Higher-order Arities, Signatures and Equations via Modules

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Keywords associated with syntax

Induction/Recursion

Substitution

Model

Syntax

Arity/Signature

Operation/Construction

This talk: give a mathematical account of this topic

Motivation: LCD

The ***differentiable λ-calculus*** (LCD) was introduced by [Ehrard-Regnier 2003].

The syntax is not straightforward, as it involves some equations.

There are alternative presentations of the syntax in later articles, more or less verbose.

Motivation: LCD

The ***differentiable λ-calculus*** (LCD) was introduced by [Ehrard-Regnier 2003].

The syntax is not straightforward, as it involves some equations.

There are alternative presentations of the syntax in later articles, more or less verbose.

The next slides give 3 variants of the syntax

Syntax of LCD: version 1/3

A **syntax** for the ***differentiable λ-calculus*** by ***mutual induction:***

[Categorical Models for Simply Typed Resource Calculi]

Simple terms:

$$\Lambda^s : \quad s, t, u, v ::= \quad x \mid \lambda x.s \mid sT \mid \mathsf{D}\, s \cdot t$$

Differential λ-terms:

$$\Lambda^d : \quad S, T, U, V ::= \quad 0 \mid s \mid s + T$$

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A **syntax** for the ***differentiable λ-calculus*** by ***mutual induction:***

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variable

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modulo α-renaming of x

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neutral element for $+$

modulo commutativity

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A syntax is specified by operations and equations.

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$$\Lambda^\mathrm{d} = \mathrm{FreeCommutativeMonoid}(\Lambda^\mathrm{s})$$

A syntax is specified by operations and equations.

But which ones are allowed ? What is the limit ?

Syntax of LCD: version 2/3

Which operations/equations are allowed to specify a syntax ?

Can we avoid mutual induction ?

A stand-alone presentation of simple terms:

Simple terms:

$$\Lambda^s : \quad s, t, u, v ::= \quad x \mid \lambda x.s \mid sT \mid \mathsf{D}\, s \cdot t$$

Differential λ-terms:

$$T \in \Lambda^{\mathsf{d}} = \text{FreeCommutativeMonoid}(\Lambda^{\mathsf{s}})$$

Syntax of LCD: version 2/3

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Can we avoid mutual induction ?

A stand-alone presentation of simple terms:

Simple terms:

$$\Lambda^s : \quad s, t, u, v ::= \quad x \mid \lambda x.s \mid sT \mid \mathrm{D}\, s \cdot t$$

as an operation: $\Lambda^{\mathrm{s}} \times \mathrm{FreeCommutativeMonoid}(\Lambda^{\mathrm{s}}) \to \Lambda^{\mathrm{s}}$

Differential λ-terms:

$$T \in \Lambda^{\mathrm{d}} = \mathrm{FreeCommutativeMonoid}(\Lambda^{\mathrm{s}})$$

Syntax of LCD: version 3/3

Which operations/equations are allowed to specify a syntax ?

A stand-alone presentation of differential λ-terms:

Allow summands everywhere (not only in the right arg of application)

Differential λ-terms:

$$\Lambda^{\mathrm{d}} : \quad S, T \quad ::= \quad x \mid \lambda x.S \mid S\,T \mid \mathsf{D}S \cdot T$$

$$\mid \; 0 \; \mid \; S + T$$

neutral element for $+$

modulo commutativity and associativity

Turn [Categorical Models for Simply Typed Resource Calculi]'s abbreviations into equations:

$$\lambda x.\Sigma_i t_i = \Sigma_i \lambda x.t_i$$

$$(\Sigma_i t_i)u = \Sigma_i t_i u$$

$$D(\Sigma_i t_i) \cdot (\Sigma_j u_j) = \Sigma_i \Sigma_j D t_i \cdot u_j$$

Syntax of LCD: Conclusion

How can we compare these different versions ?

In which sense are they syntaxes ?

Which operations/equations are we allowed to specify in a syntax ?

Syntax of LCD: Conclusion

How can we compare these different versions ?

