Higher-order Arities, Signatures and Equations via Modules

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joint work with
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Keywords associated with syntax

Induction/Recursion

Substitution

Syntax

Model

Arity/Signature

Operation/Construction

This talk: give a *discipline* for specifying syntaxes

Motivating example: dLC

syntax of dLC $=$ ***differential λ-calculus*** [Ehrhard-Regnier 2003].

- explicitly involves **equations** e.g. $s+t = t+s$
- specifically taylored: (not an *instance* of a general framework/scheme)

  inductive definition of a set + ad-hoc structure
  e.g. **unary substitution**

Our proposal $=$ a discipline for presenting syntaxes

- signature = operations + equations
- [Fiore-Hure 2010]: alternative approach, for simply typed syntaxes

$\Rightarrow$ our approach explicitly relies on monads and modules (untyped case).

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Syntax of dLC: [Ehrhard-Regnier 2003]

Let be given a denumerable set of variables. We define by induction on k an increasing family of sets (Δ_k). We set $\Delta_0 = \emptyset$ and Δ_{k+1} is defined as follows.

Monotonicity: if t belongs to Δ_k then t belongs to Δ_{k+1}.

Variable: if $n \in \mathbb{N}$, x is a variable, $i_1, \ldots, i_n \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ and $u_1, \ldots, u_n \in \Delta_k$, then

$$D_{i_1,\ldots,i_n} x \cdot (u_1, \ldots, u_n)$$

belongs to Δ_{k+1}. This term is identified with all the terms of the shape $D_{i_{\sigma(1)},\ldots,i_{\sigma(n)}} x \cdot (u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \in \Delta_{k+1}$ where σ is a permutation on $\{1, \ldots, n\}$.

Abstraction: if $n \in \mathbb{N}$, x is a variable, $u_1, \ldots, u_n \in \Delta_k$ and $t \in \Delta_k$, then

$$D_1^n \lambda x\, t \cdot (u_1, \ldots, u_n)$$

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Application: if $s \in \Delta_k$ and $t \in R\langle \Delta_k \rangle$, then

$$(s)t$$

belongs to Δ_{k+1}.

Setting $n = 0$ in the first two clauses, and restricting application by the constraint that $t \in \Delta_k \subseteq R\langle \Delta_k \rangle$, one retrieves the usual definition of lambda-terms which shows that differential terms are a superset of ordinary lambda-terms.

The permutative identification mentioned above will be called *equality up to differential permutation*. We also work up to α-conversion.

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Application: if $s \in \Delta_k$ and $t \in R\langle \Delta_k \rangle$, then

$$(s)t$$ as an operation: $\Lambda \times \text{FreeCommutativeMonoid}(\Lambda) \to \Lambda$

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Syntax of dLC: [BEM 2010]

A **syntax** for the ***differential λ-calculus*** by ***mutual induction:***

[Bucciarelli-Ehrhard-Manzonetto 2010]

Simple terms:

$$\Lambda^s : \quad s, t \quad ::= \quad x \mid \lambda x.s \mid sT \mid \mathsf{D} s \cdot t$$

Differential λ-terms:

$$\Lambda^d : \quad T \quad ::= \quad 0 \mid s \mid s + T$$

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Syntax: specified by operations and equations.

But which ones are allowed ? What is the limit ?

Syntax of dLC: Our version

Which operations/equations are allowed to specify a syntax ?

A stand-alone presentation of differential λ-terms:

Allow sums everywhere (not only in the right arg of application)

Differential λ-terms:

$$\Lambda^{\mathrm{d}} : \quad S, T \quad ::= \quad x \mid \lambda x.S \mid S\,T \mid \mathsf{D}S \cdot T$$
$$\mid\ 0\ \mid\ S + T$$

neutral element for $+$

modulo commutativity and associativity

Macros in [BEM 2010]:

$$\lambda x.\Sigma_i t_i := \Sigma_i \lambda x.t_i$$
$$(\Sigma_i t_i) u := \Sigma_i t_i u$$
$$D(\Sigma_i t_i) \cdot (\Sigma_j u_j) := \Sigma_i \Sigma_j D t_i \cdot u_j$$

Syntax of dLC: Conclusion

How can we compare these different versions ?

