i. JOINT, MARGINAL, AND CONDITIONAL DISTRIBUTIONS
ii. COVARIANCE, CORRELATION, INDEPENDENCE OF VARIABLES (STOCHASTIC INDEPENDENCE)

Q1) The joint probability function of two discrete random variables X and Y is given by $f(x,y) = cxy$ for $x = 1,2,3$ and $y = 1,2,3$ and equals zero otherwise. Find:

a. The constant c.
b. $P(X = 2, Y = 3)$.
c. $P(1 \leq X \leq 2, Y \leq 2)$.
d. $P(X \geq 2)$.
e. $P(Y < 2)$.
f. $P(X = 1)$.
g. $P(Y = 3)$.

Solution :

$$f(x,y) = cxy \quad ; x = 1,2,3 \ ; y = 1,2,3$$

a) We know that $\sum_x \sum_y f(x,y) = 1$

$\sum_{i=1}^{3} \sum_{j=1}^{3} f(x_i, y_j) = 1$

$C + 2C + 3C + 2C + 4C + 6C + 3C + 6C + 9C = 1 \Rightarrow \quad C = \frac{1}{36}$

Or $\quad 6C + 12C + 18C = 1 \ \Rightarrow C = \frac{1}{36}$

$$f(x,y) = \frac{1}{36}xy \quad ; x = 1,2,3 \ ; y = 1,2,3$$

$Y \backslash X$	1	2	3	$f_Y(y) = \sum_x f(x,y)$
1	1C	2C	3C	6C
2	2C	4C	6C	12C
3	3C	6C	9C	18C
$f_X(x) = \sum_y f(x,y)$	6C	12C	18C	1

b) $P(X = 2, Y = 3) = f(2,3) = 6C = \frac{6}{36} = \frac{1}{6}$

c) $P(1 \leq X \leq 2 \ , Y \leq 2) \qquad ; x = 1,2 \quad , y = 1,2$

$P(1 \leq X \leq 2 \ , Y \leq 2) = P(X = 1, Y = 1) + P(X = 1, Y = 2)$

$+P(X = 2, Y = 1) + P(X = 2, Y = 2)$

$= f(1,1) + f(1,2) + f(2,1) + f(2,2)$

$= \frac{1}{36}[1 + 2 + 2 + 4] = \frac{9}{36}$

d) $P(X \geq 2) \;\; ; \;\; x = 2,3$

$$P(X \geq 2) = f_X(2) + f_X(3) = \frac{12}{36} + \frac{18}{36} = \frac{30}{36}$$

e) $P(Y < 2) \;\; ; y = 1$

$$P(Y < 2) = f_Y(1) = \frac{6}{36} = \frac{1}{6}$$

f) $P(X = 1) = f_X(1) = \frac{6}{36} = \frac{1}{6}$

g) $P(Y = 3) = f_Y(3) = \frac{18}{36}$

Q2) For the random variables of **Problem 1**, find the marginal probability function of X and Y. Determine whether X and Y are independent.

Solution :

Marginal dis of X :

x	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	total
$f_X(x) = P(X = x) = \sum_y f(x,y)$	6/36 = 1/6	12/36 =2/6	18/36 =3/6	1

Marginal dis of Y :

y	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	total
$f_Y(y)$	6/36	12/36	18/36	1

Are X and Y independent ?

If $f_{XY}(x,y) = f_X(x)\, f_Y(y) \;\; \forall\, x = 1,2,3 \;\; ; y = 1,2,3$
Then X and Y are independent. But if there some values of (x) and (y) which make that $f_{XY}(x,y) \neq f_X(x)\, f_Y(y)$ then X and Y are **not** independent .

In this example, we have :

$$f(1,1) = f_X(1) f_Y(1) \Rightarrow \frac{1}{36} = \left(\frac{6}{36}\right)\left(\frac{6}{36}\right)$$

$$f(1,2) = f_X(1) f_Y(2) \Rightarrow \frac{2}{36} = \left(\frac{6}{36}\right)\left(\frac{12}{36}\right)$$

$$\vdots$$

$$f(3,3) = f_X(3) f_Y(3) \Rightarrow \frac{9}{36} = \left(\frac{18}{36}\right)\left(\frac{18}{36}\right)$$

So as $f(x,y) = f_X(x) f_Y(y) \;\; \forall\, x = 1,2,3 \;\; ; y = 1,2,3$, then X and Y are independent.

Q3) For the distribution of **Problem 1**, find the **conditional** probability function of X given Y, Y given X.