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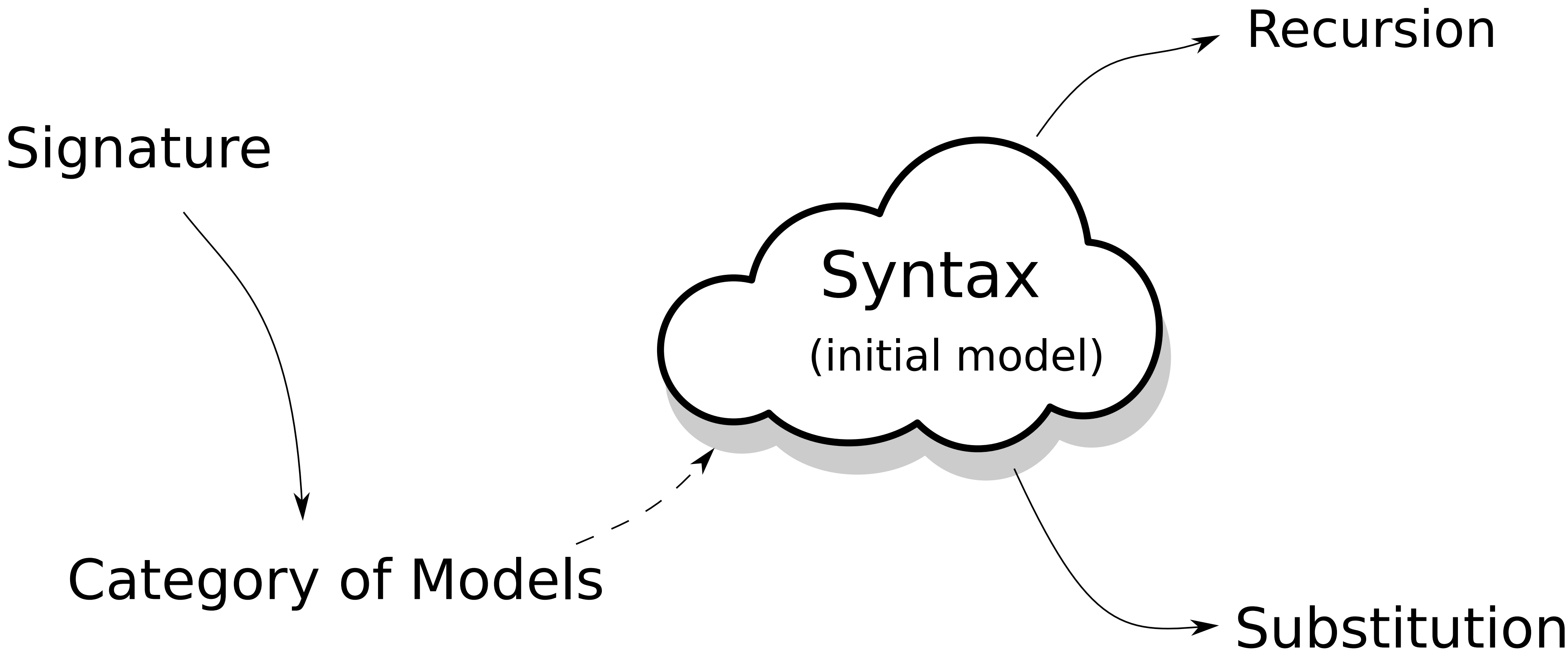
Which operations/equations are we allowed to specify in a syntax ?

What is a syntax ?

What is a syntax?



generates a syntax = existence of the initial model

Overview

Topic: specification and construction of untyped syntaxes with variables and a well-behaved substitution (e.g. differential λ-calculus).

Our work:

1. general notion of ***1-signature*** based on ***monads*** and ***modules***.
   - *Caveat:* Not all of them do ***generate a syntax***
   - special case: classical ***algebraic 1-signatures*** generate a syntax
2. notion of **2-signature**: a pair of a 1-signature and a set of equations.
   - special case: ***algebraic 2-signatures*** generate a syntax

Previous work of Fiore-Hur 2010

[Fiore-Hur 2010]: presentations of simply typed languages by generating *binding* operations (e.g. λ-abstraction) and equations among them.

Our work: for the untyped setting, a variant of their approach where monads and modules over them are the central notions.

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Categorical formulation of a term language

Example: syntax with a binary operation ★, a constant 0, and variables

$$\begin{aligned} \mathrm{expr} ::= {} & x && \textit{(variable)} \\ \mid {} & t_1 \bigstar t_2 && \textit{(binary operation)} \\ \mid {} & 0 && \textit{(constant)} \end{aligned}$$

The syntax can be considered as the endofunctor B (on Set):

$$B : X \mapsto \{\text{expressions over } X\}$$

For example:

$$B(\emptyset) = \{0, 0 \bigstar 0, \dots\}$$

$$B(\{x, y\}) = \{0, 0 \bigstar 0, \dots, x, y, x \bigstar y, \dots\}$$

Categorical formulation of a term language

Then we have:

$$\begin{aligned} \star :&\ B \times B \overset{\cdot}{\to} B \\ 0 :&\quad\ 1 \quad\ \overset{\cdot}{\to} B \\ \mathrm{var} :&\quad \mathrm{Id}_{\mathrm{Set}} \overset{\cdot}{\to} B \end{aligned}$$

