

Higher-order Arities, Signatures and Equations via Modules

Ambroise Lafont

joint work with
Benedikt Ahrens, André Hirschowitz, Marco Maggesi

Keywords associated with syntax

Induction/Recursion

Substitution

Model

Syntax

Operation/Construction

Arity/Signature

This talk: give a *discipline* for specifying syntaxes

Motivating example: dLC

syntax of dLC = **differential λ -calculus** [Ehrhard-Regnier 2003].

- explicitly involves **equations** e.g. $s+t = t+s$
- specifically tailored: (not an *instance* of a general framework/scheme)
inductive definition of a set + ad-hoc structure
e.g. **unary substitution**

Bien dire que je n'y connais rien à la sémantique

Our proposal = a discipline for presenting syntaxes

- signature = operations + equations
- [Fiore-Hur 2010]: alternative approach, for simply typed syntaxes
⇒ our approach explicitly relies on monads and modules (untyped case).

Syntax of dLC: [Ehrhard-Regnier 2003]

Let be given a denumerable set of variables. We define by induction on k an increasing family of sets (Δ_k) . We set $\Delta_0 = \emptyset$ and Δ_{k+1} is defined as follows.

Monotonicity: if t belongs to Δ_k then t belongs to Δ_{k+1} .

Variable: if $n \in \mathbb{N}$, x is a variable, $i_1, \dots, i_n \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ and $u_1, \dots, u_n \in \Delta_k$, then

$$D_{i_1, \dots, i_n} x \cdot (u_1, \dots, u_n)$$

belongs to Δ_{k+1} . This term is identified with all the terms of the shape $D_{i_{\sigma(1)}, \dots, i_{\sigma(n)}} x \cdot (u_{\sigma(1)}, \dots, u_{\sigma(n)}) \in \Delta_{k+1}$ where σ is a permutation on $\{1, \dots, n\}$.

Abstraction: if $n \in \mathbb{N}$, x is a variable, $u_1, \dots, u_n \in \Delta_k$ and $t \in \Delta_k$, then

$$D_1^n \lambda x t \cdot (u_1, \dots, u_n)$$

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Application: if $s \in \Delta_k$ and $t \in R\langle \Delta_k \rangle$, then

$$(s)t$$

belongs to Δ_{k+1} .

Setting $n=0$ in the first two clauses, and restricting application by the constraint that $t \in \Delta_k \subseteq R\langle \Delta_k \rangle$, one retrieves the usual definition of lambda-terms which shows that differential terms are a superset of ordinary lambda-terms.

The permutative identification mentioned above will be called *equality up to differential permutation*. We also work up to α -conversion.

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$$(s)t \leftarrow \text{as an operation: } \Lambda \times \text{FreeCommutativeMonoid}(\Lambda) \rightarrow \Lambda$$

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Syntax of dLC: [BEM 2010]

A **syntax** for the ***differential λ -calculus*** by ***mutual induction***:

[Bucciarelli-Ehrhard-Manzonetto 2010]

Simple terms:

$$\Lambda^s : \quad s, t \quad ::= \quad x \mid \lambda x. s \mid sT \mid D s \cdot t$$

Differential λ -terms:


$$\Lambda^d : \quad T \quad ::= \quad 0 \mid s \mid s + T$$

Syntax of dLC: [BEM 2010]



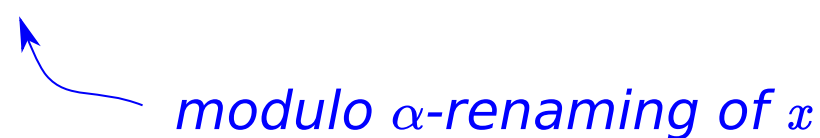
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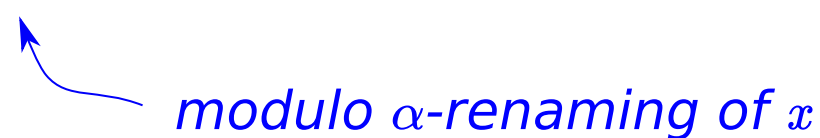
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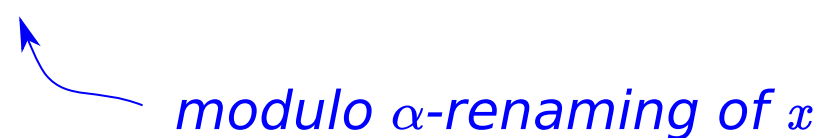
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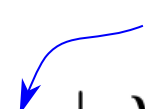
Syntax: specified by operations and **equations**.