Solution :

Distribution of $X|Y: f_{X|Y}(x)=\frac{f(x,y)}{f_y(y)}$; $x=1,2,3$

If $Y=1:$ $f_{X|Y=1}(x)=\frac{f(x,1)}{f_y(1)}=\frac{f(x,1)}{6/36}$

x	$x_1=1$	$x_2=2$	$x_3=3$	total
$f_{X\|Y=1}(y)$	$\frac{f(1,1)}{6/36}=\frac{1/36}{6/36}=\mathbf{1/6}$	$\frac{f(2,1)}{6/36}=\frac{2/36}{6/36}=\frac{2}{6}=\mathbf{1/3}$	$\frac{f(3,1)}{6/36}=\frac{3}{6}=\mathbf{1/2}$	1

If $Y=2:$ $f_{X|Y=2}(x)=\frac{f(x,2)}{f_y(2)}=\frac{f(x,2)}{12/36}$

x	$x_1=1$	$x_2=2$	$x_3=3$	total
$f_{X\|Y=2}(y)$	$\frac{f(1,2)}{12/36}=\frac{2/36}{12/36}=\mathbf{1/6}$	$\frac{f(2,2)}{12/36}=\frac{4/36}{12/36}=\mathbf{2/6}$	$\frac{f(3,2)}{12/36}=\frac{6/36}{12/36}=\mathbf{3/6}$	1

If $Y=3:$ $f_{X|Y=2}(x)=\frac{f(x,3)}{f_y(3)}=\frac{f(x,3)}{18/36}$

x	$x_1=1$	$x_2=2$	$x_3=3$	total
$f_{X\|Y=3}(y)$	$\frac{f(1,3)}{18/36}=\frac{3/36}{18/36}=\mathbf{1/6}$	$\frac{f(2,3)}{18/36}=\frac{6/36}{18/36}=\mathbf{2/6}$	$\frac{f(3,3)}{18/36}=\frac{9/36}{18/36}=\mathbf{3/6}$	1

Note: Since X and Y are independent so $f_{X|Y}(x)=\frac{f(x,y)}{f_y(y)}=\frac{f_X(x)f_Y(y)}{f_y(y)}=f_X(x)=\frac{x}{6}$

Distribution of $Y|X$: $f_{Y|X}(y) = \frac{f(x,y)}{f_X(x)}$; $y = 1, 2, 3$

If $X = 1$: $f_{Y|X=1}(y) = \frac{f(1,y)}{f_X(1)} = \frac{f(1,y)}{6/36} \Longrightarrow$

y	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	total
$f_{Y|X=1}(y)$	$\frac{f(1,1)}{6/36} = \frac{1/36}{6/36} = \mathbf{1/6}$	$\frac{f(1,2)}{6/36} = \frac{2/36}{6/36} = \mathbf{2/6}$	$\frac{f(1,3)}{6/36} = \frac{3/36}{6/36} = \mathbf{3/6}$	1

If $X = 2$: $f_{Y|X=2}(y) = \frac{f(2,y)}{f_X(2)} = \frac{f(2,y)}{12/36} \Longrightarrow$

y	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	total
$f_{Y|X=2}(y)$	$\frac{f(2,1)}{12/36} = \frac{2/36}{12/36} = \mathbf{2/12}$	$\frac{f(2,2)}{12/36} = \frac{4/36}{12/36} = \mathbf{4/12}$	$\frac{f(2,3)}{12/36} = \frac{6/36}{12/36} = \mathbf{6/12}$	1

If $X = 3$: $f_{Y|X=3}(y) = \frac{f(3,y)}{f_X(3)} = \frac{f(3,y)}{18/36} \Longrightarrow$

y	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	total
$f_{Y|X=2}(y)$	$\frac{f(3,1)}{18/36} = \frac{2/36}{18/36} = \mathbf{3/18}$	$\frac{f(3,2)}{18/36} = \frac{4/36}{18/36} = \mathbf{6/18}$	$\frac{f(3,3)}{18/36} = \frac{6/36}{18/36} = \mathbf{9/18}$	1

Note: Since X and Y are independent so $f_{Y|X}(y) = \frac{f(x,y)}{f_X(x)} = \frac{f_X(x)f_Y(y)}{f_X(x)} = f_Y(y) = \frac{y}{6}$

Q4) Let X and Y be continuous random variables having joint density function

$$f(x,y) = \begin{cases} c(x^2 + y^2) & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

Determine:

a. The constant c.

b. $P\left(X < \frac{1}{2}, Y > \frac{1}{2}\right)$

c. $P\left(\frac{1}{4} < X < \frac{3}{4}\right)$.

d. $P\left(Y<\frac{1}{2}\right)$.

e. Whether X and Y are independent.

Solution :

$$f(x,y)=C(x^2+y^2)\ ,\qquad 0<x<1\ ,\ 0<y<1$$

a) We know that $\iint_{(x,y)\epsilon\, C} f(x,y)\,dx\,dy=1$

$$\int_0^1\int_0^1 C\,(x^2+y^2)dx\,dy=1 \;\Rightarrow\; C\int_0^1\left[\int_0^1 (x^2+y^2)\,dx\right]dy=1$$

$$\Rightarrow\; C\int_0^1\left[\frac{x^3}{3}+xy^2\right]_0^1 dy=1 \;\Rightarrow\; C\int_0^1\left(\frac{1}{3}+y^2\right)=1 \;\Rightarrow\; C\left[\frac{1}{3}y+\frac{y^3}{3}\right]_0^1=1$$

$$\Rightarrow C\left[\frac{1}{3}+\frac{1}{3}\right]=1 \;\Rightarrow\; \boldsymbol{C=\frac{3}{2}}$$

b)