Putting all together:

$$B \times B + 1 + \mathrm{Id}_{\mathrm{Set}} \overset{\cdot}{\to} B$$

i.e. B is an algebra for the endofunctor $F \mapsto F \times F + 1 + \mathrm{Id}_{\mathrm{Set}}$ on the category $\mathrm{End}_{\mathrm{Set}}$.

Actually, B can be **characterized** as the initial algebra.

Binding Signatures

Definition

Binding signature = a family of lists of natural numbers.

Each list specifies one operation in the syntax:

- length of the list = number of arguments of the operation
- natural number in the list = number of bound variables in the corresponding argument

Syntax with 0, ★:

$$\big(\underbrace{()}_{0}, \underbrace{(0, 0)}_{\star}\big)$$

Lambda calculus:

$$\big(\underbrace{(0 , 0)}_{app} , \underbrace{(1)}_{abs}\big)$$

Initial semantics for binding signatures

Reminder

The syntax $(0,\bigstar)$ is the initial algebra for the endofunctor:

$$F \mapsto F \times F + 1 + \mathrm{Id}_{\mathrm{Set}}$$

More generally, any binding signature gives rise to an endofunctor Σ.

Definition

Model = $(\Sigma + \mathrm{Id}_{\mathrm{Set}})$-algebra

Classical Theorem

The initial $(\Sigma + \mathrm{Id}_{\mathrm{Set}})$-algebra of a binding signature Σ always exists.

Question: Does this initial algebra come with a well-behaved substitution?

Answer: Yes: see e.g. [Fiore, Plotkin, Turi 1999], [Ghani & Uustalu 2003]

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The Big Picture of 1-signatures and models

Binding signatures $\hookrightarrow$ Our 1-signatures

A **1-signature** Σ is a functorial assignment:

$$R \mapsto \Sigma(R)$$

A **model of Σ** is a pair:

$$(R, \quad \rho : \Sigma(R) \to R)$$

monad	:=	endofunctor with substitution
module over a monad	:=	endofunctor with substitution
module morphism	:=	natural transformation preserving substitution

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Substitution and monads

Reminder:

- $B(X)$ = expressions built out of 0, ★ and variables taken in X
- Variables induce a natural transformation $\mathrm{var} : \mathrm{Id}_{\mathrm{Set}} \to B$

Substitution:

$$\mathrm{bind} : B(X) \to (X \to B(Y)) \to B(Y)$$

+ laws

A triple (B, var, bind) is called a **monad**.

monad morphism = mapping preserving var and bind.

Monads

1. $\mathrm{B} : \mathrm{Set} \to \mathrm{Set}$

   $B(X)$ = expressions built out of 0, ★ and variables taken in X

2. A collection of functions $(\mathrm{var}_{\mathrm{X}} : X \to B(X))_X$

   Variables are expressions

3. For each function $\mathrm{u} : \mathrm{X} \to \mathrm{B(Y)}$, a function $\mathrm{bind}_{\mathrm{u}} : \mathrm{B(X)} \to \mathrm{B(Y)}$

   Parallel substitution

   Notation: $\mathbf{bind_u(t) = t[x \mapsto u(x)]}$

4. Monadic laws:

$$\mathrm{var(y)[x \mapsto u(x)] = u(y)}$$
$$\mathrm{t[x \mapsto var(x)] = t}$$
$$\mathrm{t[x \mapsto f(x)][y \mapsto g(y)] = t[x \mapsto f(x)[y \mapsto g(y)]\ ]}$$

Preview: Operations are module morphisms

$\star$ commutes with substitution

$$(t \star u)[x \mapsto v_x] = t[x \mapsto v_x] \star u[x \mapsto v_x]$$

Categorical formulation

$B \times B$ supports B-substitution $\rightsquigarrow$ $B \times B$ is a **module over B**

$\star$ commutes with substitution $\rightsquigarrow$ $\star : B \times B \to B$ is a **module morphism**