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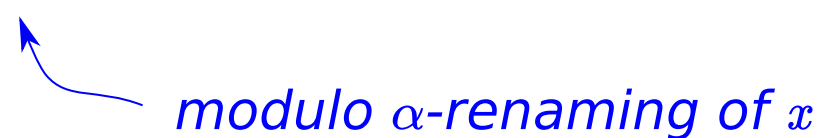
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Syntax: specified by operations and **equations**.

But which ones are allowed ? What is the limit ?

Syntax of dLC: Our version

Which operations/equations are allowed to specify a syntax ?

A stand-alone presentation of differential λ -terms:

Allow sums everywhere (not only in the right arg of application)

Differential λ -terms:

$$\Lambda^d : S, T ::= x \mid \lambda x. S \mid S T \mid D S \cdot T$$

Apparemment c'est une des versions de Lionel Va

neutral element for +

modulo commutativity and associativity

Macros in [BEM 2010]:

$$\lambda x. \sum_i t_i := \sum_i \lambda x. t_i$$

$$(\sum_i t_i) u := \sum_i t_i u$$

$$D(\sum_i t_i) \cdot (\sum_j u_j) := \sum_i \sum_j D t_i \cdot u_j$$

Syntax of dLC: Conclusion

How can we compare these different versions ?

In which sense are they syntaxes ?

Which operations/equations are we allowed to specify in a syntax ?

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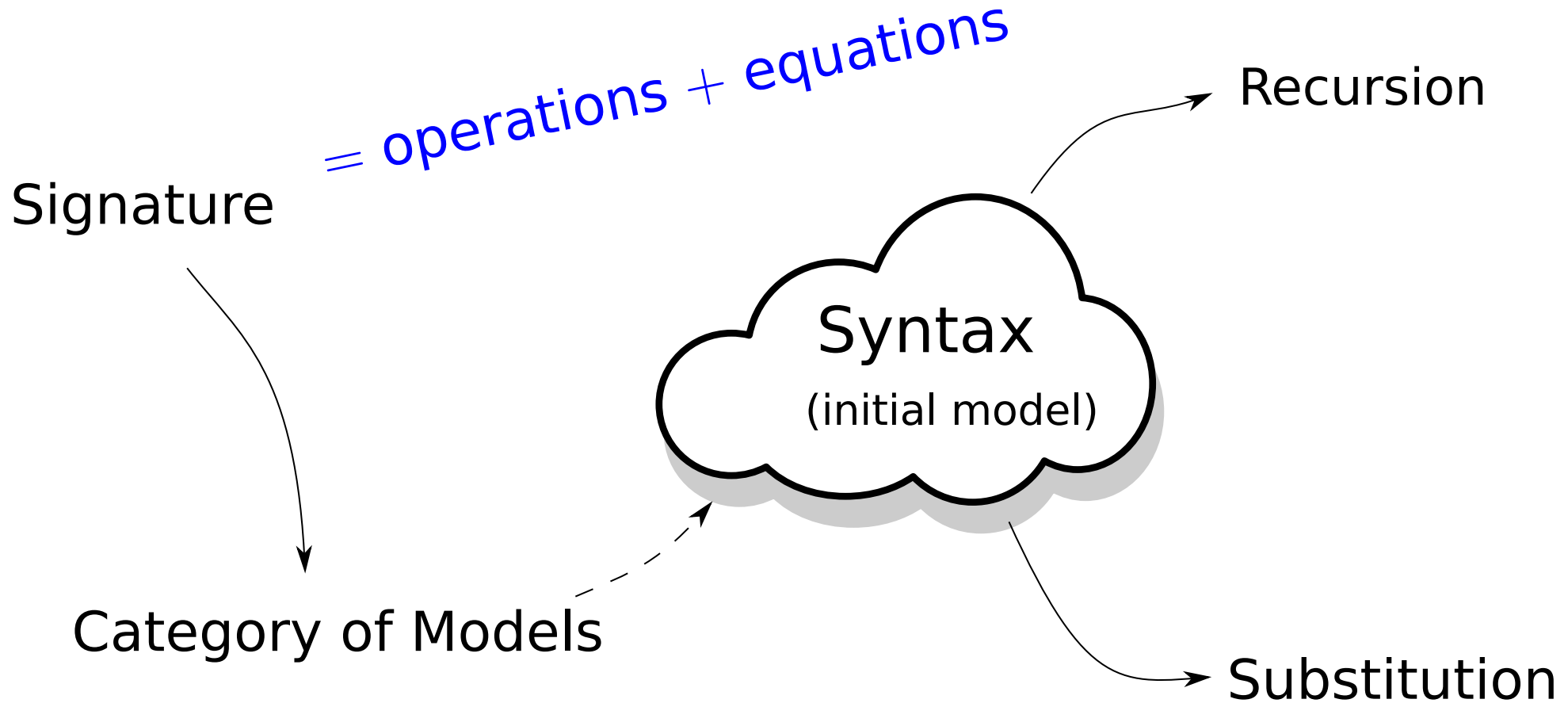
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What is a syntax ?