$$P\left(X<\frac{1}{2}\,,Y>\frac{1}{2}\right)=\frac{3}{2}\int_{1/2}^1\left[\int_0^{1/2}(x^2+y^2)dx\right]dy=\frac{3}{2}\int_{1/2}^1\left[\frac{x^3}{3}+xy^2\right]_0^{1/2}dy$$

$$=\frac{3}{2}\int_{1/2}^1\left(\frac{1}{24}+\frac{1}{2}y^2\right)dy=\frac{3}{2}\left[\frac{y}{24}+\frac{y^3}{6}\right]_{\frac{1}{2}}^1=\frac{3}{2}\left[\left(\frac{1}{24}+\frac{1}{6}\right)-\left(\frac{1}{48}+\frac{1}{48}\right)\right]=\frac{1}{4}$$

c)

$$P\left(\frac{1}{4}<X<\frac{3}{4}\right)=\frac{3}{2}\int_0^1\left[\int_{\frac{1}{4}}^{\frac{3}{4}}(x^2+y^2)dx\right]dy=\frac{3}{2}\int_0^1\left[\frac{x^3}{3}+xy^2\right]_{\frac{1}{4}}^{\frac{3}{4}}dy$$

$$=\frac{3}{2}\int_0^1\left[\left(\frac{9}{64}+\frac{3}{4}y^2\right)-\left(\frac{1}{192}+\frac{1}{4}y^2\right)\right]dy=\frac{3}{2}\int_0^1\left(\frac{13}{96}+\frac{1}{2}y^2\right)dy$$

$$=\frac{3}{2}\left[\frac{13}{96}y+\frac{1}{6}y^3\right]_0^1=\frac{3}{2}\left(\frac{13}{96}+\frac{1}{6}\right)=\frac{29}{64}$$

d)

$$P\left(Y<\frac{1}{2}\right)=\frac{3}{2}\int_0^{\frac{1}{2}}\left[\int_0^1(x^2+y^2)dx\right]dy=\frac{3}{2}\int_0^{\frac{1}{2}}\left[\frac{x^3}{3}+xy^2\right]_0^1 dy$$

$$=\frac{3}{2}\int_0^{1/2}\left(\frac{1}{3}+y^2\right)dy=\frac{3}{2}\left[\frac{y}{3}+\frac{y^3}{3}\right]_0^{\frac{1}{2}}=\frac{3}{2}\left[\frac{1}{6}+\frac{1}{24}\right]=\frac{5}{16}$$

e) X and Y are independent if satisfy :

1) $f(x,y)=f_X(x)f_Y(y)\ \forall\, x,y$.
2) the ranges of X and Y are independent.

In this example , we can see that

$$f(x,y) \neq f_X(x)f_Y(y) \quad \therefore X \text{ and } Y \text{ are not independent.}$$

$$f_Y(y) = \frac{3}{2}\int_0^1 (x^2+y^2)dx = \frac{3}{2}\left[\frac{x^3}{3} + xy^2\right]_0^1 = \frac{3}{2}\left(\frac{1}{3}+y^2\right) = \frac{1}{2} + \frac{3}{2}y^2$$

$$f_X(x) = \frac{3}{2}\int_0^1 (x^2+y^2)dy = \frac{3}{2}\left[x^2y + \frac{y^3}{3}\right]_0^1 = \frac{3}{2}\left(x^2+\frac{1}{3}\right) = \frac{3}{2}x^2 + \frac{1}{2}$$

$$f_X(x)f_Y(y) = \left(\frac{3}{2}x^2+\frac{1}{2}\right)\left(\frac{3}{2}y^2+\frac{1}{2}\right) = \frac{9}{4}x^2y^2 + \frac{3}{4}x^2 + \frac{3}{4}y^2 + \frac{1}{4} \neq f(x,y)$$

Q5) For the random variables of **Problem 4**, find the marginal probability function of X and Y.

Solution :

Marginal distribution of X : $f(x) = \int_0^1 f(x,y)dy$

$$f(x) = \frac{3}{2}\int_0^1 (x^2+y^2)dy = \frac{3}{2}\left[x^2y + \frac{y^3}{3}\right]_0^1 = \frac{3}{2}\left[x^2 + \frac{1}{3}\right] = \frac{3}{2}x^2 + \frac{1}{2}$$

$$\therefore f_X(x) = \frac{3}{2}x^2 + \frac{1}{2} \quad ; 0 < x < 1$$

Marginal distribution of Y : $f(y) = \int_0^1 f(x,y)dx$

$$f(y) = \frac{3}{2}\int_0^1 (x^2+y^2)dx = \frac{3}{2}\left[\frac{x^3}{3} + y^2x\right]_0^1 = \frac{3}{2}\left(\frac{1}{3}+y^2\right) = \frac{3}{2}y^2 + \frac{1}{2}$$

$$\therefore f_Y(y) = \frac{3}{2}y^2 + \frac{1}{2} \quad ; \quad 0 < y < 1$$

Q6) For the distribution of **Problem 4**, find the conditional probability function of X given Y, Y given X.

