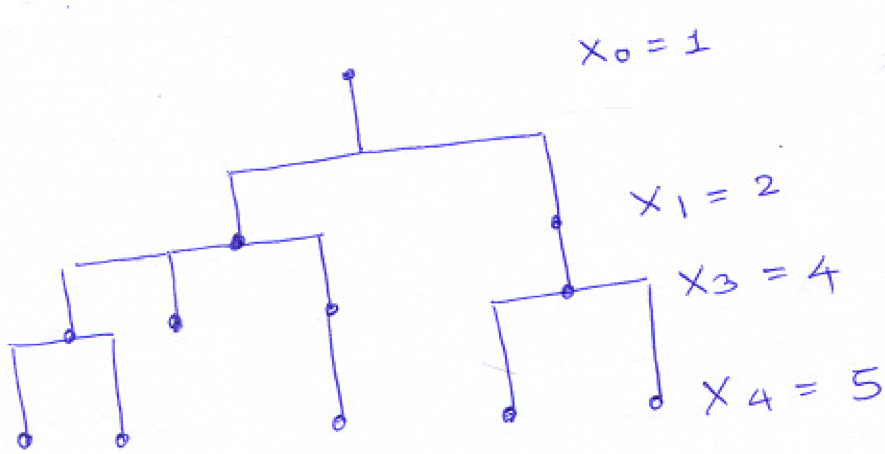
a) The family sizes of the individuals form a collection of independent r.v.s

b) The family sizes have same distribution function

Then since the family size of the $(n+1)^{st}$ generation is dependent only on the family size of the n^{th} generation.

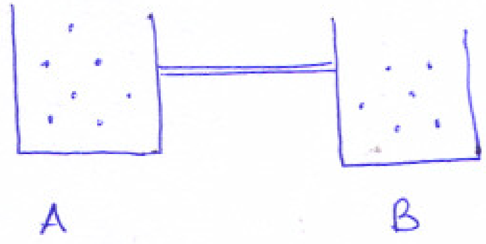
ie $P(X_n = \alpha_n \mid X_{n-1} = \alpha_{n-1}, \ldots X_0 = \alpha_0) = P(X_n = \alpha_n \mid X_{n-1} = \alpha_{n-1})$

$\{X_n : n \geq 0\}$ forms a Markov chain.



A realization of a branching process

Eg: 3 Erhenfest Model of Diffusion



Containers A and B contain a total number of m molecules. They are connected by a small aperture. At each epoch of time one molecule is picked uniformly from the m available and passed through the aperture. Let X_n be the number of molecules in container A after n time units; then $\{X_n : n \geq 0\}$ is a Markov chain. The transition probabilities are given by

$$P_{i,i+1} = \frac{m-i}{m}, \quad P_{i,i-1} = \frac{i}{m}$$

Eg 4 : Land of Oz

The Land of Oz is blessed with a lot of things but not good weather. There are never two nice days in a row. If they have a nice day, they are just as likely to have snow or rain the next day. If they have snow or rain they have even chance of the same the next day. If there is a change from snow or rain only half the time it is to a nice day.

The transition matrix of this Markov chain is

$$\begin{array}{c|ccc} & R & N & S \\ \hline R & & & \\ N & & & \\ S & & & \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} R & N & S \end{array} \\ \begin{array}{c} R \\ N \\ S \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \end{array}$$

Now, if it is rainy day today what is the probability that it will be snowing two days from now?

This event is a dijoint union of 3 events

1) It is rainy tommorrow and snowy in two days
2) It is nice " " " " " " —(*)
3) It is sunny " " " " " "

Let $P_{13}^{(2)}$:= Denote the two step transition probability that it is ~~rainy~~ $\ast$ snowy in two days given that it is rainy today

$(2) \to$ in two days; $1 \to$ rainy; $3 \to$ snowy

Then from (*)

$$P_{13}^{(2)} = P_{11} \cdot P_{13} + P_{12} P_{23} + P_{13} \cdot P_{33}$$

$$= (P^2)_{13}$$

$\swarrow$ Square of the one-step prob. matrix

This suggests the following

Chapman Kolmogorov Eqns

Let $(P^{(n)})_{ij} := P(X_{n+k}=j \mid X_k=i)$ (n-step transition prob.)