What is a syntax?



generates a syntax = existence of the initial model

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1. 1-Signatures and models based on monads and modules

2. Equations

3. Recursion

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- Substitution and monads
- 1-Signatures and their models

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Substitution and monads

Example: differential λ -calculus

$$\Lambda^d : S, T ::= x \mid \lambda x. S \mid S T \mid DS \cdot T \\ \mid 0 \mid S + T$$

Free variable indexing:

$$dLC : X \mapsto \{\text{terms taking free variables in } X\}$$

$$dLC(\emptyset) = \{0, \lambda z.z, \dots\}$$

$$dLC(\{x, y\}) = \{0, \lambda z.z, \dots, x, y, x + y, \dots\}$$

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Parallel substitution:

$$t \mapsto t[\mathbf{x} \mapsto \mathbf{f}(\mathbf{x})]$$

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Parallel substitution:

$$\begin{array}{ccc} \text{bind}_f : dLC(X) & \rightarrow & dLC(Y) \\ t & \mapsto & t[x \mapsto f(x)] \end{array} \quad \text{where } f : X \rightarrow dLC(Y)$$

$\Rightarrow (dLC, \text{var}_X : X \subset dLC(X), \text{bind}) = \mathbf{monad\ on\ Set}$

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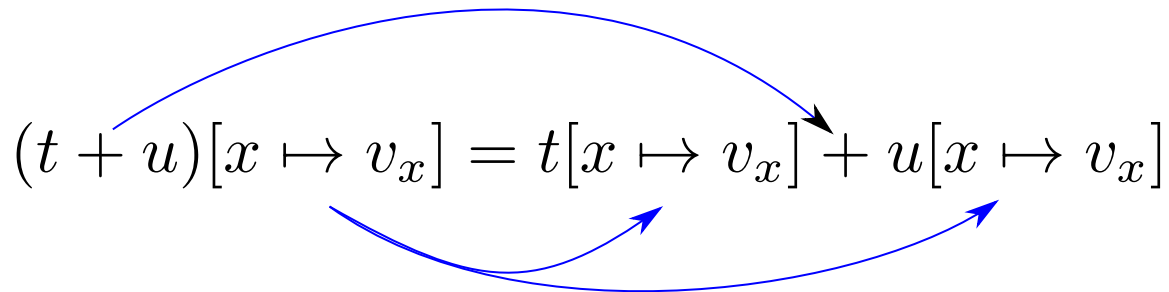
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$\Rightarrow (dLC, \text{var}_X : X \subset dLC(X), \text{bind}) = \mathbf{monad\ on\ Set}$

monad morphism = mapping preserving variables and substitutions.

Preview: Operations are module morphisms

+ commutes with substitution

$$(t + u)[x \mapsto v_x] = t[x \mapsto v_x] + u[x \mapsto v_x]$$


Dire que je ne vais pas donner la def de module

Categorical formulation

$dLC \times dLC$ supports
 dLC -substitution



$dLC \times dLC$ is a **module over** dLC

+ commutes
with substitution



$+ : dLC \times dLC \rightarrow dLC$ is a
module morphism

Building blocks for specifying operations

Essential constructions of **modules over a monad** R :

- R itself

- $M \times N$ for any modules M and N

e.g. $R \times R$: $f: X \rightarrow R(Y)$

$$(t, u)[x \mapsto f(x)] := (t[x \mapsto f(x)], u[x \mapsto f(x)])$$

- $M' = \mathbf{derivative\ of\ a\ module}\ M$: $M'(X) = M(X \coprod \{\diamond\})$.

disjoint union
fresh variable

used to model an operation binding a variable (Cf next slide).