Solution :

Conditional distribution $X|Y$:

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}y^2+\frac{1}{2}} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}\left(y^2+\frac{1}{3}\right)} = \frac{x^2+y^2}{y^2+\frac{1}{3}}$$

for $0 < x < 1$ *wher* $0 < y < 1$ fixed value .

Conditional distribution $Y|X$:

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}x^2+\frac{1}{2}} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}\left(x^2+\frac{1}{3}\right)} = \frac{x^2+y^2}{x^2+\frac{1}{3}}$$

for $0 < y < 1$ $wher$ $0 < x < 1$ fixed value .

Q7) Let $f(x,y) = \begin{cases} x+y & 0 \le x \le 1\,,\ 0 \le y \le 1 \\ 0 & otherwise \end{cases}$

Find the conditional probability function of X given Y, Y given X.

Solution : H.W

$$f(x,y) = x+y \quad ;0 \le x \le 1\ ,0 \le y \le 1$$

Marginal pdf of X : $f(x) = \int_0^1 f(x,y)dy = \int_0^1 (x+y)dy = \left[xy+\frac{y^2}{2}\right]_0^1 = x+\frac{1}{2}$

Marginal pdf of Y : $f(y) = y+\frac{1}{2}$

Conditional distribution $X|Y$:

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{x+y}{y+\frac{1}{2}} = \frac{2(x+y)}{(2y+1)}$$

Conditional distribution $Y|X$:

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{x+y}{x+\frac{1}{2}} = \frac{2(x+y)}{(2x+1)}$$

Q8) Let $f(x,y) = \begin{cases} e^{-(x+y)} & x \ge 0\,, y \ge 0 \\ 0 & otherwise \end{cases}$

be the joint density function of X and Y. Find the conditional probability function of X given Y, Y given X.

Solution :

Marginal pdf of X : $f(x) = \int_0^\infty e^{-(x+y)}\,dy = e^{-x}[-e^{-y}]_0^\infty = e^{-x}[0+1] = e^{-x}$

Marginal pdf of Y : $f(y) = \int_0^\infty e^{-(x-y)}\,dx = e^{-y}$

Conditional distribution $X|Y$: $f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-(x+y)}}{e^{-y}} = e^{-x}$

Conditional distribution $Y|X$: $f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-(x-y)}}{e^{-x}} = e^{-y}$

Q9) Let X and Y be random variables having joint density function

$$f(x,y) = \begin{cases} c(2x+y) & 0 < x < 1\,, \quad 0 < y < 2 \\ 0 & otherwise \end{cases}$$

Find:

a. The constant c.
b. $P\left(X > \frac{1}{2}, Y < \frac{3}{2}\right)$.
c. The (marginal) density function of X.
d. The (marginal) density function of Y.

Solution : H.W

Q10) The joint probability function for the random variables X and Y is given in following table, then find:

X \ Y	0	1	2
0	1/18	1/9	1/6
1	1/9	1/18	1/9
2	1/6	1/6	1/18

a. The marginal probability functions of X and Y.
b. $P(1 \leq X < 3, Y \geq 1)$.
c. Determine whether X and Y are independent.

Solution : H.W

Q11) Let X and Y be random variables having joint density function

$$f(x,y) = \begin{cases} x+y & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

Find: a. Var(X). b. Var(Y). c. σ_X. d. σ_Y. e. σ_{XY}. f. ρ.

Solution :

$f(x) = \int_0^1 f(x,y)dy = \int_0^1 (x+y)dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}$

$f(x) = x + \frac{1}{2} \quad ,0 < x < 1$

$f(y) = y + \frac{1}{2} \quad ,0 < y < 1$

a, b)

$$E(X) = \int_0^1 x\, f(x)dx = \int_0^1 x\left(x + \frac{1}{2}\right)dx$$

$$= \int_0^1 \left(x^2 + \frac{x}{2}\right)dx = \left[\frac{x^3}{3} + \frac{1}{4}x^2\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$E(Y) = \int_0^1 y\, f(y)dy = \frac{7}{12}$$

$$E(X^2) = \int_0^1 x^2\, f(x)dx = \int_0^1 x^2\left(x + \frac{1}{2}\right)dx$$

$$= \int_0^1 \left(x^3 + \frac{x^2}{2}\right)dx = \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

$$E(Y^2) = \int_0^1 y^2\, f(y)dy = \frac{5}{12}$$

$$var(X) = E(X^2) - [E(X)]^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$

$$var(Y) = E(Y^2) - [E(Y)]^2 = \frac{11}{144}$$

c, d)

$$\sigma_X = \sqrt{var(X)} = \sqrt{11/144} = 0.2764$$

$$\sigma_Y = \sqrt{var(Y)} = \sqrt{11/144} = 0.2764$$

e) $\sigma_{XY} = cov(x,y) = E(XY) - E(X)E(Y)$

$$E(XY) = \int_0^1 \int_0^1 xy\,(x+y)dx\,dy = \int_0^1 \left[\frac{x^3}{3}y + \frac{x^2}{2}y^2\right]_0^1 dy$$

$$= \int_0^1 \left(\frac{1}{3}y + \frac{y^2}{2}\right) dy = \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

$$cov(x,y) = \frac{2}{6} - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right) = -\frac{1}{144}$$

f) $\rho = cor(x,y) = \frac{cov(x,y)}{\sqrt{var(x)var(y)}} = \frac{-1/144}{\sqrt{\left(\frac{11}{144}\right)\left(\frac{11}{144}\right)}} = \frac{-1}{11} = -\,0.091$

(Weak negative correlation)

Q12) The joint density function is

$$f(x,y) = \begin{cases} e^{-(x+y)} & x \geq 0\,, y \geq 0 \\ 0 & otherwise \end{cases}$$

Find: a. Var(X). b. Var(Y). c. σ_X. d. σ_Y. e. σ_{XY}. f. ρ.

