

A more general categorical framework for congruence of applicative bisimilarity

Tom Hirschowitz 

Univ. Savoie Mont Blanc, CNRS, LAMA,
73000, Chambéry, France

Ambroise Lafont 

University of Cambridge, United Kingdom

Abstract

We prove a general congruence result for bisimilarity in higher-order languages, which generalises previous work [7, 18] to languages specified by a labelled transition system in which programs may occur as labels, and which may rely on operations on terms other than capture-avoiding substitution. This is typically the case for PCF, λ -calculus with delimited continuations, and early-style bisimilarity in higher-order process calculi.

2012 ACM Subject Classification Theory of computation \rightarrow Denotational semantics; Theory of computation \rightarrow Operational semantics; Theory of computation \rightarrow Process calculi; Theory of computation \rightarrow Categorical semantics

Keywords and phrases applicative bisimilarity, higher-order languages, congruence, category theory

Digital Object Identifier 10.4230/LIPIcs.FSCD.2023.23

1 Introduction

General congruence results for bisimilarity based on category theory date back at least to Turi and Plotkin’s seminal paper [32], which covers labelled transition systems in a categorical version of the Positive GSOS format [6]. The result was then extended to languages with variable binding and renaming like the π -calculus [12, 31]. More recently, Borthelle et al. [7, 18] managed to deal with a wider class of languages, whose operational semantics may rely not only on renaming, but also on capture-avoiding substitution.

However, their result fails to cover significant languages to which Howe’s method has been adapted, such as (variants of) PCF [17], λ -calculus with delimited continuations [9, 5], or (early-style) higher-order process calculi [30, 23]. The reason Borthelle et al.’s framework does not cover such applications is that they are specified by labelled transition systems

- in which programs may occur as labels, or
- which rely on operations on terms other than capture-avoiding substitution.

In this paper, we extend Borthelle et al.’s result to such languages, which requires a non-trivial extension of the proof method, essentially abstracting over ideas from Bernstein [4].

We introduce **algebraic transition systems**, which model transition systems whose vertices (=states) bear some algebraic structure, and which may have arbitrary vertices as labels. For such transition systems, we define **enhanced bisimilarity** as an abstract counterpart to applicative bisimilarity

We introduce **operational signatures**, which allow us to generate algebraic transition systems of interest, including all above-mentioned languages. Following initial-algebra semantics [15], an operational signature specifies algebraic structure and transition rules, and, in applications, the initial object in the category of models of an operational signature is the desired syntactic transition system.

Finally, we prove (Theorem 52) that, under suitable conditions, enhanced bisimilarity in the algebraic transition system generated by an operational signature is a congruence for the



© Tom Hirschowitz and Ambroise Lafont;
licensed under Creative Commons License CC-BY 4.0

FSCD 2023.

Editors: John Q. Open and Joan R. Access; Article No. 23; pp. 23:1–23:42

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

considered algebraic structure. We also exhibit (Theorem 61) sufficient conditions that are easier to check in practice. This covers all above-mentioned applications, except higher-order process calculi, whose operational signatures fail to satisfy the required conditions.

Related work

Beyond Borthelle et al. [18], which was discussed above, the most closely related work is Goncharov et al.’s [16] bialgebraic framework for higher-order operational semantics. They upgrade Turi and Plotkin’s [32] original presentation of operational semantics as a natural transformation into a dinatural transformation, which allows them to cover transitions with programs as labels. Their main applications are strong variants of applicative bisimilarity for pure λ -calculus (call-by-name and call-by-value). In its current state, their framework cannot handle non-deterministic computation, hence in particular weak variants of bisimilarity.

Plan

We start in §2 with an overview of the development. In §3, we then present our running example, which will be used as the basis of our abstraction process. We then introduce our abstract notions of transition systems (§4), and algebraic transition systems (§5), together with bisimilarity and its enhanced variant. Finally, we introduce operational signatures and state our main results in §6, and conclude in §7.

Prerequisites and notations

We assume some basic knowledge of category theory [24], notably including factorisation systems and monad distributive laws [3]. Additionally, we rely in places on locally presentable categories [1], but this may be taken as technical, and ignored on a first reading. We often conflate natural numbers n with sets $\{1, \dots, n\}$. We denote by \mathfrak{n} the corresponding ordinal viewed as a category, so that, e.g., \mathcal{C}^2 is the usual category of morphisms in \mathcal{C} . We let **CAT** denote the category of locally small categories. Moreover, we denote by $\widehat{\mathbb{C}}$ the category of (contravariant) presheaves over a given category \mathbb{C} , and by $\mathbf{y}: \mathbb{C} \rightarrow \widehat{\mathbb{C}}$ the Yoneda embedding. Furthermore, we recall that endofunctor algebras differ from monad algebras. (A monad algebra structure must be suitably compatible with unit and multiplication.) We write $F\text{-}\mathbf{alg}$ for endofunctor algebras, and $T\text{-}\mathbf{Alg}$ for monad algebras (capital ‘A’!). Finally, for any endofunctor F on a sufficiently nice category, e.g., a presheaf category, we write F^* for the free monad on F , which is furthermore **algebraically free** in the sense that $F\text{-}\mathbf{alg} \cong F^*\text{-}\mathbf{Alg}$.

2 Overview

The development roughly follows [18]. We summarise it here, emphasising the differences. Our running example throughout is a pure λ -calculus with delimited continuations [5].

2.1 Transition systems

Let us first sketch our notion of transition system, starting from graphs. Consider the diagonal functor $\Delta: \mathbf{Set} \rightarrow \mathbf{Set}$, defined by $\Delta(X) = X^2$. A graph consists of two sets E and V , equipped with two maps $E \rightarrow V$, or equivalently a map $E \rightarrow \Delta(V)$.

Hirschowitz and Lafont [18] propose a “typed” generalisation: they postulate a category \mathbf{VT} of **vertex types**, a category \mathbf{ET} of **edge types**, and two functors $\mathbf{s}, \mathbf{t}: \mathbf{ET} \rightarrow \mathbf{VT}$ associating to each edge type the types of its source and target. A transition system in

their sense consists of a **vertex object** V in $\widehat{\mathbb{V}\mathbb{T}}$, a **edge object** $E \in \widehat{\mathbb{E}\mathbb{T}}$, and a morphism $E \rightarrow \Delta(V)$, where $\Delta: \widehat{\mathbb{V}\mathbb{T}} \rightarrow \widehat{\mathbb{E}\mathbb{T}}$ maps any V to $\Delta(V)(\alpha) = V(\mathbf{s}(\alpha)) \times V(\mathbf{t}(\alpha))$, for all $\alpha \in \mathbb{E}\mathbb{T}$. Taking $\mathbb{V}\mathbb{T} = \mathbb{E}\mathbb{T} = 1$, one recovers plain graphs.

In this paper, in order to account for labels, we generalise this by adding a functor \mathbf{l} associating to each edge type $\alpha \in \mathbb{E}\mathbb{T}$ a sequence $\mathbf{l}(\alpha) = (\mathbf{l}_1^\alpha, \dots, \mathbf{l}_{n_\alpha}^\alpha)$ of vertex types. A tuple $(\mathbb{V}\mathbb{T}, \mathbb{E}\mathbb{T}, \mathbf{s}, \mathbf{t}, \mathbf{l})$ is called a **Howe context**. Let us fix one for the rest of this section.

We modify Δ accordingly, defining it by

$$\Delta(V)(\alpha) = V(\mathbf{s}(\alpha)) \times \left(\prod_{i=1}^{n_\alpha} V(\mathbf{l}_i^\alpha) \right) \times V(\mathbf{t}(\alpha)).$$

A transition system again consists of objects $V \in \widehat{\mathbb{V}\mathbb{T}}$ and $E \in \widehat{\mathbb{E}\mathbb{T}}$, together with a morphism $E \rightarrow \Delta(V)$, which means that, to each edge, we associate a source, a target, and a sequence of labels of suitable types. For such transition systems, we define a generalisation of bisimulation, straightforwardly.

2.2 Algebraic transition systems and enhanced bisimilarity

Let us now briefly explain the notion of algebraic structure that we adopt. Following Fiore et al. [11, 13], Borthelle et al. [7, 18] use Σ -monoids, which are designed to model syntax with substitution. In this paper, relying on Hirschowitz and Lafont [19], we adopt a different notion of algebraic structure designed to cover syntax with more general additional operations.

► **Definition 1.** An *enhanced syntax* (on $\widehat{\mathbb{V}\mathbb{T}}$) consists of

- finitary functors $\Sigma: \widehat{\mathbb{V}\mathbb{T}} \rightarrow \widehat{\mathbb{V}\mathbb{T}}$ and $\Gamma: \widehat{\mathbb{V}\mathbb{T}}^2 \rightarrow \widehat{\mathbb{V}\mathbb{T}}$ such that Γ is **left-cocontinuous**, i.e., cocontinuous in its first argument, equipped with
- a distributive law $\delta: T \circ S \rightarrow S \circ T$, where $S = \Sigma^*$ denotes the monad freely generated by Σ and $T = \Gamma_S^*$ the one generated by $X \mapsto \Gamma(X, S(X))$.

Here, Σ models basic syntax, and Γ models additional operations like substitution. The fact that Γ is a bifunctor is for distinguishing a “main” occurrence in its arity, which is used below in the definition of enhanced bisimilarity. The distributive law models commutation of additional operations with basic ones, at the main occurrence (typically $(M N)[\sigma] = M[\sigma] N[\sigma]$).

Following initial-algebra semantics [15], the main object of interest here is the initial Σ -algebra $S(\emptyset)$, and the main point is that it automatically possesses T -algebra structure, given by the composite $T(S(\emptyset)) \xrightarrow{\delta_\emptyset} S(T(\emptyset)) \cong S(\emptyset)$ (the initial object is a T -algebra by cocontinuity, hence $T(\emptyset) \cong \emptyset$). This algebra structure in fact makes $S(\emptyset)$ into an initial algebra for the composite monad ST .

Fixing some enhanced syntax $\sigma = (\Sigma, \Gamma, \delta)$, for us, an algebraic transition system is thus a transition system $E \rightarrow \Delta(V)$, equipped with ST -algebra structure on V . We call such transition systems **σ -algebraic**.

Finally, for any σ -algebraic transition system $G = (E, V, \partial)$, we define **enhanced bisimilarity**, denoted by \sim_G^σ , as the greatest bisimulation R which is **enhanced**, in the sense that $\Gamma(R, V) \subseteq R$ – this is where we use the fact that Γ is a bifunctor. In concrete instances, as noticed by Borthelle et al. [7, 18], enhanced bisimilarity agrees with applicative bisimilarity.

The goal is then to prove that, in algebraic transition systems G of interest, enhanced bisimilarity \sim_G^σ is a **congruence**, i.e., $\Sigma(\sim_G^\sigma) \subseteq \sim_G^\sigma$.

2.3 Operational signatures

For this, we restrict attention to algebraic transition systems generated by a suitable notion of **operational signature**, which we now describe. Operational signatures comprise two components, one for generating an enhanced syntax, the other for specifying transition rules.

► **Definition 2.** A *syntactic signature* is an endofunctor Σ equipped with a sequence

$$T_0 = \text{id} \xrightarrow{(\Gamma_1, d_1)} T_1 \quad \dots \quad T_{n-1} \xrightarrow{(\Gamma_n, d_n)} T_n \quad (1)$$

of **incremental structural laws** [19]. An incremental structural law $T \rightarrow T'$ consists of a finitary, left-cocontinuous bifunctor $\Gamma: \widehat{\mathbb{T}}^2 \rightarrow \widehat{\mathbb{T}}$, together with a natural transformation $d_{X,Y}: \Gamma(\Sigma(X), Y) \rightarrow S(T(\Gamma(X, S(T(Y)))) + X + Y)$, such that $T' = T \oplus \Gamma_S^*$, where \oplus denotes monad coproduct.

In examples, a natural transformation $d_{X,Y}$ amounts to a definition by structural recursion, where the first argument of Γ models the decreasing occurrence of the argument, and the second argument models other occurrences. Given any syntactic signature (1), the given incremental structural laws induce distributive laws $\delta_i: T_i \circ S \rightarrow S \circ T_i$, hence in particular $\delta_n: T_n \circ S \rightarrow S \circ T_n$, and furthermore we have $T_n = (\sum_i \Gamma_i)_S^*$. Thus, letting \mathbf{d} denote the given syntactic signature, the triple $\sigma(\mathbf{d}) = (\Sigma, \sum_i \Gamma_i, \delta_n)$ forms an enhanced syntax. As a bonus, one can show that algebras for the composite monad ST_n are equivalently objects equipped with suitably coherent algebra structure for Σ and each functor $X \mapsto \Gamma_i(X, X)$.

The next step is to specify the dynamics of algebraic transition systems of interest. This is done by introducing **dynamic signatures**. Roughly, a dynamic signature over an enhanced syntax σ is an endofunctor on σ -algebraic transition systems, which is required to preserve the vertex object and satisfy a suitable “structuralness” condition inspired by structural operational semantics [26]. Intuitively, a dynamic signature Σ_1 is a family of transition rules, and structuralness demands that, in each transition rule, the source of the conclusion has depth at most one.

Pursuing the analogy, σ -algebraic transition systems satisfying the rules are a special kind of Σ_1 -algebras which we call **vertical**. Verticality means that the algebra structure is trivial on vertices: this enforces that satisfying the rules is only about edges, not vertices.

Finally, an **operational signature** consists of a syntactic signature \mathbf{d} , and a dynamic signature Σ_1 on $\sigma(\mathbf{d})$. The real object of interest is here the initial vertical Σ_1 -algebra, say $\mathbf{Z} = (E_{\mathbf{Z}}, V_{\mathbf{Z}}, \partial_{\mathbf{Z}})$, which in applications is the desired syntactic transition system.

2.4 Congruence of enhanced bisimilarity

Our goal is then to prove that, under suitable hypotheses, enhanced bisimilarity $\sim_{\mathbf{Z}}^{\sigma(\mathbf{d})}$ in the initial vertical Σ_1 -algebra is a congruence. For this, abstracting over Bernstein’s [4] proof, we start by defining **flexible bisimulation**, a variant of Sangiorgi’s BA-bisimulation [29]. Flexible bisimulation is like plain bisimulation: given related elements e and e' , any transition from e should be matched by some transition from e' . The difference is that, instead of having the same label, the matching transition should exist for any related label. Defining **functional flexible bisimulations** to be morphisms of algebraic transition systems whose graph is a flexible bisimulation, our main result (Theorem 52) states that if the dynamic signature Σ_1 preserves functional flexible bisimulations, then $\sim_{\mathbf{Z}}^{\sigma(\mathbf{d})}$ is a congruence.

Finally, preservation of functional flexible bisimulations is quite an abstract condition, so we set out to design a more concrete criterion for making the result easier to apply. In

fact, if the considered dynamic signature Σ_1 is **familial** [10, 8, 33, 14], then preservation of functional flexible bisimulations becomes quite tractable, as we now explain. Following Joyal et al. [22], we first characterise functional flexible bisimulations as the right class of a weak factorisation system [20, 28] – we call the left class **cofibrations**. Furthermore, when the dynamic signature is familial, a transition rule with conclusion of type any α , is intuitively an element of $\Sigma_1(1)(\alpha)$, and we extract for each rule two algebraic transition systems A and B , and a morphism $\varphi: A \rightarrow B$, such that, intuitively, A describes the metavariables occurring in the source and label of the conclusion, B describes all metavariables in the rule, including transition premises, and φ embeds the former into the latter. We call φ the **border arity** of the rule. The main point is then that a familial Σ_1 preserves functional flexible bisimulations iff all border arities are cofibrations (Theorem 61). How is this any more concrete? Well, cofibrations are well-known to be closed under composition and cobase change, so in order to check preservation of functional flexible bisimulations, it suffices to reconstruct the border arity of each rule from generating cofibrations, by composition and cobase change. This reconstruction process is close in spirit to usual acyclicity criteria [21, 4].

► **Example 3.** Taking algebraic transition systems to be just plain graphs, for a rule like $\frac{a \rightarrow b \quad b \rightarrow c}{a \rightarrow c}$, A would be the one-vertex graph, B would consist of two composable edges $x \rightarrow y \rightarrow z$, and φ would pick x . To check that it is a cofibration, we reconstruct it as the bottom composite in

$$A = [0] \xrightarrow{s} [1] \xrightarrow{\quad} B.$$

$\begin{array}{ccc} [0] & \xrightarrow{s} & [1] \\ \downarrow & & \downarrow \\ [0] & \xrightarrow{s} & [1] \end{array}$

As an application, we recover congruence of applicative bisimilarity in the considered λ -calculus with delimited continuations [5].

3 A concrete example

As a concrete example result that we want to abstract over, let us recall the case of λ -calculus with delimited continuations. We present it in a non-standard way in order for it to fit the abstract framework. Indeed, the framework is based on **structural** operational semantics [26], in the sense that, in each transition rule, the source of the conclusion has depth at most one. Following [7, 18], we also present the definition of the open extension of applicative bisimilarity to make it compatible with the abstract developments to come.

The syntax, presented in the usual, informal way, is as below left,

$$\text{Values } \ni v ::= x \mid \lambda x.e \qquad \Box[e] = e \qquad (2)$$

$$\text{Programs } \ni e ::= v \mid e_1 \ e_2 \mid \mathcal{S}x.e \mid \langle e \rangle \qquad (v \ E)[e] = v \ E[e] \qquad (3)$$

$$\text{Evaluation contexts } \ni E ::= \Box \mid E \ e \mid v \ E \qquad (E \ e')[e] = E[e] \ e'. \qquad (4)$$

where x binds in e , in both $\lambda x.e$ and $\mathcal{S}x.e$. Capture-avoiding substitution and context application are defined as usual. E.g., context application is defined as above right. The dynamics are governed by the rules in Figure 1. There are three kinds of transitions, of types $e \xrightarrow{\tau} e'$, $e \xrightarrow{v} e'$, $e \xrightarrow{E} e'$, where all expressions are closed. The first four rules deal with functions. The first two rules suffice to make (β) derivable, as shown in Figure 1. The next two rules are the usual context rules. The last three rules, where α ranges over all labels, enforce that we work with weak bisimulation: we close transitions under composition with silent transitions. The remaining rules describe the dynamics of $\mathcal{S}x.e$ and $\langle e \rangle$, which are

$$\begin{array}{c}
 \frac{e_1 \xrightarrow{v} e_2}{e_1 \ v \xrightarrow{\tau} e_2} (\beta') \quad \frac{}{\lambda x.e \xrightarrow{v} e[x \mapsto v]} \quad \frac{e_1 \xrightarrow{\tau} e'_1}{e_1 \ e_2 \xrightarrow{\tau} e'_1 \ e_2} \quad \frac{e_2 \xrightarrow{\tau} e'_2}{v \ e_2 \xrightarrow{\tau} v \ e'_2} \quad \frac{}{\langle v \rangle \xrightarrow{\tau} v} \\
 \\
 \frac{e \xrightarrow{\tau} e'}{\langle e \rangle \xrightarrow{\tau} \langle e' \rangle} \quad \frac{e \xrightarrow{\square} e'}{\langle e \rangle \xrightarrow{\tau} e'} \quad \frac{e_1 \xrightarrow{E[\square] e_2} e_3}{e_1 \ e_2 \xrightarrow{E} e_3} \quad \frac{e_1 \xrightarrow{E[v \ \square]} e_2}{v \ e_1 \xrightarrow{E} e_2} \quad \frac{}{\mathcal{S}k.e \xrightarrow{E} \langle e[k \mapsto \lambda x.\langle E[x] \rangle] \rangle} \quad \text{(SA)} \\
 \\
 \frac{}{e \xrightarrow{\tau} e} \quad \frac{e_1 \xrightarrow{\tau} e_2 \xrightarrow{\alpha} e_3}{e_1 \xrightarrow{\alpha} e_3} \quad \frac{e_1 \xrightarrow{\alpha} e_2 \xrightarrow{\tau} e_3}{e_1 \xrightarrow{\alpha} e_3} \quad \text{Deriving } \beta: \quad \frac{\lambda x.e \xrightarrow{v} e[x \mapsto v]}{(\lambda x.e) \ v \xrightarrow{\tau} e[x \mapsto v]} (\beta')
 \end{array}$$

■ **Figure 1** Transition rules

208 respectively called **shift** and **reset**. The first two of them enforce that silent computation
 209 occurs normally inside any reset, and if it succeeds, i.e., if it results in a value, then the
 210 reset disappears. The next rules describe how shift captures the ambient context up to the
 211 enclosing reset, say E , and substitutes its reification $\lambda k.\langle E[k] \rangle$ as a value for the bound
 212 variable, placing a new reset around the result.

213 Bisimulation is then as expected:

214 ► **Definition 4.** A binary relation R between closed programs is a **simulation** iff for all
 215 $e \ R \ e'$ and transitions $e \xrightarrow{\alpha} e_1$, there exists a transition $e' \xrightarrow{\alpha} e'_1$ such that $e_1 \ R \ e'_1$. A
 216 **bisimulation** is a simulation whose converse relation also is a simulation.

217 ► **Definition 5** ([7, 18]). A relation R on potentially open expressions is **enhanced** iff it is
 218 closed under substitution, context composition, and context application, i.e., a $R \ a'$ entails
 219 $a[\sigma] \ R \ a'[\sigma]$ for all substitutions σ , $E \ R \ E'$ entails $E[e] \ R \ E'[e]$ and $E[E''] \ R \ E'[E'']$, for
 220 all e and E'' .

221 An **enhanced bisimulation** is an enhanced relation R whose restriction to closed
 222 programs is a bisimulation.

223 ► **Proposition 6.** There is a largest enhanced bisimulation, called **applicative bisimilarity**.

224 The result that we want to abstract over is:

225 ► **Theorem 7** (generalised variant of [5, Theorem 1]). Applicative bisimilarity is a **congruence**,
 226 in the sense that it is preserved by all constructions of the language.

