

Higher-order Arities, Signatures and Equations via Modules

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joint work with
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Keywords associated with syntax

Induction/Recursion

Substitution

Model

Syntax

Operation/Construction

Arity/Signature

This talk: give a *discipline* for specifying syntaxes

Motivating example: dLC

syntax of dLC = **differential λ -calculus** [Ehrhard-Regnier 2003].

- explicitly involves **equations** e.g. $s+t = t+s$
- specifically tailored: (not an *instance* of a general framework/scheme)
inductive definition of a set + ad-hoc structure
 e.g. **unary substitution**

Our proposal = a discipline for presenting syntaxes

- signature = operations + equations
- [Fiore-Hure 2010]: alternative approach, for simply typed syntaxes
 \Rightarrow our approach explicitly relies on monads and modules (untyped case).

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Syntax of dLC: [Ehrhard-Regnier 2003]

Let be given a denumerable set of variables. We define by induction on k an increasing family of sets (Δ_k) . We set $\Delta_0 = \emptyset$ and Δ_{k+1} is defined as follows.

Monotonicity: if t belongs to Δ_k then t belongs to Δ_{k+1} .

Variable: if $n \in \mathbb{N}$, x is a variable, $i_1, \dots, i_n \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ and $u_1, \dots, u_n \in \Delta_k$, then

$$D_{i_1, \dots, i_n} x \cdot (u_1, \dots, u_n)$$

belongs to Δ_{k+1} . This term is identified with all the terms of the shape $D_{i_{\sigma(1)}, \dots, i_{\sigma(n)}} x \cdot (u_{\sigma(1)}, \dots, u_{\sigma(n)}) \in \Delta_{k+1}$ where σ is a permutation on $\{1, \dots, n\}$.

Abstraction: if $n \in \mathbb{N}$, x is a variable, $u_1, \dots, u_n \in \Delta_k$ and $t \in \Delta_k$, then

$$D_1^n \lambda x t \cdot (u_1, \dots, u_n)$$

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Application: if $s \in \Delta_k$ and $t \in R\langle \Delta_k \rangle$, then

$$(s)t$$

belongs to Δ_{k+1} .

Setting $n=0$ in the first two clauses, and restricting application by the constraint that $t \in \Delta_k \subseteq R\langle \Delta_k \rangle$, one retrieves the usual definition of lambda-terms which shows that differential terms are a superset of ordinary lambda-terms.

The permutative identification mentioned above will be called *equality up to differential permutation*. We also work up to α -conversion.

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Application: if $s \in \Delta_k$ and $t \in R\langle \Delta_k \rangle$, then

$$(s)t \leftarrow \text{as an operation: } \Lambda \times \text{FreeCommutativeMonoid}(\Lambda) \rightarrow \Lambda$$

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Syntax of dLC: [BEM 2010]

A **syntax** for the **differential λ -calculus** by **mutual induction**:

[Bucciarelli-Ehrhard-Manzonetto 2010]

Simple terms:

$$\Lambda^s : \quad s, t \quad ::= \quad x \mid \lambda x. s \mid sT \mid D s \cdot t$$

Differential λ -terms:


$$\Lambda^d : \quad T \quad ::= \quad 0 \mid s \mid s + T$$

Syntax of dLC: [BEM 2010]



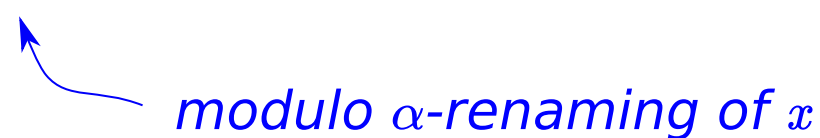
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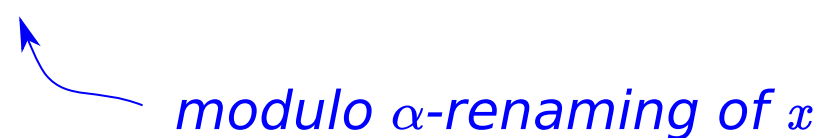
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modulo α -renaming of x

neutral element for $+$

modulo commutativity


$$\Lambda^d = \mathbf{FreeCommutativeMonoid}(\Lambda^s)$$

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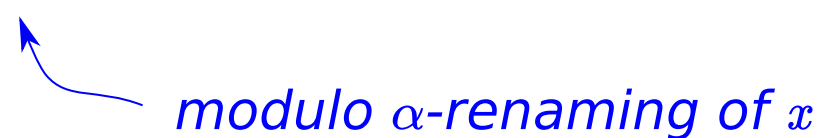
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Syntax: specified by operations and **equations**.