then

$$(P^{(m+n)})_{ij} = \sum_k P^{(n)}_{ik} \cdot P^{(m)}_{kj}$$

In other words

$$P^{(m+n)} = P^{(n)} \cdot P^{(m)}$$ (Matrix Multiplication)

Proof :

$$(P^{(m+n)})_{ij} = P\{X_{m+n}=j \mid X_0=i\}$$

$$= \sum_k P\{X_{m+n}=j, X_n=k \mid X_0=i\}$$

$$= \sum_k P\{X_{m+n}=j \mid X_n=k, X_0=i\} \cdot P\{X_n=k \mid X_0=i\}$$

(Check
$P(A, B \mid C) = P(A \mid B, C) \cdot P(B \mid C)$)

$$= \sum_k P\{X_{m+n}=j \mid X_n=k\} \cdot P\{X_n=k \mid X_0=i\}$$

By Markov prop.

$$= \sum_k P^{(m)}_{jk} \; P^{(n)}_{ik}$$

Classification of States

Defn: A state i is *persistent* (recurrent) if $P\{X_n = i \text{ for some } n \mid X_0 = i\} = 1$

If this probability is strictly less than 1 then the state is called *transient*.

Eg: In the random walk if $p = q = \frac{1}{2}$ all states are persistent. If $p \neq q \neq \frac{1}{2}$ all states are transient.

persistent $\Longrightarrow$ visitation to this state occurs infinitely often.

Proposition:

(1) A state i is persistent iff $\sum_{n=1}^{\infty} P_{ii}^{n} = \infty$

(and if this holds then $\sum_{n} P_{ji} = \infty$ for all j)

(2) A state i is transient if $\sum_{n=1}^{\infty} P_{ii}^{n} < \infty$

(and if this holds then $\sum_{n} P_{ji}^{n} < \infty$ for all n)

(Note: This implies that $P_{ii}^{n} \xrightarrow[n\to\infty]{} 0$ and $P_{ji}^{(n)} \xrightarrow[n\to\infty]{} 0$)

(For Proof see Grimmett & Stirzaker pg 221)

This implies that the n-step transition probability ~~is~~ matrix is just the one-step transition matrix raised to the n^{th} power. Since

$$P^{(1)} = P$$

$$P^{(2)} = P^{(1+1)} = P^{(1)} \cdot P^{(1)} = P^2 \quad \left(P^{(m+n)} = P^{(m)} P^{(n)} \right)$$

$$P^{(n)} = P^{(n-1+1)} = P^{(n-1)} \cdot P^{(1)}$$

$$= P^{n-1} \cdot P = P^n \text{ (By Induction)}$$

<u>Lemma</u>:

Let $\mu_i^{(n)} = P(X_n = i)$ ($\mu^{(n)}$ is a row vector)

then $\mu^{(m+n)} = \mu^{(m)} P^n$

and hence $\mu^{(n)} = \mu^{(0)} P^n$

<u>Proof</u>:

$$\mu_j^{(m+n)} = P(X_{m+n} = j)$$

$$= \sum_i P(X_{m+n} = j \mid X_m = i) \cdot P(X_m = i)$$

$$= \sum_i \mu_i^{(m)} (P^n)_{ij}$$

$$\therefore \mu^{(m+n)} = \mu^{(m)} \cdot P^n$$

Defn: Mean recurrence time

$$T_i = \min\{n \geq 1 : X_n = i\}$$

$\mu_i = \mathbb{E}(T_i \mid X_0 = i)$ is called the mean reccurrence time of the state i.

A persistent state is called null-persistent if $\mu_i = \infty$ and non-null (positive) persistent if $\mu_i < \infty$.

(Note: for a transient state $P(T_j = \infty \mid X_0 = i) > 0$ therefor $\mu_i = \infty$ if i is transient)

Defn Period of a state i

Period of a state $d(i) := \gcd\{n : P_{ii}(n) > 0\}$

A state is called aperiodic if it has period 1.