In which sense are they syntaxes ?

Which operations/equations are we allowed to specify in a syntax ?

Syntax of dLC: Conclusion

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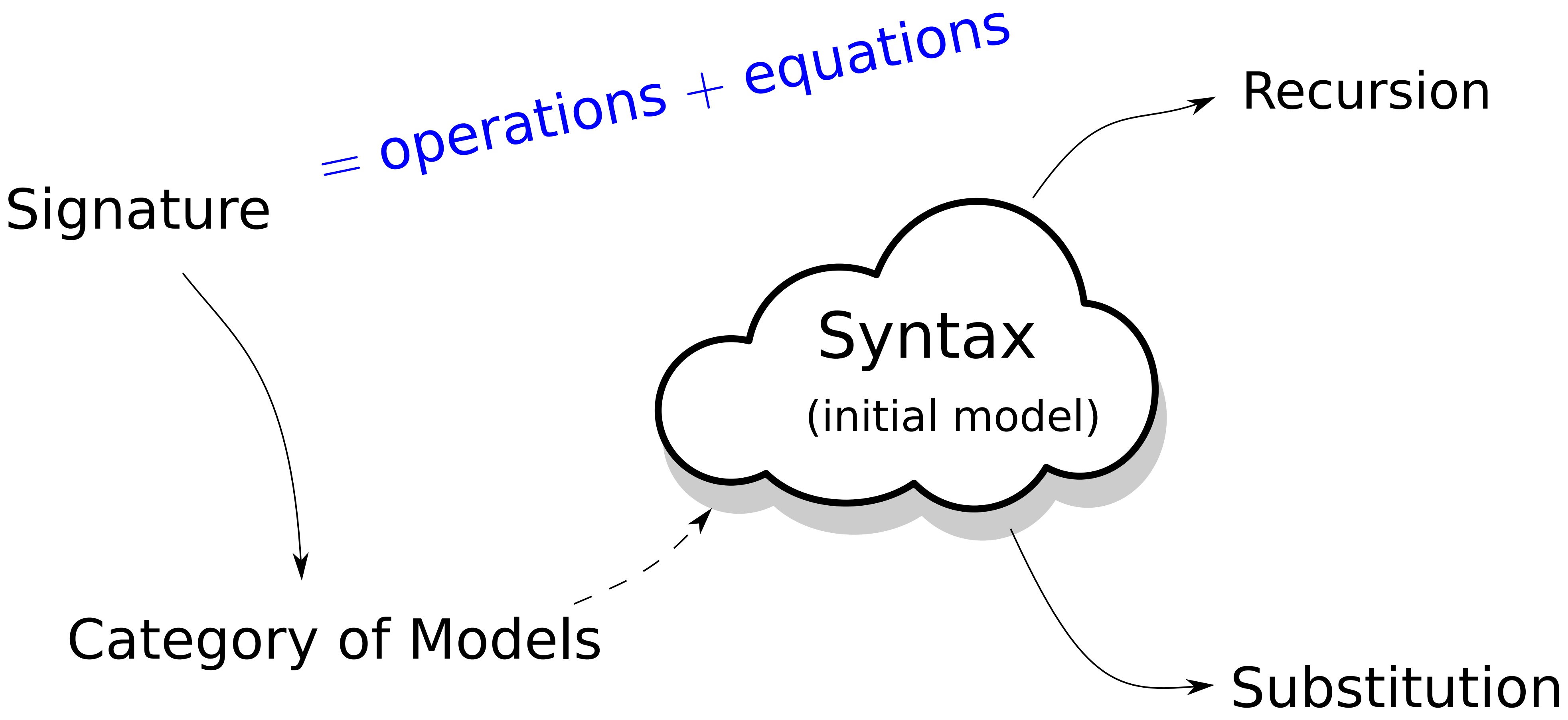
What is a syntax ?