Modules VS Monads

Monad

1. $\mathrm{B} : \mathrm{Set} \to \mathrm{Set}$

   $B(X)$ = expressions built out of 0, ★ and variables taken in X

2. A collection of functions $(\mathrm{var}_\mathrm{X} : X \to B(X))_X$

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   Parallel substitution

   Notation: $\mathbf{bind_u(t)} = \mathbf{t[x \mapsto u(x)]^B}$

4. Substitution laws:

$$\mathrm{var(y)[x \mapsto u(x)]^B = u(y)}$$
$$\mathrm{t[x \mapsto var(x)]^B = t}$$
$$\mathrm{t[x \mapsto f(x)]^B[y \mapsto g(y)]^B = t[x \mapsto f(x)[y \mapsto g(y)]^B\ ]^B}$$

Modules VS Monads

~~**Monad**~~ **Module over a monad** B (e.g. $\mathrm{B} \times \mathrm{B}, \mathbf{2}, \ldots$)

1. $\mathrm{M} : \mathrm{Set} \to \mathrm{Set}$

   M(X) = expressions taking variables in X

2. ~~A collection of functions $(\mathrm{var}_X : X \to M(X))_X$~~

3. For each function $\mathrm{u} : \mathrm{X} \to \mathrm{B}(\mathrm{Y})$, a function $\mathrm{bind}_\mathrm{u} : \mathrm{M}(\mathrm{X}) \to \mathrm{M}(\mathrm{Y})$

   Parallel substitution

   Notation: $\mathbf{bind_u(t)}$ **=** $\mathbf{t[x \mapsto u(x)]^{M}}$

4. Substitution laws:

$$\cancel{\mathrm{var}(\mathrm{y})[\mathrm{x} \mapsto \mathrm{u}(\mathrm{x})] = \mathrm{u}(\mathrm{y})}$$

$$\mathrm{t}[\mathrm{x} \mapsto \mathrm{var}(\mathrm{x})]^{\mathrm{M}} = \mathrm{t}$$

$$\mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})]^{\mathrm{M}}[\mathrm{y} \mapsto \mathrm{g}(\mathrm{y})]^{\mathrm{M}} = \mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})[\mathrm{y} \mapsto \mathrm{g}(\mathrm{y})]^{\mathrm{B}}\,]^{\mathrm{M}}$$

Building blocks for binding signatures

Essential constructions of **modules over a monad $\boldsymbol{R}$**:

- R itself
- M x N for any modules M and N (in particular, R x R)
- The **derivative of a module $\boldsymbol{M}$** is the module M' defined by M'(X) = M($X + \{\diamond\}$).

  The derivative is used to model an operation binding a variable

  (Cf next slide).

Syntactic operations are module morphisms

module morphism = maps commuting with substitution.

$$id_M : M \to M$$

$$0 : 1 \to B$$

$$\star : B \times B \to B$$

$$app : \Lambda \times \Lambda \to \Lambda$$

$$abs : \Lambda' \to \Lambda$$

The Big Picture again

A **1-signature** Σ is a functorial assignment:

$$\underbrace{R}_{\text{monad}} \mapsto \underbrace{\Sigma(R)}_{\text{module over } R}$$

A **model of Σ** is a pair:

$$(\underbrace{R}_{\text{monad}}, \quad \underbrace{\rho : \Sigma(R) \to R}_{\text{module morphism}})$$

A **model morphism** $m : (R,\rho) \to (S,\sigma)$ is a monad morphism commuting with the module morphism:

$$\begin{array}{ccc} \Sigma(R) & \xrightarrow{\rho} & R \\ {\scriptstyle \Sigma(m)}\downarrow & & \downarrow{\scriptstyle m} \\ \Sigma(S) & \xrightarrow[\sigma]{} & S \end{array}$$

Syntax

Definition

Given a 1-signature Σ, its **syntax** is an initial object in its category of models.

Question: Does the syntax exist for every 1-signature?

Answer: No.

Counter-example: the 1-signature $R \mapsto \mathscr{P} \circ R$

powerset endofunctor on $\mathbb{S}\mathrm{et}$

Examples of 1-signatures generating syntax

- **(0,★) language**:

  Signature: $R \mapsto 1 + R \times R$

  Model: $(R\,,\quad 0 : 1 \to R\,,\quad \bigstar : R \times R \to R)$

  Syntax: $(B\,,\quad 0 : 1 \to B\,,\quad \bigstar : B \times B \to B)$

- **lambda calculus**:

  Signature: $R \mapsto R' + R \times R$

  Model: $(R\,,\quad abs : R' \to R\,,\quad app : R \times R \to R)$

  Syntax: $(\Lambda\,,\quad abs : \Lambda' \to \Lambda\,,\quad app : \Lambda \times \Lambda \to \Lambda)$

Can we generalize this pattern?