For eg: In the case of R.W. all states have period (2) since $P_{ii}^{(2n)} > 0$ and $P_{ii}^{(2n+1)} = 0$ (for $n = 1, 2 \ldots$).

Defn Ergodic State.

A state is called ergodic if it is persistent, non-null and aperiodic.

Theorem:

In A one-dimensional r.w. is all ~~state~~ states ~~is~~ are persistent if $p = q = \frac{1}{2}$ and transient if $p \neq q \neq \frac{1}{2}$

Proof:

Let P_{00}^{2n} be the probability of returning to 0 in 2n steps. Note when n is odd $P_{00}^{2n+1} = 0$.

Now $P_{00}^{2n} = \binom{2n}{n} p^n (1-p)^{n}$. (The r.w. returns to origin iff number of ~~all~~ heads equal number of tails

Using Stirling's appx. for $n! \sim \frac{n^n \sqrt{2\pi n}}{e^n}$

we get $P_{00}^{2n} \sim \frac{(4p(1-p))^n}{\sqrt{\pi n}}$ (check!)

Now using proposition state 0 is persistent if $\sum_n P_{00}^{2n} = \infty$ and transient if $\sum_n P_{00}^{2n} < \infty$. But $\sum_n P_{00}^{2n} = \infty$ iff $p = q = \frac{1}{2}$ (when $P_{00}^{2n} = \frac{1}{\sqrt{\pi n}}$) else it is $< \infty$. (Since $\sum_n \frac{a^n}{\sqrt{\pi n}}$ $a > 0$ diverges.)

$\therefore$ R.W. is persistent for $p = q = \frac{1}{2}$ & transient if $p \neq q \neq \frac{1}{2}$

Classification of Chains & Decomposition thm:

<u>Defn</u> "i <u>communicates</u> with j" if $(i \Rightarrow j)$

$P_{ij}^{n} > 0$ for some n. ($\exists$ a positive prob. of reaching j from i)

<u>Defn</u> "state i and j intercommunicate" if $i \Rightarrow j$ and $j \Rightarrow i$ (written as $i \Leftrightarrow j$).

<u>Defn</u> A set of states $\mathcal{C}$ is called

(a) <u>Closed</u> if $P_{ij} = 0$ for all $i \in \mathcal{C}$ and $j \notin \mathcal{C}$

(b) <u>Irreducible</u> if $i \Longleftrightarrow j$ for all $i, j \in \mathcal{C}$

<u>Theorem</u> If $i \Leftrightarrow j$ then

a) i & j have the same period

b) i is transient iff j is transient

c) i is null persistent iff j is null persistent

<u>Proof</u>:

(a) By Chapman-Kolmogorov eqns.

$$P_{ii}^{(m+r+n)} \geq P_{ij}^{m} \cdot P_{jj}^{(r)}\, P_{ji}^{(n)} \quad \text{for all } m, r, n \geq 0$$

Since $i \Leftrightarrow j$ pick m & n s.t

$\alpha = P_{ij}^{(m)} \cdot P_{ji}^{(n)} > 0$. Then

$$P_{ii}(m+r+n) \geq \alpha\, P_{jj}^{(r)}$$

Setting $r=0$ we get

$$p_{ii}^{(m+n)} > 0 \implies d(i) \mid (m+n).$$

Now suppose for any r $d(i) \nmid r$ then since $d(i) \mid (m+n)$ implies $p_{ii}^{(m+n+r)} = 0$ so

$p_{jj}^{(r)} = 0. \implies d(j) \nmid r$

ie. $d(i) \nmid r \implies d(j) \nmid r$ or $d(i) \mid r \Rightarrow d(j) \mid r$