What is a syntax?



generates a syntax = existence of the initial model

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Substitution and monads

Example: differential λ-calculus

$$\Lambda^{\mathrm{d}} : \quad S, T \quad ::= \quad x \mid \lambda x. S \mid S\,T \mid \mathrm{D}S \cdot T \\ \mid 0 \mid S + T$$

Free variable indexing:

$$dLC : X \mapsto \{\text{terms taking free variables in } X\}$$

$$dLC(\emptyset) = \{0, \lambda z.z, \ldots\}$$
$$dLC(\{x, y\}) = \{0, \lambda z.z, \ldots, x, y, x + y, \ldots\}$$

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Parallel substitution:

$$\begin{array}{rcl} \mathrm{bind_f} : dLC(\mathrm{X}) & \to & dLC(\mathrm{Y}) \\ \mathrm{t} & \mapsto & \mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})] \end{array}$$

where $\mathrm{f} : \mathrm{X} \to dLC(\mathrm{Y})$

$\Rightarrow$ (dLC, var_X : X ⊂ dLC(X) , bind) $=$ **monad on Set**

Substitution and monads

Example: differential λ-calculus

$$\Lambda^{\mathrm{d}}: \quad S, T \quad ::= \quad x \mid \lambda x.\, S \mid S\,T \mid \mathrm{D}S \cdot T \mid 0 \mid S + T$$

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where $\mathrm{f} : \mathrm{X} \to \color{red}{dLC(\mathrm{Y})}$

$\Rightarrow$ (dLC, $\mathrm{var}_X : X \subset \mathrm{dLC}(X)$, bind) $=$ **monad on $\mathbf{Set}$**

monad morphism $=$ mapping preserving variables and substitutions.

Preview: Operations are module morphisms

+ commutes with substitution

$$(t + u)[x \mapsto v_x] = t[x \mapsto v_x] + u[x \mapsto v_x]$$

Categorical formulation

$dLC \times dLC$ supports dLC-substitution	$\rightsquigarrow$	$dLC \times dLC$ is a **module over** $\boldsymbol{dLC}$
+ commutes with substitution	$\rightsquigarrow$	$+ : dLC \times dLC \to dLC$ is a **module morphism**

Building blocks for specifying operations

Essential constructions of **modules over a monad $\boldsymbol{R}$**:

- R itself
- $M \times N$ for any modules M and N

  e.g. $\mathrm{R} \times \mathrm{R}$:

  $(\mathrm{t},\mathrm{u})[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})] := (\mathrm{t}[\mathrm{x}\mapsto \mathrm{f}(\mathrm{x})], \mathrm{u}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})])$ where $f : X \to R(Y)$

- M' = **derivative of a module $\boldsymbol{M}$**: $M'(X) = M(X \coprod \{ \diamond \})$.

  used to model an operation binding a variable (Cf next slide).

Syntactic operations are module morphisms

operations = module morphisms = maps commuting with substitution.

$$\mathrm{app} : \mathrm{dLC} \times \mathrm{dLC} \to \mathrm{dLC}$$
$$\mathrm{abs} : \mathrm{dLC}' \to \mathrm{dLC}$$

$$0 : 1 \to \mathrm{dLC}$$
$$+ : \mathrm{dLC} \times \mathrm{dLC} \to \mathrm{dLC}$$

$$\begin{aligned} \mathrm{abs}_X : \mathrm{dLC}(\mathrm{X} + \{\diamond\}) &\to \mathrm{dLC}(X) \\ t &\mapsto \lambda\diamond.t \end{aligned}$$