Initial semantics for algebraic 1-signatures

Theorem [Hirschowitz & Maggesi 2007]

Syntax exists for any **algebraic 1-signature**, i.e. 1-signature built out of derivatives, products, coproducts, and the trivial 1-signature $R \mapsto R$.

Algebraic 1-signatures correspond to binding signatures through the embedding:

$$\text{Binding signatures} \hookrightarrow \text{Our 1-signatures}$$

Question: Can we enforce some equations in the syntax ?
For example: lambda calculus modulo beta and eta.

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Example: a commutative binary operation

Specification of a binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R\,,\ + : R \times R \to R)$

What is an appropriate notion of model for a commutative binary operation ?

Example: a commutative binary operation

Specification of a commutative binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R\,,\ + : R \times R \to R)$ s.t. $t + u = u + t$ (1)

What is an appropriate notion of model for a commutative binary operation ?

Answer: a monad equipped with a commutative binary operation

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What is an appropriate notion of model for a commutative binary operation ?

Answer: a monad equipped with a commutative binary operation

Equation (1) states an equality between R-module morphisms:

$$\begin{tikzcd}
\mathrm{R} \times \mathrm{R} \arrow[rr, bend left, "+"] \arrow[r, bend right, "\text{swap}"'] & \mathrm{R} \times \mathrm{R} \arrow[r, bend right, "+"'] & \mathrm{R}
\end{tikzcd}$$

Review: Signatures with equations

- [Fiore-Hur 2010]: existence of an initial model for an inductively defined (with a specific syntax) set of possible equations.

- [AHLM CSL 2018]: "quotients" of algebraic 1-signatures generate a syntax (e.g. a binary commutative operation).

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Our work: alternative approach where monads and modules are the central notions.

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- [AHLM CSL 2018]: "quotients" of algebraic 1-signatures generate a syntax (e.g. a binary commutative operation).

This work: more general equations (e.g. λ-calculus modulo βη).

Equations

Given a 1-signature Σ, a **Σ-equation** A $\rightrightarrows$ B is a functorial assignment

$$R \mapsto \left(\begin{tikzcd} A(R) \arrow[r, shift left] \arrow[r, shift right] & B(R) \end{tikzcd} \right)$$

model of Σ

parallel pair of module

morphisms over R

A **2-signature** is a pair

$$(\Sigma, \mathrm{E})$$

1-signature

set of Σ-equations

***model* of a 2-signature $(\boldsymbol{\Sigma}, \mathbf{E})$**:

- a model R of Σ
- s.t. ∀ (A $\rightrightarrows$ B) $\in$ E, the two morphisms $A(R) \rightrightarrows \mathrm{B}(\mathrm{R})$ are equal

Algebraic 2-signatures

Given a 1-signature Σ, a Σ-equation A $\rightrightarrows$ B is **elementary** if:

1. A "preserves pointwise epimorphisms"

(e.g., any "algebraic 1-signature")

2. B is of the form R ↦ R'$^{\cdots}$' (e.g. R ↦ R)

Algebraic 2-signature:

$$(\Sigma, \mathrm{E})$$

algebraic 1-signature

set of **elementary** Σ-equations

Theorem

Syntax exists for any algebraic 2-signature

Example: λ-calculus modulo βη

The algebraic 2-signature $(\Sigma_{\mathrm{LC\beta\eta}}, E_{\mathrm{LC\beta\eta}})$ of λ-calculus modulo βη:

$$\boldsymbol{\Sigma}_{\mathbf{LC\beta\eta}}(R) := \Sigma_{\mathrm{LC}}(R) = R \times R + R'$$

model of $\boldsymbol{\Sigma}_{\mathbf{LC}}$ = monad R with module morphisms:

$$\mathrm{app} : R \times R \to R \qquad \mathrm{abs} : R' \to R$$

β-equation: $(\lambda x.t)\, u = \underbrace{t[x \mapsto u]}_{\sigma_R(t,u)}$

η-equation: $t = \lambda x.(t\ x)$

$$\mathbf{E}_{\mathbf{LC\beta\eta}} = \{\ \beta\text{-equation},\ \eta\text{-equation}\ \}$$