► **Remark 8.** It is not entirely trivial that this agrees with Biernacki and Lenglet's presentation.
 In fact, their transition system only differs in that they replace rule (β') with the standard rule
 (β) . We have already seen that (β) is derivable from (β') , and conversely (β') is admissible in
 their transition system. Indeed, suppose given any transition $e_1 \xrightarrow{v} e_2$. By an easy induction,
 there exist transitions $e_1 \xrightarrow{\tau} \lambda x.e_3 \xrightarrow{v} e_3[x \mapsto v] \xrightarrow{\tau} e_2$. Hence, grouping saturation rules, we
 derive (β) as follows.

$$\frac{\frac{e_1 \xrightarrow{\tau} \lambda x.e_3}{e_1 \ v \xrightarrow{\tau} (\lambda x.e_3) \ v} \quad (\lambda x.e_3) \ v \xrightarrow{\tau} e_3[x \mapsto v] \quad e_3[x \mapsto v] \xrightarrow{\tau} e_2}{e_1 \ v \xrightarrow{\tau} e_2}$$

Our problem is that this result is not an instance of Borthelle et al.'s [18, Theorem 6.15], because the dynamics rely on two features that are not handled:

- (a) operations on terms, context application and composition, which differ from substitution,
- (b) and contexts and values occurring as labels.

For (a), context application and composition might be encodable in Borthelle et al.'s setting, perhaps by resorting to the skew monoidal variant [7]. But this is quite artificial, and requires extra work that should not be necessary. For (b), it appears to be a hard obstruction.

4 Transition systems in the abstract

In this section, we start to abstract over the development of §3, by introducing a notion of labelled transition system, together with its associated notion of bisimilarity.

4.1 Howe contexts

Let us start by formally introducing Howe contexts, as sketched in §2.

► **Definition 9.** A *Howe context* consists of

- a small category $\mathbb{V}\mathbb{T}$ of *state types*,
- a small category $\mathbb{E}\mathbb{T}$ of *transition types*,
- *source* and *target* functors $\mathbf{s}, \mathbf{t}: \mathbb{E}\mathbb{T} \rightarrow \mathbb{V}\mathbb{T}$, and
- a *label* functor $\mathbf{l}: \mathbb{E}\mathbb{T} \rightarrow \widehat{\mathbb{V}\mathbb{T}}$, such that each $\mathbf{l}(c)$ is a finite coproduct of representables.

► **Example 10.** For plain graphs, we would take:

- $\mathbb{V}\mathbb{T}$ to be the terminal category, because there is just one kind of vertex,
- $\mathbb{E}\mathbb{T}$ to also be the terminal category, because there is just one kind of edge,
- the source and target functors both are the unique functor $1 \rightarrow 1$, and
- the label functor to map the unique object to the empty coproduct, i.e., \emptyset .

► **Example 11.** For modelling the transition system of §3, we need a presheaf on $\mathbb{V}\mathbb{T}$ to be equivalent to a triple of functors $V_p, V_v, V_c: \mathbb{F} \rightarrow \mathbf{Set}$, where \mathbb{F} denotes a skeleton of the category of finite sets, e.g., finite ordinals and all maps between them, equipped with a natural transformation $\iota: V_v \rightarrow V_p$, or otherwise said to a functor $\mathbb{F} \rightarrow \mathbf{Set}^{1+2}$. We think of $V_p(n)$, $V_v(n)$, and $V_c(n)$ as sets of programs, values, and contexts with n free variables, respectively. For making this into a presheaf category, let us first observe that such tuples (V_p, V_v, V_c, ι) are precisely the objects of the oplax limit of the functor $\Delta_{\mathbf{y}in_1}: \mathbb{F}^{op} + \mathbb{F}^{op} \rightarrow \widehat{\mathbb{F}^{op}}$ mapping any copairing $[V_p, V_c]$ to V_p . But, as we now recall, oplax limits of this form are equivalent to presheaf categories.

► **Definition 12.** For any small categories \mathbb{X} and \mathbb{Y} , and functor $F: \mathbb{X} \rightarrow \widehat{\mathbb{Y}}$, the *collage* of F , denoted by $\mathbb{Y}[\mathbb{X}]_F$, or merely $\mathbb{Y}[\mathbb{X}]$ when F is clear from context, has as objects the disjoint union of those of \mathbb{X} and \mathbb{Y} , and morphisms defined by cases as follows.

$$\begin{array}{lll} \mathbb{X}[\mathbb{Y}](x, x') & = & \mathbb{X}(x, x') & \mathbb{X}[\mathbb{Y}](y, x) & = & F(x)(y) \\ \mathbb{X}[\mathbb{Y}](y, y') & = & \mathbb{Y}(y, y') & \mathbb{X}[\mathbb{Y}](x, y) & = & \emptyset \end{array}$$

Composition is defined as in \mathbb{X} and \mathbb{Y} in both left-hand cases, and otherwise by action of F .

► **Proposition 13** ([8, Lemma 4.9]). For any small categories \mathbb{X} and \mathbb{Y} , and functor $F: \mathbb{X} \rightarrow \widehat{\mathbb{Y}}$, letting $\Delta_F(Y)(x) = \widehat{\mathbb{Y}}(F(x), Y)$ denote the induced *nerve* functor $\widehat{\mathbb{Y}} \rightarrow \widehat{\mathbb{X}}$, the oplax limit $\widehat{\mathbb{X}}/\Delta_F$ is equivalent to the category $\widehat{\mathbb{Y}[\mathbb{X}]}$ of presheaves on the collage of F .

Now, the above functor $\Delta_{\mathbf{y}in_1}$ is indeed the nerve of $\mathbb{F}^{op} \xrightarrow{in_1} \mathbb{F}^{op} + \mathbb{F}^{op} \xrightarrow{\mathbf{y}} \widehat{\mathbb{F}^{op} + \mathbb{F}^{op}}$.

► **Corollary 14.** *Letting $\mathbb{V}\mathbb{T} = (\mathbb{F}^{op} + \mathbb{F}^{op})[\mathbb{F}^{op}]_{\mathbf{y}in_1}$, we have $[\mathbb{F}, \mathbf{Set}^{1+2}] \simeq \widehat{\mathbb{V}\mathbb{T}}$.*

► **Notation 1.** *We denote objects in_1n , in_2n , and in_3n of $\mathbb{V}\mathbb{T}$ by $n_{\mathbf{v}}$, $n_{\mathbf{p}}$, $n_{\mathbf{c}}$, respectively, for values, programs, and contexts. For any $V \in \widehat{\mathbb{V}\mathbb{T}}$, we denote the corresponding functors $\mathbb{F} \rightarrow \mathbf{Set}$ by $V_{\mathbf{v}}$, $V_{\mathbf{p}}$, and $V_{\mathbf{c}}$, so that, e.g., $V(n_{\mathbf{v}}) = V_{\mathbf{v}}(n)$.*

Let us now define $\mathbb{E}\mathbb{T} = 3 = \{[\tau], [\mathbf{v}], [\mathbf{c}]\}$, where $[\alpha]$ indicates a label of type α . Accordingly, writing $c : a \xrightarrow{L} b$ for $\mathbf{s}(c) = a$, $\mathbf{l}(c) = L$, and $\mathbf{t}(c) = b$, and respectively interpreting τ , \mathbf{v} , and \mathbf{c} as \emptyset , $\mathbf{y}_{\mathbf{v}}$, and $\mathbf{y}_{\mathbf{c}}$, we put: $[\alpha] : 0_{\mathbf{p}} \xrightarrow{\alpha} 0_{\mathbf{p}}$, for all $\alpha \in \{\tau, \mathbf{v}, \mathbf{c}\}$.

4.2 Generalised transition systems

Let us now introduce transition systems. Let us fix a Howe context $\mathbb{H} = (\mathbb{V}\mathbb{T}, \mathbb{E}\mathbb{T}, \mathbf{s}, \mathbf{t}, \mathbf{l})$ for the whole subsection.

We start by defining define functors relating both presheaf categories $\widehat{\mathbb{V}\mathbb{T}}$ and $\widehat{\mathbb{E}\mathbb{T}}$.

► **Definition 15.** *We define four functors $\widehat{\mathbb{V}\mathbb{T}} \rightarrow \widehat{\mathbb{E}\mathbb{T}}$ as follows, for all $V \in \widehat{\mathbb{V}\mathbb{T}}$ and $\alpha \in \mathbb{E}\mathbb{T}$.*

$$\begin{array}{ll} \Delta_{\mathbf{s}}(V)(\alpha) &= V(\mathbf{s}(\alpha)) & \Delta_{\mathbf{l}}(V)(\alpha) &= \widehat{\mathbb{V}\mathbb{T}}(\mathbf{l}(\alpha), V) \\ \Delta_{\mathbf{t}}(V)(\alpha) &= V(\mathbf{t}(\alpha)) & \Delta_{\mathbb{H}}(V) &= \Delta_{\mathbf{s}}(V) \times \Delta_{\mathbf{l}}(V) \times \Delta_{\mathbf{t}}(V), \end{array}$$

► **Notation 2.** *We often abbreviate $\Delta_{\mathbb{H}}$ to Δ when \mathbb{H} is clear from context. We also use juxtaposition of indices to denote product of the corresponding functors, e.g., $\Delta_{\mathbf{s}, \mathbf{l}} := \Delta_{\mathbf{s}} \times \Delta_{\mathbf{l}}$.*

► **Definition 16.** *An \mathbb{H} -transition system G consists of a **vertex** presheaf $V_G \in \widehat{\mathbb{V}\mathbb{T}}$, an **edge** presheaf $E_G \in \widehat{\mathbb{E}\mathbb{T}}$, and a **border** natural transformation $\partial_G : E_G \rightarrow \Delta(V_G)$.*

► **Remark 17.** Letting $\mathbf{l}(\alpha) = \sum_{i \in n_{\alpha}} \mathbf{y}_{\mathbf{l}_i}^{\alpha}$, we have $\Delta_{\mathbf{l}}(V)(\alpha) = [\sum_{i \in n_{\alpha}} \mathbf{y}_{\mathbf{l}_i}^{\alpha}, V] \cong \prod_{i \in n_{\alpha}} V(\mathbf{l}_i^{\alpha})$ for any $\alpha \in \mathbb{E}\mathbb{T}$ and $V \in \widehat{\mathbb{V}\mathbb{T}}$. The border natural transformation thus has type

$$E(\alpha) \rightarrow V(\mathbf{s}(\alpha)) \times (\prod_{i \in n_{\alpha}} V(\mathbf{l}_i^{\alpha})) \times V(\mathbf{t}(\alpha)).$$

► **Example 18.** Let us unfold the definition for the Howe context of Example 11: a transition system consists of presheaves $V \in \widehat{\mathbb{V}\mathbb{T}}$ and $E \in \widehat{\mathbb{E}\mathbb{T}}$, equipped with maps

$$E[\tau] \rightarrow V_{\mathbf{p}}(0)^2 \quad E[\mathbf{v}] \rightarrow V_{\mathbf{p}}(0) \times V_{\mathbf{v}}(0) \times V_{\mathbf{p}}(0) \quad E[\mathbf{c}] \rightarrow V_{\mathbf{p}}(0) \times V_{\mathbf{c}}(0) \times V_{\mathbf{p}}(0).$$

We now equip \mathbb{H} -transition systems with morphisms:

► **Proposition 19.** *\mathbb{H} -transition systems are precisely the objects of the oplax limit category $\widehat{\mathbb{E}\mathbb{T}}/\Delta$ of the functor $\widehat{\mathbb{V}\mathbb{T}} \xrightarrow{\Delta} \widehat{\mathbb{E}\mathbb{T}}$ in \mathbf{CAT} , or equivalently the comma category $\text{id}_{\widehat{\mathbb{E}\mathbb{T}}} \downarrow \Delta$.*

Proof. An object of the oplax limit is by definition a triple $(E, V, \partial : E \rightarrow \Delta(V))$. ◀

► **Definition 20.** *Let $\mathbb{H}\text{-Trans} = \widehat{\mathbb{E}\mathbb{T}}/\Delta_{\mathbb{H}}$.*

4.3 Bisimulation and bisimilarity

We now want to define bisimulation and bisimilarity, for any fixed Howe context $\mathbb{H} = (\mathbb{V}\mathbb{T}, \mathbb{E}\mathbb{T}, \mathbf{s}, \mathbf{t}, \mathbf{l})$. Let us start with the notion of simulation.

► **Notation 3.** A **span** is a pair of morphisms with the same source. In a category with binary products, we often write spans $X \leftarrow R \rightarrow Y$ as their pairings $R \rightarrow X \times Y$. The **converse** of a span $\langle f, g \rangle: R \rightarrow X \times Y$ is the composite $\langle g, f \rangle: R \rightarrow X \times Y$.

In a presheaf category $\widehat{\mathbb{C}}$, for any span $j: R \rightarrow X \times Y$, object $c \in \mathbb{C}$, and element $r \in R(c)$, we write $r: x R y$ when $j_c(r) = (x, y)$. We call r a **witness** that x and y are related by R .

Finally, in any \mathbb{H} -transition system G , for any transition type α with $\mathbf{l}(\alpha) \cong \sum_{i \in n_\alpha} \mathbf{y}_i^\alpha$, we write $e: x \xrightarrow{\alpha(l_1, \dots, l_{n_\alpha})} y$ to mean that $e \in E_G(\alpha)$ and $\partial_G(e) = (x, (l_1, \dots, l_{n_\alpha}), y)$.

► **Definition 21.** For any \mathbb{H} -transition system $G = (V, E, \partial: E \rightarrow \Delta V)$, a given span $j: R \rightarrow V^2$ is a **simulation** when, for any transition $e: x \xrightarrow{\alpha(l_1, \dots, l_{n_\alpha})} x'$ and witness $r: x R y$, there exists a transition $f: y \xrightarrow{\alpha(l_1, \dots, l_{n_\alpha})} y'$ and a witness $r': x' R y'$, as in

$$\begin{array}{ccccc} & x & R(\mathbf{s}(\alpha)) & y & \\ & \downarrow e: \alpha(l_1, \dots, l_{n_\alpha}) & & \downarrow f: \alpha(l_1, \dots, l_{n_\alpha}) & \\ & x' & R(\mathbf{t}(\alpha)) & y' & \end{array} \quad (5)$$

A span is a **bisimulation** when it is a simulation and so is its converse. A **bisimulation relation** is a bisimulation which is also a relation, i.e., a mono $R \hookrightarrow V^2$.

► **Proposition 22.** The full subcategory $\mathbf{Bisim}(G)$ of $\widehat{\mathbb{V}\mathbb{T}}/V^2$ spanning bisimulations admits a terminal object, which we call **bisimilarity** and denote by \sim_G .

Proof. Bisimulation relations are stable under unions, so that a terminal object is given by the union of them all. ◀

5 Algebraic transition systems

In this section, we explain enhanced syntax, algebraic transition systems, and enhanced bisimulation in a bit more detail than in §2.2. The notion of enhanced syntax has already been introduced (Definition 1), and we fix a Howe context $\mathbb{H} = (\mathbb{V}\mathbb{T}, \mathbb{E}\mathbb{T}, \mathbf{s}, \mathbf{t}, \mathbf{l})$ and an enhanced syntax $\sigma = (\Sigma, \Gamma, \delta: TS \rightarrow ST)$, where, we recall, $S = \Sigma^*$ and $T = \Gamma_S^*$.

5.1 Enhanced syntax

► **Definition 23.** We call ST -algebras σ -**algebras** for short, and let $\sigma\text{-Alg} = ST\text{-Alg}$.

► **Proposition 24.** The initial Σ -algebra $S\emptyset$ is automatically a T -algebra, with structure map $TS\emptyset \xrightarrow{\delta_\emptyset} ST\emptyset \xrightarrow{\cong} S\emptyset$.

Proof. By cocontinuity, \emptyset is a Γ_S -algebra: we have $\Gamma(\emptyset, S(\emptyset)) \cong \emptyset$. It is thus an initial Γ_S -algebra, hence an initial T -algebra since $\Gamma_S\text{-alg} \cong T\text{-Alg}$. ◀

► **Example 25.** Following up on Example 11, the syntax and additional operations of §3 may be presented by an incremental structural law on $\widehat{\mathbb{V}\mathbb{T}}$, as follows. First, basic operations are specified by the endofunctor Σ_0 defined as follows (recalling original notation on the right).

$$\begin{array}{ll} \Sigma_0(X)_v(n) = n + X_p(n+1) & v ::= x \mid \lambda x.e \\ \Sigma_0(X)_p(n) = \Sigma_0(X)_v(n) + X_v(n) + X_p(n)^2 + X_p(n+1) + X_p(n) & e ::= v \mid e_1 e_2 \mid \mathcal{S}x.e \mid \langle e \rangle \\ \Sigma_0(X)_c(n) = 1 + X_v(n) \times X_c(n) + X_c(n) \times X_p(n) & E ::= \square \mid E e \mid v E \end{array}$$

We then want to define the arity of additional operations, namely substitution, context application, and context composition. Since these three additional operations are independent, we may specify them at once by the bifunctor $\Gamma: \widehat{\mathbb{V}\mathbb{T}}^2 \rightarrow \widehat{\mathbb{V}\mathbb{T}}$ defined as follows.

$$\begin{array}{lll}
 \Gamma(X, Y)_v(n) & = & \sum_{m \in \mathbb{N}} X_v(m) \times Y_v(n)^m & v + ::= v[\sigma] \\
 \Gamma(X, Y)_p(n) & = & \sum_{m \in \mathbb{N}} X_p(m) \times Y_v(n)^m + X_c(n) \times Y_p(n) & e + ::= e[\sigma] \mid E[e] \\
 \Gamma(X, Y)_c(n) & = & X_c(n) \times Y_c(n) & E + ::= E[E']
 \end{array}$$

That the actual definition of additional operations induces a distributive law of Γ_S^* over Σ^* is harder to see, and will follow from the theory of syntactic signatures below (Example 33).

5.2 Algebraic transition systems

Let us now introduce algebraic transition systems.

► **Definition 26.** A σ -*transition system* is an \mathbb{H} -transition systems equipped with σ -algebra structure on its vertex object. A σ -transition system morphism is a morphism of \mathbb{H} -transition systems whose vertex component is a σ -algebra morphism. Let σ -**Trans** denote the category of σ -transition systems and morphisms between them.

► **Proposition 27.** The forgetful functor \mathcal{U} has a left adjoint, say $\mathcal{L}: \mathbb{H}\text{-Trans} \rightarrow \sigma\text{-Trans}$.

Proof. The left adjoint maps any $\partial: E \rightarrow \Delta(V)$ to $E \xrightarrow{\partial} \Delta(V) \xrightarrow{\Delta(\eta_V^{ST})} \Delta(S(T(V)))$. ◀

We conclude this section by defining the notion of congruence.

► **Definition 28.** For any σ -transition system $G = (V, E, \partial)$, a *congruence* is a span $R \rightarrow V^2$ for which there exists a morphism $\Sigma(R) \rightarrow R$ making the first diagram of Figure 2 commute.

$$\begin{array}{ccc}
 \Sigma(R) & \xrightarrow{\quad\quad\quad} & R \\
 \downarrow & & \downarrow \\
 \Sigma(V^2) & \xrightarrow{\quad\quad\quad} & V^2 \\
 \langle \Sigma(\pi_1), \Sigma(\pi_2) \rangle & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Gamma(R, V) & \xrightarrow{\quad\quad\quad} & R \\
 \downarrow & & \downarrow \\
 \Gamma(V^2, V) & \xrightarrow{\quad\quad\quad} & V^2 \\
 \langle \Gamma(\pi_1, V), \Gamma(\pi_2, V) \rangle & &
 \end{array}$$

■ **Figure 2** Congruence and enhancement

5.3 Enhanced bisimilarity

► **Definition 29.** For any σ -algebra V , a span $p: R \rightarrow V^2$ is *enhanced* when there exists a morphism $\Gamma(R, V) \rightarrow R$ making the second diagram of Figure 2 commute.

► **Definition 30.** For any σ -transition system G , let $\mathbf{Bisim}^\sigma(G)$ denote the full subcategory of $\mathbf{Bisim}(G)$ on enhanced spans. We call such spans *enhanced bisimulations*.

► **Proposition 31.** For any σ -transition system G , $\mathbf{Bisim}^\sigma(G)$ admits a terminal object, which we call *enhanced bisimilarity* and denote by \sim_G^σ .

Proof. Similar to Proposition 22, using left-cocontinuity of Γ . ◀

► **Example 32.** In the setting of Example 25, enhanced bisimilarity is applicative bisimilarity.

6 Signatures for operational semantics

6.1 Syntactic signatures for enhanced syntax

Syntactic signatures have already been introduced in Definition 2.

► **Example 33.** Following up on Example 25, the syntax and additional operations of §3 may be presented as an incremental structural law $d_{X,Y}: \Gamma_Y(\Sigma(X)) \rightarrow S(\Gamma_{S(Y)}(X) + X + Y)$ (taking $T_1 = \text{id}$) on $\widehat{\mathbb{V}\mathbb{T}}$, as follows. For context application, Equations (2)–(4) may be interpreted as the component $\Sigma(X)_c(n) \times Y_p(n) \rightarrow S(\Gamma_{S(Y)}(X) + X + Y)_p(n)$, namely we take them to mean

$$\begin{aligned} (in_1(\star), y) &\mapsto in'_3(y) \\ (in_2(v, E), y) &\mapsto \iota(in'_2(v)) in'_1(E, y) \\ (in_3(E, e), y) &\mapsto in'_1(E, y) in'_2(e), \end{aligned}$$

where $in'_i = \eta^S \circ in_i$. For context composition, we define the component at \mathbf{c} (for any n), by the exact same formulas, only with $y \in Y_c(n)$. Substitution is defined similarly [11, 7, 18].