Combining operations into a single one using disjoint union

$$[\mathrm{app}, \mathrm{abs}] : (\mathrm{dLC} \times \mathrm{dLC}) \coprod \mathrm{dLC}' \to \mathrm{dLC}$$

$$[0, +] : 1 \coprod (\mathrm{dLC} \times \mathrm{dLC}) \to \mathrm{dLC}$$

The Big Picture of 1-signatures and models

A **1-signature** Σ = functorial assignment:

$$R \mapsto \Sigma(R)$$

Example: (0,+)

$$\Sigma_{0,+}(R) = 1 \coprod (R \times R)$$

A **model of Σ** is a pair:

$$(R, \quad \rho : \Sigma(R) \to R)$$

dLC $=$ model of $\Sigma_{0,+}$

$$[0,+] : 1 \coprod (dLC \times dLC) \to dLC$$

A **model morphism** $m : (R,\rho) \to (S,\sigma) =$ monad morphism commuting with the module morphism:

$$\begin{tikzcd} \Sigma(R) \arrow[r, "\rho"] \arrow[d, "\Sigma(m)"'] & R \arrow[d, "m"] \\ \Sigma(S) \arrow[r, "\sigma"'] & S \end{tikzcd}$$

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Syntax

Definition

Given a 1-signature Σ, its **syntax** is an initial object in its category of models.

Question: Does the syntax exist for every 1-signature?

Answer: No.

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powerset endofunctor on $\mathbb{S}\mathrm{et}$

Counter-example: Suppose the 1-signature $R \mapsto \mathscr{P} \circ R$ has a syntax S.

Then, $\mathscr{P} \circ \mathrm{S} \hookrightarrow \mathrm{Id} + \mathscr{P} \circ \mathrm{S} \cong \mathrm{S}$

variables

operations

Lemma

Given a 1-signature Σ with syntax **S**. Then, $\mathrm{S} \cong \mathrm{Id} + \Sigma(\mathrm{S})$.

But, $\mathscr{P} \circ \mathrm{S} \hookrightarrow \mathrm{S}$ is impossible (by strict cardinality inequality)

Examples of 1-signatures generating syntax

- **(0,+) language**:

  Signature: $R \mapsto 1 \coprod (R \times R)$

  Model: $(R\ ,\ \ 0 : 1 \to R\ ,\ \ + : R \times R \to R)$

  Syntax: $(B\ ,\ \ 0 : 1 \to B\ ,\ \ + : B \times B \to B)$

- **lambda calculus**:

  Signature: $R \mapsto R' \coprod (R \times R)$

  Model: $(R\ ,\ \ abs : R' \to R\ ,\ \ app : R \times R \to R)$

  Syntax: $(\Lambda\ ,\ \ abs : \Lambda' \to \Lambda\ ,\ \ app : \Lambda \times \Lambda \to \Lambda)$

Can we generalize this pattern?

Initial semantics for algebraic 1-signatures

Theorem [Hirschowitz & Maggesi 2007]

Syntax exists for any **algebraic 1-signature**, i.e. 1-signature built out of derivatives, products, disjoint unions, and the 1-signature $R \mapsto R$.

Algebraic 1-signatures correspond to the binding signatures described in [Fiore-Plotkin-Turi 1999]

(binding signatures: lists of natural numbers specify n-ary operations, possibly binding variables)

Question: Can we enforce some equations in the syntax ?
For example: commutativity of $+$ for the differential λ-calculus.

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Example: a commutative binary operation

Specification of a binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R\,,\ + : R \times R \to R)$

What is an appropriate notion of model for a commutative binary operation ?

Example: a commutative binary operation

Specification of a commutative binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R\ ,\ + : R \times R \to R)$ s.t. $t + u = u + t$ (1)

What is an appropriate notion of model for a commutative binary operation ?

Answer: a monad equipped with a commutative binary operation

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Equation (1) states an equality between R-module morphisms:

$$
\begin{tikzcd}
\mathrm{R} \times \mathrm{R} \arrow[rr, bend left, "+"] \arrow[dr, "\text{swap}"'] & & \mathrm{R} \\
& \mathrm{R} \times \mathrm{R} \arrow[ur, "+"'] &
\end{tikzcd}
$$

Review: Signatures with equations

- [Fiore-Hur 2010]: inductively defined set of possible equations.

- [AHLM CSL 2018]: "quotients" of algebraic 1-signatures

  Examples:

  - a binary commutative operation

  - application of the simple terms of differential λ-calculus (2^{nd} variant)

    app : dLC $\times$ FreeCommutativeMonoid(dLC) $\to$ dLC

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This work: more general equations (e.g. associativity of a binary op).

Equations

Given a 1-signature Σ, (e.g. binary operation: $\Sigma(\mathrm{R}) = \mathrm{R} \times \mathrm{R}$)

a Σ-**equation** A $\rightrightarrows$ B is a functorial assignment: e.g. commutativity:

$$R \mapsto \left(A(R) \rightrightarrows B(R) \right) \qquad R \mapsto \left(R \times R \underset{+\circ swap}{\overset{+}{\rightrightarrows}} R \right)$$

model of Σ

parallel pair of module morphisms over R

A **2-signature** is a pair

$$(\Sigma, \mathrm{E})$$

1-signature

set of Σ-equations

***model* of a 2-signature $(\mathbf{\Sigma}, \mathbf{E})$**:

- a model R of Σ
- s.t. ∀ (A $\rightrightarrows$ B) $\in$ E, the two morphisms $A(R) \rightrightarrows B(R)$ are equal

Initial semantics for algebraic 2-signatures

Algebraic 2-signature: (Σ, E)

algebraic 1-signature

set of **elementary** Σ-equations

Theorem

Syntax exists for any algebraic 2-signature.

Main instances of **elementary** Σ-equations A $\rightrightarrows$ B:

- $A =$ algebraic 1-signature e.g. A$(\mathrm{R}) = \mathrm{R} \times \mathrm{R}$
- $B(R) = R$

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- $A =$ algebraic 1-signature e.g. A$(\mathrm{R}) = \mathrm{R}\times\mathrm{R}$
- $B(R) = R$

Sketch of the construction of the syntax:

Quotient the initial model R of Σ by the following relation:

$\mathrm{x} \sim \mathrm{y}$ in $\mathrm{R}(\mathrm{X})$ iff for any model S of (Σ, E) , $\mathrm{i}(\mathrm{x}) = \mathrm{i}(\mathrm{y})$

initial Σ-model morphism $\mathrm{i} : \mathrm{R} \to \mathrm{S}$

Example: λ-calculus modulo βη

The algebraic 2-signature $(\Sigma_{LC\beta\eta}, E_{LC\beta\eta})$ of λ-calculus modulo βη:

$$\boldsymbol{\Sigma_{LC\beta\eta}}\,(R) := \Sigma_{LC}(R) = (R \times R) \coprod R'$$

model of $\boldsymbol{\Sigma_{LC}}$ = monad R with module morphisms:

$$\mathrm{app} : R \times R \to R \qquad \mathrm{abs} : R' \to R$$

β-equation: $(\lambda x.t)\, u = \underbrace{t[x \mapsto u]}_{\sigma_R(t,u)}$

η-equation: $t = \lambda x.(t\ x)$

$$\mathbf{E_{LC\beta\eta}} = \{\ \beta\text{-equation},\ \eta\text{-equation}\ \}$$

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$$\begin{tikzcd} R' \times R \arrow[r, "\mathrm{abs} \times R"'] \arrow[rr, bend left, "\sigma_R"] & R \times R \arrow[r, "\mathrm{app}"'] & R \end{tikzcd}$$

η-equation: $t = \lambda x.(t\; x)$

$$\begin{tikzcd} R \arrow[r, "R\iota_1"'] \arrow[rr, bend left, "\mathrm{id}_R"] & R' \arrow[r, "\mathrm{abs}"'] & R \end{tikzcd}$$

$$\mathbf{E_{LC\beta\eta}} = \{\ \beta\text{-equation},\ \eta\text{-equation}\ \}$$

Example: fixpoint operator

Definition [AHLM CSL 2018]

A **fixpoint operator** in a monad R is a module morphism $\mathrm{f} : \mathrm{R}' \to \mathrm{R}$ s.t.

for any term $\mathrm{t} \in \mathrm{R}(\mathrm{X} \coprod \{ \diamond \})$, $\mathrm{f}(\mathrm{t}) = \mathrm{t}[\diamond \mapsto \mathrm{f}(\mathrm{t})]$

i.e. (1)
$$\begin{tikzcd} R' \arrow[rr, bend left, "\mathrm{f}"] \arrow[r, "(\mathrm{id}_{\mathrm{R}'},\mathrm{f})"'] & R' \times R \arrow[r, "\sigma_{\mathrm{R}}"'] & R \end{tikzcd}$$
commutes.

The algebraic 2-signature $(\Sigma_{\mathrm{fix}}, \mathrm{E}_{\mathrm{fix}})$ of a fixpoint operator:

$$\Sigma_{\mathrm{fix}}(\mathrm{R}) := \mathrm{R}' \qquad \mathrm{E}_{\mathrm{fix}} = \{ (1) \}$$

Proposition [AHLM CSL 2018]

Fixpoint operators in $\mathrm{LC}_{\beta\eta}$ are in one to one correspondance with fixpoint combinators (i.e. λ-terms Y s.t. $t\,(Y\,t) = Y\,t$ for any t).

Combining algebraic 2-signatures

Algebraic 2-signatures can be combined:

fixpoint operator

λ-calculus modulo βη

$$(\Sigma_{\text{fix}}, E_{\text{fix}}) \quad + \quad (\Sigma_{\text{LC}\beta\eta}, E_{\text{LC}\beta\eta})$$

$$=$$

$$(\Sigma_{\text{fix}} \coprod \Sigma_{\text{LC}\beta\eta} \;,\; E_{\text{fix}} \cup E_{\text{LC}\beta\eta})$$

λ-calculus modulo βη with an explicit fixpoint operator

Example: free commutative monoid

An algebraic 2-signature $(\Sigma_{\mathrm{mon}}, \mathrm{E}_{\mathrm{mon}})$ for the free commutative monoid monad:
$$\boldsymbol{\Sigma}_{\mathrm{mon}}(\mathrm{R}) := 1 \coprod (\mathrm{R} \times \mathrm{R})$$

model of Σ_{mon} = monad R with module morphisms:

$$0 : 1 \to \mathrm{R} \qquad + : \mathrm{R} \times \mathrm{R} \to \mathrm{R}$$

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4 elementary Σ-equations:

$$\begin{tikzcd} \underset{t}{R} \arrow[r, bend left, "0+t"] \arrow[r, bend right, "t"'] & R \end{tikzcd} \qquad \begin{tikzcd} \underset{s,t,u}{R \times R \times R} \arrow[r, bend left, "(s+t)+u"] \arrow[r, bend right, "s+(t+u)"'] & R \end{tikzcd}$$

$$\begin{tikzcd} \underset{t}{R} \arrow[r, bend left, "t+0"] \arrow[r, bend right, "t"'] & R \end{tikzcd} \qquad \begin{tikzcd} \underset{s,t}{R \times R} \arrow[r, bend left, "s+t"] \arrow[r, bend right, "t+s"'] & R \end{tikzcd}$$

Our target: dLC

Syntax of the *differential λ-calculus*:

Differential λ-terms

$$\begin{array}{rcll} s,t & ::= & \mathrm{x} & \\ & \mid & \lambda \mathrm{x.t} & \multirow{2}{*}{$\Big\}$ λ-calculus} \\ & \mid & \mathrm{s\ t} & \\ & \mid & \mathrm{Ds} \cdot \mathrm{t} & \\ & \mid & \mathrm{s} + \mathrm{t} & \multirow{2}{*}{$\Big\}$ free commutative monoid} \\ & \mid & 0 & \end{array}$$

and (bi)linearity of constructors with respect to +:

$$\lambda \mathrm{x.(s+t)} = \lambda \mathrm{x.s} + \lambda \mathrm{x.t} \qquad \ldots$$

Algebraic 1-signature for dLC

Syntax of the *differential λ-calculus*:

Differential λ-terms

Corresponding 1-signature

$$
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& | & \mathrm{Ds} \cdot \mathrm{t} \\
& | & \mathrm{s} + \mathrm{t} \\
& | & 0
\end{array}
$$

$$\Sigma_{\mathrm{LC}}(\mathrm{R}) = \mathrm{R}' \coprod (\mathrm{R} \times \mathrm{R})$$

$$\mathrm{R} \mapsto \mathrm{R} \times \mathrm{R}$$

$$\Sigma_{\mathrm{mon}}(\mathrm{R}) = 1 \coprod (\mathrm{R} \times \mathrm{R})$$

Algebraic 1-signature for dLC

Syntax of the *differential λ-calculus*:

Differential λ-terms — Corresponding 1-signature

$$s,t ::= x$$

$$\left.\begin{array}{l} \mid \ \lambda x.t \\ \mid \ s\ t \end{array}\right\} \quad \Sigma_{LC}(R) = R' \coprod (R \times R)$$

$$\mid \ Ds \cdot t \qquad R \mapsto R \times R$$

$$\left.\begin{array}{l} \mid \ s + t \\ \mid \ 0 \end{array}\right\} \quad \Sigma_{mon}(R) = 1 \coprod (R \times R)$$

Resulting algebraic 1-signature: $\Sigma_{dLC}(R) = \Sigma_{LC}(R) \coprod (R \times R) \coprod \Sigma_{mon}(R)$

Elementary equations for dLC

Commutative monoidal structure:

$$E_{\mathrm{mon}} \begin{cases} s + t = t + s & R \times R \rightrightarrows R \\ s + (t + u) = (s + t) + u & R \times R \times R \rightrightarrows R \\ 0 + t = t & R \rightrightarrows R \\ t + 0 = t & R \rightrightarrows R \end{cases}$$

Linearity:

$$\lambda x.(s+t) = \lambda x.s + \lambda x.t \qquad R \times R \rightrightarrows R$$

$$D(s+t) \cdot u = Ds \cdot u + Dt \cdot u \qquad R \times R \times R \rightrightarrows R$$

$$Ds \cdot (t+u) = Ds \cdot t + Ds \cdot u \qquad R \times R \times R \rightrightarrows R$$

$$\ldots$$

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Initiality of R $\Rightarrow$ model morphism $\mathrm{R} \to \mathrm{S}$ $\Rightarrow$ monad morphism $\mathrm{R} \to \mathrm{S}$

Example: Computing the set of free variables

LC = initial model of $(\Sigma_{\mathrm{LC}}\,,\varnothing)$ $\quad\quad$ $\Sigma_{\mathrm{LC}}(\mathrm{R}) = (\mathrm{R} \times \mathrm{R}) \coprod \mathrm{R}'$

$\mathcal{P}$ = power set monad

Definition of a (monad) morphism $\mathrm{fv} : \mathrm{LC} \to \mathcal{P}$ **s.t.**

$$\mathrm{fv}(\mathrm{app}(\mathrm{t},\mathrm{u})) = \mathrm{fv}(\mathrm{t}) \cup \mathrm{fv}(\mathrm{u}) \qquad \mathrm{fv}(\mathrm{abs}(\mathrm{t})) = \mathrm{fv}(\mathrm{t}) \setminus \{\diamond\}$$

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$\Rightarrow$ make $\mathcal{P}$ a model of Σ_{LC}:

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Initiality of $\mathrm{LC} \Rightarrow$ $\mathrm{fv} : \mathrm{LC} \to \mathcal{P}$ satisfying the above equations (as a model morphism).

Example: Translating λ-calculus with fixpoint

$\mathrm{LC}_{\beta\eta\mathrm{fix}}$ = initial model of $(\Sigma_{\mathrm{LC}\beta\eta}\,, \mathrm{E}_{\mathrm{LC}\beta\eta}) + (\Sigma_{\mathrm{fix}}\,,\ \mathrm{E}_{\mathrm{fix}})$

λ-calculus modulo βη with a fixpoint operator $\mathrm{fix} : \mathrm{LC}_{\beta\eta\mathrm{fix}}{}' \to \mathrm{LC}_{\beta\eta\mathrm{fix}}$

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λ-calculus modulo βη

monad morphism

Definition of a translation $f : \mathrm{LC}_{\beta\eta\mathrm{fix}} \to \mathrm{LC}_{\beta\eta}$ **s.t.**

$$f(u) = \text{"}u[\ \mathrm{fix}(t) \mapsto \mathrm{app}(Y, \mathrm{abs}(t))\ ]\text{"}$$

a chosen fixpoint combinator

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$\mathrm{app, abs}$

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Initiality of $\mathrm{LC}_{\beta\eta\mathrm{fix}} \ \Rightarrow\ \mathrm{f} : \mathrm{LC}_{\beta\eta\mathrm{fix}} \to \mathrm{LC}_{\beta\eta}$

Example: Computing the size of a term

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Definition of a (monad) morphism $\mathrm{s} : \mathrm{LC} \to \mathbb{N}$ **s.t.**

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Solution [CSL AHLM 2018]:

continuation monad $\mathrm{C(X)} = \mathbb{N}^{(\mathbb{N}^{X})}$

1. define $\mathrm{f} : \mathrm{LC} \to \mathrm{C}$ by recursion
2. deduce $\mathrm{s} : \mathrm{LC} \to \mathbb{N}$

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Solution [CSL AHLM 2018]: continuation monad $\mathrm{C}(\mathrm{X}) = \mathbb{N}^{(\mathbb{N}^{\mathrm{X}})}$

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affects an arbitrary size to each variable

Intuition: uncurrying $\mathrm{f}_{\mathrm{X}} : \mathrm{LC}(\mathrm{X}) \to \mathbb{N}^{(\mathbb{N}^{\mathrm{X}})}$ yields $\mathrm{g} : \mathrm{LC}(\mathrm{X}) \times \mathbb{N}^{\mathrm{X}} \to \mathbb{N}$

Example: Computing the size of a term

LC = initial model of $(\Sigma_{\mathrm{LC}}\,, \emptyset)$ $\qquad\qquad$ $\Sigma_{\mathrm{LC}}(\mathrm{R}) = (\mathrm{R} \times \mathrm{R}) \coprod \mathrm{R}'$

Definition of a (~~monad~~) morphism $\mathrm{s} : \mathrm{LC} \to \mathbb{N}$ **s.t.**

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$$\mathrm{s}(\mathrm{t}) = \mathrm{g}(\mathrm{t}, (\mathrm{x} \mapsto 0))$$

variables are of size 0

Conclusion

Summary of the talk:

- presented a notion of 1-signature and models
- defined a 2-signature as a 1-signature and a set of equations
- identified a class of 2-signatures that generate a syntax

The main theorem has been formalized in Coq using the UniMath library.

Future work:

- add the notion of reductions;
- extend our work to simply typed syntaxes.

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Thank you!