Example: λ-calculus modulo βη

The algebraic 2-signature $(\Sigma_{\mathrm{LC\beta\eta}}, E_{\mathrm{LC\beta\eta}})$ of λ-calculus modulo βη:

$$\mathbf{\Sigma_{LC\beta\eta}}\,(\mathrm{R}) := \Sigma_{\mathrm{LC}}(\mathrm{R}) = \mathrm{R} \times \mathrm{R} + \mathrm{R}'$$

model of $\mathbf{\Sigma_{LC}}$ = monad R with module morphisms:

$$\mathrm{app} : \mathrm{R} \times \mathrm{R} \to \mathrm{R} \qquad \mathrm{abs} : \mathrm{R}' \to \mathrm{R}$$

β-equation: $(\lambda x.t)\ u = \underbrace{t[x \mapsto u]}_{\sigma_{\mathrm{R}}(t,u)}$ **η-equation**: $t = \lambda x.(t\ x)$

$$\begin{tikzcd} R' \times R \arrow[rr, bend left, "\sigma_R"] \arrow[r, "\mathrm{abs} \times R"'] & R \times R \arrow[r, "\mathrm{app}"'] & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[rr, bend left, "\mathrm{id}_R"] \arrow[r, "R\iota_1"'] & R' \arrow[r, "\mathrm{abs}"'] & R \end{tikzcd}$$

$$\mathbf{E_{LC\beta\eta}} = \{\ \beta\text{-equation},\ \eta\text{-equation}\ \}$$

Example: fixpoint operator

Definition [AHLM CSL 2018]

A **fixpoint operator** in a monad R is a module morphism $f : R' \to R$ s.t.

$$(1) \qquad \begin{tikzcd} R' \arrow[rr, bend left, "f"] \arrow[r, bend right, "(\mathrm{id}_{R'}, f)"'] & R' \times R \arrow[r, bend right, "\sigma_R"'] & R \end{tikzcd} \qquad \text{commutes.}$$

The algebraic 2-signature $(\Sigma_{\mathrm{fix}}, E_{\mathrm{fix}})$ of a fixpoint operator:

$$\Sigma_{\mathrm{fix}}(R) := R' \qquad\qquad E_{\mathrm{fix}} = \{ (1) \}$$

Proposition [AHLM CSL 2018]

Fixpoint operators in $LC_{\beta\eta}$ are in one to one correspondance with fixpoint combinators (i.e. λ-terms Y s.t. $t\,(Y\,t) = Y\,t$ for any t).

Combining algebraic 2-signatures

Algebraic 2-signatures can be combined:

fixpoint operator

λ-calculus modulo βη

$$(\Sigma_{\mathrm{fix}}, \mathrm{E}_{\mathrm{fix}}) \quad + \quad (\Sigma_{\mathrm{LC}\beta\eta}, \mathrm{E}_{\mathrm{LC}\beta\eta})$$

$$=$$

$$(\Sigma_{\mathrm{fix}} + \Sigma_{\mathrm{LC}\beta\eta} \;,\; \mathrm{E}_{\mathrm{fix}} \cup \mathrm{E}_{\mathrm{LC}\beta\eta})$$

λ-calculus modulo βη with an explicit fixpoint operator

Example: free monoid

An algebraic 2-signature $(\Sigma_{\mathrm{mon}}, E_{\mathrm{mon}})$ for the free monoid monad $X \mapsto \coprod_n X^n$

$$\boldsymbol{\Sigma}_{\mathrm{mon}}(R) := 1 + R \times R$$

model of Σ = monad R with module morphisms:

$$\varepsilon : 1 \to R \qquad m : R \times R \to R$$

3 elementary Σ-equations:

$$\begin{tikzcd}
 & R^2 \arrow[dr, bend left, "m"] & \\
R^3 \arrow[ur, bend left, "R\times m"] \arrow[dr, bend right, "m\times R"'] & & R \\
 & R^2 \arrow[ur, bend right, "m"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R^2 \arrow[dr, bend left, "m"] & \\
R \arrow[ur, bend left, "\epsilon, id"] \arrow[rr, bend right, "id"'] & & R
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R^2 \arrow[dr, bend left, "m"] & \\
R \arrow[ur, bend left, "id, \epsilon"] \arrow[rr, bend right, "id"'] & & R
\end{tikzcd}$$

associativity   left unit   right unit

Our target: LCD

Syntax of the *differentiable λ-calculus*:

Simple terms $\mathrm{s},\mathrm{t} \in \Lambda$

$$
\begin{array}{rl}
s,t \; ::= & \left.\begin{array}{l} \mathrm{x} \\ \mid \;\; \lambda \mathrm{x}.\mathrm{t} \\ \mid \;\; \mathrm{s}\;\mathrm{t} \end{array}\right\} \; \lambda\text{-calculus} \\
& \mid \;\; \mathrm{Ds} \cdot \mathrm{t} \\
& \left.\begin{array}{l} \mid \;\; \mathrm{s} + \mathrm{t} \\ \mid \;\; 0 \end{array}\right\} \; \text{free commutative monoid}
\end{array}
$$

and (bi)linearity of constructors with respect to +:

$$\lambda \mathrm{x}.(\mathrm{s}+\mathrm{t}) = \lambda \mathrm{x}.\mathrm{s} + \lambda \mathrm{x}.\mathrm{t} \qquad \ldots$$

Algebraic 1-signature for LCD

Syntax of the *differentiable λ-calculus*:

Simple terms $s,t \in \Lambda$ — Corresponding 1-signature

$$\left.\begin{array}{rcl} s,t & ::= & x \\ & \mid & \lambda x.t \\ & \mid & s\ t \end{array}\right\} \quad \Sigma_{\mathrm{LC}}(R) = R' + R \times R$$

$$\begin{array}{rcl} & \mid & Ds \cdot t \end{array} \qquad R \mapsto R \times R$$

$$\left.\begin{array}{rcl} & \mid & s + t \\ & \mid & 0 \end{array}\right\} \quad \Sigma_{\mathrm{mon}}(R) = 1 + R \times R$$

Algebraic 1-signature for LCD

Syntax of the *differentiable λ-calculus*:

Simple terms $s,t \in \Lambda$ — Corresponding 1-signature

$$
\left.
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 & | & s\ t
\end{array}
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$$
\left.
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 & | & 0
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\right\}
\quad \Sigma_{mon}(R) = 1 + R \times R
$$

Resulting algebraic 1-signature: $\Sigma_{LCD}(R) = \Sigma_{LC}(R) + R \times R + \Sigma_{mon}(R)$

Elementary equations for LCD

Commutative monoidal structure:

$$s + t = t + s \qquad R{\times}R \rightrightarrows R$$

$$E_{\mathrm{mon}} \left\{ \begin{array}{ll} s + (t + u) = (s + t) + u & R{\times}R{\times}R \rightrightarrows R \\ 0 + t = t & R \rightrightarrows R \\ t + 0 = t & R \rightrightarrows R \end{array} \right.$$

Linearity:

$$\lambda x.(s{+}t) = \lambda x.s + \lambda x.t \qquad R{\times}R \rightrightarrows R$$

$$D(s{+}t){\cdot}u = Ds{\cdot}u + Dt{\cdot}u \qquad R{\times}R{\times}R \rightrightarrows R$$

$$Ds{\cdot}(t{+}u) = Ds{\cdot}t + Ds{\cdot}u \qquad R{\times}R{\times}R \rightrightarrows R$$

$$\ldots$$

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2. Show that all the equations in E are satisfied for this model

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Initiality of R $\Rightarrow$ model morphism $R \to S$ $\Rightarrow$ monad morphism $R \to S$

Example: Computing the set of free variables

LC = initial model of $(\Sigma_{\mathrm{LC}}, \varnothing)$ $\qquad$ $\Sigma_{\mathrm{LC}}(\mathrm{R}) = \mathrm{R} \times \mathrm{R} + \mathrm{R}'$

$\mathcal{P}$ = power set monad

Definition of a (monad) morphism $\mathrm{fv} : \mathrm{LC} \to \mathcal{P}$ **s.t.**

$$\mathrm{fv}(\mathrm{app}(\mathrm{t},\mathrm{u})) = \mathrm{fv}(\mathrm{t}) \cup \mathrm{fv}(\mathrm{u}) \qquad \mathrm{fv}(\mathrm{abs}(\mathrm{t})) = \mathrm{fv}(\mathrm{t}) \setminus \{\diamond\}$$

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$\Rightarrow$ make $\mathcal{P}$ a model of Σ_{LC}:

$$\cup : \mathcal{P} \times \mathcal{P} \to \mathcal{P} \qquad \_\setminus\{\diamond\} : \mathcal{P}' \to \mathcal{P}$$

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Initiality of $\mathrm{LC} \Rightarrow$ $\mathrm{fv} : \mathrm{LC} \to \mathcal{P}$ satisfying the above equations (as a model morphism).