► **Proposition 34.** For any syntactic signature $\mathbf{d} = (\Sigma, (\Gamma_i, d_i)_{i \in n})$ as in (1), the given incremental structural laws induce distributive laws $\delta_i: T_i \circ S \rightarrow S \circ T_i$, hence in particular $\delta_n: T_n \circ S \rightarrow S \circ T_n$, and furthermore we have $T_n = (\sum_i \Gamma_i)_S^*$. Thus, the triple $\sigma(\mathbf{d}) = (\Sigma, \sum_i \Gamma_i, \delta_n)$ forms an enhanced syntax.

Proof. By [19, Theorem 4.2], each incremental structural law d_i induces a distributive law of $(T_{i-1} \oplus \Gamma_S^*)$ over S , i.e., of T_i over S by definition, using $(F + G)^* \cong F^* \oplus G^*$. ◀

Let us conclude this subsection by giving an explicit description of the algebras of the composite monad ST_n generated by a syntactic signature.

► **Definition 35.** Consider any syntactic signature $\mathbf{d} = (\Sigma, (\Gamma_i, d_i)_{i \in n})$. For $i \in n$, an **enhanced algebra** is an object equipped with algebra structures $a: \Sigma X \rightarrow X$, $b_1: \Gamma_1(X, X) \rightarrow X$, ..., $b_n: \Gamma_n(X, X) \rightarrow X$ such that for all $i \in n$ the following diagram commutes,

$$\begin{array}{ccc} \Gamma_i(\Sigma X, X) & \xrightarrow{(d_i)_{X,X}} ST_i(\Gamma_i(X, ST_i X) + X + X) & \xrightarrow{ST_i(\Gamma_i(X, \bar{a} \circ S \bar{a}_i) + [X, X])} ST_i(\Gamma_i(X, X) + X) \xrightarrow{ST_i[b_i, X]} ST_i X \\ \Gamma_i(a, X) \downarrow & & \downarrow \bar{a} \circ S \bar{a}_i \\ \Gamma_i(X, X) & \xrightarrow{\quad b_i \quad} & X \end{array}$$

where $\bar{a}_i: T_i X \rightarrow X$ and $\bar{a}: SX \rightarrow X$ denote the algebra structures induced by $(b_j)_{j < i}$, and a . Let $\mathbf{d}\text{-Alg}$ denote the full subcategory of $(\Sigma + \sum_{i \in n} \Gamma_i)\text{-alg}$ spanned by enhanced algebras.

► **Proposition 36.** Let $\mathbf{d} = (\Sigma, (\Gamma_i, d_i)_{i \in n})$ denote any syntactic signature. The forgetful functor $\sigma(\mathbf{d})\text{-Alg} \rightarrow (\Sigma + \sum_{i \in n} \Gamma_i)\text{-alg}$ lifts to $\mathbf{d}\text{-Alg}$, and the lifting is an isomorphism. In short, we have $\sigma(\mathbf{d})\text{-Alg} \cong \mathbf{d}\text{-Alg}$ over $\widehat{\mathbb{C}}$.

Proof. By induction on n and [19, Theorem 4.13]. ◀

6.2 Dynamic signatures

Let us now introduce signatures for the dynamical part of an operational semantics. We want a dynamic signature to be something like an endofunctor on $\sigma\text{-Trans}$, with built-in structuralness. For this, we introduce a variant of \mathbb{H} -transition systems called diplopic \mathbb{H} -transition systems, which feature an object of distinguished vertices, among which all sources of transitions must lie. This will enable structuralness, by allowing sources of conclusions of transition rules to have a distinguished head constructor. We fix an enhanced syntax $\sigma = (\Sigma, \Gamma, \delta: TS \rightarrow ST)$ for this subsection.

► **Definition 37.** A *diplopic \mathbb{H} -transition system* G consists of a *vertex* object $V_G \in \widehat{\mathbb{V}\mathbb{T}}$, a *distinguished vertex* object $D_G \in \widehat{\mathbb{V}\mathbb{T}}$, an *edge* object $E_G \in \widehat{\mathbb{E}\mathbb{T}}$, together with morphisms $\gamma_G: D_G \rightarrow V_G$ and $\partial_G: E_G \rightarrow \Delta_s(D_G) \times \Delta_{l,t}(V_G)$.

A *diplopic σ -transition system* is a diplopic \mathbb{H} -transition system G equipped with σ -algebra structure on V_G .

As before, we organise both notions into categories $\mathbb{H}\text{-}\mathbf{Trans}_2 = \widehat{\mathbb{E}\mathbb{T}}/\Delta_2$ and $\sigma\text{-}\mathbf{Trans}_2 = \mathbb{H}\text{-}\mathbf{Trans}_2 \times_{\widehat{\mathbb{V}\mathbb{T}}} \sigma\text{-}\mathbf{Alg}$, where Δ_2 denotes the composite $\widehat{\mathbb{V}\mathbb{T}}^2 \xrightarrow{\langle \pi_1, \pi_2, \pi_2 \rangle} \widehat{\mathbb{V}\mathbb{T}}^3 \xrightarrow{\Delta_s \times \Delta_l \times \Delta_t} \widehat{\mathbb{E}\mathbb{T}}$.

► **Definition 38.** A *dynamic signature* over σ is a functor $\Sigma_1: \sigma\text{-}\mathbf{Trans} \rightarrow \sigma\text{-}\mathbf{Trans}_2$ such that, for all $G \in \sigma\text{-}\mathbf{Trans}$, $V_{\Sigma_1(G)} = V_G$, $D_{\Sigma_1(G)} = V_G + \Sigma(V_G)$, and $\gamma_G: V_G + \Sigma(V_G) \rightarrow V_G$ is the canonical morphism (and similarly on morphisms).

► **Example 39.** Letting \mathbf{d} denote the syntactic signature of Example 33. The transition rules of §3 define a dynamic signature $\Sigma_1: \sigma(\mathbf{d})\text{-}\mathbf{Trans} \rightarrow \sigma(\mathbf{d})\text{-}\mathbf{Trans}_2$. Its behaviour on the underlying $\sigma(\mathbf{d})$ -algebra is fixed, so we merely need to define it on transitions. For any $G = (D, V, E, \partial) \in \sigma(\mathbf{d})\text{-}\mathbf{Trans}$ and $\alpha \in \mathbb{E}\mathbb{T}$, we define $\Sigma_1(G)(\alpha)$ to be a coproduct over all rules ρ producing a transition of type α , of a set describing the premises of ρ . One non-trivial rule is (SA), whose set of premises is $E[\mathbf{c}] \times_{V(0_c)} (V(0_c) \times V(0_v))$. Concretely, it is the set of tuples $(r, (E, v))$, where r is a transition $e_1 \xrightarrow{E'} e_2$, and the pullback condition imposes $E' = E[v \sqcap]$. (We take the pullback of $E[\mathbf{c}] \xrightarrow{\pi_2 \partial} V(0_c) \xleftarrow{E[v \sqcap] \hookrightarrow E, v} V(0_c) \times V(0_v)$.) We define the source of $(r, (E, v))$ to be $in_2(in_2(\iota(v), e_1)) \in V_p(0) + \Sigma_0(V)_p(0)$, (i.e., recalling Σ_0 from Example 25, the formal application $\iota(v)$ its label to be E , and its target to be e_2).

Returning to the abstract setting, let us now define the category of models of a dynamic signature Σ_1 . For this, we need to build an endofunctor out of Σ_1 , hence a link between $\sigma\text{-}\mathbf{Trans}$ and $\sigma\text{-}\mathbf{Trans}_2$.

► **Definition 40.** Let $\iota\text{-}\mathbf{Trans}: \sigma\text{-}\mathbf{Trans} \rightarrow \sigma\text{-}\mathbf{Trans}_2$ map any $E \rightarrow \Delta(V)$ to itself (with underlying arrow $V \rightarrow V$).

► **Proposition 41.** The functor $\iota\text{-}\mathbf{Trans}: \sigma\text{-}\mathbf{Trans} \rightarrow \sigma\text{-}\mathbf{Trans}_2$ is a (full) reflective embedding. The left adjoint, say $\rho\text{-}\mathbf{Trans}: \sigma\text{-}\mathbf{Trans}_2 \rightarrow \sigma\text{-}\mathbf{Trans}$ maps any $E \rightarrow \Delta_s(D) \times \Delta_{l,t}(V)$ to the composite $E \rightarrow \Delta_s(D) \times \Delta_{l,t}(V) \rightarrow \Delta(V)$.

► **Definition 42.** For any Σ_1 , let $\check{\Sigma}_1$ be the composite $\sigma\text{-}\mathbf{Trans} \xrightarrow{\Sigma_1} \sigma\text{-}\mathbf{Trans}_2 \xrightarrow{\rho\text{-}\mathbf{Trans}} \sigma\text{-}\mathbf{Trans}$.

Models of Σ_1 will almost be $\check{\Sigma}_1$ -algebras. The problem is that a $\check{\Sigma}_1$ -algebra structure includes in particular algebra structure for the action of $\check{\Sigma}_1$ on the underlying σ -algebra, i.e., algebra structure $V \rightarrow V$ for the identity endofunctor on σ -algebras. This structure is not relevant for our purposes, so we require it to be the canonical candidate, i.e., the identity on V .

► **Definition 43.** A $\check{\Sigma}_1$ -algebra structure $\check{\Sigma}_1(G) \rightarrow G$ is *vertical* when its image under the forgetful functor $\sigma\text{-}\mathbf{Trans} \rightarrow \sigma\text{-}\mathbf{Alg}$ is the identity. A $\check{\Sigma}_1$ -algebra is called *vertical* accordingly. Let $\check{\Sigma}_1\text{-}\mathbf{alg}_v$ denote the full subcategory of $\check{\Sigma}_1\text{-}\mathbf{alg}$ spanning all vertical algebras.

► **Theorem 44.** The forgetful functor $\check{\Sigma}_1\text{-}\mathbf{alg}_v \rightarrow \sigma\text{-}\mathbf{Trans}$ is monadic, and the initial $\check{\Sigma}_1$ -algebra, say \mathbf{Z}_{Σ_1} , or \mathbf{Z} for short when Σ_1 is clear from context, may be chosen to be vertical, hence in particular to also be initial in $\check{\Sigma}_1\text{-}\mathbf{alg}_v$. (In this case, $V_{\mathbf{Z}}$ is an initial σ -algebra.)

Proof. Same as [18, Theorem 5.18 and Proposition 5.19]. ◀

► **Example 45.** For Σ_1 as in Example 39, \mathbf{Z} is the syntactic transition system of §3.

Let us now collect the static and dynamic part of signatures and their models.

► **Definition 46.** An *operational signature* consists of a syntactic signature \mathbf{d} , together with a dynamic signature $\Sigma_1: \sigma(\mathbf{d})\text{-Trans} \rightarrow \sigma(\mathbf{d})\text{-Trans}_2$ over the generated enhanced syntax $\sigma(\mathbf{d})$ (Proposition 34). The category of (\mathbf{d}, Σ_1) -algebras is $\check{\Sigma}_1\text{-alg}_v$.

By definition, we have:

► **Proposition 47.** The initial vertical $\check{\Sigma}_1$ -algebra is an initial (\mathbf{d}, Σ_1) -algebra.

6.3 Congruence of enhanced bisimilarity

In this subsection, we state our main congruence result. For this, we need to make an important hypothesis involving so-called functional flexible bisimulations. These are like a functional version of bisimulations, where labels are required to be related instead of identical, much as in Sangiorgi's BA-bisimulation [29], which we need to define both for algebraic transition systems and their diplopic variant. The hypothesis will then require that the considered dynamic signature Σ_1 preserve functional flexible bisimulations. We again fix a Howe context $\mathbb{H} = (\mathbb{V}\mathbb{T}, \mathbb{E}\mathbb{T}, \mathbf{s}, \mathbf{t}, \mathbf{l})$ and an enhanced syntax σ over it.

► **Definition 48.** A morphism $f: R \rightarrow X$ in $\mathbb{H}\text{-Trans}_2$ is a **functional flexible bisimulation** iff for any $\alpha \in \mathbb{E}\mathbb{T}$, $r \in D_R(\mathbf{s}(\alpha))$, $(r_1, \dots, r_{n_\alpha}) \in \Delta_{\mathbf{l}}(V_R)(\alpha)$, and transition $e': f_D(r) \xrightarrow{\alpha(f_V(r_1), \dots, f_V(r_{n_\alpha}))} x'$ there exists $e: r \xrightarrow{\alpha(r_1, \dots, r_{n_\alpha})} r'$ such that $f_E(e) = e'$.

A morphism in $\sigma\text{-Trans}_2$ is a functional flexible bisimulation iff the underlying morphism in $\mathbb{H}\text{-Trans}_2$ is. A morphism in $\sigma\text{-Trans}$ is a functional flexible bisimulation iff its embedding into $\sigma\text{-Trans}_2$ (by $\iota\text{-Trans}$) is. In any of these categories \mathcal{C} , let $\mathbf{FFBisim}(\mathcal{C})$ denote the class of all functional flexible bisimulations.

► **Definition 49.** A dynamic signature $\Sigma_1: \sigma\text{-Trans} \rightarrow \sigma\text{-Trans}_2$ **preserves functional flexible bisimulations** iff for all morphisms f in $\sigma\text{-Trans}$, if f is a functional flexible bisimulation, then so is $\Sigma_1(f)$.

Let us introduce a last hypothesis before stating the main result:

► **Definition 50.** A functor is **algebraic** iff it is finitary and preserves wide pullbacks and reflexive coequalisers. A syntactic signature (Σ, Γ, δ) is algebraic if the endofunctor Σ is.

► **Remark 51.** Algebraicity is straightforward to verify in all our applications.

► **Theorem 52.** For any operational signature (\mathbf{d}, Σ_1) , if \mathbf{d} is algebraic and Σ_1 preserves functional flexible bisimulations, then enhanced bisimilarity on the initial vertical $\check{\Sigma}_1$ -algebra is a congruence.

Proof. See Appendix A. ◀

6.4 Preservation of functional flexible bisimulations

In this section, we exhibit a sufficient condition for a dynamic signature to preserve functional flexible bisimulations, slightly generalising [18, §7]. Fixing a Howe context $\mathbb{H} = (\mathbb{V}\mathbb{T}, \mathbb{E}\mathbb{T}, \mathbf{s}, \mathbf{t}, \mathbf{l})$ and an enhanced syntax $\sigma = (\Sigma_0, \Gamma, \delta: TS \rightarrow ST)$ on $\mathbb{V}\mathbb{T}$, we first characterise $\mathbb{H}\text{-Trans}$ and $\mathbb{H}\text{-Trans}_2$ as presheaf categories, which allows us to characterise functional flexible bisimulations as the right class of a weak factorisation system [20, 28] – we call the left class **cofibrations**. We then recall familial functors, and define the notion of rule of a dynamic

signature Σ_1 , and the **border arity** of any rule. We finally show that a familial Σ_1 preserves functional flexible bisimulations iff the border arities of all rules are cofibrations.

Let us characterise transitions systems as presheaves, recalling Definition 12:

► **Proposition 53.** *We have $\mathbb{H}\text{-Trans} \simeq \widehat{\mathbb{VT}[\mathbb{ET}]_{\mathbf{y}_s + \mathbf{l} + \mathbf{y}_t}}$, where $\mathbf{y}_s + \mathbf{l} + \mathbf{y}_t: \mathbb{ET} \rightarrow \widehat{\mathbb{VT}}$.*

Proof. The functor $\Delta_{\mathbb{H}}$ is the nerve of $\mathbf{y}_s + \mathbf{l} + \mathbf{y}_t$, so we conclude by [8, Lemma 4.9]. ◀

Doing the same for $\mathbb{H}\text{-Trans}_2$ leads to considering the functor $\mathbb{ET} \rightarrow \widehat{\mathbb{VT}}^2$ mapping any α to the arrow $\mathbf{y}_{s(\alpha)} \rightarrow \mathbf{y}_{s(\alpha)} + \mathbf{l}(\alpha) + \mathbf{y}_{t(\alpha)}$. But for [8, Lemma 4.9] to apply, we need the codomain of this functor to be a presheaf category. This is in fact the case up to equivalence:

► **Lemma 54.** *We have $\widehat{\mathbb{VT}}^2 \simeq \widehat{\mathbb{VT}[\mathbb{VT}]_{\mathbf{y}}}$, where $\mathbf{y}: \mathbb{VT} \rightarrow \widehat{\mathbb{VT}}$.*

► **Notation 4.** *For each state type $b \in \mathbb{VT}$, the category $\mathbb{VT}[\mathbb{VT}]$ has an object b_V corresponding to the vertex object, an object b_D for the distinguished vertex object, and a morphism $b_V \rightarrow b_D$.*

Gluing along the obtained functor $\alpha \mapsto \mathbf{y}_{s(\alpha)_D} + \sum_{i \in n_\alpha} \mathbf{y}_{(l_i^\alpha)_V} + \mathbf{y}_{t(\alpha)_V}$, we obtain:

► **Proposition 55.** *We have $\mathbb{H}\text{-Trans}_2 \simeq \mathbb{VT}[\widehat{\mathbb{VT}}][\mathbb{ET}]$.*

Let us now characterise functional flexible bisimulations by a lifting property.

► **Definition 56.** *In a category \mathcal{C} , given a class \mathbb{J} of morphisms, let \mathbb{J}^\natural consist of morphisms $f: X \rightarrow Y$ such that for any $j: A \rightarrow B$ in \mathbb{J} , any $(u, v): j \rightarrow f$ in \mathcal{C}^2 admits a **lifting**, i.e., a morphism $k: B \rightarrow X$ such that $k \circ j = u$ and $f \circ k = v$. Let ${}^\natural\mathbb{J}$ consist of all f such that any $(u, v): f \rightarrow j$ in \mathcal{C}^2 admits a lifting. A **\mathbb{J} -cofibration** is an element of ${}^\natural(\mathbb{J}^\natural)$.*

► **Proposition 57.** *For any \mathbb{J} , \mathbb{J} -cofibrations are closed under cobase change and composition.*

For any $\alpha \in \mathbb{ET}$, the element $(in_1(id_{s(\alpha)})) \in (\mathbf{y}_s + \mathbf{l} + \mathbf{y}_t)(\alpha)(s(\alpha))$, corresponds to a morphism $s_\alpha: s(\alpha) \rightarrow \alpha$ in $\mathbb{VT}[\mathbb{ET}]$, and similarly we get morphisms $l_i^\alpha: l_i^\alpha \rightarrow \alpha$ for all $i \in n_\alpha$.

► **Definition 58.** *Let \mathbb{J}_σ denote the set of all maps $\mathcal{L}'(j_\alpha)$ in $\sigma\text{-Trans}$, where $\mathcal{L}': \widehat{\mathbb{VT}[\mathbb{ET}]} \rightarrow \sigma\text{-Trans}$ is left adjoint to the forgetful functor, and $j_\alpha: \mathbf{y}_{s(\alpha)} + \sum_{i \in n_\alpha} \mathbf{y}_{l_i^\alpha} \rightarrow \mathbf{y}_\alpha$ denotes the cotupling $[\mathbf{y}_{s_\alpha}, [\mathbf{y}_{l_i^\alpha}]_{i \in n_\alpha}]$, for all α .*

Let $\mathbb{J}_{2,\sigma}$ denote the set of all maps $\mathcal{L}'_2(j_{2,\alpha})$ in $\sigma\text{-Trans}_2$, where $\mathcal{L}'_2: \widehat{\mathbb{VT}[\mathbb{VT}][\mathbb{ET}]} \rightarrow \sigma\text{-Trans}_2$ is left adjoint to the forgetful functor, say \mathcal{U}'_2 , and $j_{2,\alpha}: \mathbf{y}_{s(\alpha)_D} + \sum_{i \in n_\alpha} \mathbf{y}_{(l_i^\alpha)_V} \rightarrow \mathbf{y}_\alpha$ denotes the analogous cotupling $[\mathbf{y}_{s_{2,\alpha}}, [\mathbf{y}_{l_i^{2,\alpha}}]_{i \in n_\alpha}]$, for all α .

► **Proposition 59.** *We have $\mathbf{FFBisim}(\sigma\text{-Trans}) = \mathbb{J}_\sigma^\natural$ and $\mathbf{FFBisim}(\sigma\text{-Trans}_2) = \mathbb{J}_{2,\sigma}^\natural$.*

Let us now introduce border arities. A functor $F: \mathcal{C} \rightarrow \widehat{\mathbb{D}}$ to some presheaf category is **familial** iff there exists a functor $E: \text{el}(F(1)) \rightarrow \mathcal{C}$ from the **category of elements** [25, §I.5] of $F(1)$, called the **exponent** of F , such that, we have a natural isomorphism

$$F(C)(d) \cong \sum_{o \in F(1)(d)} \mathcal{C}(E(d, o), C).$$

Intuitively, elements $o \in F(1)(d)$ are operations of output arity d , and $E(d, o)$ gives their input arity. Morphisms $u: d \rightarrow d'$ of \mathbb{D} act on $F(C)$ by precomposition: for any $o' \in F(1)(d')$, we have a morphism $(d, o) \xrightarrow{u \uparrow o'} (d', o')$ in $\text{el}(F(1))$, where $o = F(1)(u)(o')$ – which we write $o = o' \cdot u$; and the map $F(C)(u): F(C)(d') \rightarrow F(C)(d)$ sends any $(o', \varphi: E(d', o') \rightarrow C)$ to $(o, E(d, o) \xrightarrow{u \uparrow o'} E(d', o') \xrightarrow{\varphi} C)$. This is the basis for defining border arities.

499 ▶ **Definition 60.** Consider a dynamic signature Σ_1 such that the composite $\sigma\text{-Trans} \xrightarrow{\Sigma_1}$
 500 $\sigma\text{-Trans}_2 \xrightarrow{\mathcal{U}'_2} \widehat{\mathbb{V}\mathbb{T}[\mathbb{V}\mathbb{T}][\mathbb{E}\mathbb{T}]}$ is familial with exponent E . Let us fix $\alpha \in \mathbb{E}\mathbb{T}$ and $r \in \mathcal{U}'_2 \Sigma_1(1)(\alpha)$.
 501 For any $k: A_k \rightarrow \alpha$ among $I_\alpha := \{s_{2,\alpha}, l_1^{2,\alpha}, \dots, l_{n_\alpha}^{2,\alpha}\}$, we have $E(k \upharpoonright r): E(A_k, r \cdot k) \rightarrow$
 502 $E(\alpha, r)$. The **border arity** \mathbf{b}_r of r is the cotupling $[E(k \upharpoonright r)]_{k \in I_\alpha}: \sum_{k \in I_\alpha} E(A_k, r \cdot k) \rightarrow E(\alpha, r)$.