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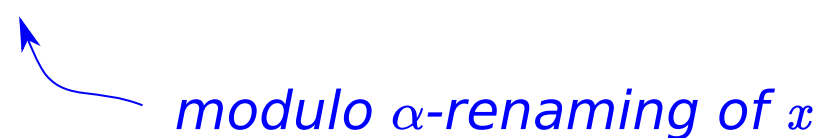
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Syntax: specified by operations and **equations**.

But which ones are allowed ? What is the limit ?

Syntax of dLC: Our version

Which operations/equations are allowed to specify a syntax ?

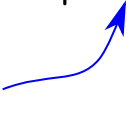

A stand-alone presentation of differential λ -terms:

Allow sums everywhere (not only in the right arg of application)

Differential λ -terms:

$$\Lambda^d : S, T ::= x \mid \lambda x. S \mid S T \mid D S \cdot T$$

$$\mid 0 \mid S + T$$

neutral element for +  *modulo commutativity and associativity* 

Macros in [BEM 2010]:

$$\lambda x. \Sigma_i t_i := \Sigma_i \lambda x. t_i$$

$$(\Sigma_i t_i) u := \Sigma_i t_i u$$

$$D(\Sigma_i t_i) \cdot (\Sigma_j u_j) := \Sigma_i \Sigma_j D t_i \cdot u_j$$

Syntax of dLC: Conclusion

How can we compare these different versions ?

In which sense are they syntaxes ?

Which operations/equations are we allowed to specify in a syntax ?

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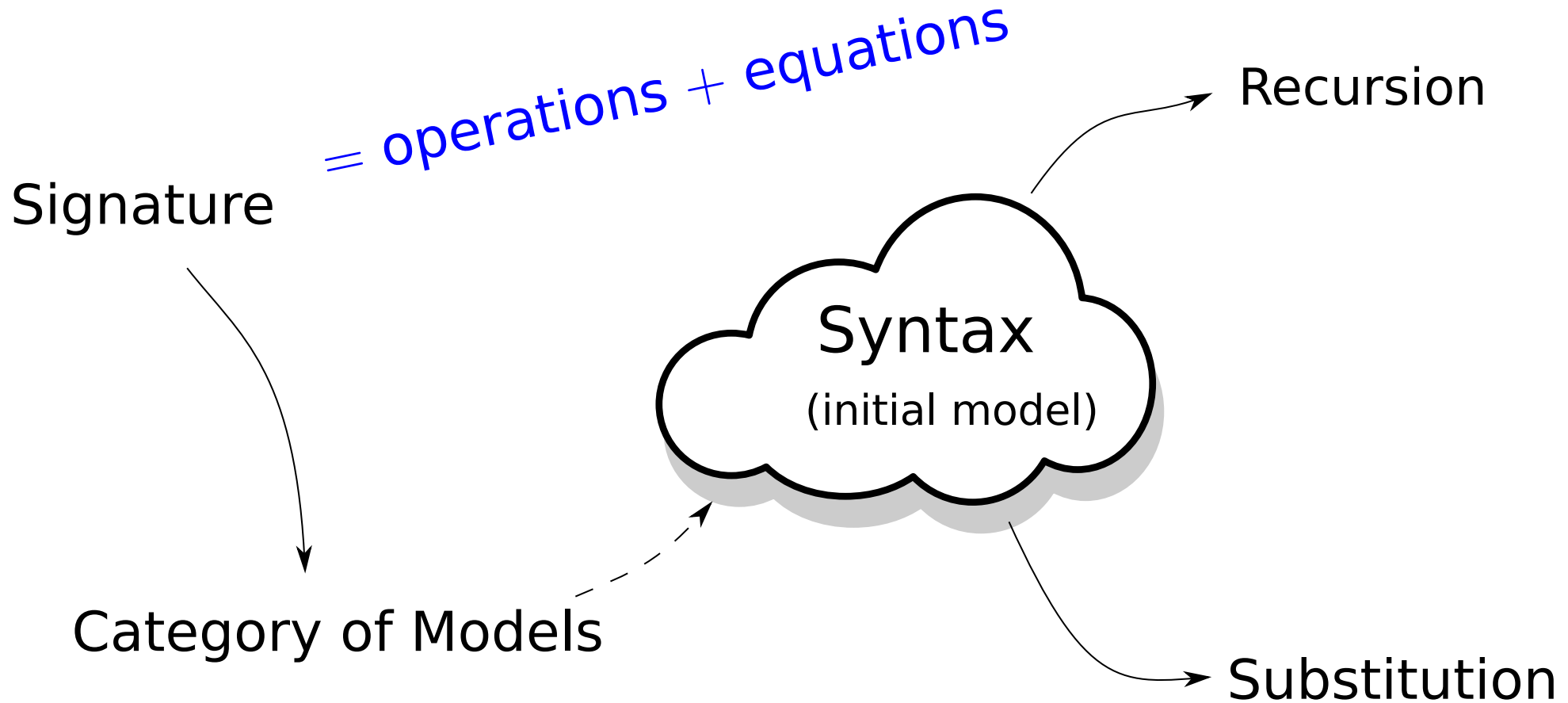
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What is a syntax ?