Example: Translating λ-calculus with fixpoint

$\mathrm{LC}_{\beta\eta\mathrm{fix}}$ = initial model of $(\Sigma_{\mathrm{LC}\beta\eta}, \mathrm{E}_{\mathrm{LC}\beta\eta}) + (\Sigma_{\mathrm{fix}}, \mathrm{E}_{\mathrm{fix}})$

λ-calculus modulo βη with a fixpoint operator $\mathrm{fix} : \mathrm{LC}_{\beta\eta\mathrm{fix}}' \to \mathrm{LC}_{\beta\eta\mathrm{fix}}$

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λ-calculus modulo βη

monad morphism

Definition of a translation $\mathrm{f} : \mathrm{LC}_{\beta\eta\mathrm{fix}} \to \mathrm{LC}_{\beta\eta}$ **s.t.**

$$\mathrm{f}(\mathrm{u}) = \text{"}\mathrm{u}[\,\mathrm{fix}(\mathrm{t}) \mapsto \mathrm{app}(\mathrm{Y}, \mathrm{abs}(\mathrm{t}))\,]\text{"}$$

a chosen fixpoint combinator

Example: Translating λ-calculus with fixpoint

$LC_{\beta\eta fix}$ = initial model of $(\Sigma_{LC\beta\eta}, E_{LC\beta\eta}) + (\Sigma_{fix}, E_{fix})$

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λ-calculus modulo βη

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Definition of a translation $f : LC_{\beta\eta fix} \to LC_{\beta\eta}$ **s.t.**

$$f(u) = \text{"}u[\ \mathrm{fix}(t) \mapsto \mathrm{app}(Y, \mathrm{abs}(t))\ ]\text{"}$$

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$\Rightarrow$ make $LC_{\beta\eta}$ a model of $(\Sigma_{LC\beta\eta}, E_{LC\beta\eta}) + (\Sigma_{fix}, E_{fix})$:

app, abs

$\hat{Y} : LC_{\beta\eta}' \to LC_{\beta\eta}$

$t \mapsto \mathrm{app}(Y, \mathrm{abs}(t))$

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$\mathrm{app}, \mathrm{abs}$

$$\hat{\mathrm{Y}} : \mathrm{LC}_{\beta\eta}' \to \mathrm{LC}_{\beta\eta}$$
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Solution [CSL AHLM 2010]:

continuation monad $\mathrm{C}(\mathrm{X}) = \mathbb{N}^{(\mathbb{N}^{\mathrm{X}})}$

1. define $\mathrm{f} : \mathrm{LC} \to \mathrm{C}$ by recursion
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Solution [CSL AHLM 2010]:

continuation monad C(X) = $\mathbb{N}^{(\mathbb{N}^{\mathrm{X}})}$

1. define $\mathrm{f} : \mathrm{LC} \to \mathrm{C}$ by recursion
2. deduce $\mathrm{s} : \mathrm{LC} \to \mathbb{N}$

affects an arbitrary size to each variable

Intuition: uncurrying $\mathrm{f}_{\mathrm{X}} : \mathrm{LC}(\mathrm{X}) \to \mathbb{N}^{(\mathbb{N}^{\mathrm{X}})}$ yields $\mathrm{g} : \mathrm{LC}(\mathrm{X}) \times \mathbb{N}^{\mathrm{X}} \to \mathbb{N}$

Example: Computing the size of a term

LC = initial model of $(\Sigma_{\mathrm{LC}}, \emptyset)$ $\Sigma_{\mathrm{LC}}(R) = R \times R + R'$

Definition of a ~~(monad)~~ morphism $s : \mathrm{LC} \to \mathbb{N}$ **s.t.**

$$s(\mathrm{app}(t,u)) = 1 + s(t) + s(u) \qquad s(\mathrm{abs}(t)) = 1 + s(t)$$

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Solution [CSL AHLM 2010]: continuation monad C(X) = $\mathbb{N}^{(\mathbb{N}^X)}$

1. define $f : \mathrm{LC} \to C$ by recursion
2. deduce $s : \mathrm{LC} \to \mathbb{N}$

affects an arbitrary size to each variable

Intuition: uncurrying $f_X : \mathrm{LC}(X) \to \mathbb{N}^{(\mathbb{N}^X)}$ yields $g : \mathrm{LC}(X) \times \mathbb{N}^X \to \mathbb{N}$

$$s(t) = g(t, (x \mapsto 0))$$

variables are of size 0

Conclusion

Summary of the talk:

- presented a notion of 1-signature and models
- defined a 2-signature as a 1-signature and a set of equations
- identified a class of 2-signatures that generate a syntax

The main theorem has been formalized in Coq using the UniMath library.

Future work:

- add the notion of reductions;
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Thank you!