503 ▶ **Theorem 61.** For any dynamic signature $\Sigma_1: \sigma\text{-Trans} \rightarrow \sigma\text{-Trans}_2$ such that $\mathcal{U}'_2 \Sigma_1$ is
 504 familial, Σ_1 preserves functional flexible bisimulations iff all border arities are \mathbb{J}_σ -cofibrations.

505 **Proof sketch for “if”, see §B.** Consider any $(u, v): \mathcal{L}'_2(j_{2,\alpha}) \rightarrow \Sigma_1(f)$, with $f: A \rightarrow B$ in
 506 **FFBisim**($\sigma\text{-Trans}$). By adjunction, we get $(\tilde{u}, \tilde{v}): j_{2,\alpha} \rightarrow \mathcal{U}'_2(\Sigma_1(f))$. Letting r be the
 507 composite $\mathbf{y}_\alpha \xrightarrow{\tilde{v}} \mathcal{U}'_2(\Sigma_1(B)) \xrightarrow{\mathcal{U}'_2(\Sigma_1(!))} \mathcal{U}'_2(\Sigma_1(1))$, we use familiarity to factor (\tilde{u}, \tilde{v}) as the
 508 solid part below. The result follows from finding k as shown, by $\mathbf{b}_r \in {}^\mathfrak{h}(\mathbb{J}_\sigma{}^\mathfrak{h})$ and $f \in \mathbb{J}_\sigma{}^\mathfrak{h}$.

$$\begin{array}{c}
 \mathbf{y}_{s(\alpha)_D} + \sum_{i \in n_\alpha} \mathbf{y}_{(l_i^\alpha)_V} \xrightarrow{\quad} \mathcal{U}'_2(\Sigma_1(\sum_{k \in I_\alpha} E(A_k, r \cdot k))) \xrightarrow{\mathcal{U}'_2(\Sigma_1(\psi))} \mathcal{U}'_2(\Sigma_1(A)) \\
 \downarrow j_{2,\alpha} \quad \downarrow [\mathcal{U}'_2(\Sigma_1(in_k)) \circ (r \cdot k, \text{id})]_{k \in I_\alpha} \quad \downarrow \mathcal{U}'_2(\Sigma_1(\mathbf{b}_r)) \quad \downarrow \mathcal{U}'_2(\Sigma_1(k)) \quad \downarrow \mathcal{U}'_2(\Sigma_1(f)) \\
 \mathbf{y}_\alpha \xrightarrow{(r, \text{id})} \mathcal{U}'_2(\Sigma_1(E(\alpha, r))) \xrightarrow{\mathcal{U}'_2(\Sigma_1(\varphi))} \mathcal{U}'_2(\Sigma_1(B)) \quad \blacktriangleleft
 \end{array}$$

510 ▶ **Example 62.** Let us now sketch a proof of Theorem 7. By Theorems 52 and 61 and Pro-
 511 position 57, it suffices to reconstruct the border arity of each rule. We only treat rule (SA)
 512 for lack of space: its border arity is the bottom morphism in

$$\begin{array}{ccc}
 \mathcal{L}(0_p + 0_c) & \xrightarrow{\mathcal{L}[s[c], t[c]]} & \mathcal{L}[c] \\
 (e_1, E[v \sqcap]) \downarrow & & \downarrow \\
 \mathcal{L}(0_v + 0_p + 0_c) & \xrightarrow{\quad} & A,
 \end{array}$$

514 with hopefully clear notation.

515 ▶ **Example 63.** This also works for PCF as in [17], which we omit for lack of space.

7 Conclusion and perspectives

517 We have introduced a categorical framework for applicative bisimilarity in the presence of
 518 operations on terms other than substitution, and of terms as labels. We have furthermore
 519 provided a notion of signature for generating instances of this framework, and proved
 520 that under suitable hypotheses, notably preservation of functional flexible bisimulations,
 521 applicative bisimilarity in the generated instance is a congruence. We have finally exhibited
 522 a more concrete sufficient condition in terms of border arities being cofibrations, which has
 523 allowed us to recover congruence of applicative bisimilarity for λ -calculus with delimited
 524 control operators and PCF.

525 For future work, we would be interested in further generalising the framework to cover
 526 a kind of adaptation of Howe’s method that still eludes our abstraction efforts, namely
 527 (early-style) higher-order process calculi [23].

References

- 529 1 J. Adámek and J. Rosický. *Locally Presentable and Accessible Categories*. Cambridge University
 530 Press, 1994. doi:10.1017/CB09780511600579.
- 531 2 J. Adámek, J. Rosický, and E. M. Vitale. *Algebraic Theories: A Categorical Introduction*
 532 *to General Algebra*. Cambridge Tracts in Mathematics. Cambridge University Press, 2010.
 533 doi:10.1017/CB09780511760754.

- 534 3 Jon M. Beck. Distributive laws. In Beno Eckmann and Myles Tierney, editors, *Seminar*
535 *on Triples and Categorical Homology Theory*, volume 80 of *Lecture Notes in Mathematics*.
536 Springer, 1969.
- 537 4 Karen L. Bernstein. A congruence theorem for structured operational semantics of higher-order
538 languages. In *Proc. 13th Symposium on Logic in Computer Science*, pages 153–164. IEEE,
539 1998. doi:10.1109/LICS.1998.705652.
- 540 5 Dariusz Biernacki and Sergueï Lenglet. Applicative bisimulations for delimited-control operat-
541 ors. In Lars Birkedal, editor, *Proc. 15th Foundations of Software Science and Computational*
542 *Structures*, volume 7213 of *Lecture Notes in Computer Science*, pages 119–134. Springer, 2012.
543 doi:10.1007/978-3-642-28729-9_8.
- 544 6 B. Bloom, S. Istrail, and A. Meyer. Bisimulation can’t be traced. *Journal of the ACM*,
545 42:232–268, 1995. doi:10.1145/200836.200876.
- 546 7 Peio Borthelle, Tom Hirschowitz, and Ambroise Lafont. A cellular Howe theorem. In Holger
547 Hermanns, Lijun Zhang, Naoki Kobayashi, and Dale Miller, editors, *Proc. 35th ACM/IEEE*
548 *Symposium on Logic in Computer Science*. ACM, 2020. doi:10.1145/3373718.3394738.
- 549 8 Aurelio Carboni and Peter Johnstone. Connected limits, familial representability and Artin
550 glueing. *Mathematical Structures in Computer Science*, 5(4):441–459, 1995. doi:10.1017/
551 S0960129500001183.
- 552 9 Olivier Danvy and Andrzej Filinski. Abstracting control. In Gilles Kahn, editor, *Proc.*
553 *ACM Conference on LISP and Functional Programming (LFP)*, pages 151–160. ACM, 1990.
554 doi:10.1145/91556.91622.
- 555 10 Yves Diers. Spectres et localisations relatifs à un foncteur. *Comptes rendus hebdomadaires des*
556 *séances de l’Académie des sciences*, 287(15):985–988, 1978.
- 557 11 Marcelo Fiore, Gordon Plotkin, and Daniele Turi. Abstract syntax and variable binding. In *Proc.*
558 *14th Symposium on Logic in Computer Science*. IEEE, 1999. doi:10.1109/LICS.1999.782615.
- 559 12 Marcelo Fiore and Daniele Turi. Semantics of name and value passing. In *Proc. 16th Symposium*
560 *on Logic in Computer Science*, pages 93–104. IEEE, 2001. doi:10.1109/LICS.2001.932486.
- 561 13 Marcelo P. Fiore. Second-order and dependently-sorted abstract syntax. In *Proc. 23rd*
562 *Symposium on Logic in Computer Science*, pages 57–68. IEEE, 2008. doi:10.1109/LICS.2008.
563 38.
- 564 14 Richard H. G. Garner and Tom Hirschowitz. Shapely monads and analytic functors. *Journal*
565 *of Logic and Computation*, 28(1):33–83, 2018. doi:10.1093/logcom/exx029.
- 566 15 Joseph A Goguen and James W Thatcher. Initial algebra semantics. In *15th Annual Symposium*
567 *on Switching and Automata Theory (SWAT)*, pages 63–77. IEEE, 1974.
- 568 16 Sergey Goncharov, Stefan Milius, Lutz Schröder, Stelios Tsampas, and Henning Urbat. Towards
569 a higher-order mathematical operational semantics. *Proceedings of the ACM on Programming*
570 *Languages*, 7(POPL), jan 2023. doi:10.1145/3571215.
- 571 17 Andrew D. Gordon. Bisimilarity as a theory of functional programming. *Theoretical Computer*
572 *Science*, 228(1-2):5–47, 1999. doi:10.1016/S0304-3975(98)00353-3.
- 573 18 Tom Hirschowitz and Ambroise Lafont. A categorical framework for congruence of applicative
574 bisimilarity in higher-order languages. *Logical Methods in Computer Science*, 18(3), 2022.
575 URL: <https://lmcs.episciences.org/10066>, doi:10.46298/lmcs-18(3:37)2022.
- 576 19 Tom Hirschowitz and Ambroise Lafont. A unified treatment of structural definitions on
577 syntax for capture-avoiding substitution, context application, named substitution, partial
578 differentiation, and so on. 2022. URL: <https://hal.archives-ouvertes.fr/hal-03633933>.
- 579 20 Mark Hovey. *Model Categories*, volume 63 of *Mathematical Surveys and Monographs, Volume*
580 *63, AMS (1999)*. American Mathematical Society, 1999. doi:10.1090/surv/063.
- 581 21 Douglas J. Howe. Proving congruence of bisimulation in functional programming languages.
582 *Information and Computation*, 124(2):103–112, 1996. doi:10.1006/inco.1996.0008.
- 583 22 André Joyal, Mogens Nielsen, and Glynn Winskel. Bisimulation and open maps. In *Proc. 8th*
584 *Symposium on Logic in Computer Science*, pages 418–427. IEEE, 1993. doi:10.1109/LICS.
585 1993.287566.

- 586 23 Sergueï Lenglet and Alan Schmitt. Howe’s method for contextual semantics. In Luca Aceto and
 587 David de Frutos-Escrig, editors, *Proc. 26th International Conference on Concurrency Theory*,
 588 volume 42 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 212–225. Schloss
 589 Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015. doi:10.4230/LIPIcs.CONCUR.2015.212.
- 590 24 Saunders Mac Lane. *Categories for the Working Mathematician*. Number 5 in Graduate Texts
 591 in Mathematics. Springer, 2nd edition, 1998. doi:10.1007/978-1-4757-4721-8.
- 592 25 Saunders Mac Lane and Ieke Moerdijk. *Sheaves in Geometry and Logic: A First Introduction*
 593 *to Topos Theory*. Universitext. Springer, 1992. doi:10.1007/978-1-4612-0927-0.
- 594 26 Gordon D. Plotkin. A structural approach to operational semantics. DAIMI Report FN-19,
 595 Computer Science Department, Aarhus University, 1981.
- 596 27 Jan Reiterman. A left adjoint construction related to free triples. *Journal of Pure and Applied*
 597 *Algebra*, 10:57–71, 1977. doi:10.1016/0022-4049(77)90028-7.
- 598 28 Emily Riehl. *Categorical Homotopy Theory*. Number 24 in New Mathematical Monographs.
 599 Cambridge University Press, 2014.
- 600 29 Davide Sangiorgi, Naoki Kobayashi, and Eijiro Sumii. Logical bisimulations and functional
 601 languages. In Farhad Arbab and Marjan Sirjani, editors, *Proc. International Symposium on*
 602 *Fundamentals of Software Engineering (FSEN)*, volume 4767 of *Lecture Notes in Computer*
 603 *Science*, pages 364–379. Springer, 2007. doi:10.1007/978-3-540-75698-9_24.
- 604 30 Davide Sangiorgi and David Walker. *The π -calculus – A Theory of Mobile Processes*. Cambridge
 605 University Press, 2001.
- 606 31 Sam Staton. General structural operational semantics through categorical logic. In *Proc. 23rd*
 607 *Symposium on Logic in Computer Science*, pages 166–177, 2008. doi:10.1109/LICS.2008.43.
- 608 32 Daniele Turi and Gordon Plotkin. Towards a mathematical operational semantics. In
 609 *Proc. 12th Symposium on Logic in Computer Science*, pages 280–291. IEEE, 1997. doi:
 610 10.1109/LICS.1997.614955.
- 611 33 Mark Weber. *Symmetric Operads for Globular Sets*. PhD thesis, Macquarie University, 2001.

A Proof of Theorem 52

We assume given a Howe context $\mathbb{H} = (\mathbb{V}\mathbb{T}, \mathbb{E}\mathbb{T}, \mathbf{s}, \mathbf{t}, \mathbf{l})$. To ease readability, we introduce some notations.

► **Notation 5.** For any $G = (D, V, E, \gamma, \partial) \in \mathbb{H}\text{-Trans}_2$, we let $G_{D,V}$ denote the underlying triple $(D, V, \gamma: D \rightarrow V) \in \widehat{\mathbb{V}\mathbb{T}}^2$, G_0 denote V , G_1 denote E , and G_s denote D . Furthermore, following Notation 2, we denote, e.g., by $\Delta_{2,s,1}$ the functor mapping any $\gamma: D \rightarrow V$ to $\Delta_s(D) \times \Delta_1(V)$. Finally, we sometimes treat the projection $G \mapsto G_{D,V}$ as an implicit coercion. E.g., we write $\Delta_{2,s,1}(G)$ for $\Delta_s(D) \times \Delta_1(V)$.

A.1 Basic properties of flexible bisimulation

In this section, we establish basic properties of flexible bisimulations.

► **Proposition 64.** The functor Δ_1 is a right adjoint, hence in particular it preserves all limits.

Proof. The functor Δ_1 is the nerve functor of \mathbf{l} . It is right adjoint to the left Kan extension of \mathbf{l} along the Yoneda embedding, as in the following diagram.

$$\begin{array}{ccc}
 \mathbb{E}\mathbb{T} & \xrightarrow{y} & \widehat{\mathbb{E}\mathbb{T}} \\
 \searrow \mathbf{l} & \swarrow \bar{\mathbf{l}} & \nearrow \Delta_1 \\
 & \widehat{\mathbb{V}\mathbb{T}} &
 \end{array}$$

Regarding preservation of colimits, the fact that any $\mathbf{l}(c)$ is a finite coproduct of representables entails:

► **Proposition 65.** The functor Δ_1 is algebraic, and preserves epimorphisms.

Proof. Just for making the proof slicker, we rely on the well-known facts [2] that in presheaf categories preserving filtered colimits and reflexive coequalisers is equivalent to preserving sifted colimits. Furthermore, just as the covariant hom of any finitely presentable object preserves filtered colimits, in a presheaf category the covariant hom of any finite coproduct of representable objects preserves sifted colimits, hence epimorphisms. The latter fact deals with the second statement.

For the first, for any sifted colimit $\text{colim}_i X_i$ and $c \in \widehat{\mathbb{C}}$:

$$\begin{aligned}
 \Delta_1(\text{colim}_i X_i)(c) &= \widehat{\mathbb{V}\mathbb{T}}(\mathbf{l}(c), \text{colim}_i X_i) \\
 &= \text{colim}_i \widehat{\mathbb{V}\mathbb{T}}(\mathbf{l}(c), X_i) \quad (\mathbf{l}(c) \text{ a finite coproduct of representables}) \\
 &= \text{colim}_i \Delta_1(X_i)(c).
 \end{aligned}$$

► **Proposition 66.** All functors $\Delta, \Delta_1, \Delta_s, \Delta_t, \Delta_{s,1}, \dots$ are algebraic right adjoints (and preserve epimorphisms).

Proof. Let us first deal with algebraicity. Because algebraic functors are closed under pointwise products, it suffices to deal with each of Δ_1 , Δ_s , and Δ_t in isolation: Δ_s and Δ_t are, as restriction functors; Δ_1 is by Proposition 65. Finally, in presheaf categories, being algebraic entails preservation of epimorphisms.

For right adjointness, as right adjoints are closed under pointwise products (under (co)completeness conditions satisfied here), it suffices to show that each of $\Delta_{\mathbf{l}}$, $\Delta_{\mathbf{s}}$, and $\Delta_{\mathbf{t}}$ is a right adjoint. Again, $\Delta_{\mathbf{s}}$ and $\Delta_{\mathbf{t}}$ are, as restriction functors; and $\Delta_{\mathbf{l}}$ is by Proposition 64. ◀

► **Proposition 67.** *All functors $\Delta_2, \Delta_{2,\mathbf{l}}, \Delta_{2,\mathbf{s}}, \Delta_{2,\mathbf{t}}, \Delta_{2,\mathbf{s},\mathbf{l}}, \dots$ are algebraic right adjoints and preserve epimorphisms.*

Proof. Algebraic functors between presheaf categories automatically preserve epimorphisms, so it suffices to prove that all these functors are algebraic right adjoints.

Algebraic right adjoints being closed under pointwise finite products, it further suffices to prove that each of $\Delta_{2,\mathbf{l}}$, $\Delta_{2,\mathbf{s}}$, and $\Delta_{2,\mathbf{t}}$ is an algebraic right adjoint. Now each of these functors $\Delta_{2,x}$ is the corresponding functor Δ_x , precomposed with one of the projections $\widehat{\mathbb{V}\mathbb{T}}^2 \rightarrow \widehat{\mathbb{V}\mathbb{T}}$. But each Δ_x is an algebraic right adjoint by Proposition 66, and projections, being restriction functors, are left and right adjoints, hence algebraic right adjoints, hence the result. ◀

► **Lemma 68.** *In any presheaf category, for any commuting diagram of the form*

$$\begin{array}{ccccc} A & \longrightarrow & A' & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ C' & \twoheadrightarrow & C & \longrightarrow & D \end{array}$$

if the exterior rectangle is a pointwise weak pullback and the marked morphism is epi, then so is the right-hand square.

Proof. Straightforward, using the fact that any morphism $\mathbf{y}_C \rightarrow C$ from some representable presheaf lifts to C' because epis are pointwise in presheaf categories. ◀

► **Proposition 69.** *A morphism $R \rightarrow X$ of diplopic \mathbb{H} -transition systems is a functional flexible bisimulation iff the following square is a pointwise weak pullback.*

$$\begin{array}{ccc} R_1 & \xrightarrow{\quad} & X_1 \\ \downarrow & & \downarrow \\ \Delta_{\mathbf{s}}(R_{\mathbf{s}}) \times \Delta_{\mathbf{l}}(R_0) & \xrightarrow{\quad} & \Delta_{\mathbf{s}}(X_{\mathbf{s}}) \times \Delta_{\mathbf{l}}(X_0) \end{array}$$

► **Lemma 70.** *For any morphisms $R \xrightarrow{f} S \xrightarrow{g} X$ in $\mathbb{H}\text{-Trans}_2$ such that $f_{D,V}: R_{D,V} \rightarrow S_{D,V}$ is an epi, if gf is a functional flexible bisimulation, then so is g .*

Proof. We have a diagram

$$\begin{array}{ccccc} R_1 & \xrightarrow{\quad} & S_1 & \xrightarrow{\quad} & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_{2,\mathbf{s},\mathbf{l}}R_{D,V} & \xrightarrow{\quad} & \Delta_{2,\mathbf{s},\mathbf{l}}S_{D,V} & \xrightarrow{\quad} & \Delta_{2,\mathbf{s},\mathbf{l}}X_{D,V}, \end{array}$$

and want to prove that the right-hand square is a pointwise weak pullback, knowing that the outer rectangle is one: this follows readily by Lemma 68 and Proposition 67. ◀

► **Corollary 71.** *For any $X \in \mathbb{H}\text{-Trans}_2$ and span morphism $f: R \rightarrow S$ in $\mathbb{H}\text{-Trans}_2/X^2$ such that $f_{D,V}$ is an epi, if R is a (bi)simulation, then so is S .*

► **Proposition 72.** *The projection functors $\mathbb{H}\text{-Trans} \rightarrow \widehat{\mathbb{V}\mathbb{T}}$ and $\mathbb{H}\text{-Trans}_2 \rightarrow \widehat{\mathbb{V}\mathbb{T}}^2$ are Grothendieck fibrations.*

23:20 A more general categorical framework for congruence of applicative bisimilarity

676 **Proof.** This follows readily from the next lemma. ◀

677 ► **Lemma 73.** *For any functor $F: \mathbf{A} \rightarrow \mathbf{B}$ to some category \mathbf{B} with pullbacks, the projection*
 678 *functor $p: \mathbf{B}/F \rightarrow \mathbf{A}$, mapping any object $b \rightarrow Fa$ to a , is a Grothendieck fibration.*

679 **Proof.** Given any object $x: b \rightarrow Fa$ and morphism $f: a' \rightarrow a$, a cartesian lifting is given by
 680 the following pullback,

$$\begin{array}{ccc}
 b|_{a'} & \xrightarrow{x|f} & b \\
 x|f \downarrow & \lrcorner & \downarrow x \\
 Fa' & \xrightarrow{Ff} & Fa
 \end{array}$$

681

682 cartesianness being ensured by universal property of pullback. ◀

683 ► **Definition 74.** *A span $R \rightarrow X \times Y$ of diplopic \mathbb{H} -transition systems (resp. diplopic σ -*
 684 *transition systems for any enhanced syntax σ) is a **flexible simulation** if its left-hand leg*
 685 *$R \rightarrow X$ is a functional flexible bisimulation, and a **flexible bisimulation** when both of its*
 686 *legs are.*

687 *By convention, for any $R \in \widehat{\mathbb{V}\mathbb{T}}^2$ and $X \in \mathbb{H}\text{-Trans}_2$, a span $R \rightarrow X^2_{D,V}$ is a **flexible***
 688 *bisimulation when the cartesian lifting $R^\uparrow \rightarrow X^2$ (in the sense of Proposition 72) of X along*
 689 *$R \rightarrow X^2_{D,V}$ is.*

690 ► **Proposition 75.** *If a span $R \rightarrow X^2$ is a flexible (bi)simulation, then so is the cartesian*
 691 *lifting $R^\uparrow_{D,V} \rightarrow X^2$.*

692 **Proof.** By Corollary 71 applied to the span morphism $R \rightarrow R^\uparrow_{D,V}$. ◀

693 ► **Lemma 76.** *Consider any pullback-preserving functor $F: \mathbf{A} \rightarrow \mathbf{B}$ between categories with*
 694 *pullbacks and (strong epi-mono) factorisations. Then:*

- 695 (i) *A morphism (f, g) in \mathbf{B}/F is monic iff both f and g are.*
- 696 (ii) *A morphism (f, g) in \mathbf{B}/F is a strong epi iff both f and g are.*
- 697 (iii) *The forgetful functor $\mathbf{B}/F \rightarrow \mathbf{B} \times \mathbf{A}$ creates, hence preserves, (strong epi-mono) factor-*
 698 *isations.*

699 **Proof.** First of all, the forgetful functor creates all colimits, and all limits that F preserves,
 700 hence in particular pullbacks. Furthermore, in any category \mathbf{C} , a morphism $f: X \rightarrow Y$ is
 701 monic iff its **self square**

$$\begin{array}{ccc}
 X & \xlongequal{\quad} & X \\
 \parallel & & \downarrow f \\
 X & \xrightarrow{f} & Y
 \end{array}$$

702

703 is a pullback. Thus, a morphism (f, g) in \mathbf{B}/F is mono iff its self square is a pullback, iff the
 704 self squares of f and g are both pullbacks, iff f and g are both monic. This settles (i).