What is a syntax?



generates a syntax = existence of the initial model

Table of contents

1. 1-Signatures and models based on monads and modules

2. Equations

3. Recursion

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- Substitution and monads
- 1-Signatures and their models

2. Equations

3. Recursion

Substitution and monads

Example: differential λ -calculus

$$\Lambda^d : S, T ::= x \mid \lambda x. S \mid S T \mid DS \cdot T \\ \mid 0 \mid S + T$$

Free variable indexing:

$$dLC : X \mapsto \{\text{terms taking free variables in } X\}$$

$$dLC(\emptyset) = \{0, \lambda z.z, \dots\}$$

$$dLC(\{x, y\}) = \{0, \lambda z.z, \dots, x, y, x + y, \dots\}$$

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Parallel substitution:

$$\begin{array}{ll} \text{bind}_f : dLC(X) \rightarrow dLC(Y) & \text{where } f : X \rightarrow dLC(Y) \\ t \mapsto t[x \mapsto f(x)] \end{array}$$

$\Rightarrow (\text{dLC}, \text{var}_X : X \subset \text{dLC}(X), \text{bind}) = \mathbf{monad \ on \ Set}$

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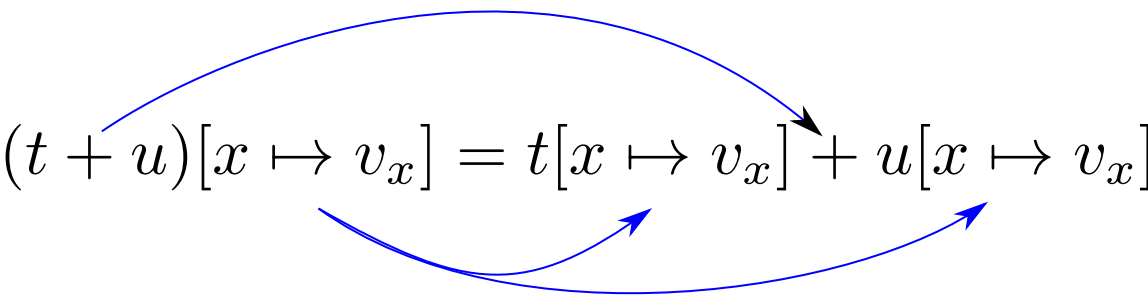
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$\Rightarrow (dLC, \text{var}_X : X \subset dLC(X), \text{bind}) = \mathbf{monad\ on\ Set}$

monad morphism = mapping preserving variables and substitutions.

Preview: Operations are module morphisms

+ commutes with substitution

$$(t + u)[x \mapsto v_x] = t[x \mapsto v_x] + u[x \mapsto v_x]$$


Categorical formulation

$dLC \times dLC$ supports
 dLC -substitution



$dLC \times dLC$ is a **module over dLC**

+ commutes
with substitution



$+ : dLC \times dLC \rightarrow dLC$ is a
module morphism

Building blocks for specifying operations

Essential constructions of **modules over a monad R** :

- R itself

- $M \times N$ for any modules M and N

e.g. $R \times R$:

$$(t, u)[x \mapsto f(x)] := (t[x \mapsto f(x)], u[x \mapsto f(x)]) \quad \text{where } f: X \rightarrow R(Y)$$

- $M' = \text{derivative of a module } M$: $M'(X) = M(X \amalg \{\diamond\})$.

used to model an operation binding a variable (Cf next slide).