Solution : H.W

Q13) Find a. The covariance. b. The correlation coefficient of two random variables X and Y. If $E(X) = 2,\ E(Y) = 3,\ E(XY) = 10,\ E(X^2) = 9,\ E(Y^2) = 16.$

Solution :

a) $cov(x,y) = E(XY) - E(X)E(Y) = 10 - (2)(3) = 4$

b) $var(x) = E(x^2) - [E(x)]^2 = 9 - 2^2 = 5$

$$var(y) = E(y^2) - [E(y)]^2 = 16 - 3^2 = 7$$

$\rho = \frac{cov(x,y)}{\sqrt{var(x)var(y)}} = \frac{4}{\sqrt{(5)(7)}} = \frac{4}{\sqrt{35}} = 0.676123$ (Moderate positive correlation)

Q14) The correlation coefficient of two random variables **X** and **Y** is **(-1/4)** while their variances are **3** and **5**. Find the covariance.

Solution : H.W

$$\rho = -\frac{1}{4} \quad ; var(x) = 3\,, var(y) = 5$$

$$\rho = cor(x,y) = \frac{cov(x,y)}{\sqrt{var(x)var(y)}}$$

$$-\frac{1}{4} = \frac{cov(x,y)}{\sqrt{(3)(5)}} \Rightarrow -\frac{1}{4}\sqrt{15} = cov(x,y) \Longrightarrow cov(x,y) = -0.9682$$

(Strong negative correlation)

Q15) The joint probability function of two **discrete** random variables X and Y is given by $f(x,y) = c(2x + y)$, where x and y can assume all integers such that $0 < x < 1,\ 0 < y < 3$, and $f(x,y) = 0$ otherwise. Find:

a. The value of the constant c.
b. $P(X = 2, Y = 1)$.
c. $P(X \geq 1, Y \leq 2)$

Solution : H.W

$$f(x,y) = c(2x + y) \quad ; x = 0,1 \ ; \ \ y = 0,1,2,3$$

a) $\sum_{i=0}^{3} \sum_{j=0}^{1} f(x_i, y_j) \ = 1$

$$6c + 14c = 1 \ \Rightarrow \ 20c = 1 \Rightarrow c = \frac{1}{20}$$

$y \backslash x$	0	1	$f_Y(y)$
0	0	2c	2c
1	c	3c	4c
2	2c	4c	6c
3	3c	5c	8c
$f_X(x)$	6c	14c	1

b) $P(X = 2, Y = 1) = 0$

c) $P(X \geq 1, Y \leq 2) = P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 1, Y = 2)$

$$= \frac{1}{20}(2 + 3 + 4) = \frac{9}{20}$$

Q16) For the **Problem 15**, find a. E(X). b. E(Y). c. E(XY). d. $E(X^2)$. e. $E(Y^2)$. f. Var(X). g. Var(Y). h. Cov(X,Y). i. ρ.

Solution :

$$E(X) = \sum_{x=0}^{1} x\, f(x) = \frac{1}{20}[0 * 6 + 1 * 14] = \frac{14}{20} = \frac{7}{10}$$

$$E(X^2) = \sum_{x=0}^{1} x^2\, f(x) = \frac{1}{20}[0 * 6 + 1 * 14] = \frac{14}{20} = \frac{7}{10}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{7}{10} - \left(\frac{7}{10}\right)^2 = \frac{21}{100}$$

$$E(Y) = \sum_{y=0}^{3} y\, f(y) = \frac{1}{20}[0*2+1*4+2*6+3*8] = \frac{40}{20} = 2$$

$$E(Y^2) = \sum_{y=0}^{3} y^2\, f(y) = \frac{1}{20}[0*2+1*4+4*6+9*8] = \frac{100}{20} = 5$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = 5 - (2)^2 = 1$$

$$E(XY) = \sum_{y=0}^{3}\sum_{x=0}^{1} yx\, f(x,y) = \frac{1}{20}(1*1*3+1*2*4+1*3*5) = \frac{26}{20} = \frac{13}{10}$$

$$\sigma_{XY} = cov(x,y) = E(XY) - E(X)E(Y) = \frac{13}{10} - \left(\frac{7}{10}\right)(2) = -0.1$$

$$\rho = \frac{cov(x,y)}{\sqrt{var(x)var(y)}} = \frac{-0.1}{\sqrt{\frac{21}{100}*1}} = -0.2182$$

weak negative correlation.