705 We next deal with the ‘if’ part of (ii), consider any diagram like the solid part in

$$\begin{array}{ccccc}
b & \xrightarrow{f} & d & & \\
\downarrow x & \searrow e & \nearrow k & \searrow y & \swarrow m \\
& b' & \xrightarrow{h} & d' & \\
\downarrow Fg & & \downarrow Fl & & \downarrow Fs \\
Fa & \xrightarrow{Fg} & Fc & & \\
\downarrow Fr & & \downarrow Fj & & \\
& Fa' & \xrightarrow{Fj} & Fc' &
\end{array}$$

where e and r are strong epis and m and s are monos. By orthogonality, we find unique liftings k and l making all four triangles commute (without F on the bottom face). It remains to show that the vertical, diagonal square commutes: this follows by orthogonality using the fact that Fs is monic (because F preserves pullbacks, hence monos).

For (iii), consider any objects $x: b \rightarrow Fa$ and $x': b' \rightarrow Fa'$, and let $f: b \rightarrow b'$ and $g: a \rightarrow a'$ make the following diagram commute.

$$\begin{array}{ccc}
b & \xrightarrow{f} & b' \\
x \downarrow & & \downarrow x' \\
Fa & \xrightarrow{Fg} & Fa'
\end{array}$$

Let now $b \xrightarrow{e} b'' \xrightarrow{m} b'$ and $a \xrightarrow{r} a'' \xrightarrow{s} a'$ be (strong epi-mono) factorisations of f and g , respectively. Because F preserves monos, Fs is a mono, hence by orthogonality we find a unique lifting making both squares commute in

$$\begin{array}{ccccc}
b & \xrightarrow{e} & b'' & \xrightarrow{m} & b' \\
x \downarrow & & \downarrow x'' & & \downarrow x' \\
Fa & \xrightarrow{Fr} & Fa'' & \xrightarrow{Fs} & Fa'
\end{array}$$

Furthermore, by (i) and (ii), this lifting is in fact a (strong epi-mono) factorisation of (f, g) , as desired. Preservation follows from (strong epi-mono) factorisations being unique up to unique isomorphism and existing in $\mathbf{B} \times \mathbf{A}$ by hypothesis.

Finally, for the ‘only if’ part of (ii): a morphism is a strong epi iff the monic part of its (strong epi-mono) factorisation is an isomorphism. So given a strong epi (e, r) in \mathbf{B}/F , we compute its (strong epi-mono) factorisations $m \circ e'$ and $s \circ r'$ of e and r , respectively. By (iii), they lift uniquely to a (strong epi-mono) factorisation $(m, s) \circ (e', r')$ of (e, r) in \mathbf{B}/F . But (e', r') is a strong epi by (ii), and so is (e, r) by hypothesis, and (m, s) is a mono between them, hence an isomorphism by Lemma 83. Thus, m and s are both isomorphisms, and hence e and r are both strong epis, as desired. \blacktriangleleft

► **Lemma 77.** *The forgetful functor*

$$\mathbb{H}\text{-}\mathbf{Trans}_2 \rightarrow \widehat{\mathbb{E}\mathbf{T}} \times \widehat{\mathbb{V}\mathbf{T}}^2$$

creates all colimits and limits, as well as (strong epi)-mono factorisations.

Proof. This is clear for colimits. For limits, the projection $\mathbb{H}\text{-}\mathbf{Trans}_2 \rightarrow \widehat{\mathbb{E}\mathbf{T}} \times \widehat{\mathbb{V}\mathbf{T}}^2$ creates all limits that the functor Δ_2 preserves (because $\mathbb{H}\text{-}\mathbf{Trans}_2$ is its lax limit), i.e., all of them by Proposition 67. For (strong epi)-mono factorisations, this follows by Lemma 76. \blacktriangleleft

23:22 A more general categorical framework for congruence of applicative bisimilarity

► **Lemma 78.** *For any diplopic \mathbb{H} -transition system X , the forgetful functor*

$$\mathbb{H}\text{-}\mathbf{Trans}_2/X^2 \rightarrow \mathbb{H}\text{-}\mathbf{Trans}_2 \rightarrow \widehat{\mathbb{E}\mathbb{T}} \times \widehat{\mathbb{V}\mathbb{T}}^2$$

creates all colimits and connected limits.

Proof. The projection $(\mathbb{H}\text{-}\mathbf{Trans}_2)/X^2 \rightarrow \mathbb{H}\text{-}\mathbf{Trans}_2$ creates colimits and connected limits, as any projection from a slice category does. The result thus follows by Lemma 77. ◀

► **Lemma 79.** *Flexible bisimulations are closed under filtered colimits in $\mathbb{H}\text{-}\mathbf{Trans}_2^2$, i.e., in the arrow category of $\mathbb{H}\text{-}\mathbf{Trans}_2$.*

Proof. Let $(r_\infty: R_\infty \rightarrow X_\infty) = \text{colim}_j (r_j: R_j \rightarrow X_j)$ denote the colimit of any filtered digram of functional flexible bisimulations. By Lemma 78 and the fact that colimits are pointwise in the arrow category, we have

$$(R_\infty)_0 \cong \text{colim}_j (R(j)_0) \quad (R_\infty)_s \cong \text{colim}_j (R(j)_s) \quad (R_\infty)_1 \cong \text{colim}_j (R(j)_1)$$

and

$$(X_\infty)_0 \cong \text{colim}_j (X(j)_0) \quad (X_\infty)_s \cong \text{colim}_j (X(j)_s) \quad (X_\infty)_1 \cong \text{colim}_j (X(j)_1).$$

Furthermore, all morphisms

$$\begin{aligned} \gamma_{R_\infty}: (R_\infty)_s &\rightarrow (R_\infty)_0 & \partial_{R_\infty}: (R_\infty)_1 &\rightarrow \Delta_2(R_\infty) \\ \gamma_{X_\infty}: (X_\infty)_s &\rightarrow (X_\infty)_0 & \partial_{X_\infty}: (X_\infty)_1 &\rightarrow \Delta_2(X_\infty) \end{aligned}$$

$$(r_\infty)_1: (R_\infty)_1 \rightarrow (X_\infty)_1 \quad (r_\infty)_s: (R_\infty)_s \rightarrow (X_\infty)_s \quad (r_\infty)_0: (R_\infty)_0 \rightarrow (X_\infty)_0$$

are induced by universal property.

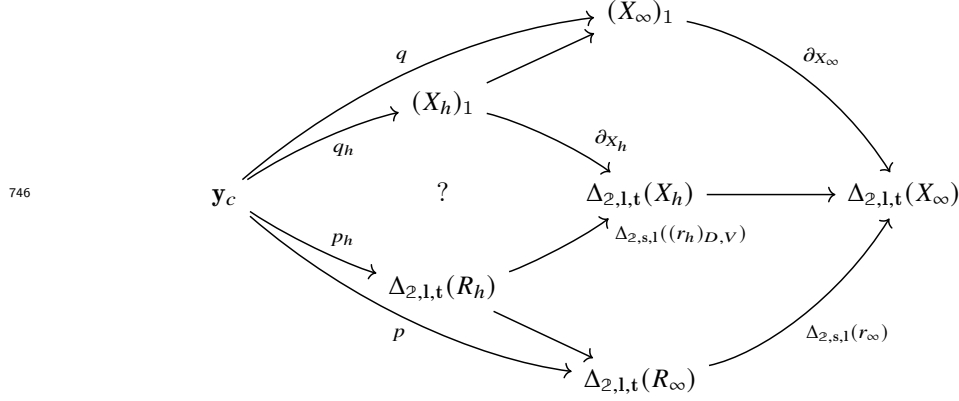
Now consider any p and q making the following diagram commute.

$$\begin{array}{ccc} \mathbf{y}_c & \xrightarrow{q} & (X_\infty)_1 \\ p \downarrow & & \downarrow \partial_{X_\infty} \\ \Delta_{2,s,1}(R_\infty) & \xrightarrow{\Delta_{2,s,1}r_\infty} & \Delta_{2,s,1}(X_\infty) \end{array}$$

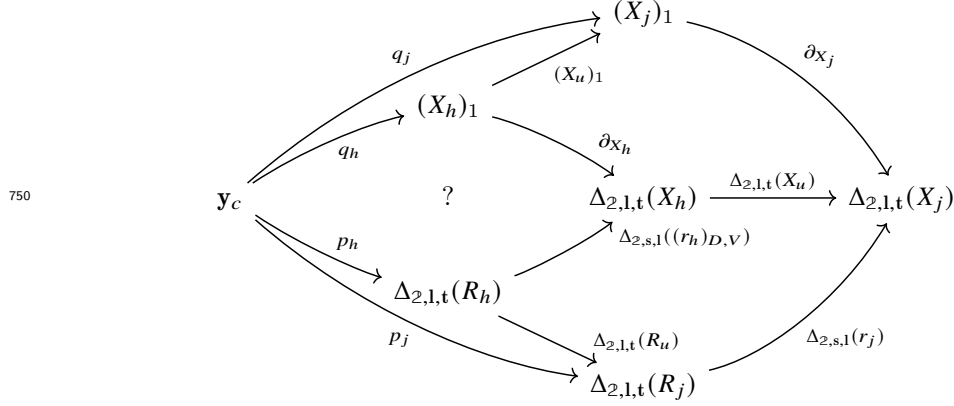
The functor $\Delta_{2,s,1}$ is finitary, and the object \mathbf{y}_c finitely presentable, so p factors through some $\Delta_{2,s,1}(R_k)$, say as p_k , and q factors through some $(X_l)_1$, say as q_l . Furthermore, by filteredness, we find h and morphisms $k \xrightarrow{f} h \xleftarrow{g} l$, so that we may define p_h and q_h as in the following diagram.

$$\begin{array}{ccccc} & & & (X_h)_1 & \\ & & q_h \curvearrowright & \nearrow & \\ & & (X_l)_1 & \xrightarrow{(X_g)_1} & \\ & q_l \nearrow & & & \partial_{X_h} \searrow \\ \mathbf{y}_c & \xrightarrow{p_k} & \Delta_{2,s,1}(R_k) & \xrightarrow{\Delta_{2,s,1}(R_f)} & \Delta_{2,1,t}(X_h) \\ & p_h \searrow & & \nearrow \Delta_{2,s,1}((r_h)_{D,V}) & \\ & & \Delta_{2,1,t}(R_h) & & \end{array}$$

745 Because the following diagram commutes,

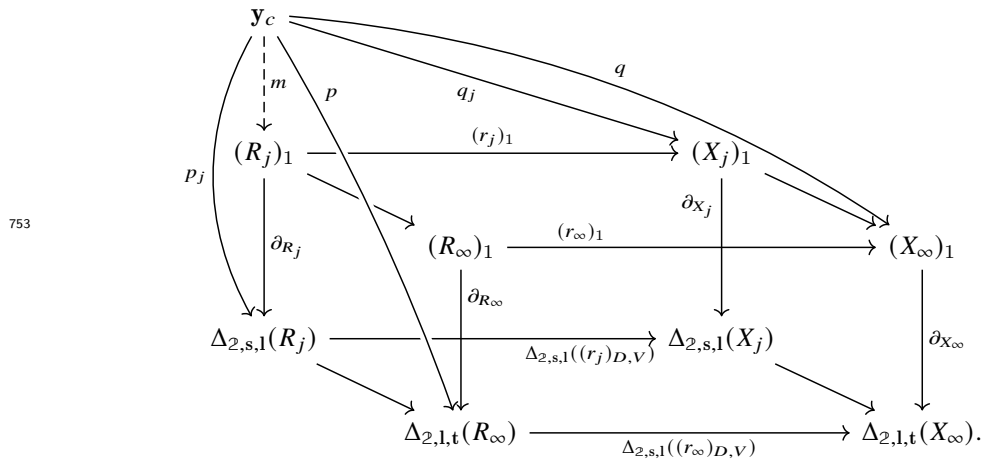


747 by filteredness, we find some j and morphism $u: h \rightarrow j$ such that $\Delta_{2,s,1}X_u$ coequalises the
 748 question marked parallel pair above. We then define p_j and q_j by composition to obtain a
 749 commuting diagram as the following



751 (where again the question marked parallel pair may not commute but the exterior does).

752 We thus obtain a situation like



But $r_j: R_j \rightarrow X_j$ is a functional flexible bisimulation, so we find a mediating arrow m as shown. The composite

$$y_c \xrightarrow{m} (R_j)_1 \rightarrow (R_\infty)_1$$

finally provides the desired mediating arrow. \blacktriangleleft

► **Corollary 80.** *For any diplopic \mathbb{H} -transition system X , flexible bisimulations $R \rightarrow X^2$ over X are closed under filtered colimits in $\mathbb{H}\text{-Trans}_2/X^2$.*

► **Lemma 81.** *For any diplopic \mathbb{H} -transition system X , flexible bisimulations $R \rightarrow X^2$ over X are closed under span composition.*

Proof. We need to show that the square

$$\begin{array}{ccc} (R; S)_1 & \xrightarrow{\pi_1} & X_1 \\ \downarrow & & \downarrow \\ \Delta_{2,s,1}(R; S) & \xrightarrow{\Delta_{2,s,1}\pi_1} & \Delta_{2,s,1}X \end{array}$$

is a pointwise weak pullback. By construction, this square factors as

$$\begin{array}{ccccc} (R; S)_1 & \xrightarrow{\pi_1} & R_1 & \xrightarrow{\pi_1} & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_{2,s,1}(R; S) & \xrightarrow{\Delta_{2,s,1}\pi_1} & \Delta_{2,s,1}R & \xrightarrow{\Delta_{2,s,1}\pi_1} & \Delta_{2,s,1}X \end{array}$$

where the right-hand square is a pointwise weak pullback by hypothesis. Now the left-hand square is the left-hand face in the following diagram,

$$\begin{array}{ccccc} (R; S)_1 & \xrightarrow{\quad} & S_1 & \xrightarrow{\pi_1} & X_1 \\ & \searrow & \downarrow & \dashrightarrow & \downarrow \\ & & R_1 & \xrightarrow{\pi_2} & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_{2,s,1}(R; S) & \xrightarrow{\quad} & \Delta_{2,s,1}S & \xrightarrow{\Delta_{2,s,1}\pi_1} & \Delta_{2,s,1}X \\ & \searrow & \downarrow & & \downarrow \\ & & \Delta_{2,s,1}R & \xrightarrow{\Delta_{2,s,1}\pi_2} & \Delta_{2,s,1}X \end{array}$$

whose top and bottom faces are pullbacks by Lemma 78 and the fact that $\Delta_{2,s,1}: \widehat{\mathbb{V}\mathbb{T}}^2 \rightarrow \widehat{\mathbb{E}\mathbb{T}}$, being a right adjoint, is continuous. Since the right-hand face is a weak pullback by hypothesis, so is the left face by [18, Lemma 9.26, (i), then (ii)]. We finally conclude by [18, Lemma 9.26, (i)]. \blacktriangleleft

► **Lemma 82.** *For any diplopic \mathbb{H} -transition system X , flexible bisimulations $R \rightarrow X_{D,V}^2$ over X are closed under span composition.*

Proof. Given any two flexible bisimulations, say R and S , we observe that there is a projection $\pi: R^\uparrow; S^\uparrow \rightarrow (R; S)^\uparrow$ making the following diagram commute.

$$\begin{array}{ccccc} R^\uparrow; S^\uparrow & \xrightarrow{\quad} & (R; S)^\uparrow & \xrightarrow{\quad} & X_1^2 \\ \downarrow & \searrow \pi & \downarrow & \lrcorner & \downarrow X_\partial \\ \Delta_2(R); \Delta_2(S) & \xrightarrow{\cong} & \Delta_2(R; S) & \longrightarrow & \Delta_2 X^2 \end{array}$$

To see this, we observe that by interchange of limits $R^\uparrow; S^\uparrow$ is the limit of

$$\begin{array}{ccccc}
X_1 & & X_1 & & X_1 \\
\downarrow X_\partial & & \downarrow X_\partial & & \downarrow X_\partial \\
\Delta_2 X & \xleftarrow{\Delta_2 \pi_1} & \Delta_2 X & \xleftarrow{\Delta_2 \pi_1} & \Delta_2 X \\
& \nwarrow \Delta_2 \pi_2 & \nearrow \Delta_2 \pi_2 & \nwarrow \Delta_2 \pi_2 & \nearrow \Delta_2 \pi_2 \\
& \Delta_2 R & & \Delta_2 S & \\
& \nwarrow \Delta_2 \pi_1 & & \nearrow \Delta_2 \pi_1 &
\end{array}$$

while $(R; S)^\sharp$ is the limit of the following subdiagram.

$$\begin{array}{ccccc}
X_1 & & X_1 & & X_1 \\
\downarrow X_\partial & & \downarrow X_\partial & & \downarrow X_\partial \\
\Delta_2 X & \xleftarrow{\Delta_2 \pi_1} & \Delta_2 X & \xleftarrow{\Delta_2 \pi_1} & \Delta_2 X \\
& \nwarrow \Delta_2 \pi_2 & \nearrow \Delta_2 \pi_2 & \nwarrow \Delta_2 \pi_2 & \nearrow \Delta_2 \pi_2 \\
& \Delta_2 R & & \Delta_2 S &
\end{array}$$

Finally, by Lemma 81, we know that $R^\sharp; S^\sharp$ is a flexible bisimulation, hence so is $R; S$ by Lemma 68. \blacktriangleleft

► **Lemma 83.** *For any strong epis $e: X \rightarrow Y$ and $e': X \rightarrow Z$, any mono $f: Y \rightarrow Z$ such that $f \circ e = e'$ is an isomorphism.*

Proof. We find a section of f by lifting as in

$$\begin{array}{ccc}
X & \xrightarrow{e} & Y \\
e' \downarrow & \lrcorner l & \downarrow f \\
Z & \xlongequal{\quad} & Z.
\end{array}$$

But l is in fact an inverse by uniqueness of lifting in

$$\begin{array}{ccccc}
X & \xrightarrow{e} & Y & & \\
\downarrow e & \searrow e' & \nearrow l & \downarrow f & \\
& Z & & & \\
\uparrow f & \nearrow & \downarrow & \uparrow & \\
Y & \xrightarrow{f} & Z & &
\end{array}$$

► **Lemma 84.** *For any diplopic \mathbb{H} -transition system X , flexible bisimulations $R \rightarrow X^2$ over X are closed under images.*

Proof. Both $\widehat{\mathbb{E}\mathbb{T}}$ and $\widehat{\mathbb{V}\mathbb{T}}^2$ are (isomorphic to) presheaf categories, hence images are computed as (strong epi-mono) factorisations. Furthermore, Δ_2 preserves pullbacks by Proposition 67, hence by Lemma 76 the forgetful functor $\mathbb{H}\text{-}\mathbf{Trans}_2 \rightarrow \widehat{\mathbb{E}\mathbb{T}} \times \widehat{\mathbb{V}\mathbb{T}}^2$ creates (strong epi-mono) factorisations, hence images.

Now, consider any flexible bisimulation $p: R \rightarrow X^2$. As we just saw, we obtain a (strong epi-mono) factorisation of p by factoring p_1 and $p_{D,v}$. We then need to show that the square

$$\begin{array}{ccc}
\text{im}(R_1) & \xrightarrow{\quad} & X_1 \\
\downarrow & & \downarrow X_\partial \\
\Delta_{2,s,1}(\text{im}(R_{D,v})) & \xrightarrow{\Delta_{2,s,1}\pi_1} & \Delta_{2,s,1}X
\end{array}$$

is a pointwise weak pullback. But by Proposition 67, $\Delta_{2,1,s}$ preserves epimorphisms. Thus, since the exterior of

$$\begin{array}{ccccc} R_1 & \longrightarrow & \text{im}(R_1) & \longrightarrow & X_1 \\ \downarrow & & \downarrow & & \downarrow X_\theta \\ \Delta_{2,s,1}(R_D, V) & \twoheadrightarrow & \Delta_{2,s,1}(\text{im}(R_D, V)) & \xrightarrow{\Delta_{2,s,1}\pi_1} & \Delta_{2,s,1}X \end{array}$$

is a pointwise weak pullback by hypothesis, we conclude by Lemma 68. \blacktriangleleft

A.2 Composition of flexible and rigid simulations

Our goal in this subsection is to prove the following.