Syntactic operations are module morphisms

operations = module morphisms = maps commuting with substitution.

$$\text{app} : \text{dLC} \times \text{dLC} \rightarrow \text{dLC}$$

$$0 : 1 \rightarrow \text{dLC}$$

$$\text{abs} : \text{dLC}' \rightarrow \text{dLC}$$

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Combining operations into a single one using disjoint union

$$[\text{app}, \text{abs}] : (\text{dLC} \times \text{dLC}) \coprod \text{dLC}' \rightarrow \text{dLC}$$

$$[0, +] : 1 \coprod (\text{dLC} \times \text{dLC}) \rightarrow \text{dLC}$$

1-signatures and their models

A **1-signature** Σ = functorial assignment:

$$R \mapsto \Sigma(R)$$

Example: $(0, +)$

$$\Sigma_{0,+}(R) = 1 \coprod (R \times R)$$

A **model of Σ** is a pair:

$$(R, \rho : \Sigma(R) \rightarrow R)$$

dLC = model of $\Sigma_{0,+}$

$$[0, +] : 1 \coprod (dLC \times dLC) \rightarrow dLC$$

A **model morphism** $m : (R, \rho) \rightarrow (S, \sigma)$ = monad morphism commuting with the module morphism:

$$\begin{array}{ccc} \Sigma(R) & \xrightarrow{\rho} & R \\ \Sigma(m) \downarrow & & \downarrow m \\ \Sigma(S) & \xrightarrow{\sigma} & S \end{array}$$

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Syntax

Definition

Given a 1-signature Σ , its **syntax** is an initial object in its category of models.

Question: Does the syntax exist for every 1-signature?

Answer: No.

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Counter-example: The 1-signature $R \mapsto \mathcal{P} \circ R$ has a syntax S .

powerset endofunctor on Set



Examples of 1-signatures generating syntax

- **(0,+) language:**

Signature: $R \mapsto 1 \coprod (R \times R)$

Model: $(R, \quad 0 : 1 \rightarrow R, \quad + : R \times R \rightarrow R)$

Syntax: $(B, \quad 0 : 1 \rightarrow B, \quad + : B \times B \rightarrow B)$

- **lambda calculus:**

Signature: $R \mapsto R' \coprod (R \times R)$

Model: $(R, \quad abs : R' \rightarrow R, \quad app : R \times R \rightarrow R)$

Syntax: $(\Lambda, \quad abs : \Lambda' \rightarrow \Lambda, \quad app : \Lambda \times \Lambda \rightarrow \Lambda)$

Can we generalize this pattern?

Initial semantics for algebraic 1-signatures

Theorem [Hirschowitz & Maggesi 2007]

Syntax exists for any **algebraic 1-signature**, i.e. 1-signature built out of derivatives, products, disjoint unions, and the 1-signature $R \mapsto R$.

Algebraic 1-signatures correspond to the binding signatures described in [Fiore-Plotkin-Turi 1999]

(binding signatures = lists of natural numbers specify n-ary operations, possibly binding variables)

Question: Can we enforce some equations in the syntax ?

e.g. **associativity** and **commutativity** of $+$ for the differential λ -calculus.

Quotients of algebraic 1-signatures

More sophisticated 1-signatures: ***quotients*** of algebraic 1-signatures.

Theorem [AHLM CSL 2018]

Syntax exists for any "***quotient***" of algebraic 1-signature.

Examples:

- a **commutative** binary operation
- application of the simple terms of differential λ -calculus (2nd variant)

app : $\text{dLC} \times \text{FreeCommutativeMonoid}(\text{dLC}) \rightarrow \text{dLC}$

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... but not enough for the differential λ -calculus:

- **associativity** of $+$
- **linearity** of the operations

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Example: a commutative binary operation

Specification of a binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R, + : R \times R \rightarrow R)$

What is an appropriate notion of model for a commutative binary operation ?

Example: a commutative binary operation

Specification of a **commutative** binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R, + : R \times R \rightarrow R)$ s.t. $t + u = u + t$ (1)

What is an appropriate notion of model for a commutative binary operation ?

Answer: a monad equipped with a **commutative** binary operation

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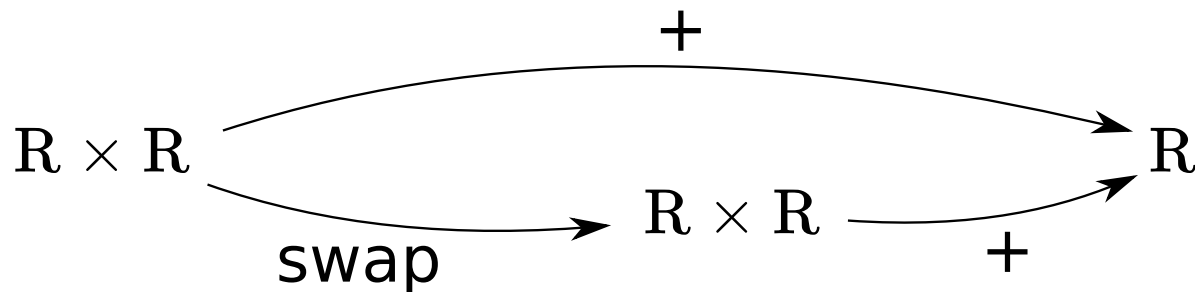
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Equation (1) states an equality between R -module morphisms:



Equations

Given a 1-signature Σ , (e.g. binary operation: $\Sigma(R) = R \times R$)

a Σ -**equation** $A \rightrightarrows B$ is a functorial assignment: e.g. commutativity:

$$R \mapsto \left(A(R) \rightrightarrows B(R) \right)$$

model of Σ

parallel pair of module morphisms over R

$$R \mapsto \left(R \times R \xrightarrow[+ \circ swap]{+} R \right)$$

A **2-signature** is a pair

$$(\Sigma, E)$$

1-signature

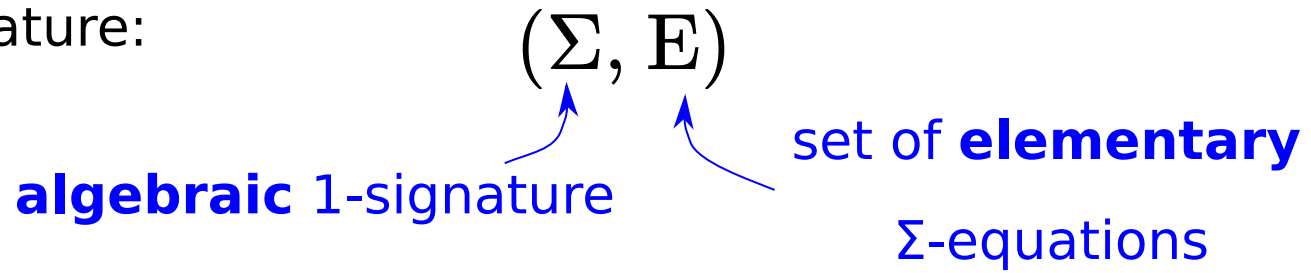
set of Σ -equations

model of a 2-signature (Σ, E) :

- a model R of Σ
- s.t. $\forall (A \rightrightarrows B) \in E$, the two morphisms $A(R) \rightrightarrows B(R)$ are equal

Initial semantics for algebraic 2-signatures

Algebraic 2-signature:



Theorem

Syntax exists for any algebraic 2-signature.

Main instances of **elementary** Σ -equations $A \Rightarrow B$:

- $A =$ algebraic 1-signature e.g. $A(R) = R \times R$
- $B(R) = R$

Initial semantics for algebraic 2-signatures

Algebraic 2-signature:

(Σ, E)
↑ ↑
algebraic 1-signature set of **elementary**
 Σ -equations

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- $A =$ algebraic 1-signature e.g. $A(R) = R \times R$
- $B(R) = R$

Sketch of the construction of the syntax:

Quotient the initial model R of Σ by the following relation:

$x \sim y$ in $R(X)$ iff for any model S of (Σ, E) , $i(x) = i(y)$

initial Σ -model morphism $i : R \rightarrow S$

Example: λ -calculus modulo $\beta\eta$

The algebraic 2-signature $(\Sigma_{\text{LC}\beta\eta}, E_{\text{LC}\beta\eta})$ of λ -calculus modulo $\beta\eta$:

$$\Sigma_{\text{LC}\beta\eta}(\mathbf{R}) := \Sigma_{\text{LC}}(\mathbf{R}) = (\mathbf{R} \times \mathbf{R}) \amalg \mathbf{R}'$$

model of Σ_{LC} = monad \mathbf{R} with module morphisms:

$$\text{app} : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \qquad \text{abs} : \mathbf{R}' \rightarrow \mathbf{R}$$

β -equation: $(\lambda x.t) u = \underbrace{t[x \mapsto u]}_{\sigma_{\mathbf{R}}(t,u)}$

η -equation: $t = \lambda x.(t x)$

$$E_{\text{LC}\beta\eta} = \{ \beta\text{-equation}, \eta\text{-equation} \}$$

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$$\text{app} : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \quad \text{abs} : \mathbf{R}' \rightarrow \mathbf{R}$$

β -equation: $(\lambda x.t) u = t[\underbrace{x \mapsto u}_{\sigma_{\mathbf{R}}(t,u)}]$