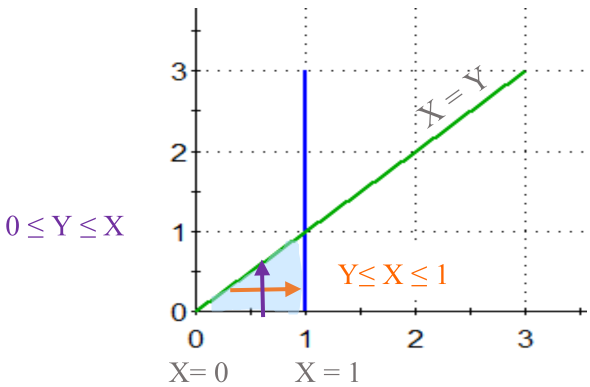
Q17) The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} 8xy & 0 \le x \le 1\ , 0 \le y \le x \\ 0 & otherwise \end{cases}$$

Find:

a. The marginal density of X.
b. The marginal density of Y.
c. The conditional density of X.
d. The conditional density of Y.

Solution :



a) $f_X(x) = \int_0^x f(x,y)dy$

$$= 8\int_0^x xy\, dy = 8x\left[\frac{y^2}{2}\right]_0^x = 8x\left[\frac{x^2}{2}\right] = 4x^3 \quad for\ \ 0 < x < 1$$

b) $f_Y(y) = \int_y^1 f(x,y)dx$

$$= 8y\int_y^1 x\, dx = 8y\left[\frac{x^2}{2}\right]_y^1 = 4y[1-y^2] = 4y - 4y^3$$

$$f_Y(y) = 4(y - y^3) \quad for\ \ 0 < y < 1$$

c) Conditional distribution $X|Y$: $f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)} = \frac{8\,x\,y}{4(y-y^3)} = \frac{2xy}{y(1-y^2)} = \frac{2x}{1-y^2}$

$for\ \ y < x < 1\ wher\ 0 < y < 1\ fixed\ value\ .$

d) Conditional distribution $Y|X$: $f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{8\,x\,y}{4\,x^3} = \frac{2y}{x^2}$

$for\ \ 0 < y < x\ \ wher\ 0 < x < 1\ fixed\ value\ .$

Q18) Find the conditional expectation of X given Y and Y given X in **Problem 17**.

Solution :

$$E(X|Y=y) = \int_y^1 x\, f_{X|Y=Y}(x)dx = \frac{2}{1-y^2}\int_y^1 x^2\, dx$$

$$= \frac{2}{1-y^2}\left[\frac{x^3}{3}\right]_y^1 = \frac{2}{3}\frac{1}{1-y^2}(1-y^3) = \frac{2}{3}\frac{(1-y^3)}{(1-y^2)}$$

$$E(Y|X=x) = \int_0^x y\, f_{Y|X=x}(y)dy = \frac{2}{x^2}\int_0^x y^2\, dy \ = \frac{2}{x^2}\left[\frac{y^3}{3}\right]_0^x = \frac{2}{3}\frac{1}{x^2}[x^3] \ = \frac{2}{3}\,x$$

Q19) Find the conditional variance of Y given X for **Problem 17**.

Solution :

$$var(Y|X) = E(Y^2|X) - E(Y|X)^2$$

$$E(Y^2|X) = \int_0^x y^2 f_{Y|X=x}(y)dy = \ \frac{2}{x^2}\int_0^x y^3 dy = \ \frac{2}{x^2}\left[\frac{y^4}{4}\right]_0^x = \frac{2}{4}\frac{1}{x^2}\ [x^4] = \frac{x^2}{2}$$

$$var(Y|X) = \frac{x^2}{2} - \left[\frac{2}{3}x\right]^2 = \left(\frac{1}{2} - \frac{2^2}{3^2}\right)x^2 = \frac{1}{18}x^2$$

Note: Q13,Q14,Q16,Q18,Q19, Q20, Q21, Q22 ⇒ we will solve it later.

Q20) The joint pdf of (X,Y) is given by $f(x,y) = \frac{e^{-y}}{y}$; $0 < x < y,\ 0 < y < \infty$.
Find $E(X), E(Y), V(X), V(Y)$ and $Cov(X,Y)$.

Solution : H.W

$$\boldsymbol{E(X)} = \int_0^\infty \int_0^y x \frac{1}{y} e^{-y}\ dx\, dy = \int_0^\infty \frac{1}{y} e^{-y} \left[\frac{x^2}{2}\right]_0^y dy\ = \int_0^\infty \frac{1}{2} y\, e^{-y} dy$$