Syntactic operations are module morphisms

operations = module morphisms = maps commuting with substitution.

$$0 : 1 \rightarrow \text{dLC}$$

$$\text{app} : \text{dLC} \times \text{dLC} \rightarrow \text{dLC}$$

$$+ : \text{dLC} \times \text{dLC} \rightarrow \text{dLC}$$

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Combining operations into a single one using disjoint union

$$[0, +] : 1 \coprod (\text{dLC} \times \text{dLC}) \rightarrow \text{dLC}$$

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1-signatures and their models

A **1-signature** Σ = functorial assignment:

$$R \mapsto \Sigma(R)$$

Example: $(0, +)$

$$\Sigma_{0,+}(R) = 1 \coprod (R \times R)$$

A **model of Σ** is a pair:

$$(R, \rho : \Sigma(R) \rightarrow R)$$

dLC = model of $\Sigma_{0,+}$

$$[0, +] : 1 \coprod (dLC \times dLC) \rightarrow dLC$$

A **model morphism** $m : (R, \rho) \rightarrow (S, \sigma)$ = monad morphism commuting with the module morphism:

$$\begin{array}{ccc} \Sigma(R) & \xrightarrow{\rho} & R \\ \Sigma(m) \downarrow & & \downarrow m \\ \Sigma(S) & \xrightarrow{\sigma} & S \end{array}$$

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Syntax

Definition

Given a 1-signature Σ , its **syntax** is an initial object in its category of models.

Question: Does the syntax exist for every 1-signature?

Answer: No.

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Counter-example: the 1-signature $R \mapsto \mathcal{P} \circ R$.


powerset endofunctor on Set

Examples of 1-signatures generating syntax

- **(0,+) language:**

Signature: $R \mapsto 1 \coprod (R \times R)$

Model: $(R, \quad 0 : 1 \rightarrow R, \quad + : R \times R \rightarrow R)$

Syntax: $(B, \quad 0 : 1 \rightarrow B, \quad + : B \times B \rightarrow B)$

- **lambda calculus:**

Signature: $R \mapsto R' \coprod (R \times R)$

Model: $(R, \quad abs : R' \rightarrow R, \quad app : R \times R \rightarrow R)$

Syntax: $(\Lambda, \quad abs : \Lambda' \rightarrow \Lambda, \quad app : \Lambda \times \Lambda \rightarrow \Lambda)$

Can we generalize this pattern?

Initial semantics for algebraic 1-signatures

Theorem [Hirschowitz & Maggesi 2007]

Syntax exists for any **algebraic 1-signature**, i.e. 1-signature built out of derivatives, products, disjoint unions, and the 1-signature $R \mapsto R$.

Algebraic 1-signatures correspond to the binding signatures described in [Fiore-Plotkin-Turi 1999]

(binding signature = lists of natural numbers specify n-ary operations, possibly binding variables)

Question: Can we enforce some equations in the syntax ?

e.g. **associativity** and **commutativity** of $+$ for the differential λ -calculus.

Quotients of algebraic 1-signatures

[AHLM CSL 2018]: notion of **quotients** of 1-signatures.

Theorem [AHLM CSL 2018]

Syntax exists for any "**quotient**" of algebraic 1-signature.

Examples:

- a **commutative** binary operation
- application of the differential λ -calculus (original variant)

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... but not enough for the differential λ -calculus:

- **associativity** of $+$
- **linearity** of the operations

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Example: a commutative binary operation

Specification of a binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R, + : R \times R \rightarrow R)$

What is an appropriate notion of model for a commutative binary operation ?

Example: a commutative binary operation

Specification of a **commutative** binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R, + : R \times R \rightarrow R)$ s.t. $t + u = u + t$ (1)

What is an appropriate notion of model for a commutative binary operation ?