$\therefore$ $d(i) \mid d(j)$. Similarly $d(j) \mid d(i)$ giving

$d(i) = d(j)$

(b) Again as in (a) if $i \iff j$ there exist $m, n \geq 0$ s.t $\alpha = p_{ij}^{(m)} \cdot p_{ji}^{(n)} > 0$. By Chapman Kolmogorov equations

$$p_{ii}^{(m+r+n)} \geq p_{ij}^{(m)} p_{jj}^{(r)} p_{ji}^{(n)} = \alpha\, p_{jj}^{(r)}$$

Now sum over r

if $\sum_r p_{ii}^{(m+r+n)} < \infty$ then $\sum_r \alpha\, p_{jj}^{(r)} < \infty$

ie $\sum_r p_{jj}^{(r)} < \infty$

$\therefore$ i transient $\Rightarrow$ j transient. Similarly we can show j transient $\Rightarrow$ i transient.

(c) Use fact that a persistent state is null iff $p_{ii}^{(n)} \xrightarrow[n\to\infty]{} 0$. (Also $p_{ji}^{(n)} \xrightarrow[n\to\infty]{} 0$ for all j)

Decomposition thm: The state space S can be partitioned uniquely as $S = T \cup C_1 \cup C_2 \cup \cdots$ where T is the set of transient states, and C_i are irreducible closed sets of persistent states.

Proof:

$\Leftrightarrow$ is an equivalence relation on the state space ($i \Leftrightarrow i$, $i \Leftrightarrow j$, $j \Leftrightarrow k$ implies $i \Leftrightarrow k$, $i \Leftrightarrow j$ implies $j \Leftrightarrow i$)

Therefore it partitions the state space into Transient and persistent irreducible states (By the previous thm. ($i \Leftrightarrow j \Rightarrow$ i transient iff j transient iff

$i \Leftrightarrow j \Rightarrow$ i null pers. iff j null pers.)

So we only need to show that states $C_1, C_2, \ldots$ are the persistent irreducible also closed. Suppose by way of contradiction C_r is not closed the $\exists$ $i \in C_r$ and $j \notin C_r$

s.t. ~~$i \Rightarrow j$~~ $P_{ij} > 0$.

Since $j \not\to i$ (otherwise $j \in C_r$)

$$P\{X_n \neq i \text{ for all } n \geq 1 \mid X_0 = i\}$$

$$= \sum_{k=1} P\{X_k = j, X_n \neq i \text{ for } n > k \mid X_0 = i\}$$

$$\geq P(X_1 = j \mid X_0 = i) > 0 \quad \text{which is}$$

a contradiction to our assumption that i is persistent.

<u>Lemma</u> If S is finite then atleast one state is persistent and all persistent states are non-null

<u>Proof</u>: If all states are transient

then $P_{ji}^{(n)} \xrightarrow[n\to\infty]{} 0$ for all i, j

by Proposition proved earlier.

$$\therefore \quad 1 = \sum_j P_{ij}^{(n)} = \lim_{n\to\infty} \sum_j P_{ij}^{(n)}$$

$$= \sum_j \lim_{n\to\infty} P_{ij}^{(n)}$$ (limit can be brought inside because state space is finite)

$$= 0$$ which is a contradiction

(For proof of non-null use prop. that if state i is persistent ~~non~~-null then $P_{ji}^{(n)} \xrightarrow[n\to\infty]{} 0$)

<u>Examples</u>

$$P = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

$\{1,2\}$, $\{5,6\}$ are closed irreducible persistent states. $\{3\}$ is transient since $3 \Rightarrow 1$ ($P_{31} = \frac{1}{4}$). and same for $\{4\}$.

The partitioning of the state space is

$$\{3,4\} \cup \{1,2\} \cup \{5,6\}$$

Transient closed, irreducible persistent.

All states are aperiodic since $P_{ii} > 0$ for all i. By the proposition since we have a finite state-space $\{1,2,5,6\}$ are all persistent non-null states and hence they are ergodic.

Stationary Distributions

Defn: A vector π is called a stationary distribution of ~~the~~ a Markov chain if

(a) $\pi_j \geq 0$ for all j $\sum_j \pi_j = 1$

(b) $\pi P = \pi$, which is to say that

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{for all } j.$$

Such a distribution is called stationary because $\pi P^2 = (\pi P)P = \pi P = \pi$

and so $\pi P^n = \pi$ for all $n \geq 0$.

which means that if X_0 has distribution π then X_n has distribution π for all n,

Thus dist. of X_n is stationary as time passes.