Note: $X \sim \text{exponential}(\lambda) \rightarrow f(x) = \lambda e^{-\lambda x}\ \ x > 0$; $E(x) = \frac{1}{\lambda}$; $V(x) = \frac{1}{\lambda^2}$

$$let\ W = e^{-y} \sim Exp(1)\quad ; E(W) = V(W) = 1$$

$$\therefore E(X) = \frac{1}{2} E(W) = \frac{1}{2}$$

$$\boldsymbol{or}\ use\ \int_0^\infty x^a\, e^{-b\,x}\ dx = \frac{\Gamma(a+1)}{b^{a+1}}\quad , \qquad \Gamma(a) = (a-1)!$$

$$E(X^2) = \int_0^\infty \int_0^y x^2\ \frac{1}{y} e^{-y}\ dx\, dy = \int_0^\infty \frac{1}{y} e^{-y} \left[\frac{x^3}{3}\right]_0^y\ dy$$

$$= \int_0^\infty \frac{1}{3} y^2 e^{-y}\ \ dy = \frac{1}{3}\ E(W^2) = \frac{1}{3}(2) = \frac{2}{3}$$

$$\boldsymbol{V(X)} = \frac{2}{3} - \left(\frac{1}{2}\right)^2 = \frac{5}{12}$$

$$\boldsymbol{E(Y)} = \int_0^\infty \int_0^y y\ \frac{1}{y} e^{-y}\ dx\, dy = \int_0^\infty e^{-y}\, dy = 1$$

$$E(Y^2) = \int_0^\infty y^2\, e^{-y}\ dy = 2 \qquad ; \qquad V(Y) = 2 - 1 = 1$$

$$E(XY) = \int_0^\infty \int_0^y xy\ \frac{1}{y}\ e^{-y}\ dx\, dy = \int_0^\infty \int_0^y x e^{-y}\ dx\, dy$$

$$= \int_0^\infty e^{-y} \left[\frac{x^2}{2}\right]_0^y\ dy = \int_0^\infty \frac{1}{2}\ y^2 e^{-y}\, dy = \frac{1}{2}\frac{\Gamma(3)}{1^3} = \frac{1}{2}(2!) = 1$$

$$cov(x,y) = E(XY) - E(X)E(Y) = 1 - \frac{1}{2}(1) = \frac{1}{2}$$

Q21) Let (X,Y) have joint density given by

$f(x,y) = 24xy \qquad ;\ 0 < x < 1,\quad 0 < y < 1,\quad x + y < 1$

Find:

A. The marginal pdf's.

B. The following expectations:

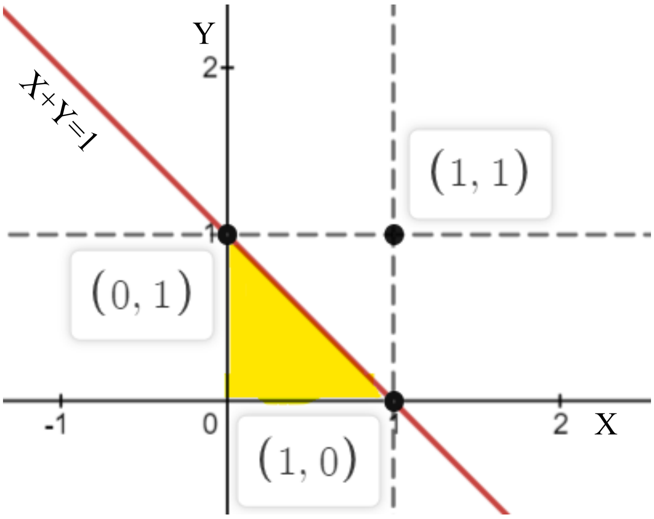
i. E(X) and E(X^2).

ii. E(Y) and E(Y^2).

iii. E(XY) and E(X^2 Y^3).

iv. V(X), V(Y), Cov(X,Y). Do X and Y have a positive or negative relationship?

Solution : H.W



a) $f(x) = \int_0^{1-x} 24xy\ dy = 24x\left[\frac{y^2}{2}\right]_0^{1-x} = 12\,x(1-x)^2\ ;\ 0 < x < 1$

$f(y) = \int_0^{1-y} 24xy\ dx = 24y\left[\frac{x^2}{2}\right]_0^{1-y} = 12\,y(1-y)^2\ \ ;\ 0 < y < 1$

b) Expectations $E(X) = \int_{-\infty}^{\infty} xf(x)\ dx$

$$E(X) = \int_0^1 12x^2(1-x)^2\ dx = \int_0^1 12x^2(1-2x+x^2)\ dx$$

$$= 12\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1 = 12\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right)$$

$$\boldsymbol{E(X) = \frac{2}{5}}$$

Or you can solve it as $E(X) = \int_0^1 \int_0^{1-y} 24\,x^2\,y\,dx\ dy = \int_0^1 24y\ \left[\frac{x^3}{3}\right]_0^{1-y} dy$

$= \int_0^1 8\,y(1-y)^3\ \ dy = \int_0^1 8\,y(1-3y+3y^2-y^3)\ dy$

$= 8\left[\frac{y^2}{2} - \frac{3y^3}{3} + \frac{3y^4}{4} - \frac{y^5}{5}\right]_0^1 = 8\left[\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right] = \frac{8}{20} = \frac{2}{5}$

$$E(X^2)=\int_0^1 12x^3(1-x)^2\ dx=\int_0^1 12x^3(1-2x+x^2)\ dx=12\left[\frac{x^4}{4}-\frac{2x^5}{5}+\frac{x^6}{6}\right]_0^1$$

$$\boldsymbol{E(X^2)=\frac{1}{5}}$$

$$V(X)=E(X^2)-\left(E(X)\right)^2=\frac{1}{5}-\left(\frac{2}{5}\right)^2=\frac{1}{25}=0.04$$