► **Lemma 85.** *For any \mathbb{H} -transition system X , diplopic flexible simulation $R \rightarrow X^2$, and simulation $S_0 \rightarrow X_0^2$, equipped with a span morphism $\rho: R_0; S_0 \rightarrow R_0$, the relation $\text{im}(R; \theta S_0)$ is a flexible simulation, hence so is $\text{im}(R_D, V; S_0)^\dagger$.*

In order to prove this smoothly, we introduce the following notion of triplopic transition system.

► **Definition 86.** *Let $\mathbb{H}\text{-Trans}_3$ denote the lax limit of $\widehat{\mathbb{V}\mathbb{T}}^3 \xrightarrow{\Delta_s \times \Delta_1 \times \Delta_t} \widehat{\mathbb{E}\mathbb{T}}$. Objects of $\mathbb{H}\text{-Trans}_3$ are called **triplopic transition systems**.*

► **Notation 6.** *We denote by $\Delta_3, \Delta_{3,s}, \Delta_{3,s,1}, \dots$ the functors analogous to $\Delta_2, \Delta_{2,s}, \Delta_{2,s,1}, \dots$, and often treat the projection $\mathbb{H}\text{-Trans}_3 \rightarrow \widehat{\mathbb{V}\mathbb{T}}^3$ as an implicit coercion, thus writing, e.g., $\Delta_{3,s,1}X$ for any $X \in \mathbb{H}\text{-Trans}_3$, meaning $\Delta_s(X_s) \times \Delta_1(X_l)$.*

A triplopic transition system X thus consists of presheaves $X_s, X_l, X_t \in \widehat{\mathbb{V}\mathbb{T}}$ and $X_1 \in \widehat{\mathbb{E}\mathbb{T}}$, together with a morphism $X_1 \rightarrow \Delta_s(X_s) \times \Delta_l(X_l) \times \Delta_t(X_t)$.

► **Remark 87.** We use a boldface 2 in $\mathbb{H}\text{-Trans}_2$ and a normal 3 in $\mathbb{H}\text{-Trans}_3$, to reflect the fact that any diplopic transition system $X \in \mathbb{H}\text{-Trans}_2$ comes with a morphism $X_s \rightarrow X_0$, while there is no such requirement for triplopic transition systems.

Let us readily notice the following useful facts.

► **Proposition 88.** *All functors $\Delta_3, \Delta_{3,1}, \Delta_{3,s}, \Delta_{3,t}, \Delta_{3,s,1}, \dots$ are algebraic right adjoints and preserve epimorphisms.*

Proof. Algebraic functors between presheaf categories automatically preserve epimorphisms, so it suffices to prove that all these functors are algebraic right adjoints.

Algebraic right adjoints being closed under pointwise finite products, it further suffices to prove that each of $\Delta_{3,1}$, $\Delta_{3,s}$, and $\Delta_{3,t}$ is an algebraic right adjoint. Now each of these functors $\Delta_{3,x}$ is the corresponding functor Δ_x , precomposed with one of the projections $\widehat{\mathbb{V}\mathbb{T}}^3 \rightarrow \widehat{\mathbb{V}\mathbb{T}}$. But each Δ_x is an algebraic right adjoint by Proposition 66, and projections, being restriction functors, are left and right adjoints, hence algebraic right adjoints, hence the result. \blacktriangleleft

► **Lemma 89.** *The forgetful functor*

$$\mathbb{H}\text{-Trans}_3 \rightarrow \widehat{\mathbb{E}\mathbb{T}} \times \widehat{\mathbb{V}\mathbb{T}}^2$$

creates all colimits and limits, as well as (strong epi)-mono factorisations.

Proof. Just as Lemma 77. \blacktriangleleft

The idea of triplopic transition systems is to unify flexible and rigid bisimulation into a single framework, while allowing maximal flexibility in the choice of input and output states, and labels. Let us now define (bi)simulation in triplopic transition systems. We will then describe embeddings of transition systems and diplopic transition systems into triplopic transition systems, proving in each case that the embedding preserves and reflects bisimulation.

► **Definition 90.** A morphism $f: R \rightarrow X$ of triplopic transition systems is a **functional bisimulation** iff the square

$$\begin{array}{ccc} R_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ \Delta_{3,s,1}R & \longrightarrow & \Delta_{3,s,1}X \end{array}$$

is a pointwise weak pullback. Spans and relations in $\mathbb{H}\text{-}\mathbf{Trans}_3$ are called simulations and bisimulations analogously to the case of $\mathbb{H}\text{-}\mathbf{Trans}_2$.

► **Proposition 91.** Mapping any diplopic transition system

$$(X_1, \gamma: X_s \rightarrow X_0, \partial: X_1 \rightarrow \Delta_s(X_s) \times \Delta_{l,t}(X_0))$$

to

$$(X_1, X_s, X_0, \partial: X_1 \rightarrow \Delta_s(X_s) \times \Delta_{l,t}(X_0))$$

yields an embedding $\iota: \mathbb{H}\text{-}\mathbf{Trans}_2 \rightarrow \mathbb{H}\text{-}\mathbf{Trans}_3$.

Proof. Straightforward. ◀

► **Notation 7.** By composition with $\mathbb{H}\text{-}\mathbf{Trans} \hookrightarrow \mathbb{H}\text{-}\mathbf{Trans}_2$, we obtain a further embedding $\mathbb{H}\text{-}\mathbf{Trans} \hookrightarrow \mathbb{H}\text{-}\mathbf{Trans}_3$. Treating the former as an implicit coercion, we thus often also merely denote the composite by ι .

► **Proposition 92.** A morphism (resp. a span) of diplopic transition systems is a functional bisimulation (resp. a simulation or bisimulation) iff its embedding into triplopic transition systems is.

Proof. Straightforward. ◀

Beyond the embedding $\mathbb{H}\text{-}\mathbf{Trans} \hookrightarrow \mathbb{H}\text{-}\mathbf{Trans}_3$ that we saw above, there is the following embedding of spans:

► **Proposition 93.** For any $X \in \mathbb{H}\text{-}\mathbf{Trans}$, mapping any span $R_0 \rightarrow X_0^2$ in $\widehat{\mathbb{V}\mathbb{T}}$ to the triplopic transition system $\theta(R_0)$ given by (R_0, X_0, R_0) and $\theta(R_0)_1 = R_0^\uparrow$, i.e., given by the pullback

$$\begin{array}{ccc} R_0^\uparrow & \longrightarrow & X_1^2 \\ \downarrow \lrcorner & & \downarrow \\ \Delta_s(R_0) \times \Delta_l(X_0) \times \Delta_t(R_0) & \longrightarrow & \Delta X_0^2, \end{array}$$

extends to an embedding $\theta: \widehat{\mathbb{V}\mathbb{T}}/X_0^2 \rightarrow \mathbb{H}\text{-}\mathbf{Trans}_3/X^2$, which we call the **thin** embedding.

► **Remark 94.** Thinness here refers to labels, which are forced to agree on both sides of any transition in $\theta(R_0)$.

23:28 A more general categorical framework for congruence of applicative bisimilarity

The thin embedding enables the following characterisation of bisimulation in \mathbb{H} -transition systems in terms of bisimulation in triplopic \mathbb{H} -transition systems:

► **Proposition 95.** *For any $X \in \mathbb{H}\text{-Trans}$, a span $R_0 \rightarrow X_0^2$ is a simulation (resp. bisimulation) iff $\theta(R_0) \rightarrow X^2$ is one.*

Proof. Both statements mean that the square

$$\begin{array}{ccc} R_0^\uparrow & \xrightarrow{\pi_1} & X_1 \\ \downarrow & & \downarrow \\ \Delta_s(R_0) \times \Delta_l(X_0) & \xrightarrow{\pi_1} & \Delta_{s,l}X_0 \end{array}$$

is a pointwise weak pullback. ◀

Finally, we have the easy

► **Proposition 96.** *(Bi)simulations are closed under span composition in $\mathbb{H}\text{-Trans}_3$.*

Proof. By symmetry it suffices to show that simulations are closed under span composition. Let us thus consider any simulations R and S over some $X \in \mathbb{H}\text{-Trans}_3$. We must show that the square

$$\begin{array}{ccc} (R; S)_1 & \xrightarrow{\pi_1} & X_1 \\ \downarrow & & \downarrow \\ \Delta_{3,s,l}(R; S) & \xrightarrow{\Delta_{3,s,l}\pi_1} & \Delta_{3,s,l}X \end{array}$$

is a pointwise weak pullback. This square factors as

$$\begin{array}{ccccc} (R; S)_1 & \xrightarrow{\pi_1} & R_1 & \xrightarrow{\pi_1} & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_{3,s,l}(R; S) & \xrightarrow{\Delta_{3,s,l}\pi_1} & \Delta_{3,s,l}(R) & \xrightarrow{\Delta_{3,s,l}\pi_1} & \Delta_{3,s,l}X, \end{array}$$

where the right-hand square is a pointwise weak pullback by hypothesis, and the left-hand square is the left-hand face in

$$\begin{array}{ccccc} (R; S)_1 & \xrightarrow{\quad} & S_1 & \xrightarrow{\pi_1} & X_1 \\ \downarrow & \searrow & \downarrow & \dashrightarrow & \downarrow \\ & R_1 & \xrightarrow{\pi_2} & & X_1 \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \Delta_{3,s,l}(R; S) & \xrightarrow{\quad} & \Delta_{3,s,l}S & \xrightarrow{\Delta_{3,s,l}\pi_1} & \Delta_{3,s,l}X \\ & \searrow & \downarrow & & \downarrow \\ & \Delta_{3,s,l}R & \xrightarrow{\Delta_{3,s,l}\pi_2} & & \Delta_{3,s,l}X \end{array}$$

whose top and bottom faces are pullbacks by Lemma 78 and the fact that $\Delta_{3,s,l}$, being a right adjoint, is continuous. Since the right-hand face is a pointwise weak pullback by hypothesis, so is the left-hand face by [18, Lemma 9.26, (i), then (ii)]. The whole rectangle thus is a pointwise weak pullback by [18, Lemma 9.26, (i)], as desired. ◀

► **Proposition 97.** *Triplopic (bi)simulations are closed under images.*

Proof. By symmetry it suffices to treat the case of simulations. Let $R \rightarrow X^2$ be any triplopic simulation. Then by Proposition 89 we need to prove that the right-hand square below is a pointwise weak pullback,

$$\begin{array}{ccccc}
R_1 & \longrightarrow & \text{im}(R_1) & \hookrightarrow & X_1^2 \\
\downarrow & & \downarrow & & \downarrow \\
\Delta_{3,s,1}R & \longrightarrow & \Delta_{3,s,1}\text{im}(R) & \hookrightarrow & \Delta_{3,s,1}X^2
\end{array}$$

which is the case by Lemma 68 and the fact that $\Delta_{3,s,1}$ preserves epis by algebraicity (Lemma 88). \blacktriangleleft

► **Lemma 98.** *Given a retraction $R \twoheadrightarrow S$ over any X^2 in $\mathbb{H}\text{-Trans}_3$, if S is a simulation, then so is $\text{im}(R)$.*

Proof. The given retraction and its section yield morphisms

$$\text{im}(R) \rightarrow \text{im}(S) \quad \text{and} \quad \text{im}(S) \rightarrow \text{im}(R),$$

hence $\text{im}(R) \cong \text{im}(S)$, so we conclude by Lemma 97. \blacktriangleleft

Proof of Lemma 85. The morphism $\tilde{\rho}: R; \iota S \rightarrow R; \theta S$ defined by the triple

$$\text{id}: R_0; S_0 \rightarrow R_0; S_0 \quad \rho: R_0; S_0 \rightarrow R_0 \quad \text{id}: R_0; S_0 \rightarrow R_0; S_0$$

admits a section, namely the morphism $R; \theta S \rightarrow R; \iota S$ defined by

$$\text{id}: R_0; S_0 \rightarrow R_0; S_0 \quad R_0 \cong R_0; X_0 \rightarrow R_0; S_0 \quad \text{id}: R_0; S_0 \rightarrow R_0; S_0$$

(induced by reflexivity of S_0). Thus, $\text{im}(R; \iota S)$ is a triplopic simulation by Lemma 98, hence a diplopic one by Proposition 92. Finally, $\text{im}(R_{D,V}; S_0)^\uparrow$ is a flexible simulation by Proposition 75. \blacktriangleleft

A.3 Fundamental property of flexible bisimulation

In this section, we reduce the theorem to a certain result involving flexible bisimulations, using the following fundamental property of flexible bisimulation:

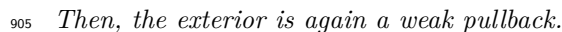
► **Proposition 99.** *For any $X \in \mathbb{H}\text{-Trans}$ and reflexive, flexible bisimulation $R \rightarrow X^2$, $R_0 \rightarrow X_0^2$ is a bisimulation.*

We need the following lemma.

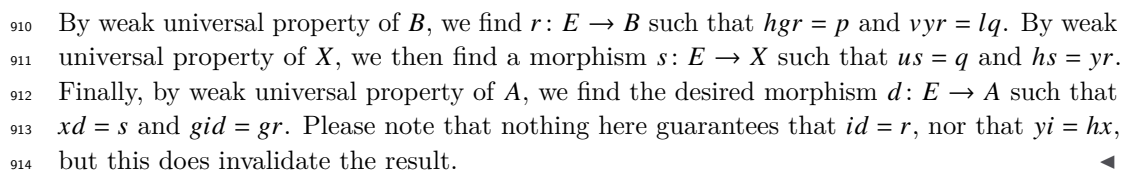
► **Lemma 100.** *Consider any commuting diagram of the following form*

$$\begin{array}{ccccccc}
& & & f & & & \\
& & & \curvearrowright & & & \\
A & & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\
x \downarrow & & y \downarrow & & z \downarrow & & \downarrow w \\
X & \xrightarrow{j} & Y & \xrightarrow{k} & Z & & \\
t \downarrow & & u \downarrow & & v \downarrow & & \\
T & \xrightarrow{l} & U & \xrightarrow{m} & V & \xrightarrow{n} & W
\end{array}$$

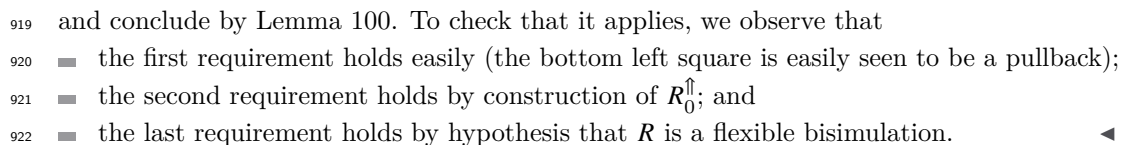
(i.e., all three squares and the rectangle commute, plus $zf = k j x$), such that all three squares below are weak pullbacks.



908 Now, consider any cone (p, q) as shown below.



917



925 **► Corollary 101.** *Consider any syntactic signature $\mathbf{d} = (\Sigma, (\Gamma_i, d_i)_{i \in n})$. Let σ denote the*
926 *generated enhanced syntax $\sigma(\mathbf{d})$. Let X be any σ -transition system, and suppose that there*
927 *exists a reflexive, enhanced, flexible bisimulation relation $R \rightarrow X^2$ such that $\sim_X^\sigma \subseteq R_0$ and R_0*
928 *is a congruence. Then enhanced bisimilarity $\sim_{\mathbf{y}}^\sigma$ is a congruence.*

Proof. Consider any reflexive, enhanced, flexible bisimulation relation $R \rightarrow X^2$ such that R_0 contains enhanced bisimilarity and is a congruence. By Proposition 99, R_0 is an enhanced bisimulation, so by terminality of \sim_X^σ , we have $R_0 \subseteq \sim_X^\sigma$, hence morphisms

$$\Sigma_0(\sim_X^\sigma) \rightarrow \Sigma_0(R_0) \rightarrow R_0 \rightarrow \sim_X^\sigma$$

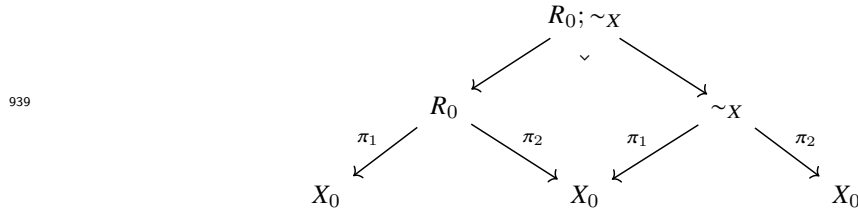
929 over X_0^2 . ◀

930 A.4 Howe closure: basic properties

931 In this section, we introduce our candidate reflexive, enhanced, flexible bisimulation relation
932 $R \rightarrow \mathbf{Z}^2$ such that $\sim_{\mathbf{Z}} \subseteq R_0$ and R_0 is a congruence. As is standard, we

- 933 ■ construct it directly as a congruence,
- 934 ■ prove that it is reflexive and enhanced (relatively easily), and, finally,
- 935 ■ struggle to prove that it (or rather its transitive closure) is a flexible bisimulation.

936 ► **Definition 102.** Let the **Howe functor** $\Sigma_0^{+;\sim X}: \widehat{\mathbb{V}\mathbb{T}}/X_0^2 \rightarrow \widehat{\mathbb{V}\mathbb{T}}/X_0^2$ map any $R_0 \rightarrow X_0^2$ to
937 the coproduct span $\Sigma_0(R_0) + (R_0; \sim_X)$, where the second term more concretely denotes the
938 following composite span.



940 Let the **proof-relevant Howe closure** R_0° be the free $\Sigma_0^{+;\sim X}$ -algebra on R_0 , and the
941 (**proof-irrelevant, or relational**) **Howe closure** R_0^\bullet denote the image of $R_0^\circ \rightarrow X_0^2$.

942 The Howe functor is a finitary endofunctor on a presheaf category, so we have [27]:

943 ► **Proposition 103.** The free $\Sigma_0^{+;\sim X}$ -algebra on any R_0 exists and is computed by the standard
944 initial chain, and the forgetful functor $\Sigma_0^{+;\sim X} \text{-alg} \rightarrow \widehat{\mathbb{V}\mathbb{T}}/X_0^2$ is finitary monadic.

945 ► **Proposition 104.** Let $\mathcal{U}: \mathbf{Sub}(X_0^2) \hookrightarrow \widehat{\mathbb{V}\mathbb{T}}/X_0^2$ denote the canonical embedding, and let
946 $\Sigma = \Sigma_0^{+;\sim X}$ just for this proposition. The composite endofunctor $\text{im} \circ \Sigma^* \circ \mathcal{U}$ on $\mathbf{Sub}(X_0^2)$ is a
947 monad, which is in fact the free monad on $\text{im} \circ \Sigma \circ \mathcal{U}$. Consequently, the relational Howe
948 closure $(\mathcal{U}R)^\bullet$ on a relation $R \in \mathbf{Sub}(X_0^2)$ is the free $(\text{im} \circ \Sigma \circ \mathcal{U})$ -algebra over R .

949 **Proof.** Using algebraicity of Σ_0 , it is straightforward to show that Σ preserves epimorphisms.
950 For any $R \in \widehat{\mathbb{V}\mathbb{T}}/X_0^2$, letting $T = \mathcal{U} \circ \text{im}$ denote the monad induced by the adjunction $\text{im} \dashv \mathcal{U}$,
951 we thus have by unique lifting a morphism $\delta_R: \Sigma TR \rightarrow T\Sigma R$ as in the following diagram,

952

$$\begin{array}{ccc} \Sigma R & \xrightarrow{\quad} & \mathcal{U} \text{im } \Sigma R \\ \Sigma e \downarrow & \nearrow \delta_R & \downarrow \\ \Sigma \mathcal{U} \text{im } R & \xrightarrow{\Sigma m} & \Sigma(X_0^2) \longrightarrow X_0^2 \end{array}$$

where $R \rightarrow X_0^2$ factors as $e \circ m$ and the last horizontal morphism is

$$\Sigma_0^{+;\sim X}(X_0^2) = \Sigma_0(X_0^2) + (X_0^2); \sim_X \xrightarrow{[\langle a \circ \Sigma_0(\pi_1), a \circ \Sigma_0(\pi_2) \rangle, \langle \pi_1 \circ \pi_1, \pi_2 \circ \pi_2 \rangle]} X_0^2.$$

953 The result thus follows from the next lemma. ◀

► **Lemma 105.** Consider a full, reflective embedding $U: \mathcal{D} \hookrightarrow \mathcal{C}$ from some poset \mathcal{D} into a locally finitely presentable category \mathcal{C} , say with left adjoint $L: \mathcal{C} \rightarrow \mathcal{D}$, together with a finitary endofunctor Σ on \mathcal{C} . Furthermore, assume given a functor distributive law, i.e., a natural transformation $\delta: \Sigma T \rightarrow T\Sigma$, where $T := UL$ denotes the induced monad. Then, $L\Sigma^*U$ is the free monad on $L\Sigma U$, hence in particular the free $L\Sigma U$ -algebra on any $D \in \mathcal{D}$ is $L\Sigma^*UD$.

► **Lemma 106.** In the setting of Lemma 105, all objects of the form $UD \in \mathcal{C}$ are *subterminal*, in the sense that any two parallel morphisms to UD are equal.

Proof. Consider any $f, g: C \rightarrow UD$. By adjunction, these correspond bijectively to morphisms $\tilde{f}, \tilde{g}: LC \rightarrow D$, which, because \mathcal{D} is a poset, are equal. ◀

Proof of Lemma 105. By [27], Σ admits a free monad Σ^* .

Furthermore, by Lemma 106, the given functor distributive law δ is in fact a **functor-monad distributive law**, in the sense that it commutes with the unit and multiplication of T .