Answer: a monad equipped with a **commutative** binary operation

Example: a commutative binary operation

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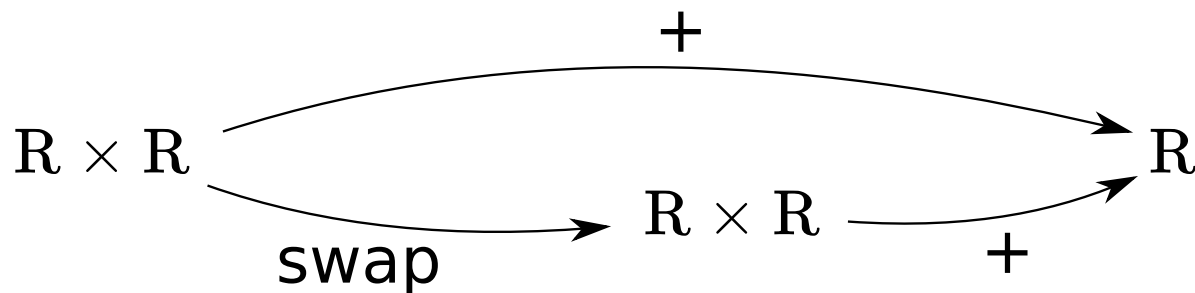
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Equation (1) states an equality between R -module morphisms:



Equations

Given a 1-signature Σ , (e.g. binary operation: $\Sigma(R) = R \times R$)

a Σ -**equation** $A \rightrightarrows B$ is a functorial assignment: e.g. commutativity:

$$R \mapsto \left(A(R) \rightrightarrows B(R) \right)$$

model of Σ (points to R)

parallel pair of module morphisms over R (points to $A(R) \rightrightarrows B(R)$)

$$R \mapsto \left(R \times R \xrightarrow[+\circ swap]{+} R \right)$$

A **2-signature** is a pair

$$(\Sigma, E)$$

1-signature (points to Σ)

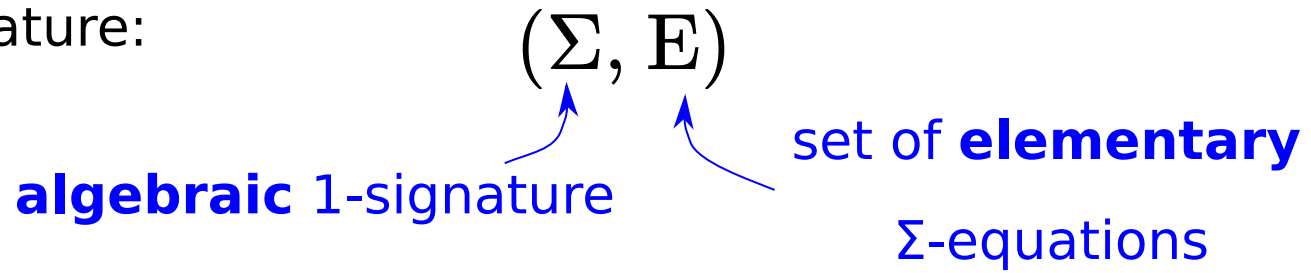
set of Σ -equations (points to E)

model of a 2-signature (Σ, E) :

- a model R of Σ
- s.t. $\forall (A \rightrightarrows B) \in E$, the two morphisms $A(R) \rightrightarrows B(R)$ are equal

Initial semantics for algebraic 2-signatures

Algebraic 2-signature:



Theorem

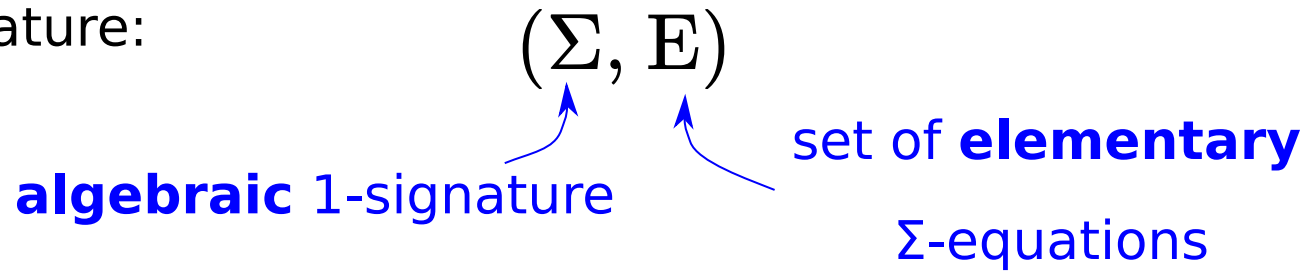
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Main instances of **elementary** Σ -equations $A \Rightarrow B$:

- $A =$ algebraic 1-signature e.g. $A(R) = R \times R$
- $B(R) = R$

Initial semantics for algebraic 2-signatures

Algebraic 2-signature:



Theorem

Syntax exists for any algebraic 2-signature.

