The Fourier Series

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1 The Basis Functions

We will consider the following differential equation

$$u'' = \lambda u$$

Where we consider boundary conditions as given

$$u(0) = 0$$
$$u(L) = 0$$

These boundary conditions force λ to be negative. If it was zero, we would arrive at a null solution. This is because u would be a linear function, a linear function cannot satisfy the given boundary conditions. The same is true if λ is positive, we obtain real exponential solutions which also cannot satisfy the boundary conditions. Thus, λ has to be negative. Following conventions of standard ODE solving procedure, we will also let it be a square number. Hence,

$$u'' = -k^2 u$$

Where k is a constant. We find the solutions to be

$$\psi_n = \sin(kx) = \sin\left(\frac{n\pi x}{L}\right)$$

Notice how I have defined a new function, ψ_n. This is a set of basis functions. Any oscillatory solution to the initial ODE may be defined as a linear combination of ψ_n. Hence

$$u = \sum_{n=0}^{\infty} a_n \psi_n$$

2 The Fourier Series

But what is a_n? We will exploit the orthogonality of the sine function, using the inner product. Recall the orthogonality relations

$$\left\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right) \right\rangle = \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}\delta_{mn}$$

The inner product will be taken with a separately indexed sine function, in order to obtain a_n.

$$\left\langle u\,,\,\sin\left(\frac{m\pi x}{L}\right)\right\rangle = \sum_{n=0}^{\infty} a_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=0}^{\infty} a_n \frac{L}{2}\delta_{nm} = a_m \frac{L}{2}$$

So, we now have an expression explicitly for an indexed a. (We will now relabel all indices back to n to remove confusion.)

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{1}$$

$$f(x) = \sum_{n=0}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \tag{2}$$

One may begin to see the framework of a Fourier series coming together, we see that the function $f(x)$ defines itself, i.e. the Fourier series is an exact definition for $f(x)$. We can see logically now that $f(x)$ may be any well-behaved function, an the above relationship holds.