η -equation: $t = \lambda x.(t x)$

$$\begin{array}{ccccc}
 & & \sigma_{\mathbf{R}} & & \\
 & \curvearrowright & & \curvearrowright & \\
 \mathbf{R}' \times \mathbf{R} & & & & \mathbf{R} \\
 & \searrow \text{abs} \times \mathbf{R} & \mathbf{R} \times \mathbf{R} & \xrightarrow{\text{app}} & \\
 & & & &
 \end{array}$$

$$\begin{array}{ccccc}
 & & \text{id}_{\mathbf{R}} & & \\
 & \curvearrowright & & \curvearrowright & \\
 \mathbf{R} & & & & \mathbf{R} \\
 & \searrow \mathbf{R}t_1 & \mathbf{R}' & \xrightarrow{\text{abs}} &
 \end{array}$$

$$E_{\text{LC}\beta\eta} = \{ \beta\text{-equation}, \eta\text{-equation} \}$$

Example: fixpoint operator

Definition [AHLM CSL 2018]

A **fixpoint operator** in a monad R is a module morphism $\text{fix}: R' \rightarrow R$ s.t. for any term $t \in R(X \amalg \{\diamond\})$, $\text{fix}(t) = t[\diamond \mapsto \text{fix}(t)]$

Intuition: $\text{fix}(t) := \text{let rec } \diamond = t \text{ in } \diamond$

Algebraic 2-signature $(\Sigma_{\text{fix}}, E_{\text{fix}})$ of a fixpoint operator:

$$\Sigma_{\text{fix}}(R) := R'$$

$$E_{\text{fix}} = \left\{ \begin{array}{ccc} & \xrightarrow{\text{fix}(t)} & \\ R' & & R \\ & \xleftarrow{t[\diamond \mapsto \text{fix}(t)]} & \\ & \text{t} & \end{array} \right\}$$

Proposition [AHLM CSL 2018]

Fixpoint operators in $\text{LC}_{\beta\eta}$ are in one to one correspondance with fixpoint combinators (i.e. λ -terms Y s.t. $t(Yt) = Yt$ for any t).

Combining algebraic 2-signatures

Algebraic 2-signatures can be combined:

fixpoint operator

λ -calculus modulo $\beta\eta$

$(\Sigma_{\text{fix}}, E_{\text{fix}})$

+

$(\Sigma_{\text{LC}\beta\eta}, E_{\text{LC}\beta\eta})$

=

$(\Sigma_{\text{fix}} \amalg \Sigma_{\text{LC}\beta\eta}, E_{\text{fix}} \cup E_{\text{LC}\beta\eta})$

λ -calculus modulo $\beta\eta$ with an explicit fixpoint operator

Example: free commutative monoid

An algebraic 2-signature $(\Sigma_{\text{mon}}, E_{\text{mon}})$ for the free commutative monoid monad:

$$\Sigma_{\text{mon}}(\mathbf{R}) := 1 \coprod (\mathbf{R} \times \mathbf{R})$$

model of Σ_{mon} = monad \mathbf{R} with module morphisms:

$$0 : 1 \rightarrow \mathbf{R} \quad + : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$$

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4 elementary Σ -equations:

$$\begin{array}{ccc} R & \xrightarrow{0+t} & R \\ & \searrow t & \nearrow t \\ & R & \end{array}$$

$$\begin{array}{ccc} R \times R \times R & \xrightarrow{(s+t)+u} & R \\ & \searrow s, t, u & \nearrow s+(t+u) \\ & R & \end{array}$$

$$\begin{array}{ccc} R & \xrightarrow{t+0} & R \\ & \searrow t & \nearrow t \\ & R & \end{array}$$

$$\begin{array}{ccc} R \times R & \xrightarrow{s+t} & R \\ & \searrow s, t & \nearrow t+s \\ & R & \end{array}$$

Our target: dLC

Syntax of the *differential λ -calculus*:

Differential λ -terms

$$\begin{array}{lcl} s, t & ::= & x \\ & | & \lambda x. t \\ & | & s \ t \\ & | & Ds \cdot t \\ & | & s + t \\ & | & 0 \end{array} \quad \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} \lambda\text{-calculus} \\ \\ \\ \text{free commutative monoid} \end{array}$$

and (bi)linearity of constructors with respect to +:

$$\lambda x. (s + t) = \lambda x. s + \lambda x. t \quad \dots$$

Algebraic 1-signature for dLC

Syntax of the *differential λ -calculus*:

Differential λ -terms

Corresponding 1-signature

$s, t ::= x$

| $\lambda x. t$

| $s \ t$

| $Ds \cdot t$

| $s + t$

| 0

}

$\Sigma_{\text{LC}}(\mathbf{R}) = \mathbf{R}' \coprod (\mathbf{R} \times \mathbf{R})$