Now, by a reasoning analogous to [3], functor-monad distributive laws $\delta: \Sigma T \rightarrow T\Sigma$ correspond bijectively to liftings of the monad T to Σ -**alg**, i.e., monads T^δ on Σ -**alg** making the following square commute,

$$\begin{array}{ccc} \Sigma\text{-alg} & \xrightarrow{T^\delta} & \Sigma\text{-alg} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{T} & \mathcal{C} \end{array}$$

whose multiplication and unit are mapped by the forgetful functor to those of T . The given functor-monad distributive law δ thus corresponds to such a lifting. But $\Sigma\text{-alg} \cong \Sigma^*\text{-Alg}$ over \mathcal{C} , hence we get a lifting of T to $\Sigma^*\text{-Alg}$, which by [3] again amounts to a monad distributive law, say $\bar{\delta}: \Sigma^*T \rightarrow T\Sigma^*$.

From this, using the fact that the counit is an isomorphism (which follows from full faithfulness of U), we equip the composite $L\Sigma^*U$ with monad structure:

- the unit is the composite $R \xrightarrow{(\varepsilon^T)^{-1}} LUR \xrightarrow{L\eta^{\Sigma^*}} L\Sigma^*UR$,
- the multiplication is

$$L\Sigma^*UL\Sigma^*UR = L\Sigma^*T\Sigma^*UR \xrightarrow{L\bar{\delta}} LT\Sigma^*\Sigma^*UR \xrightarrow{\varepsilon U \mu^{\Sigma^*} UR} L\Sigma^*UR,$$

- and the monad laws hold automatically since \mathcal{D} is a poset.

Moreover, given any $R \in \mathcal{D}$, the following are equivalent

- Σ^* -algebra structure (in the monad sense) on UR ,
- Σ^* -algebra structure (in the functor sense) on UR ,
- Σ -algebra structure on UR ,
- $L\Sigma^*U$ -algebra structure (in the monad sense) on R ,
- $L\Sigma^*U$ -algebra structure (in the functor sense) on R ,
- $L\Sigma U$ -algebra structure on R .

Indeed,

- Σ -algebra structure $\Sigma UR \rightarrow UR$ corresponds by adjunction to $L\Sigma U$ -algebra structure $L\Sigma UR \rightarrow R$;
- Σ -algebra structure $\Sigma UR \rightarrow UR$ corresponds by universal property of Σ^* to Σ^* -algebra structure $\Sigma^*UR \rightarrow UR$ in the monad sense;

- 992 ■ by subterminality, Σ^* -algebra structures $\Sigma^*UR \rightarrow UR$ in the monad and functor sense
- 993 are equivalent;
- 994 ■ by adjunction again, Σ^* -algebra structure $\Sigma^*UR \rightarrow UR$ in the functor sense is equivalent
- 995 to $L\Sigma^*U$ -structure $L\Sigma^*UR \rightarrow R$ in the functor sense;
- 996 ■ and finally, because \mathcal{D} is a poset, $L\Sigma^*U$ -structures $L\Sigma^*UR \rightarrow R$ in the functor and monad
- 997 sense are equivalent.

998 We thus in particular get $(L\Sigma^*U)\text{-Alg} \cong (L\Sigma U)\text{-alg}$ over \mathcal{D} , hence the result. ◀

999 ▶ **Definition 107.** Let $S^{+;\sim x}$ denote the monad induced by $\Sigma_0^{+;\sim x}$ on $\widehat{\mathbb{V}\mathbb{T}}/X_0^2$.

1000 ▶ **Lemma 108.** Let R'_0 be the proof-relevant (resp. proof-irrelevant) Howe closure R_0° (resp. R_0^\bullet) of (resp. a relation) R_0 . It satisfies the following properties.

- 1002 (i) R'_0 is a Σ_0 -algebra;
- 1003 (ii) there exists an action $R'_0; \sim_X \rightarrow R'_0$ over X_0^2 .
- 1004 Furthermore, if $X_0 = \Sigma_0^*(\emptyset)$ is the initial Σ_0 -algebra, we have:
- 1005 (iii) R'_0 is reflexive,
- 1006 (iv) there exists a morphism $\sim_X \rightarrow R'_0$ over X_0^2 .

1007 **Proof.** We prove the properties for the proof-relevant Howe closure – they follow easily for

1008 the proof-irrelevant one.

- 1009 (i) By definition R_0° is an $\Sigma_0^{+;\sim x}$ -algebra, hence in particular a Σ_0 -algebra, or more correctly
- 1010 an algebra for the obvious lifting of Σ_0 to $\widehat{\mathbb{V}\mathbb{T}}/X_0^2$.
- 1011 (ii) As an $\Sigma_0^{+;\sim x}$ -algebra, R_0° is an algebra for the second term functor, i.e., a morphism of
- 1012 the desired form $R_0^\circ; \sim_X \rightarrow R_0^\circ$.

1013 Let us now assume that X_0 is the initial Σ_0 -algebra. Then, by initiality of X_0 and (i), there

1014 is a unique Σ_0 -algebra morphism $X_0 \rightarrow R_0^\circ$, which witnesses reflexivity.

We then use reflexivity and (ii) to construct the following composite

$$\sim_X \cong X_0; \sim_X \rightarrow R_0^\circ; \sim_X \rightarrow R_0^\circ,$$

1015 which proves the second point. ◀

1016 A further crucial property is:

1017 ▶ **Proposition 109.** If X_0 is an ST-algebra, then the proof-relevant Howe closure R_0° on any

1018 R_0 is an ST-algebra, and $R_0^\circ \rightarrow X_0^2$ is a morphism of ST-algebras. Furthermore, the relational

1019 Howe closure R_0^\bullet is enhanced.

1020 In order to prove this, we need a few intermediate steps.

▶ **Definition 110.** For any bifunctor F on a category \mathbf{C} and $F\Delta$ -algebra X , let \bar{F} denote the lifting of F to \mathbf{C}/X^2 , which maps any $U \rightarrow X^2$ and $V \rightarrow X^2$ to the composite

$$F(U, V) \rightarrow F(X^2, X^2) \rightarrow F(X, X)^2 \rightarrow X^2.$$

▶ **Lemma 111.** For any bifunctor Γ on a category \mathbf{C} with pullbacks, object $X \in \mathbf{C}$, and spans $u_i: U_i \rightarrow X^2$, for $i \in 3$, there is a morphism

$$\Gamma((U_1; U_2), U_3) \rightarrow \Gamma(U_1, U_3); \Gamma(U_2, X)$$

1021 of spans over X .

1022 **Proof.** We construct the desired morphism by universal property of pullback, as in the
 1023 following diagram.

$$\begin{array}{ccccc}
 \Gamma((U_1; U_2), U_3) & \xrightarrow{\quad} & \Gamma(U_2, U_3) & & \\
 \downarrow & \searrow \text{dashed} & \downarrow \Gamma(\pi_1, U_3) & \searrow \Gamma(U_2, \pi_2) & \\
 & \Gamma(U_1, U_3); \Gamma(U_2, X) & \xrightarrow{\pi_2} & \Gamma(U_2, X) & \\
 \downarrow \Gamma(\pi_2, U_3) & \downarrow \pi_1 & \downarrow \Gamma(X, \pi_2) & \downarrow \Gamma(\pi_1, X) & \\
 \Gamma(U_1, U_3) & \xrightarrow{\Gamma(\pi_2, U_3)} & \Gamma(X, U_3) & \xrightarrow{\Gamma(X, \pi_2)} & \Gamma(X, X) \\
 \parallel & \downarrow \Gamma(\pi_2, \pi_2) & \downarrow \Gamma(\pi_2, \pi_2) & & \\
 \Gamma(U_1, U_3) & \xrightarrow{\Gamma(\pi_2, \pi_2)} & \Gamma(X, X) & &
 \end{array}$$

1025

► **Lemma 112.** Assume that X_0 is a σ -algebra with structure given by

$$a: \Sigma_0 X_0 \rightarrow X_0 \quad \dots \quad b_i: \Gamma_i(X_0, X_0) \rightarrow X_0 \quad \dots,$$

and let the derived monad algebra structures be as follows.

$$\bar{a}: SX_0 \rightarrow X_0 \quad \dots \quad \bar{b}_{<i}: T_i X_0 \rightarrow X_0 \quad \dots$$

Then, for all $i \in n$, the incremental structural law

$$d_i: \Gamma_i(\Sigma_0 A, B) \rightarrow ST_i(\Gamma_i(A, ST_i B) + A + B)$$

lifts to an incremental structural law

$$\bar{d}_i: \bar{\Gamma}_i(\Sigma_0^{+;\sim x} A, B) \rightarrow S^{+;\sim x} \bar{T}_i(\bar{\Gamma}_i(A, S^{+;\sim x} \bar{T}_i B) + A + B).$$

1026 **Proof.** By Lemma 111, using left-cocontinuity of Γ_i , and the fact that \sim_X is enhanced. ◀

Proof of Proposition 109. By Proposition 34, there exists a distributive law

$$\bar{T}_{n+1} S^{+;\sim x} \rightarrow S^{+;\sim x} \bar{T}_{n+1}$$

1027 and \bar{T}_{n+1} is constant-free, hence the natural transformation $S^{+;\sim x} \rightarrow S^{+;\sim x} \bar{T}_{n+1}$ is an iso-
 1028 morphism at \emptyset . The proof-relevant Howe closure $R_0^\odot = S^{+;\sim x} \emptyset$ thus acquires a canonical
 1029 $S^{+;\sim x} \bar{T}_{n+1}$ -algebra structure. The terminal object also is one, of course, and the unique
 1030 morphism to it is a $S^{+;\sim x} \bar{T}_{n+1}$ -algebra morphism, which completes the proof of the first point.

The proof-relevant Howe closure is in particular enhanced via

$$\Gamma_i(R_0^\odot, X) \rightarrow \Gamma_i(R_0^\odot, R_0^\odot) \rightarrow R_0^\odot,$$

1031 which entails enhancedness for the relational Howe closure by the fact that each Γ_i , being
 1032 left-cocontinuous, preserves epimorphisms in its first argument, and that all epimorphisms
 1033 are strong in presheaf categories. Indeed, we find the desired morphism by lifting as in the
 1034 following diagram.

$$\begin{array}{ccccccc}
 \Gamma(R_0^\odot, X_0) & \longrightarrow & \Gamma(R_0^\odot, R_0^\odot) & \longrightarrow & R_0^\odot & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \Gamma(R_0^\bullet, X_0) & \xleftarrow{\quad} & \Gamma(R_0^\bullet, R_0^\bullet) & \xrightarrow{\quad} & R_0^\bullet & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \Gamma(X_0^2, X_0) & \longrightarrow & \Gamma(X_0^2, X_0^2) & \longrightarrow & \Gamma(X_0, X_0)^2 & \longrightarrow & X_0^2
 \end{array}$$

1035

1036

1037 A final basic property is about symmetry of the relational transitive closure of the
 1038 relational Howe closure on the syntactic transition system (Proposition 116 below).

1039 ► **Definition 113** ([18, Definition 9.5]). The **relational transitive closure** R_0^+ of a span
 1040 $R_0 \rightarrow X_0^2$ is the union $\bigcup_{n>0} \text{im}(R_0^n)$, where $(-)^n$ denotes iterated self-composition of spans.

1041 ► **Proposition 114.** For any span $R_0 \rightarrow X_0^2$, the relational transitive closure R_0^+ is equipped
 1042 with an action $R_0; R_0^+ \rightarrow R_0^+$ over X_0^2 .

1043 The proof relies on the following lemma.

1044 ► **Lemma 115.** In any complete, cocomplete, regular, and locally cartesian closed category,
 1045 hence in particular in any presheaf category,

- 1046 (i) span composition preserves all colimits, on both sides, and
- 1047 (ii) sequential composition of relations preserves all unions, on both sides.

1048 **Proof.** The pullback functor (along the relevant projection), being a left adjoint, is cocon-
 1049 tinuous, which directly entails the first point. For the second point, in a regular category,
 1050 the pullback functor preserves regular epis and monos, hence image factorisations. ◀

Proof of Proposition 114. We have

$$R_0; \bigcup_{n>0} \text{im}(R_0^n) \twoheadrightarrow \text{im}(R_0); \bigcup_{n>0} \text{im}(R_0^n) \cong \bigcup_{n>1} \text{im}(R_0^n) \hookrightarrow \bigcup_{n>0} \text{im}(R_0^n),$$

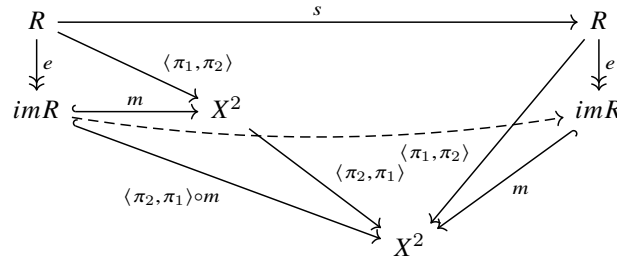
1051 where the isomorphism holds by Lemma 115(ii). ◀

1052 ► **Proposition 116.** Let again $X_0 = \Sigma_0^*(\emptyset)$. Then the relational transitive closure $\emptyset^{\bullet+}$ of the
 1053 proof-irrelevant Howe closure of \emptyset is symmetric.

1054 ► **Lemma 117** ([18, Lemma 9.10]). For any span $R_0 \rightarrow X_0^2$, if there exists a span morphism
 1055 $R_0 \rightarrow R_0^+$, then R_0^+ is symmetric.

1056 ► **Lemma 118.** If a span R is symmetric, in the sense that there is a morphism $R^+ \rightarrow R$
 1057 over X_0^2 , then so is its induced relation.

1058 **Proof.** We proceed as in the following diagram.



1059

1060 **Proof of Proposition 116.** By the lemma, it suffices to construct a morphism $\emptyset^\bullet \rightarrow \emptyset^{\bullet+}$.
 1061 Thus, by Proposition 104, it suffices to endow $\emptyset^{\bullet+}$ with algebra structure for the endofunctor
 1062 $S \mapsto \text{im}(\Sigma_0^{+;\sim X}(S))$ on $\mathbf{Sub}(X_0^2)$. For this, because $\Sigma_0^{+;\sim X}$ is algebraic, it suffices to endow $\emptyset^{\bullet+}$
 1063 with $\Sigma_0^{+;\sim X}$ -algebra structure.

23:36 A more general categorical framework for congruence of applicative bisimilarity

We first equip it with $(-; \sim_X)$ -algebra structure. We need to find a morphism $\mathcal{O}^{\bullet\ddagger}; \sim_X \rightarrow \mathcal{O}^{\bullet\ddagger}$ over X_0^2 , or equivalently by applying the involution $(-)^{\dagger}$, a morphism $\sim_X^{\dagger}; \mathcal{O}^{\bullet\ddagger} \rightarrow \mathcal{O}^{\bullet\ddagger}$. We pick the composite

$$\sim_X^{\dagger}; \mathcal{O}^{\bullet\ddagger} \rightarrow \sim_X; \mathcal{O}^{\bullet\ddagger} \rightarrow \mathcal{O}^{\bullet}; \mathcal{O}^{\bullet\ddagger} \rightarrow \mathcal{O}^{\bullet\ddagger},$$

1064 where

- 1065 ■ the first morphism is symmetry of \sim_X ,
- 1066 ■ the second morphism is that of Lemma 108,
- 1067 ■ the last morphism is the action from Proposition 114.

1068 This leaves us with the task of equipping $\mathcal{O}^{\bullet\ddagger}$ with algebra structure for the lifting of Σ_0
 1069 to $\mathbf{Sub}(X_0^2)$, for which it suffices, by algebraicity of Σ_0 , to equip it with algebra structure for
 1070 the lifting of Σ_0 to $\widehat{\mathbb{V}\mathbb{T}}/X_0^2$, say $\bar{\Sigma}_0$. By [18, Corollary 9.8], we have $\mathcal{O}^{\bullet\ddagger} \cong \mathcal{O}^{\bullet\ddagger}$, and by [18,
 1071 Lemma 9.9], $\mathcal{O}^{\bullet\ddagger}$ is the colimit of the chain

$$1072 \quad X_0 \rightarrow \text{im}(\mathcal{O}^{\bullet\ddagger}) \cong \text{im}(\mathcal{O}^{\bullet\ddagger}; X_0) \rightarrow \text{im}(\mathcal{O}^{\bullet\ddagger}; \mathcal{O}^{\bullet\ddagger}) \cong \text{im}(\mathcal{O}^{\bullet\ddagger}; \mathcal{O}^{\bullet\ddagger}; X_0) \rightarrow \text{im}(\mathcal{O}^{\bullet\ddagger}; \mathcal{O}^{\bullet\ddagger}; \mathcal{O}^{\bullet\ddagger}) \rightarrow \dots$$

1073 in $\widehat{\mathbb{V}\mathbb{T}}/X_0^2$. But $\bar{\Sigma}_0$ is algebraic, hence the forgetful functor $\bar{\Sigma}_0\text{-alg} \rightarrow \widehat{\mathbb{V}\mathbb{T}}/X_0^2$ creates filtered
 1074 colimits, hence in particular colimits of chains. It thus suffices to lift the above chain to
 1075 $\bar{\Sigma}_0\text{-alg}$. Furthermore, because all objects of the chain are relations, they are subterminal,
 1076 hence all morphisms will automatically lift to $\bar{\Sigma}_0\text{-alg}$ if the objects do. Finally, the forgetful
 1077 functor $\bar{\Sigma}_0\text{-alg} \rightarrow \widehat{\mathbb{V}\mathbb{T}}/X_0^2$ creates limits, and \mathcal{O}^{\bullet} possesses Σ_0 -algebra structure by Lemma 108,
 1078 hence so does $\mathcal{O}^{\bullet\ddagger}$. ◀

1079 To conclude this section, we use the basic facts we just proved to reduce the main result
 1080 to the fact that \mathcal{O}^{\bullet} is a flexible simulation.

1081 ► **Proposition 119.** *Consider any syntactic signature $\mathbf{d} = (\Sigma, (\Gamma_i, d_i)_{i \in n})$, and suppose that*
 1082 *$\mathcal{O}_{\mathbf{Z}}^{\bullet}$ is a flexible simulation. Then enhanced bisimilarity on \mathbf{Z} is a congruence.*

1083 ► **Remark 120.** Let us recall that by Definition 74, $R_0 \rightarrow X_0^2$ in $\widehat{\mathbb{V}\mathbb{T}}$ is a flexible simulation
 1084 when its cartesian lifting $R_0^{\uparrow} \rightarrow X^2$ is.

1085 We will rely on the following lemmas.

1086 ► **Lemma 121.** *Consider any commutative diagram of functors between locally small categories*

$$1087 \quad \begin{array}{ccc} \mathbf{A} & \xrightarrow{U} & \mathbf{B} \\ & \searrow V & \swarrow W \\ & \mathbf{C} & \end{array}$$

1088 *If V and W create colimits of a certain shape D , and W preserves them (typically if \mathbf{C} has*
 1089 *them), then U creates them.*

1090 **Proof.** Consider any functor $J: D \rightarrow \mathbf{A}$ and colimiting cocone $K: D^{\top} \rightarrow \mathbf{B}$ for $U \circ J$. Because
 1091 W preserves colimits of shape D , $W \circ K$ is colimiting for $W \circ U \circ J$, hence because V creates
 1092 colimits, we find a unique lifting J^{\uparrow} such that $J^{\uparrow} \circ I = J$ and $V \circ J^{\uparrow} = L := W \circ K$, as in the
 1093 following diagram.

$$1094 \quad \begin{array}{ccccc} D & \xrightarrow{J} & \mathbf{A} & \xrightarrow{U} & \mathbf{B} \\ \downarrow I & \nearrow J^{\uparrow} & \downarrow V & \nearrow & \downarrow W \\ D^{\top} & \xrightarrow{K} & \mathbf{C} & \xlongequal{\quad} & \mathbf{C} \\ & \searrow L & & & \end{array}$$

1095 But now $U \circ J^\uparrow$ and K both are candidate liftings for the outer rectangle, so by uniqueness in
 1096 the creation of colimits by W they are equal, and thus J^\uparrow is a lifting for the original square
 1097 (J, K) .

1098 Furthermore, any lifting for (J, K) induces one for (J, L) , hence should be equal to J^\uparrow ,
 1099 which proves uniqueness.

1100 Finally, J^\uparrow is colimiting because V creates colimits of shape D . ◀

1101 ▶ **Definition 122.** Given a bifunctor $\Gamma: \mathbf{C}^2 \rightarrow \mathbf{C}$ and an object X , a (Γ, X) -premodule is
 1102 an object M equipped with an **action**, i.e., a morphism $r: \Gamma(M, X) \rightarrow M$. A morphism of
 1103 (Γ, X) -premodules is a morphism commuting with action. We let (Γ, X) -Mod denote the
 1104 category of (Γ, X) -premodules.

1105 ▶ **Terminology 1.** When Γ is clear from context, we often omit it and talk about X -premodules
 1106 and X -Mod.

1107 ▶ **Remark 123.** An enhanced span $R \rightarrow X^2$ as in Definition 29 is a span in the category of
 1108 X -premodules.

1109 ▶ **Lemma 124.** If Γ is left-cocontinuous and \mathbf{C} is locally finitely presentable and regular,
 1110 then the category X -Mod is regular and the forgetful functor X -Mod $\rightarrow \mathbf{C}$ creates all limits
 1111 and colimits, as well as image factorisations.

1112 **Proof.** Creation of limits and colimits follows easily from the fact that X -Mod is the category
 1113 of algebras for the cocontinuous endofunctor $\Gamma(-, X)$.

1114 In particular, X -Mod is complete and cocomplete, hence regularity reduces to showing
 1115 that regular epis are stable under pullback.

1116 Let us first appeal to [18, §1.6.5] for definitions and preliminary results about images.
 1117 Notably, in a locally finitely presentable category, (strong epi)-mono factorisations yield
 1118 image factorisations, and union may be computed by cotupling followed by (strong epi)-mono
 1119 factorisation.