Main instances of **elementary** Σ -equations $A \Rightarrow B$:

- $A =$ algebraic 1-signature e.g. $A(R) = R \times R$
- $B(R) = R$

Sketch of the construction of the syntax:

Quotient the initial model R of Σ by the following relation:

$x \sim y$ in $R(X)$ iff for any model S of (Σ, E) , $i(x) = i(y)$

initial Σ -model morphism $i : R \rightarrow S$

Example: λ -calculus modulo $\beta\eta$

The algebraic 2-signature $(\Sigma_{\text{LC}\beta\eta}, E_{\text{LC}\beta\eta})$ of λ -calculus modulo $\beta\eta$:

$$\Sigma_{\text{LC}\beta\eta}(\mathbf{R}) := \Sigma_{\text{LC}}(\mathbf{R}) = (\mathbf{R} \times \mathbf{R}) \amalg \mathbf{R}'$$

model of Σ_{LC} = monad \mathbf{R} with module morphisms:

$$\text{app} : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \qquad \text{abs} : \mathbf{R}' \rightarrow \mathbf{R}$$

β -equation: $(\lambda x.t) u = \underbrace{t[x \mapsto u]}_{\sigma_{\mathbf{R}}(t,u)}$

η -equation: $t = \lambda x.(t x)$

$$E_{\text{LC}\beta\eta} = \{ \beta\text{-equation}, \eta\text{-equation} \}$$

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$$\begin{array}{ccccc}
 & \sigma_{\mathbf{R}} & & & \\
 \mathbf{R}' \times \mathbf{R} & \xrightarrow{\quad} & \mathbf{R} & & \\
 \text{abs} \times \mathbf{R} \searrow & & \nearrow \text{app} & & \\
 & \mathbf{R} \times \mathbf{R} & & &
 \end{array}$$

$$\begin{array}{ccccc}
 & \text{id}_{\mathbf{R}} & & & \\
 \mathbf{R} & \xrightarrow{\quad} & \mathbf{R} & & \\
 \mathbf{R} \downarrow \text{t}_1 & & \nearrow \text{abs} & & \\
 & \mathbf{R}' & & &
 \end{array}$$

$$E_{\text{LC}\beta\eta} = \{ \beta\text{-equation}, \eta\text{-equation} \}$$

Example: fixpoint operator

Definition [AHLM CSL 2018]

A **fixpoint operator** in a monad R is a module morphism $\text{fix}: R' \rightarrow R$ s.t. for any term $t \in R(X \amalg \{\diamond\})$, $\text{fix}(t) = t[\diamond \mapsto f(t)]$

Intuition:

- $\text{fix}(t) := \text{let rec } \diamond = t \text{ in } \diamond$
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- [AHLM CSL 2018] Fixpoint operator in $LC_{\beta\eta} \simeq$ fixpoint combinators

Algebraic 2-signature $(\Sigma_{\text{fix}}, E_{\text{fix}})$ of a fixpoint operator:

$$\Sigma_{\text{fix}}(R) := R' \qquad E_{\text{fix}} = \left\{ \begin{array}{ccc} & \xrightarrow{\text{fix}(t)} & \\ R' & & R \\ & \xleftarrow{t[\diamond \mapsto \text{fix}(t)]} & \\ & \text{t} & \end{array} \right\}$$

Combining algebraic 2-signatures

Algebraic 2-signatures can be combined:

fixpoint operator

λ -calculus modulo $\beta\eta$

$(\Sigma_{\text{fix}}, E_{\text{fix}})$

+

$(\Sigma_{\text{LC}\beta\eta}, E_{\text{LC}\beta\eta})$

=

$(\Sigma_{\text{fix}} \amalg \Sigma_{\text{LC}\beta\eta}, E_{\text{fix}} \cup E_{\text{LC}\beta\eta})$

λ -calculus modulo $\beta\eta$ with an explicit fixpoint operator

Example: free commutative monoid

Algebraic 2-signature $(\Sigma_{\text{mon}}, \mathbf{E}_{\text{mon}})$ for the free commutative monoid monad:

$$\Sigma_{\text{mon}}(\mathbf{R}) := 1 \coprod (\mathbf{R} \times \mathbf{R})$$

model of Σ_{mon} = monad \mathbf{R} with module morphisms:

$$0 : 1 \rightarrow \mathbf{R} \quad + : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$$

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3 elementary Σ -equations:

$$\begin{array}{ccc}
 R \times R \times R & \begin{array}{c} \xrightarrow{(s+t)+u} \\ \xrightarrow{s+(t+u)} \end{array} & R \\
 \textcolor{blue}{s, t, u} & &
 \end{array}$$

$$\begin{array}{ccc}
 R \times R & \begin{array}{c} \xrightarrow{s+t} \\ \xrightarrow{t+s} \end{array} & R \\
 \textcolor{blue}{s, t} & &
 \end{array}$$

$$\begin{array}{ccc}
 R & \begin{array}{c} \xrightarrow{0+t} \\ \xrightarrow{t} \end{array} & R \\
 \textcolor{blue}{t} & &
 \end{array}$$

Our target: dLC

Syntax of the *differential λ -calculus*:

Differential λ -terms

$$\begin{array}{lcl} s, t & ::= & x \\ & | & \lambda x. t \\ & | & s \ t \\ & | & Ds \cdot t \\ & | & s + t \\ & | & 0 \end{array} \quad \left. \begin{array}{l} \} \\ \} \end{array} \right\} \begin{array}{l} \lambda\text{-calculus} \\ \text{free commutative monoid} \end{array}$$

and (bi)linearity of operations with respect to +:

$$\lambda x. (s + t) = \lambda x. s + \lambda x. t \quad \dots$$

Algebraic 1-signature for dLC

Syntax of the *differential λ -calculus*:

Differential λ -terms

Corresponding 1-signature

$s, t ::= x$

(variables $\subset \mathbf{R}$ for any monad \mathbf{R})

| $\lambda x. t$

| $s \ t$

| $\mathbf{D}s \cdot t$

| $s + t$

| 0

}

$$\Sigma_{\text{LC}}(\mathbf{R}) = \mathbf{R}' \coprod (\mathbf{R} \times \mathbf{R})$$

$$\mathbf{R} \mapsto \mathbf{R} \times \mathbf{R}$$

}

$$\Sigma_{\text{mon}}(\mathbf{R}) = 1 \coprod (\mathbf{R} \times \mathbf{R})$$

Algebraic 1-signature for dLC

Syntax of the *differential λ -calculus*:

Differential λ -terms

Corresponding 1-signature

$s, t ::= x$	(variables $\subset R$ for any monad R)
$\lambda x. t$	$\left. \begin{array}{l} \lambda x. t \\ s \ t \end{array} \right\} \Sigma_{LC}(R) = R' \coprod (R \times R)$
$s \ t$	
$Ds \cdot t$	$R \mapsto R \times R$
$s + t$	$\left. \begin{array}{l} s + t \\ 0 \end{array} \right\} \Sigma_{mon}(R) = 1 \coprod (R \times R)$
0	

Resulting algebraic 1-signature: $\Sigma_{dLC}(R) = \Sigma_{LC}(R) \coprod (R \times R) \coprod \Sigma_{mon}(R)$

Elementary equations for dLC

Commutative monoidal structure:

$$E_{\text{mon}} \left\{ \begin{array}{l} s + t = t + s \\ s + (t + u) = (s + t) + u \\ 0 + t = t \end{array} \right. \quad \begin{array}{l} R \times R \Rightarrow R \\ R \times R \times R \Rightarrow R \\ R \Rightarrow R \end{array}$$

Linearity:

$$\begin{array}{ll} \lambda x. (s + t) = \lambda x. s + \lambda x. t & R \times R \Rightarrow R \\ D(s + t) \cdot u = Ds \cdot u + Dt \cdot u & R \times R \times R \Rightarrow R \\ Ds \cdot (t + u) = Ds \cdot t + Ds \cdot u & R \times R \times R \Rightarrow R \end{array}$$

...

n-ary fixpoint operator

Reminder: unary fixpoint operator in a monad R

$$\begin{array}{ccc} R(X \amalg \{\diamond\}) & \rightarrow & R(X) \\ t & \mapsto & \bar{t} \end{array} \quad \textbf{s.t.} \quad t[\diamond \mapsto \bar{t}] = \bar{t}$$

Intuition: $\bar{t} := \text{let rec } \diamond = t \text{ in } \diamond$

n-ary fixpoint operator:

$$\forall \textcolor{red}{i} \in \{1, \dots, n\}, \quad \begin{array}{ccc} R(X \amalg \{\diamond_1, \dots, \diamond_n\})^{\textcolor{blue}{n}} & \rightarrow & R(X) \\ t_1, \dots, t_n & \mapsto & \bar{t}_{\textcolor{red}{i}} \end{array} \quad \textbf{s.t.} \quad \forall i, t_i \left[\begin{array}{c} \diamond_1 \mapsto \bar{t}_1 \\ \dots \\ \diamond_n \mapsto \bar{t}_n \end{array} \right] = \bar{t}_i$$