Theorem: An irreducible chain has a stationary distribution iff all the states are non-null persistent, in this case π is unique the stationary distribution is given by $\pi_i = \frac{1}{\mu_i}$ where μ_i is the mean recurrence time of i.

(Note: Since on a finite state space an irreducible chain has all non-null persistent states therefore a Markov chain defined on a finite state space always has a stationary distribution)

Theorem:

(a) An ergodic Markov chain has a unique stationary distribution

(b) The stationary distribution is given by $\pi_i = \frac{1}{\mu_i}$ for each i where μ_i is the mean reccurence time of i

(c) In this case $p_{ij}(n) \xrightarrow[n\to\infty]{} \pi_j = \mu_j^{-1}$

(For proofs see Grimmett & Stirzaker Chap 6.4)

Note: From c) it follows that the limit probability $\lim_{n\to\infty} p_{ij}(n)$ does not depend on the starting point $X_0 = i$ i.e. the chain forgets the origin

Eg: Consider a Markov chain (two state).

The probability that it rains tomorrow given that it has rained today is α, and the probability that it will rain tomorrow given that it has not rained today. is β.

If we say that the state is 0 when it rains and 1 when it does not.

The transition matrix is given by

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \alpha & 1-\alpha \\ \cancel{1-\beta}\,\beta & 1-\beta \end{pmatrix} \end{array}$$

The stationary distribution for this ergodic Markov chain can be found by solving the equations

$$\pi = \pi_0, \pi_1$$

$$\pi P = \pi$$

$$(\pi_0 \ \pi_1)\begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix} = (\pi_0 \ \pi_1)$$

which gives

$$\pi_0 = \alpha\pi_0 + \beta\pi_1$$

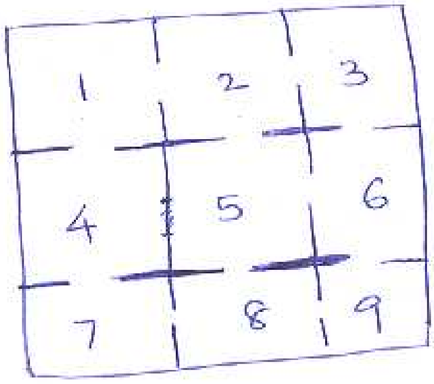
$$\pi_1 = (1-\alpha)\pi_0 + (1-\beta)\pi_1$$

and $\pi_0 + \pi_1 = 1$

which yields

$$\pi_0 = \frac{\beta}{1+\beta-\alpha}, \quad \pi_1 = \frac{1-\alpha}{1+\beta-\alpha}$$

Eg: A rat is put into a maze (as shown in figure). The rat moves through compartments at random. That is if there are k ways to leave a compartment, it choses each one with equal Probability. Find the stationary distribution if it exists.



Since this is an irreducible Markov chain on a finite state space the chain has a stationary distribution. The transition matrix of this Markov chain is

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

Solving the equation $\pi P = \pi$ will involve 10 equations and nine unknowns.

Now π_j is the long-run proportion of time the chain spends in state j. It seems reasonable to guess π by the following argument. "The times spent in each compartment in the long-run should be proportional to the number of entries to the compartment to the number of

Thus we try the vector

$$\pi = (2 \quad 3 \quad 2 \quad 3 \quad 4 \quad 3 \quad 2 \quad 3 \quad 2)$$

which is normalized as

$$\pi = \left(\frac{1}{12} \quad \frac{1}{8} \quad \frac{1}{12} \quad \frac{1}{8} \cdots \frac{1}{6} \quad \frac{1}{8} \quad \frac{1}{12} \quad \frac{1}{8} \quad \frac{1}{12} \right)$$

It is easy to check that this is indeed the stationary vector.

Eg: Random walk on a graph.