By similarly, $E(Y)=\frac{2}{5}$; $E(Y^2)=\frac{1}{5}$; $V(Y)=\frac{1}{25}$

$$E(XY)=\int_0^1\int_0^1 24\,x^2y^2\,dx\ dy=\int_0^1 24\,y^2\left[\frac{x^3}{3}\right]_0^1 dy\ =\int_0^1 8\,y^2\ dy\ =8\left[\frac{y^3}{3}\right]_0^1$$

$$\boldsymbol{E(XY)=8}$$

$$cov(x,y)=E(XY)-E(X)E(Y)=8-\left(\frac{1}{5}\right)^2=7.96$$

Since $cov(x,y)>0$, then X and Y have a positive relationship.

Q22) Let joint pdf of (X,Y) given by $f(x,y)=\frac{1}{y}\ e^{-y}\ e^{-\frac{x}{y}}\quad ;\ x\ >\ 0, y\ >\ 0$, find:

a. E(X) and E(X^2).
b. E(Y) and E(Y^2).
c. Show that Cov(X,Y)=1.
d. $\rho(X,Y)$.

Solution :

$$f(x)=\int_0^\infty \frac{1}{y}\ e^{-y}\ e^{-\frac{x}{y}}\ dy=??$$

$$f(y)=\int_0^\infty \frac{1}{y}\ e^{-y}\ e^{-\frac{x}{y}}\ dx=e^{-y}\ \left[-e^{-\frac{x}{y}}\right]_0^\infty=e^{-y}(0+1)=e^{-y}$$

$$E(X)=\int_0^\infty\int_0^\infty \frac{x}{y}e^{-y}e^{-\frac{x}{y}}\,dx\,dy=\quad\int_0^\infty \frac{1}{y}e^{-y}\left[\int_0^\infty xe^{-\frac{x}{y}}\,dx\right]dy$$

use $\int_0^\infty x^a\ e^{-b\,x}\ dx=\frac{\Gamma(a+1)}{b^{a+1}}$, $\Gamma(a)=(a-1)!$

$$=\int_0^\infty \frac{1}{y}e^{-y}\ \frac{\Gamma(2)}{\left(\frac{1}{y}\right)^2}\ dy=\int_0^\infty y\,e^{-y}\,dy\qquad \{f(y)=e^{-y}\sim\text{Exponential}(1)\,,E(y)=1\}$$

$$\boldsymbol{E(X)=1}$$

$$E(X^2) = \int_0^\infty \int_0^\infty \frac{x^2}{y} e^{-y} e^{-\frac{x}{y}}\, dx\, dy = \int_0^\infty \frac{1}{y} e^{-y} \left[\int_0^\infty x^2 e^{-\frac{x}{y}}\, dx \right] dy$$

$$= \int_0^\infty \frac{1}{y} e^{-y} \frac{\Gamma(3)}{\left(\frac{1}{y}\right)^3}\, dy = \int_0^\infty 2y^2\, e^{-y}\ \ dy = \frac{2\Gamma(3)}{(1)^3} = \mathbf{4}$$

$$E(Y) = \int_0^\infty \int_0^\infty \frac{y}{y} e^{-y} e^{-\frac{x}{y}}\, dx\, dy = \int_0^\infty e^{-y} \left[\int_0^\infty -y \left(-\frac{1}{y}\right) e^{-\frac{x}{y}}\, dx \right] dy$$

$$= \int_0^\infty y\, e^{-y} \left[-\, e^{-\frac{x}{y}} \right]_0^\infty\ dy = \int_0^\infty y\, e^{-y}\ \ dy = \ \mathbf{1}$$

$$E(Y^2) = \int_0^\infty \int_0^\infty \frac{y^2}{y} e^{-y} e^{-\frac{x}{y}}\, dx\, dy = \int_0^\infty y e^{-y} \left[\int_0^\infty -y \left(-\frac{1}{y}\right) e^{-\frac{x}{y}}\, dx \right] dy$$

$$= \int_0^\infty y^2\, e^{-y} \left[-\, e^{-\frac{x}{y}} \right]_0^\infty\ dy = \int_0^\infty y^2\, e^{-y}\ \ dy = \Gamma(3) = \ \mathbf{2}$$

$$E(XY) = \int_0^\infty \int_0^\infty \frac{xy}{y} e^{-y} e^{-\frac{x}{y}}\, dx\, dy = \int_0^\infty e^{-y} \int_0^\infty x\, e^{-\frac{x}{y}}\, dx\, dy$$

$$= \int_0^\infty e^{-y} \frac{\Gamma(2)}{\left(\frac{1}{y}\right)^2} dy = \int_0^\infty y^2\, e^{-y}\ \ dy = \Gamma(3) = \mathbf{2}$$

$$COV(X, Y) = E(XY) - E(X)E(Y) = 2 - (1)(1) = 1$$

Note: