Higher-order Arities, Signatures and Equations via Modules

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joint work with
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Keywords associated with syntax

Recursion

Substitution

Model

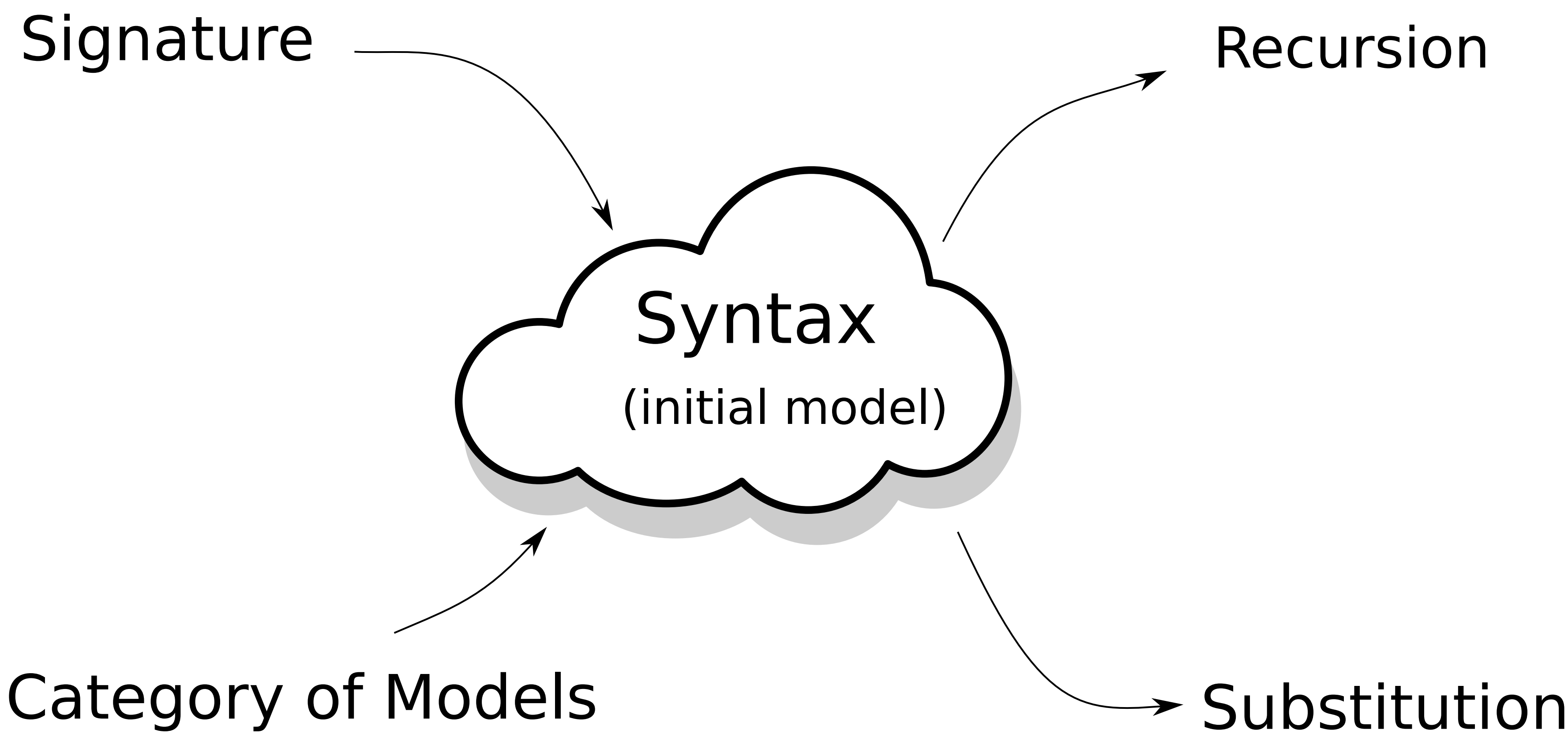
Syntax

Operation/Construction

Arity/Signature

This talk: give a mathematical framework which *catch'em all*

What is a syntax?



generates a syntax = existence of the initial model

Our target: LCD

Syntax of the *differentiable λ-calculus*:

Simple terms $\mathrm{s},\mathrm{t} \in \Lambda$

$$
\begin{array}{rcll}
s,t & ::= & \mathrm{x} & \text{(variable)} \\
 & \mid & \lambda\mathrm{x}.\mathrm{t} & \text{(modulo } \alpha\text{-renaming of x)} \\
 & \mid & \mathrm{s}\ \mathrm{t} & \\
 & \mid & \mathrm{D}\mathrm{s} \cdot \mathrm{t} & \\
 & \mid & \mathrm{s} + \mathrm{t} & \text{(modulo associativity and commutativity)} \\
 & \mid & 0 & \text{(neutral element for } +\text{)}
\end{array}
$$

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subject to the following equation:

$$\mathrm{D}(\mathrm{D}\mathrm{s} \cdot \mathrm{t}) \cdot \mathrm{u} = \mathrm{D}(\mathrm{D}\mathrm{s} \cdot \mathrm{u}) \cdot \mathrm{t}$$

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$$\mathrm{D}(\mathrm{D}\mathrm{s} \cdot \mathrm{t}) \cdot \mathrm{u} = \mathrm{D}(\mathrm{D}\mathrm{s} \cdot \mathrm{u}) \cdot \mathrm{t}$$

and (bi)linearity of constructors with respect to +:

$$\lambda \mathrm{x}.(\mathrm{s}+\mathrm{t}) = \lambda \mathrm{x}.\mathrm{s} + \lambda \mathrm{x}.\mathrm{t} \qquad \ldots$$

Overview

Topic: specification and construction of untyped syntaxes with variables and a well-behaved substitution (e.g. lambda calculus).

Our work:

1. general notion of ***1-signature*** based on ***monads*** and ***modules***.
   - *Caveat:* Not all of them do ***generate a syntax***
   - special case: classical ***algebraic 1-signatures*** generate a syntax

2. notion of **2-signature**: a pair of a 1-signature and a set of equations.
   - special case: ***algebraic 2-signatures*** generate a syntax

Some examples covered by our result

Operations:

- Commutative binary operation

$$m : T \times T \to T \qquad \text{s.t.} \qquad m(t, u) = m(u, t)$$

- Fixed point operation

Some examples covered by our result

Operations:

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$$m : T \times T \to T \qquad \text{s.t.} \qquad m(t, u) = m(u, t)$$

- Fixed point operation

More extensive examples (set of operations with equations):

- λ-calculus modulo βη

- differential λ-calculus

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Categorical formulation of a term language

Example: syntax with a binary operation ★, a constant 0, and variables

$$\begin{aligned} \mathrm{expr} ::= {} & x && \textit{(variable)} \\ \mid {} & t_1 \star t_2 && \textit{(binary operation)} \\ \mid {} & 0 && \textit{(constant)} \end{aligned}$$

The syntax can be considered as the endofunctor B (on Set):

$$B : X \mapsto \{\text{expressions over } X\}$$

For example:

$$B(\emptyset) = \{0, 0 \star 0, \dots\}$$

$$B(\{x, y\}) = \{0, 0 \star 0, \dots, x, y, x \star y, \dots\}$$

Categorical formulation of a term language

Then we have:

$$\begin{aligned} \bigstar : \; & B \times B \xrightarrow{\cdot} B \\ 0 : \; & \quad 1 \quad \xrightarrow{\cdot} B \\ \mathrm{var} : \; & \quad \mathrm{Id}_{\mathrm{Set}} \xrightarrow{\cdot} B \end{aligned}$$

Putting all together:

$$B \times B + 1 + \mathrm{Id}_{\mathrm{Set}} \xrightarrow{\cdot} B$$

i.e. B is an algebra for the endofunctor $F \mapsto F \times F + 1 + \mathrm{Id}_{\mathrm{Set}}$ on the category $\mathrm{End}_{\mathrm{Set}}$.

Actually, B can be **characterized** as the initial algebra.

Binding Signatures

Definition

Binding signature = a family of lists of natural numbers.

Each list specifies one operation in the syntax:

- length of the list = number of arguments of the operation
- natural number in the list = number of bound variables in the corresponding argument

Syntax with 0, ★:

$$\Big(\underbrace{()}_{0}, \underbrace{(0, 0)}_{\star}\Big)$$

Lambda calculus:

$$\Big(\underbrace{(0\ ,\ 0)}_{app}\ ,\ \underbrace{(1)}_{abs}\Big)$$

Initial semantics for binding signatures

Reminder

The syntax $(0,\bigstar)$ is the initial algebra for the endofunctor:

$$F \mapsto F \times F + 1 + \mathrm{Id}_{\mathrm{Set}}$$

More generally, any binding signature gives rise to an endofunctor Σ.

Definition

Model = $(\Sigma + \mathrm{Id}_{\mathrm{Set}})$-algebra

Classical Theorem

The initial $(\Sigma + \mathrm{Id}_{\mathrm{Set}})$-algebra of a binding signature Σ always exists.

Question: Does this initial algebra come with a well-behaved substitution?

Answer: Yes: see e.g. [Fiore, Plotkin, Turi 1999], [Ghani & Uustalu 2003]

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The Big Picture of 1-signatures and models

Binding signatures $\hookrightarrow$ Our 1-signatures

A **1-signature** Σ is a functorial assignment:

$$R \mapsto \Sigma(R)$$

A **model of Σ** is a pair:

$$(R, \quad \rho : \Sigma(R) \to R)$$

monad	:=	endofunctor with substitution
module over a monad	:=	endofunctor with substitution
module morphism	:=	natural transformation preserving substitution

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Substitution and monads

Reminder:

- $B(X)$ = expressions built out of 0, ★ and variables taken in X
- Variables induce a natural transformation $\mathrm{var} : \mathrm{Id}_{\mathrm{Set}} \to B$

Substitution:

$$\mathrm{bind} : B(X) \to (X \to B(Y)) \to B(Y)$$

+ laws

A triple (B, var, bind) is called a **monad**.

monad morphism = mapping preserving var and bind.

Monads

1. $\mathrm{B} : \mathrm{Set} \to \mathrm{Set}$

   $B(X)$ = expressions built out of 0, ★ and variables taken in X

2. A collection of functions $(\mathrm{var}_\mathrm{X} : X \to B(X))_X$

   Variables are expressions

3. For each function $\mathrm{u} : \mathrm{X} \to \mathrm{B(Y)}$, a function $\mathrm{bind_u} : \mathrm{B(X)} \to \mathrm{B(Y)}$

   Parallel substitution

   Notation: $\mathbf{bind_u(t) = t[x \mapsto u(x)]}$

4. Monadic laws:

$$\mathrm{var(y)[x \mapsto u(x)] = u(y)}$$

$$\mathrm{t[x \mapsto var(x)] = t}$$

$$\mathrm{t[x \mapsto f(x)][y \mapsto g(y)] = t[x \mapsto f(x)[y \mapsto g(y)]\ ]}$$

Preview: Operations are module morphisms

$\star$ commutes with substitution

$$(t \star u)[x \mapsto v_x] = t[x \mapsto v_x] \star u[x \mapsto v_x]$$

Categorical formulation

$B \times B$ supports B-substitution $\rightsquigarrow$ $B \times B$ is a **module over $\boldsymbol{B}$**

$\star$ commutes with substitution $\rightsquigarrow$ $\star : B \times B \to B$ is a **module morphism**

Modules VS Monads

Monad

1. $\mathrm{B} : \mathrm{Set} \to \mathrm{Set}$

   $B(X)$ = expressions built out of 0, ★ and variables taken in X

2. A collection of functions $(\mathrm{var}_{\mathrm{X}} : X \to B(X))_X$

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3. For each function $\mathrm{u} : \mathrm{X} \to \mathrm{B(Y)}$, a function $\mathrm{bind}_{\mathrm{u}} : \mathrm{B(X)} \to \mathrm{B(Y)}$

   Parallel substitution

   Notation: $\mathbf{bind_u(t)} = \mathbf{t[x \mapsto u(x)]^B}$

4. Substitution laws:

$$\mathrm{var(y)[x \mapsto u(x)]^B = u(y)}$$
$$\mathrm{t[x \mapsto var(x)]^B = t}$$
$$\mathrm{t[x \mapsto f(x)]^B[y \mapsto g(y)]^B = t[x \mapsto f(x)[y \mapsto g(y)]^B\,]^B}$$

Modules VS Monads

~~Monad~~ **Module over a monad** B (e.g. $\mathrm{B} \times \mathrm{B}, \mathbf{2}, \ldots$)

1. $\mathrm{M} : \mathrm{Set} \to \mathrm{Set}$

   $M(X)$ = expressions taking variables in X

2. ~~A collection of functions $(\mathrm{var}_X : X \to M(X))_X$~~

3. For each function $\mathrm{u} : \mathrm{X} \to \mathrm{B}(\mathrm{Y})$, a function $\mathrm{bind}_\mathrm{u} : \mathrm{M}(\mathrm{X}) \to \mathrm{M}(\mathrm{Y})$

   Parallel substitution

   Notation: $\mathbf{bind_u(t)} = \mathbf{t[x \mapsto u(x)]^M}$

4. Substitution laws:

$$\cancel{\mathrm{var}(\mathrm{y})[\mathrm{x} \mapsto \mathrm{u}(\mathrm{x})] = \mathrm{u}(\mathrm{y})}$$

$$\mathrm{t}[\mathrm{x} \mapsto \mathrm{var}(\mathrm{x})]^\mathrm{M} = \mathrm{t}$$

$$\mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})]^\mathrm{M}[\mathrm{y} \mapsto \mathrm{g}(\mathrm{y})]^\mathrm{M} = \mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})[\mathrm{y} \mapsto \mathrm{g}(\mathrm{y})]^\mathrm{B}\,]^\mathrm{M}$$

Building blocks for binding signatures

Essential constructions of **modules over a monad $\boldsymbol{R}$**:

- R itself
- M x N for any modules M and N (in particular, R x R)
- The **derivative of a module $\boldsymbol{M}$** is the module M' defined by M'(X) = M($X + \{\diamond\}$).

  The derivative is used to model an operation binding a variable

  (Cf next slide).

Syntactic operations are module morphisms

module morphism = maps commuting with substitution.

$$id_M : M \to M$$

$$0 : 1 \to B$$

$$\star : B \times B \to B$$

$$app : \Lambda \times \Lambda \to \Lambda$$

$$abs : \Lambda' \to \Lambda$$

The Big Picture again

A **1-signature** Σ is a functorial assignment:

$$R \mapsto \Sigma(R)$$

monad module over R

A **model of Σ** is a pair:

$$(R, \quad \rho : \Sigma(R) \to R)$$

monad module morphism

A **model morphism** $m : (R,\rho) \to (S,\sigma)$ is a monad morphism commuting with the module morphism:

$$\begin{tikzcd} \Sigma(R) \arrow[r, "\rho"] \arrow[d, "\Sigma(m)"'] & R \arrow[d, "m"] \\ \Sigma(S) \arrow[r, "\sigma"'] & S \end{tikzcd}$$

Syntax

Definition

Given a 1-signature Σ, its **syntax** is an initial object in its category of models.

Question: Does the syntax exist for every 1-signature?

Answer: No.

Counter-example: the 1-signature $R \mapsto \mathscr{P} \circ R$

powerset endofunctor on $\mathbb{S}\mathrm{et}$

Examples of 1-signatures generating syntax

- **(0,★) language**:

  Signature: $R \mapsto 1 + R \times R$

  Model: $(R\ ,\quad 0 : 1 \to R\ ,\quad \bigstar : R \times R \to R)$

  Syntax: $(B\ ,\quad 0 : 1 \to B\ ,\quad \bigstar : B \times B \to B)$

- **lambda calculus**:

  Signature: $R \mapsto R' + R \times R$

  Model: $(R\ ,\quad abs : R' \to R\ ,\quad app : R \times R \to R)$

  Syntax: $(\Lambda\ ,\quad abs : \Lambda' \to \Lambda\ ,\quad app : \Lambda \times \Lambda \to \Lambda)$

Can we generalize this pattern?

Initial semantics for algebraic 1-signatures

Theorem [Hirschowitz & Maggesi 2007]

Syntax exists for any **algebraic 1-signature**, i.e. 1-signature built out of derivatives, products, coproducts, and the trivial 1-signature $R \mapsto R$.

Algebraic 1-signatures correspond to binding signatures through the embedding:

$$\text{Binding signatures} \hookrightarrow \text{Our 1-signatures}$$

Question: Can we enforce some equations in the syntax ?
For example: lambda calculus modulo beta and eta.

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Example: a commutative binary operation

Specification of a binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R\,,\ + : R \times R \to R)$

What is an appropriate notion of model for a commutative binary operation ?

Example: a commutative binary operation

Specification of a commutative binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R\ ,\ + : R \times R \to R)$ s.t. $t + u = u + t$ (1)

What is an appropriate notion of model for a commutative binary operation ?

Answer: a monad equipped with a commutative binary operation

Example: a commutative binary operation

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What is an appropriate notion of model for a commutative binary operation ?

Answer: a monad equipped with a commutative binary operation

Equation (1) states an equality between R-module morphisms:

$$\begin{tikzcd} \mathrm{R} \times \mathrm{R} \arrow[rr, bend left, "+"] \arrow[r, bend right, "\text{swap}"'] & \mathrm{R} \times \mathrm{R} \arrow[r, bend right, "+"'] & \mathrm{R} \end{tikzcd}$$

Review: Signatures with equations

- [Fiore-Hur 2010]: existence of an initial model for an inductively defined (with a specific syntax) set of possible equations.

- [AHLM CSL 2018]: "quotients" of algebraic 1-signatures generate a syntax (e.g. a binary commutative operation).

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Our framework: alternative approach where monads and modules are the central notions.

- [AHLM CSL 2018]: "quotients" of algebraic 1-signatures generate a syntax (e.g. a binary commutative operation).

Review: Signatures with equations

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This work: more general equations (e.g. λ-calculus modulo $\beta\eta$).

Equations

Given a 1-signature Σ, a **$\boldsymbol{\Sigma}$-equation** A $\rightrightarrows$ B is a functorial assignment

$$R \mapsto \left(A(R) \rightrightarrows B(R) \right)$$

model of Σ

parallel pair of module morphisms over R

A **2-signature** is a pair

$$(\Sigma, \mathrm{E})$$

1-signature

set of Σ-equations

***model* of a 2-signature $(\boldsymbol{\Sigma}, \mathbf{E})$**:

- a model R of Σ
- s.t. ∀ (A $\rightrightarrows$ B) $\in$ E, the two morphisms $A(R) \rightrightarrows \mathrm{B}(\mathrm{R})$ are equal

Algebraic 2-signatures

Given a 1-signature Σ, a Σ-equation A $\rightrightarrows$ B is **elementary** if:

1. A "preserves pointwise epimorphisms"

   (e.g., any "algebraic 1-signature")

2. B is of the form R ↦ R'$^{\cdots}$' (e.g. R ↦ R)

Algebraic 2-signature:

$$(\Sigma, \mathrm{E})$$

algebraic 1-signature

set of **elementary**

Σ-equations

Theorem

Syntax exists for any algebraic 2-signature

Example: λ-calculus modulo βη

The algebraic 2-signature $(\Sigma_{LC\beta\eta}, E_{LC\beta\eta})$ of λ-calculus modulo βη:

$$\boldsymbol{\Sigma_{LC\beta\eta}}\,(R) := \Sigma_{LC}(R) = R \times R + R'$$

model of $\boldsymbol{\Sigma_{LC}}$ = monad R with module morphisms:

$$\mathrm{app} : R \times R \to R \qquad \mathrm{abs} : R' \to R$$

β-equation: $(\lambda x.t)\ u = \underbrace{t[x \mapsto u]}_{\sigma_R(t,u)}$

η-equation: $t = \lambda x.(t\ x)$

$$\mathbf{E_{LC\beta\eta}} = \{\ \beta\text{-equation},\ \eta\text{-equation}\ \}$$

Example: λ-calculus modulo βη

The algebraic 2-signature $(\Sigma_{\mathrm{LC\beta\eta}}, E_{\mathrm{LC\beta\eta}})$ of λ-calculus modulo βη:

$$\mathbf{\Sigma_{LC\beta\eta}}\,(\mathrm{R}) := \Sigma_{\mathrm{LC}}(\mathrm{R}) = \mathrm{R} \times \mathrm{R} + \mathrm{R}'$$

model of Σ_{LC} = monad R with module morphisms:

$$\mathrm{app} : \mathrm{R} \times \mathrm{R} \to \mathrm{R} \qquad \mathrm{abs} : \mathrm{R}' \to \mathrm{R}$$

β-equation: $(\lambda \mathrm{x.t})\ \mathrm{u} = \underbrace{\mathrm{t}[\mathrm{x} \mapsto \mathrm{u}]}_{\sigma_{\mathrm{R}}(\mathrm{t,u})}$ **η-equation**: $\mathrm{t} = \lambda \mathrm{x.}(\mathrm{t\ x})$

$$\begin{tikzcd}
R' \times R \arrow[rr, bend left, "\sigma_{\mathrm{R}}"] \arrow[r, bend right, "\mathrm{abs} \times R"'] & R \times R \arrow[r, bend right, "\mathrm{app}"'] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[rr, bend left, "\mathrm{id}_{\mathrm{R}}"] \arrow[r, bend right, "\mathrm{R}\iota_1"'] & R' \arrow[r, bend right, "\mathrm{abs}"'] & R
\end{tikzcd}$$

$$\mathbf{E_{LC\beta\eta}} = \{\ \text{β-equation, η-equation}\ \}$$

Example: fixpoint operator

Definition [AHLM CSL 2018]

A **fixpoint operator** in a monad R is a module morphism $\mathrm{f} : \mathrm{R}' \to \mathrm{R}$ s.t.

$$(1) \qquad \begin{tikzcd} R' \arrow[rr, bend left, "\mathrm{f}"] \arrow[r, "(\mathrm{id}_{\mathrm{R}'}{,}\mathrm{f})"'] & R' \times R \arrow[r, "\sigma_{\mathrm{R}}"'] & R \end{tikzcd} \qquad \text{commutes.}$$

The algebraic 2-signature $(\Sigma_{\mathrm{fix}}, \mathrm{E}_{\mathrm{fix}})$ of a fixpoint operator:

$$\Sigma_{\mathrm{fix}}\,(\mathrm{R}) := \mathrm{R}' \qquad\qquad \mathrm{E}_{\mathrm{fix}} = \{\ (1)\ \}$$

Proposition [AHLM CSL 2018]

Fixpoint operators in $\mathrm{LC}_{\beta\eta}$ are in one to one correspondance with fixpoint combinators (i.e. λ-terms Y s.t. $t\,(Y\,t) = Y\,t$ for any t).

Combining algebraic 2-signatures

Algebraic 2-signatures can be combined:

fixpoint operator

λ-calculus modulo βη

$$(\Sigma_{\mathrm{fix}}, \mathrm{E}_{\mathrm{fix}}) \quad + \quad (\Sigma_{\mathrm{LC\beta\eta}}, \mathrm{E}_{\mathrm{LC\beta\eta}})$$

$$=$$

$$(\Sigma_{\mathrm{fix}} + \Sigma_{\mathrm{LC\beta\eta}} \quad , \quad \mathrm{E}_{\mathrm{fix}} \cup \mathrm{E}_{\mathrm{LC\beta\eta}})$$

λ-calculus modulo βη with an explicit fixpoint operator

Example: free monoid

An algebraic 2-signature $(\Sigma_{\mathrm{mon}}, \mathrm{E}_{\mathrm{mon}})$ for the free monoid monad $\mathrm{X} \mapsto \coprod_{\mathrm{n}} \mathrm{X}^{\mathrm{n}}$

$$\boldsymbol{\Sigma}_{\mathrm{mon}}(\mathrm{R}) := 1 + \mathrm{R} \times \mathrm{R}$$

model of Σ = monad R with module morphisms:

$$\varepsilon : 1 \to \mathrm{R} \qquad \mathrm{m} : \mathrm{R} \times \mathrm{R} \to \mathrm{R}$$

3 elementary Σ-equations:

$$\begin{tikzcd} & R^2 \arrow[dr, bend left, "m"] & \\ R^3 \arrow[ur, bend left, "R\times m"] \arrow[dr, bend right, "m\times R"'] & & R \\ & R^2 \arrow[ur, bend right, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} & R^2 \arrow[dr, bend left, "m"] & \\ R \arrow[ur, bend left, "\epsilon, id"] \arrow[rr, bend right, "id"'] & & R \end{tikzcd} \qquad \begin{tikzcd} & R^2 \arrow[dr, bend left, "m"] & \\ R \arrow[ur, bend left, "id, \epsilon"] \arrow[rr, bend right, "id"'] & & R \end{tikzcd}$$

associativity — left unit — right unit

Our target: LCD

Syntax of the *differentiable λ-calculus*:

Simple terms $\mathrm{s,t} \in \Lambda$

$$\left.\begin{array}{rcl} s,t & ::= & \mathrm{x} \\ & | & \lambda \mathrm{x.t} \\ & | & \mathrm{s\ t} \end{array}\right\} \text{λ-calculus}$$

$$\begin{array}{rcl} & | & \mathrm{Ds} \cdot \mathrm{t} \end{array}$$

$$\left.\begin{array}{rcl} & | & \mathrm{s} + \mathrm{t} \\ & | & 0 \end{array}\right\} \text{free commutative monoid}$$

subject to the following equation:

$$\mathrm{D(Ds \cdot t) \cdot u = D(Ds \cdot u) \cdot t}$$

and (bi)linearity of constructors with respect to +:

$$\lambda \mathrm{x.(s{+}t)} = \lambda \mathrm{x.s} + \lambda \mathrm{x.t} \qquad \ldots$$

Algebraic 1-signature for LCD

Syntax of the *differentiable λ-calculus*:

Simple terms $\mathrm{s,t} \in \Lambda$ Corresponding 1-signature

$$
\begin{array}{rcll}
s,t & ::= & \mathrm{x} & \\
& | & \lambda \mathrm{x.t} & \Sigma_{\mathrm{LC}}(\mathrm{R}) = \mathrm{R}' + \mathrm{R} \times \mathrm{R} \\
& | & \mathrm{s\ t} & \\
& | & \mathrm{Ds} \cdot \mathrm{t} & \mathrm{R} \mapsto \mathrm{R} \times \mathrm{R} \\
& | & \mathrm{s} + \mathrm{t} & \\
& | & 0 & \Sigma_{\mathrm{mon}}(\mathrm{R}) = 1 + \mathrm{R} \times \mathrm{R}
\end{array}
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Algebraic 1-signature for LCD

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& | & \mathrm{s} + \mathrm{t} & \\
& | & 0 & \Sigma_{\mathrm{mon}}(\mathrm{R}) = 1 + \mathrm{R} \times \mathrm{R}
\end{array}
$$

Resulting algebraic 1-signature: $\Sigma_{\mathrm{LCD}}(\mathrm{R}) = \Sigma_{\mathrm{LC}}(\mathrm{R}) + \mathrm{R} \times \mathrm{R} + \Sigma_{\mathrm{mon}}(\mathrm{R})$

Elementary equations for LCD

Commutative monoidal structure:

$$s + t = t + s \qquad \mathrm{R}\times\mathrm{R} \rightrightarrows \mathrm{R}$$

$$\mathrm{E_{mon}} \left\{ \begin{array}{ll} s + (t + u) = (s + t) + u & \mathrm{R}\times\mathrm{R}\times\mathrm{R} \rightrightarrows \mathrm{R} \\ 0 + t = t & \mathrm{R} \rightrightarrows \mathrm{R} \\ t + 0 = t & \mathrm{R} \rightrightarrows \mathrm{R} \end{array} \right.$$

Differentiation:

$$D(Ds \cdot t) \cdot u = D(Ds \cdot u) \cdot t \qquad \mathrm{R}\times\mathrm{R}\times\mathrm{R} \rightrightarrows \mathrm{R}$$

Linearity:

$$\lambda x.(s{+}t) = \lambda x.s + \lambda x.t \qquad \mathrm{R}\times\mathrm{R} \rightrightarrows \mathrm{R}$$

$$D(s{+}t){\cdot}u = Ds{\cdot}u + Dt{\cdot}u \qquad \mathrm{R}\times\mathrm{R}\times\mathrm{R} \rightrightarrows \mathrm{R}$$

$$Ds{\cdot}(t{+}u) = Ds{\cdot}t + Ds{\cdot}u \qquad \mathrm{R}\times\mathrm{R}\times\mathrm{R} \rightrightarrows \mathrm{R}$$

...

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Initiality of R $\Rightarrow$ model morphism $\mathrm{R} \to \mathrm{S}$ $\Rightarrow$ monad morphism $\mathrm{R} \to \mathrm{S}$

Example: Computing the set of free variables

LC = initial model of $(\Sigma_{\mathrm{LC}}\, , \varnothing)$ $\Sigma_{\mathrm{LC}}(\mathrm{R}) = \mathrm{R} \times \mathrm{R} + \mathrm{R}'$

$\mathcal{P}$ = power set monad

Definition of a (monad) morphism $\mathrm{fv} : \mathrm{LC} \to \mathcal{P}$ **s.t.**

$$\mathrm{fv}(\mathrm{app}(\mathrm{t},\mathrm{u})) = \mathrm{fv}(\mathrm{t}) \cup \mathrm{fv}(\mathrm{u}) \qquad \mathrm{fv}(\mathrm{abs}(\mathrm{t})) = \mathrm{fv}(\mathrm{t}) \setminus \{\diamond\}$$

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Initiality of $\mathrm{LC} \Rightarrow$ $\mathrm{fv} : \mathrm{LC} \to \mathcal{P}$ satisfying the above equations (as a model morphism).

Example: Translating λ-calculus with fixpoint

$\mathrm{LC}_{\beta\eta\mathrm{fix}}$ = initial model of $(\Sigma_{\mathrm{LC}\beta\eta}, \mathrm{E}_{\mathrm{LC}\beta\eta}) + (\Sigma_{\mathrm{fix}}, \mathrm{E}_{\mathrm{fix}})$

λ-calculus modulo βη with a fixpoint operator $\mathrm{fix} : \mathrm{LC}_{\beta\eta\mathrm{fix}}{}' \to \mathrm{LC}_{\beta\eta\mathrm{fix}}$

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Solution [CSL AHLM 2010]:

replace $\quad s : \mathrm{LC} \to \mathbb{N}$ $\quad$ continuation monad $C(X) = \mathbb{N}^{(\mathbb{N}^X)}$

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affects an arbitrary size to each variable

Intuition: uncurrying $\mathrm{f}_X : \mathrm{LC}(X) \to \mathbb{N}^{(\mathbb{N}^X)}$ yields $\mathrm{g} : \mathrm{LC}(X) \times \mathbb{N}^X \to \mathbb{N}$

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$$\mathrm{s}(\mathrm{t}) = \mathrm{g}(\mathrm{t}, (\mathrm{x} \mapsto 0))$$

variables are of size 0

Conclusion

Summary of the talk:

- presented a notion of 1-signature and models
- defined a 2-signature as a 1-signature and a set of equations
- identified a class of 2-signatures that generate a syntax

The main theorem has been formalized in Coq using the UniMath library.

Future work:

- add the notion of reductions;
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Thank you!