$\mathbf{R} \mapsto \mathbf{R} \times \mathbf{R}$

}

$\Sigma_{\text{mon}}(\mathbf{R}) = 1 \coprod (\mathbf{R} \times \mathbf{R})$

Algebraic 1-signature for dLC

Syntax of the *differential λ -calculus*:

Differential λ -terms

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$\lambda x. t$	}
$s t$	
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0	

$\Sigma_{\text{LC}}(\mathbf{R}) = \mathbf{R}' \coprod (\mathbf{R} \times \mathbf{R})$

$\mathbf{R} \mapsto \mathbf{R} \times \mathbf{R}$

$\Sigma_{\text{mon}}(\mathbf{R}) = 1 \coprod (\mathbf{R} \times \mathbf{R})$

Resulting algebraic 1-signature: $\Sigma_{\text{dLC}}(\mathbf{R}) = \Sigma_{\text{LC}}(\mathbf{R}) \coprod (\mathbf{R} \times \mathbf{R}) \coprod \Sigma_{\text{mon}}(\mathbf{R})$

Elementary equations for dLC

Commutative monoidal structure:

$$\mathbf{E}_{\text{mon}} \left\{ \begin{array}{l} s + t = t + s \\ s + (t + u) = (s + t) + u \\ 0 + t = t \\ t + 0 = t \end{array} \right. \quad \begin{array}{l} \mathbf{R} \times \mathbf{R} \Rightarrow \mathbf{R} \\ \mathbf{R} \times \mathbf{R} \times \mathbf{R} \Rightarrow \mathbf{R} \\ \mathbf{R} \Rightarrow \mathbf{R} \\ \mathbf{R} \Rightarrow \mathbf{R} \end{array}$$

Linearity:

$$\begin{array}{ll} \lambda x. (s + t) = \lambda x. s + \lambda x. t & \mathbf{R} \times \mathbf{R} \Rightarrow \mathbf{R} \\ D(s + t) \cdot u = Ds \cdot u + Dt \cdot u & \mathbf{R} \times \mathbf{R} \times \mathbf{R} \Rightarrow \mathbf{R} \\ Ds \cdot (t + u) = Ds \cdot t + Ds \cdot u & \mathbf{R} \times \mathbf{R} \times \mathbf{R} \Rightarrow \mathbf{R} \end{array}$$

...

n-ary fixpoint operator

Reminder: unary fixpoint operator in a monad R

$$\begin{array}{ccc} R(X \amalg \{\diamond\}) & \rightarrow & R(X) \\ t & \mapsto & \bar{t} \end{array} \quad \textbf{s.t.} \quad t[\diamond \mapsto \bar{t}] = \bar{t}$$

Intuition: $\bar{t} := \text{let rec } \diamond = t \text{ in } \diamond$

n-ary fixpoint operator:

$$\forall \textcolor{red}{i} \in \{1, \dots, n\}, \quad \begin{array}{ccc} R(X \amalg \{\diamond_1, \dots, \diamond_n\})^{\textcolor{blue}{n}} & \rightarrow & R(X) \\ t_1, \dots, t_n & \mapsto & \bar{t}_{\textcolor{red}{i}} \end{array} \quad \textbf{s.t.} \quad \forall i, t_i \left[\begin{array}{c} \diamond_1 \mapsto \bar{t}_1 \\ \dots \\ \diamond_n \mapsto \bar{t}_n \end{array} \right] = \bar{t}_i$$

Intuition:

$\bar{t}_{\textcolor{red}{i}} := \text{let rec } \diamond_1 = t_1 \text{ and } \dots \text{ and } \diamond_n = t_n \text{ in } \diamond_{\textcolor{red}{i}}$

n-ary fixpoint operator

n-ary fixpoint operator:

$$\begin{array}{ll} \forall \textcolor{red}{i} \in \{1, \dots, n\}, & \\ R(X \coprod \{\diamond_1, \dots, \diamond_n\})^{\textcolor{blue}{n}} & \rightarrow R(X) \\ t_1, \dots, t_n & \mapsto \overline{t_{\textcolor{red}{i}}} \end{array} \quad \textbf{s.t.} \quad \forall i, \quad t_i \left[\begin{array}{c} \diamond_1 \mapsto \overline{t_1} \\ \dots \\ \diamond_n \mapsto \overline{t_n} \end{array} \right] = \overline{t_i}$$