1120 Let us then consider any pullback square

$$\begin{array}{ccc}
 A & \xrightarrow{u} & B \\
 v \downarrow & \lrcorner & \downarrow f \\
 C & \xrightarrow{g} & D
 \end{array}$$

1121

1122 in X -Mod, with f a regular epi, and show that v must also be a regular epi. By creation,
 1123 hence preservation, of limits and colimits, the given pullback square is also a pullback in \mathbf{C}
 1124 and f is a regular epi there too. So by regularity of \mathbf{C} , v is a regular epi in \mathbf{C} . Equivalently,
 1125 it is a coequaliser of its kernel pair. But by creation of limits the kernel pair uniquely lifts to
 1126 a kernel pair in X -Mod, and by creation of colimits v is a coequaliser there too. This shows
 1127 that X -Mod is regular.

1128 Finally, let us prove that the forgetful functor creates image factorisations. Given
 1129 $A, C \in X$ -Mod, let us consider any image factorisation $A \xrightarrow{e} B \xrightarrow{m} C$ in \mathbf{C} of a morphism
 1130 $f: A \rightarrow C$ in X -Mod, i.e., e is a regular epi and m is a mono in \mathbf{C} . In this situation, e is the
 1131 coequaliser of its kernel pair in \mathbf{C} , but, as we just saw, this kernel pair lifts to a kernel pair in
 1132 X -Mod, whose coequaliser is created by the forgetful functor, hence e is a coequaliser, hence
 1133 a regular epi in X -Mod. Finally, f also coequalises the kernel pair, hence the existence of
 1134 a unique mediating morphism $B \rightarrow C$ in X -Mod, which must be m by faithfulness of the
 1135 forgetful functor X -Mod $\rightarrow \mathbf{C}$. Thus, m is also a morphism in X -Mod. Finally, its monicity
 1136 follows again by faithfulness of the forgetful functor. ◀

1137 ► **Lemma 125.** *If \mathbf{C} is regular, then enhanced spans are stable under images, that is if*
 1138 *$p: R \rightarrow X^2$ is enhanced, then so is $\text{im}(p): \text{im}(R) \hookrightarrow X^2$.*

1139 **Proof.** By Lemma 124 (creation of image factorisations). ◀

1140 ► **Lemma 126.** *The forgetful functor $X\text{-Mod}/X^2 \rightarrow \mathbf{C}/X^2$ creates colimits. Hence, in*
 1141 *particular (by cocompleteness of \mathbf{C}/X^2), enhanced spans are closed under all colimits in*
 1142 *\mathbf{C}/X^2 .*

1143 **Proof.** Consider the following commutative diagram in **CAT**.

$$\begin{array}{ccc} X\text{-Mod}/X^2 & \longrightarrow & \mathbf{C}/X^2 \\ \downarrow & & \downarrow \\ X\text{-Mod} & \longrightarrow & \mathbf{C} \end{array}$$

1145 Colimits are created by both (vertical) projection functors, and also by the bottom functor by
 1146 Lemma 124. Furthermore, \mathbf{C} being cocomplete, the projection functor $\mathbf{C}/X^2 \rightarrow \mathbf{C}$ preserves
 1147 all colimits, hence by Lemma 121 the top functor creates them. ◀

1148 ► **Lemma 127.** *For any syntactic signature $\mathbf{d} = (\Sigma, (\Gamma_i, d_i)_{i \in n})$ and $X \in \sigma(\mathbf{d})\text{-Trans}$, if*
 1149 *$R_0 \hookrightarrow X_0^2$ in $\widehat{\mathbb{V}\mathbb{T}}$ is a reflexive, enhanced flexible simulation relation, then so is R_0^+ .*

1150 **Proof.** Reflexivity is clear. For enhancedness, we have seen in Lemmas 125 and 126 that
 1151 enhanced spans are closed under images and coproducts. Furthermore, closedness under span
 1152 composition follows directly by Lemma 111. Finally, in order to show that R_0^+ is a flexible
 1153 simulation, we adopt the characterisation of [18, Lemma 9.9], by which R_0^+ is the colimit of
 1154 the chain

$$X_0 \rightarrow \text{im}(R_0) \cong \text{im}(R_0; X_0) \rightarrow \text{im}(R_0; R_0) \cong \text{im}(R_0; R_0; X_0) \rightarrow \text{im}(R_0; R_0; R_0) \rightarrow \dots$$

1156 in $\widehat{\mathbb{V}\mathbb{T}}/X_0^2$. By Corollary 80, it suffices to show that each $\text{im}(R_0^{i;n})$ is a flexible simulation. By
 1157 Lemma 84, it further suffices to show that each $R_0^{i;n}$ is a flexible simulation. By induction
 1158 and Lemma 82, it finally suffices to show that R_0 is a flexible simulation, which it is by
 1159 hypothesis. ◀

1160 **Proof of Proposition 119.** By hypothesis $\mathbf{0}^\bullet$ is a flexible simulation. It is also enhanced
 1161 by Proposition 109. Let now $R_0 := \mathbf{0}^{\bullet+}$, which is again a flexible enhanced simulation
 1162 by Lemma 127. By Proposition 116, R_0 is moreover symmetric. But any symmetric
 1163 simulation is in fact a bisimulation, so R_0 is a flexible enhanced bisimulation. Furthermore,
 1164 R_0 contains $\sim_{\mathbf{Z}}$ by Lemma 108(iv), and is a congruence by Lemma 108(i). We thus conclude
 1165 by Corollary 101. ◀

1166 A.5 The key lemma

1167 We at last introduce the key lemma, which will directly lead us to a proof of Theorem 52.

1168 ► **Lemma 128.** *For any syntactic signature $\mathbf{d} = (\Sigma, (\Gamma_i, d_i)_{i \in n})$, if Σ_1 preserves functional*
 1169 *flexible bisimulations, then the cartesian lifting $\mathbf{0}_{\mathbf{Z}}^{\bullet\uparrow}$ of $\mathbf{0}_{\mathbf{Z}}^\bullet$ is a flexible simulation.*

1170 Before proving the lemma, let us prove the main theorem, as promised:

1171 **Proof of Theorem 52.** By Proposition 119, it suffices to prove that $\mathbf{0}_{\mathbf{Z}}^\bullet$ is a flexible simulation,
 1172 which is the case by Lemma 128. ◀

1173 The rest of this section is a proof of Lemma 128.

1174 ► **Notation 8.** We abbreviate $\emptyset_{\mathbf{Z}}^{\odot}$ to \emptyset^{\odot} and $\sim_{\mathbf{Z}}^{\sigma(\mathbf{d})}$ to \sim .

1175 In order to prove that $\emptyset_{\mathbf{Z}}^{\bullet}$ is a simulation, it suffices to prove that $\emptyset_{\mathbf{Z}}^{\odot}$ is, by Lemma 68.
Briefly, we will construct an ω -chain of flexible simulations of the form

$$\check{\Sigma}_1^n(\mathbf{Z}_0) \leftarrow R^n \rightarrow \mathbf{Z},$$

1176 whose projection to $\widehat{\mathbb{V}\mathbb{T}}$ is the constant chain on

$$1177 \quad \mathbf{Z}_0 \leftarrow \emptyset^{\odot} \rightarrow \mathbf{Z}_0. \quad (6)$$

By construction, the colimit of this chain will be a flexible simulation

$$\mathbf{Z} \leftarrow R^{\infty} \rightarrow \mathbf{Z}$$

with projection

$$\mathbf{Z}_0 \leftarrow \emptyset^{\odot} \rightarrow \mathbf{Z}_0,$$

1178 which entails by Lemma 70 that \emptyset^{\odot} is a flexible simulation as desired.

1179 For this, let us construct a category whose objects are spans of a similar form.

► **Definition 129.** Let $\mathbf{Span}/\emptyset^{\odot}$ denote the limit of the diagram

$$\widehat{\mathbb{E}\mathbb{T}}/\Delta\mathbf{Z}_0 \xleftarrow{\widehat{\mathbb{E}\mathbb{T}}/\Delta\pi_1} \widehat{\mathbb{E}\mathbb{T}}/\Delta\emptyset^{\odot} \xrightarrow{\widehat{\mathbb{E}\mathbb{T}}/\Delta\pi_2} \widehat{\mathbb{E}\mathbb{T}}/\Delta\mathbf{Z}_0 \xleftarrow{\mathbf{Z}} 1$$

weighted by

$$2 \xleftarrow{0} 1 \xrightarrow{0} 2 \xleftarrow{1} 1$$

1180 ► **Remark 130.** A weighted cone from some category A is thus a diagram of the form

$$1181 \quad \begin{array}{ccccc} & & A & \xrightarrow{!} & 1 \\ & \swarrow Y & \downarrow X & \xRightarrow{\quad} & \downarrow Z \\ \widehat{\mathbb{E}\mathbb{T}}/\Delta\mathbf{Z}_0 & \xleftarrow{\widehat{\mathbb{E}\mathbb{T}}/\Delta\pi_1} & \widehat{\mathbb{E}\mathbb{T}}/\Delta\emptyset^{\odot} & \xrightarrow{\widehat{\mathbb{E}\mathbb{T}}/\Delta\pi_2} & \widehat{\mathbb{E}\mathbb{T}}/\Delta\mathbf{Z}_0 \end{array}$$

1182 Hence, objects of the weighted limit are spans of the form $Y \leftarrow X \rightarrow \mathbf{Z}$ over (6), and a
1183 morphism from such a span to some span $Y' \leftarrow X' \rightarrow \mathbf{Z}$ is a pair $(g: Y_1 \rightarrow Y'_1, f: X_1 \rightarrow X'_1)$
1184 of morphisms in $\widehat{\mathbb{E}\mathbb{T}}$ making the following diagram commute.

$$1185 \quad \begin{array}{ccccc} Y_1 & \xleftarrow{\quad} & X_1 & \xrightarrow{\quad} & \mathbf{Z}_1 \\ & \searrow g & \downarrow & \searrow f & \downarrow \\ & & Y'_1 & \xleftarrow{\quad} & X'_1 \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ \Delta\mathbf{Z}_0 & \xleftarrow{\quad} & \Delta\emptyset^{\odot} & \xrightarrow{\quad} & \Delta\mathbf{Z}_0 \end{array}$$

1186 ► **Proposition 131.** The forgetful functor to $\widehat{\mathbb{E}\mathbb{T}}^2$ mapping any span $Y \leftarrow X \rightarrow \mathbf{Z}$ to (Y_1, X_1)
1187 creates colimits and connected limits.

1188 **Proof.** Straightforward. ◀

1189 ► **Proposition 132.** *The category $\mathbf{Span}/\emptyset^\odot$ has as initial object the span $\mathbf{Z}_0 \leftarrow \emptyset^\odot \rightarrow \mathbf{Z}$.*

1190 Since $\mathbf{Z}_0 = \check{\Sigma}_1^n(\mathbf{Z}_0)$, this span has the desired form, and its left-hand leg is trivially a
 1191 functional flexible bisimulation, so we may take it as our R^0 .

1192 ► **Definition 133.** *Let F denote the endofunctor on $\mathbf{Span}/\emptyset^\odot$ that maps any object*

$$\begin{array}{ccccc} Y_1 & \longleftarrow & X_1 & \longrightarrow & \mathbf{Z}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta \mathbf{Z}_0 & \longleftarrow & \Delta \emptyset^\odot & \longrightarrow & \mathbf{Z}_0 \end{array}$$

1194 to

$$\begin{array}{ccccc} \Sigma_1(Y)_1 & \longleftarrow & \Sigma_1(X)_1; (\sim^*)^\uparrow & \longrightarrow & \mathbf{Z}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_{2,s,l,t}(\Sigma_0^?(\mathbf{Z}_0), \mathbf{Z}_0) & \longleftarrow & \Delta_s(\Sigma_0^?(\emptyset^\odot); \sim^*) \times \Delta_l(\emptyset^\odot; \mathbf{Z}_0) \times \Delta_t(\emptyset^\odot; \sim^*) & \longrightarrow & \Delta_{2,s,l,t}(\Sigma_0^?(\mathbf{Z}_0), \mathbf{Z}_0) \\ \downarrow & & \downarrow & & \downarrow \\ \Delta \mathbf{Z}_0 & \longleftarrow & \Delta \emptyset^\odot & \longrightarrow & \mathbf{Z}_0. \end{array}$$

1197 ► **Definition 134.** *Let R^n be the initial F -chain.*

1198 ► **Lemma 135.** *The endofunctor F preserves flexible simulations.*

1199 **Proof.** Given any span $Y \leftarrow X \rightarrow \mathbf{Z}$ over (6), the left leg of its image under F is functional
 1200 flexible simulation iff the following pasting is a pointwise weak pullback.

$$\begin{array}{ccccc} \Sigma_1(X)_1; (\sim^*)^\uparrow & \longrightarrow & \Sigma_1(X)_1 & \longrightarrow & \Sigma_1(Y)_1 \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_s(\Sigma_0^?(\emptyset^\odot); \sim^*) \times \Delta_l(\emptyset^\odot; \mathbf{Z}_0) & \longrightarrow & \Delta_{2,s,l}(\Sigma_0^?(\emptyset^\odot), \emptyset^\odot) & \longrightarrow & \Delta_{2,s,l}(\Sigma_0^?(\mathbf{Z}_0), \mathbf{Z}_0) \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_{s,l}\emptyset^\odot & \longrightarrow & & \longrightarrow & \Delta_{s,l}\mathbf{Z}_0 \end{array}$$

1202 For this, by [18, Lemma 9.26,(i)], it suffices to prove that all three inner polygons are
 1203 pointwise weak pullbacks. The top right square is a pointwise weak pullback because Σ_1
 1204 preserves functional flexible bisimulations. The top left square also is a pointwise weak
 1205 pullback, as the left face of the following cube.

$$\begin{array}{ccccc} \Sigma_1(X)_1; (\sim^*)^\uparrow & \longrightarrow & (\sim^*)^\uparrow & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & \Sigma_1(X)_1 & & \mathbf{Z}_1 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ \Delta_s(\Sigma_0^?(\emptyset^\odot); \sim^*) \times \Delta_l(\emptyset^\odot; \mathbf{Z}_0) & \longrightarrow & \Delta_s(\sim^*) \times \Delta_l(\mathbf{Z}_0) & \longrightarrow & \Delta_{s,l}(\mathbf{Z}_0) \\ & \searrow & \searrow & \searrow & \\ & \Delta_{2,s,l}(\Sigma_0^?(\emptyset^\odot), \emptyset^\odot) & \longrightarrow & \Delta_{s,l}(\mathbf{Z}_0) & \end{array}$$

Indeed, the top and bottom faces are pullbacks by construction, and the right face is a pointwise weak pullback because \sim^* is a bisimulation. The left face being a pointwise weak pullback thus follows by [18, Lemma 9.26,(i)].

Finally, for the bottom rectangle, its domain is

$$\begin{aligned}
 \Delta_s(\Sigma_0^2(\emptyset^\odot); \sim^*) \times \Delta_l(\emptyset^\odot; \mathbf{Z}_0) &\cong \Delta_s(\Sigma_0^2(\emptyset^\odot); \sim^*) \times \Delta_l(\emptyset^\odot) \\
 &\cong \Delta_s(\Sigma_0(\emptyset^\odot); \sim^* + \emptyset^\odot; \sim^*) \times \Delta_l(\emptyset^\odot) \\
 &\cong (\Delta_s(\Sigma_0(\emptyset^\odot); \sim^*) + \Delta_s(\emptyset^\odot; \sim^*)) \times \Delta_l(\emptyset^\odot) \\
 &\cong \Delta_s(\Sigma_0(\emptyset^\odot); \sim^*) \times \Delta_l(\emptyset^\odot) + \Delta_s(\emptyset^\odot; \sim^*) \times \Delta_l(\emptyset^\odot) \\
 &\cong \Delta_{2,s,l}((\Sigma_0(\emptyset^\odot); \sim^*), \emptyset^\odot) + \Delta_{2,s,l}((\emptyset^\odot; \sim^*), \emptyset^\odot).
 \end{aligned}$$

Similarly, we have $\Delta_{2,s,l}(\Sigma_0^2 \mathbf{Z}_0, \mathbf{Z}_0) \cong \Delta_{2,s,l}(\Sigma_0 \mathbf{Z}_0, \mathbf{Z}_0) + \Delta_{2,s,l}(\Sigma_0^2 \mathbf{Z}_0, \mathbf{Z}_0)$. The rectangle is thus obtained by applying $\Delta_{2,s,l}$ to the (vertical) copairing of the following two squares.

$$\begin{array}{ccc}
 \Sigma_0(\emptyset^\odot); \sim^* & \longrightarrow & \Sigma_0 \mathbf{Z}_0 \\
 \cong \downarrow & & \downarrow \cong \\
 \emptyset^\odot & \longrightarrow & \mathbf{Z}_0
 \end{array}
 \quad
 \begin{array}{ccc}
 \emptyset^\odot; \sim^* & \longrightarrow & \mathbf{Z}_0 \\
 \downarrow & & \parallel \\
 \emptyset^\odot & \longrightarrow & \mathbf{Z}_0
 \end{array}$$

Because pointwise weak pullbacks are closed under (vertical) copairing and preserved by $\Delta_{2,s,l}$, it thus suffices to show that both squares are pointwise weak pullbacks. The left square is one as an isomorphism in the arrow category. The right square is one because it admits a cone morphism from the actual pullback, using reflexivity of \sim^* as in

$$\begin{array}{ccc}
 & \emptyset^\odot & \\
 & \swarrow & \searrow \\
 \emptyset^\odot & & \mathbf{Z}_0 \\
 & \searrow & \swarrow \\
 & \emptyset^\odot; \sim^* & \longrightarrow \mathbf{Z}_0 \\
 & \downarrow & \parallel \\
 & \emptyset^\odot & \longrightarrow \mathbf{Z}_0.
 \end{array}$$

B Proof of Theorem 61

We assume some basic knowledge of familial functors. In particular, there is a well-known alternative characterisation in terms of **generic-free** factorisation, across which border arities are characterised as follows.

► **Proposition 136.** *In the setting of Definition 60, for any $\alpha \in \mathbb{ET}$ and $r \in \mathcal{U}'_2 \Sigma_1(1)(\alpha)$, the border arity \mathbf{b}_r is isomorphic to the morphism φ obtained by first factoring r as $\mathbf{y}_\alpha \xrightarrow{\xi_r} \mathcal{U}'_2 \Sigma_1(B) \xrightarrow{\mathcal{U}'_2 \Sigma_1(!)} \mathcal{U}'_2 \Sigma_1(1)$ with ξ_r generic, and then $\xi_r \circ j_{2,\alpha}$ as $F(\varphi) \circ \chi_r$ with χ_r generic.*

$$\begin{array}{ccc}
 \mathbf{y}_{s(\alpha)_D} + \sum_{i \in n_\alpha} \mathbf{y}_{(l_i^\alpha)_V} & \xrightarrow{j_{2,\alpha}} & \mathbf{y}_\alpha \\
 \chi_r \downarrow & & \downarrow \xi_r \\
 \mathcal{U}'_2(\Sigma_1(A)) & \xrightarrow{\mathcal{U}'_2(\Sigma_1(\varphi))} & \mathcal{U}'_2(\Sigma_1(B))
 \end{array}$$

1228 ► **Lemma 137.** *For any categories with generating cofibrations $(\mathcal{A}, \mathbb{J})$ and $(\mathcal{B}, \mathbb{K})$, a familial*
 1229 *functor $F: \mathcal{A} \rightarrow \mathcal{B}$ preserves fibrations iff it is **cellular**, in the sense that for all commuting*
 1230 *squares*

$$\begin{array}{ccc}
 C & \xrightarrow{k} & D \\
 \xi \downarrow & & \downarrow \chi \\
 F(X) & \xrightarrow{F(\delta)} & F(Y)
 \end{array} \tag{7}$$

1232 *with $k \in \mathbb{K}$ and ξ and χ generic, δ is a cofibration (i.e., $\delta \in \mathring{\mathbb{J}}(\mathbb{J}^{\mathring{\mathbb{J}}})$).*

1233 **Proof.** This is a straightforward generalisation of [18, Lemma 7.28], whose proof applies
 1234 *mutatis mutandis.* ◀

1235 **Proof of Theorem 61.** We assume given a dynamic signature $\Sigma_1: \sigma\text{-Trans} \rightarrow \sigma\text{-Trans}_2$
 1236 such that $\mathcal{U}_2\Sigma_1$ is familial. By that that By Proposition 59, Σ_1 preserves functional flexible
 1237 bisimulations if and only if $\mathcal{U}_2\Sigma_1$ maps \mathbb{J}_σ -fibrations to $\mathbb{J}_{2,\sigma}$ -fibrations, or equivalently, by
 1238 Lemma 137, if it is cellular.

1239 Clearly, $\mathcal{U}_2\Sigma_1$ is cellular, then the border arities of all rules are \mathbb{J}_σ -cofibrations, by
 1240 a straightforward instantiation of Diagram 7. Conversely, assume that all rules are \mathbb{J}_σ -
 1241 cofibrations and consider a commuting square as in Diagram 7, taking $F = \mathcal{U}_2\Sigma_1$, specialised
 1242 to the involved sets of cofibrations:

$$\begin{array}{ccc}
 \mathbf{y}_{s(\alpha)_D} + \sum_{i \in n_\alpha} \mathbf{y}_{(l_i^\alpha)_V} & \xrightarrow{j_{2,\alpha}} & \mathbf{y}_\alpha \\
 \chi_r \downarrow & & \downarrow \xi_r \\
 F(A) & \xrightarrow{F(\varphi)} & F(B).
 \end{array}$$

1244 The result follows by considering the rule $\mathbf{y}_\alpha \xrightarrow{\xi_r} F(B) \xrightarrow{F(!)} F(1)$ and exploiting uniqueness
 1245 (up to isomorphism) of generic factorisations [18, Remark 7.19]. ◀