A particle performs a random walk on the vertex set of a connected graph G. For simplicity assume that the graph has no loops and multiple edges. At each state it moves to a neighbor of its current position, each neighbour being chosen with equal probability. If G has $n < \infty$ edges one can verify by a similar logic that the stationary distribution is given by $\pi_v = d_v/(2n)$ where d_v is the degree of vertex v.

Time Reversibility in Markov Chains

Suppose that $\{X_n : 0 \le n \le N\}$ is an irreducible non-null persistent Markov chain, with transition matrix P and stationary distribution π. We define the time reversed chain by $Y_n = X_{N-n}$ $\quad 0 \le n \le N$.

Theorem: The sequence Y_n is a Markov Chain with $P(Y_{n+1}=j \mid Y_n = i) = \left(\frac{\pi_j}{\pi_i}\right) P_{ji}$

Proof:

$$P(Y_{n+1} = i_{n+1} \mid Y_n = i_n, Y_{n-1} = i_{n-1} \dots Y_0 = i_0)$$

$$= \frac{P(Y_{n+1} = i_{n+1}, Y_n = i_n, \dots Y_0 = i_0)}{P(Y_n = i_n, Y_{n-1} = i_{n-1}, \dots Y_0 = i_0)}$$

$$= \frac{P(X_{N-n-1} = i_{n+1}, X_{N-n} = i_n, \dots X_N = i_0)}{P(X_{N-n} = i_n, \dots X_N = i_0)}$$

Since X_n is a Markov chain this gives

$$= \frac{\pi_{i_{n+1}} \cdot P_{i_{n+1}, i_n} \cdot P_{i_n, i_{n-1}} \cdots P_{i_1 i_0}}{\pi_{i_n} \cdot P_{i_n, i_{n-1}} \cdot P_{i_{n-1}, i_{n-2}} \cdots P_{i_1, i_0}} = \frac{\pi_{i_{n+1}} P_{i_{n+1}, i_n}}{\pi_{i_n}}$$

(Since for a Markov chain check that

$P(X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = P(X_0 = x_0) . P(X_1 = x_1 | X_0 = x_0) \ldots$
$\ldots P(X_n = x_n | X_{n-1} = x_{n-1})$)

<u>Defn</u>: Let $X = \{X_n : 0 \leq n \leq N\}$ be an irreducible Markov chain such that X_n has a stationary distribution. The chain is said to be reversible in equilibrium if the transision matrices of X and its time reversal Y are the same, which is to say $\pi_i P_{ij} = \pi_j P_{ji}$

<u>Theorem</u>: Let P be the transition matrix of an irreducible chain X and suppose there exists a distribution π such that $\pi_i P_{ij} = \pi_j P_{ji}$ for all $i, j \in S$ then π is a stationary distribution of the chain. Furthermore X is reversible in equilibrium.

<u>Proof</u>: Suppose

$$\pi_i P_{ij} = \pi_j P_{ji}, \text{ then summing}$$

over i

$$\sum_i \pi_i P_{ij} = \sum_i \pi_j P_{ji}$$

$$\sum_i \pi_i P_{ij} = \pi_j \sum_i P_{ji} = \pi_j$$

which implies $\pi P = \pi$

One way to think about equilibrium and reversibility in equilibrium is the following. Suppose we are provided with a Markov chain with state space S and stationary distribution π. We can associate a network with this chain with the states being the nodes and the arrow pointing from state i to j whenever $P_{ij} > 0$.

We are provided with one unit of material (disease, water) which is distributed about the nodes and allowed to flow along the arrows. The transportation rule is: at each epoch of time a proportion P_{ij} of the material flows from node i to node j. The system is in global balance if the amt. of material flowing into i is equal to the amt. of material flowing out. $\sum_j \pi_j P_{ji} = \sum_j \pi_i P_{ij} = \pi_i$. Which is $\pi P = \pi$. If there is global balance, there may or may not be local balance in the sense that amount flowing from i to j equals the amount flowing from j to i. If this occurs the system is in local balance. Local balance occurs iff

$$\pi_i P_{ij} = \pi_j P_{ji}$$