Algebraic 1-signature:

$$\Sigma_n(R) = \prod_{i=1}^n (R^{\overbrace{'\dots'}^{\textcolor{blue}{n}}})^n$$

n elementary equations $(R^{\overbrace{'\dots'}^{\textcolor{blue}{n}}})^n \Rightarrow R$

$$\forall i, \quad t_i \left[\begin{array}{c} \diamond_1 \mapsto \overline{t_1} \\ \dots \\ \diamond_n \mapsto \overline{t_n} \end{array} \right] = \overline{t_i}$$

Fixpoint operators

Syntax with fixpoint operators:

- for each n , a n -ary operator:

`let rec $\diamond_1 = t_1$ and .. and $\diamond_n = t_n$ in \diamond_i`

- compatibility between these operators [AHLM CSL 2018]

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- invariance under **permutation**:

<code>let rec $\diamond_1 = t_1$</code>		<code>let rec $\diamond_1 = t_2[\diamond_1 \leftrightarrow \diamond_2]$</code>
<code>and $\diamond_2 = t_2$</code>	<code>=</code>	<code>and $\diamond_2 = t_1[\diamond_1 \leftrightarrow \diamond_2]$</code>
<code>in \diamond_1</code>		<code>in \diamond_2</code>

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<code>in \diamond_1</code>		<code>in \diamond_2</code>

- invariance under **repetition**:

<code>let rec $\diamond_1 = t$</code>		<code>let rec $\diamond_1 = t[\diamond_2 \mapsto \diamond_1]$</code>
<code>and $\diamond_2 = t$</code>	<code>=</code>	<code>in \diamond_1</code>
<code>in \diamond_1</code>		

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`let rec $\diamond_1 = t_1$ and .. and $\diamond_n = t_n$ in \diamond_i`

- compatibility between these operators [AHLM CSL 2018]

In general:

$$\boxed{\begin{array}{l} \text{let rec } \diamond_1 = t_1 [\diamond_i \mapsto \diamond_{u(i)}] \\ \quad \dots \\ \text{and } \diamond_q = t_q [\diamond_i \mapsto \diamond_{u(i)}] \\ \text{in } \diamond_{u(j)} \end{array}} = \boxed{\begin{array}{l} \text{let rec } \diamond_1 = t_{u(1)} \\ \quad \dots \\ \text{and } \diamond_p = t_{u(p)} \\ \text{in } \diamond_j \end{array}}$$

where $u : \{1, \dots, p\} \rightarrow \{1, \dots, q\}$

$$t_1, \dots, t_q \in R(X \amalg \{\diamond_1, \dots, \diamond_p\})$$

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$$t_1, \dots, t_q \in R(X \amalg \{\diamond_1, \dots, \diamond_p\})$$

\Rightarrow Expressible as elementary equations $(R', \dots)^q \Rightarrow R$.

Table of contents

1. 1-Signatures and models based on monads and modules


2. Equations

3. Recursion

Principle of recursion

Recursion on the syntax \simeq Initiality in the category of models


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initial model of a 2-signature (Σ, E)

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


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Initiality of $R \Rightarrow$ model morphism $R \rightarrow S \Rightarrow$ monad morphism $R \rightarrow S$

Example: Computing the set of free variables

LC = initial model of (Σ_{LC}, \emptyset)

$$\Sigma_{LC}(R) = (R \times R) \amalg R'$$

\mathcal{P} = power set monad

Definition of a (monad) morphism $fv : LC \rightarrow \mathcal{P}$ **s.t.**

$$fv(\text{app}(t, u)) = fv(t) \cup fv(u)$$

$$fv(\text{abs}(t)) = fv(t) \setminus \{\diamond\}$$

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Initiality of $LC \Rightarrow$ $fv : LC \rightarrow \mathcal{P}$ satisfying the above equations (as a model morphism).

Example: Translating λ -calculus with fixpoint

$LC_{\beta\eta\text{fix}}$ = initial model of $(\Sigma_{LC\beta\eta}, E_{LC\beta\eta}) + (\Sigma_{\text{fix}}, E_{\text{fix}})$

λ -calculus modulo $\beta\eta$ with a fixpoint operator $\text{fix} : LC_{\beta\eta\text{fix}}' \rightarrow LC_{\beta\eta\text{fix}}$

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λ -calculus modulo $\beta\eta$

monad morphism

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Solution [CSL AHLM 2018]:

1. define $f : LC \rightarrow \mathbf{C}$ by recursion

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$$s(t) = g(t, (x \mapsto 0))$$

variables are of size 0

Conclusion

Summary of the talk:

- presented a notion of 1-signature and models
- defined a 2-signature as a 1-signature and a set of equations
- identified a class of 2-signatures that generate a syntax

The main theorem has been formalized in Coq using the UniMath library.

Future work:

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Thank you!