Practice 1 Answers

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1. Flux is given as
$$\Phi = G \cdot A = GA\cos(\theta)$$
A is a normal vector to the surface, so G and A are parallel, so the flux is GA. Hence,
$$\Phi = \frac{GM}{R^2} \cdot 4\pi R^2$$
Notice the radius cancels each time, so the calculation will be the same each time. This shows that the total flux remains the same through each surface, but is distributed over a greater surface each time, one that increases by a factor of R^2 each time. Therefore, we can see that the force strength decreases by a factor of R^2 each step.

2. I'd integrate across polar coordinates, hence
$$M = \int_0^{2\pi}\int_0^1 r\left((r\cos(\theta)+1)^2 + (r\sin(\theta)+1)^2\right) dr d\theta = \frac{5\pi}{2}$$
Then For the center of mass, we have
$$\bar{q} = \frac{\int q dm}{\int dm}$$
where, in this case $q = r, \theta$ and $\int dm = M$ so
$$\bar{r} = \frac{1}{M}\int_0^{2\pi}\int_0^1 r^2\left((r\cos(\theta)+1)^2 + (r\sin(\theta)+1)^2\right) dr d\theta = \frac{1}{M}\frac{26\pi}{15} = \frac{52}{75}$$
and
$$\bar{\theta} = \frac{1}{M}\int_0^{2\pi}\int_0^1 r\theta\left((r\cos(\theta)+1)^2 + (r\sin(\theta)+1)^2\right) dr d\theta = \frac{1}{15}(15\pi - 8)$$
so the coordinates for the center of mass are
$$(\bar{r}, \bar{\theta}) = \left(\frac{52}{75}, \frac{1}{15}(15\pi - 8)\right)$$

3. The sum is
$$\bar{q} = \frac{\sum q_i m_i}{\sum m_i}$$
for an arbitrary coordinate q. By interpreting the mass of each point as a frequency of sorts, and the coordinate of each point as the data, we can see the parallel of the arithmetic mean
$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$
We can generalise the mass of discrete set of points to a continuous integral form
$$\sum m_i = \int \rho dV$$
where V is the volume for which the object occupies. Using this, and knowing the expected value of a density function is
$$\int x\rho(x)dx$$
we can see that the COM calculation can be generalised to
$$\bar{q} = \frac{\int q\rho dV}{\int \rho dV}$$

4. We will integrate across a circle, without considering r as a variable, to get merely the outer ring of a circle. Where the inertia of a circle would usually be
$$I = \int_0^{2\pi} \int_0^R \rho_0 r^3 dr d\theta$$
we treat r as constant. Hence,
$$I = \int_0^{2\pi} \rho_0 r^3 d\theta = \rho_0 2\pi R^3$$
To obtain the mass, let's integrate
$$I = \int_0^{2\pi} \rho_0 r d\theta = \rho_0 2\pi r$$
So we obtain an inertia of
$$I = MR^2$$
Extra: Consider $L = r \times p$ we can consider $\dot{z}$ as the linear velocity. Hence,
$$\frac{dz}{dt} = \dot{r}e^{i\theta} + ir\dot{\theta}e^{i\theta}$$

The radial rope was described as being non extensible, so $\dot{r} = 0$, hence,

$$\dot{z} = v = ir\dot{\theta}e^{i\theta}$$

Therefore, momentum can be found to be $p = mv = imr\dot{\theta}e^{i\theta}$ Given the above expression for angular momentum, we can find

$$L = r \times p = rp\sin(90) = \mathbf{rp} = \dot{\theta}mr^2$$

Since $L = I\omega$, we can note that $\dot{\theta} = \omega$ so the inertia must be

$$I = mr^2$$

This is the same as our prior derivation. One may imagine the average mass of a rotating point of constant velocity as smeared out over the circumference of a circle, hence, the inertia is identical.

5. Angular momentum will remain conserved unless acted upon, like linear momentum. Consider $L = I\omega$ Given the mass is present at the center, on the axis of rotation, no change to the inertia is observed. Hence, ω remains constant.

6. Given that

$$KE = \frac{hc}{\lambda} - W_0 = 6.6 \times 10^{-16} J$$

We can be safe that a particle's motion is non-relativistic if its kinetic energy is less than 1% its rest energy. Let's test it

$$\frac{6.6 \times 10^{-16}}{m_e c^2} \times 100 \approx 0.8\%$$

So we can safely take a classical approach. Through the classical kinetic energy,

$$KE = \frac{1}{2}mv^2$$

We can find $v = 3.8 \times 10^7 \text{ms}^{-1}$

7. We have a square area defined as $A = a^2$, the square grows uniformly as sides move away at speed v, so rate of change is

$$\frac{dA}{dt} = 2av$$

The flux is given as

$$\Phi = B \cdot A = \mathbf{BA}$$

since the vectors are parallel. So, the change in flux is therefore

$$\frac{d\Phi}{dt} = 2avB$$

The induced emf, ε, is an induced voltage, and is equal to

$$\varepsilon = -\frac{d\Phi}{dt}$$

Recalling that $V = IR$, we can find a current as

$$I = \frac{V}{R} = -\frac{2avB}{R}$$

where R is an arbitrary resistance of the wire. The current flows clockwise, due to the negative sign. This is because the current will generate its own magnetic field as it flows, and that magnetic field (due to the right hand rule) will act in the opposite direction (antiparallel) to the initial linear magnetic field that caused it.

8. if $y = e^x$ then $\ln(y) = x$, the derivative is

$$\frac{1}{y}\frac{dy}{dx} = 1$$

so

$$\frac{dy}{dx} = y = e^x$$

9. To compute eigenvalues, we consider $\det(\mathbf{L} - \lambda\mathbf{I}) = 0$ Hence,

$$\begin{pmatrix} \cosh(\phi) - \lambda & -\sinh(\phi) \\ -\sinh(\phi) & \cosh(\phi) - \lambda \end{pmatrix}$$

The determinant can be found to be

$$(\cosh(\phi) - \lambda)^2 - \sinh^2(\phi) = 0$$

$$\cosh(\phi) + \lambda = \pm\sinh(\phi)$$

We therefore find that the eigenvalues are, through trig identities,

$$\lambda_1 = -e^x \text{ and } \lambda_2 = e^{-x}$$

We end up with two simultaneous equations as

$$A(\cosh(\phi) + e^x) - B\sinh(\phi) = 0$$

and

$$B(\cosh(\phi) + e^x) - A\sinh(\phi) = 0$$

Which, given that A and B are components of the eigenvector, implies that this particular eigenvector is

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We can consider light in an abstract sense to progress through space at the same rate as time, given the 45 degree nature of its depiction on a typical spacetime diagram, this vector spans that same direction and thus represents light on a spacetime diagram. This is reinforcing of the postulate that items at light speed to no shift speed under transformation, seen by the fact that this vector is invariant under all Lorentz transformation.