Intuition: $\bar{t}_{\textcolor{red}{i}} := \text{let rec } \diamond_1 = t_1 \text{ and } \dots \text{ and } \diamond_n = t_n \text{ in } \diamond_{\textcolor{red}{i}}$

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\Rightarrow specifiable as an algebraic 2-signature

Fixpoint operators

Syntax with fixpoint operators:

- for each n , a n -ary operator:

`let rec $\diamond_1 = t_1$ and .. and $\diamond_n = t_n$ in \diamond_i`

- compatibility between these operators [AHLM CSL 2018]

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- invariance under **permutation**:

<code>let rec $\diamond_1 = t_1$</code>		<code>let rec $\diamond_1 = t_2[\diamond_1 \leftrightarrow \diamond_2]$</code>
<code>and $\diamond_2 = t_2$</code>	<code>=</code>	<code>and $\diamond_2 = t_1[\diamond_1 \leftrightarrow \diamond_2]$</code>
<code>in \diamond_1</code>		<code>in \diamond_2</code>

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<code>in \diamond_1</code>		<code>in \diamond_2</code>

- invariance under **repetition**:

<code>let rec $\diamond_1 = t$</code>		<code>let rec $\diamond_1 = t[\diamond_2 \mapsto \diamond_1]$</code>
<code>and $\diamond_2 = t$</code>	<code>=</code>	<code>in \diamond_1</code>
<code>in \diamond_1</code>		

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- for each n , a n -ary operator:

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- compatibility between these operators [AHLM CSL 2018]

general form:

$$\begin{array}{c} \text{let rec } \diamond_1 = t_1 [\diamond_i \mapsto \diamond_{u(i)}] \\ \quad \dots \\ \text{and } \diamond_q = t_q [\diamond_i \mapsto \diamond_{u(i)}] \\ \text{in } \diamond_{u(j)} \end{array} = \begin{array}{c} \text{let rec } \diamond_1 = t_{u(1)} \\ \quad \dots \\ \text{and } \diamond_p = t_{u(p)} \\ \text{in } \diamond_j \end{array}$$

where $u : \{1, \dots, p\} \rightarrow \{1, \dots, q\}$

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\Rightarrow Expressible as elementary equations $(R', \dots)^q \Rightarrow R$.

Table of contents

1. 1-Signatures and models based on monads and modules


2. Equations

3. Recursion

Principle of recursion

Recursion on the syntax \simeq Initiality in the category of models

Recipe for constructing "by recursion" a monad morphism:

$f : R \rightarrow S$

initial model of a 2-signature (Σ, E)

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Initiality of $R \Rightarrow$ model morphism $R \rightarrow S \Rightarrow$ monad morphism $R \rightarrow S$

Example: Computing the set of free variables

λ -calculus monad

$$\begin{array}{lcl} \text{fv}_X : \text{LC}(X) & \rightarrow & \mathcal{P}(X) \\ t & \mapsto & \text{(exact) set of free variables of } t \end{array}$$

power set monad

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Equalities as a monad morphism:

$$\text{fv}(x) = \{x\}$$

$$\text{fv}(t[x \mapsto u(x)]) = \bigcup_{x \in \text{fv}(t)} \text{fv}(u(x))$$

Equalities as a model morphism:

$$\text{fv}(\text{app}(t, u)) = \text{fv}(t) \cup \text{fv}(u)$$

$$\text{fv}(\text{abs}(t)) = \text{fv}(t) \setminus \{\diamond\}$$

Example: Translating λ -calculus with fixpoint

λ -calculus modulo $\beta\eta$
+ fixpoint operator fix

compilation

\Longrightarrow

λ -calculus modulo $\beta\eta$

$\text{fix}(t) \mapsto ?$

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Proposition [AHLM CSL 2018]

Fixpoint operators in $\text{LC}_{\beta\eta}$ are in one to one correspondence with fixpoint combinators (i.e. λ -terms Y s.t. $t(Yt) = Yt$ for any t).

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λ -calculus modulo $\beta\eta$

$\text{fix}(t) \mapsto \text{app}(\mathbf{Y}, \text{abs}(t))$

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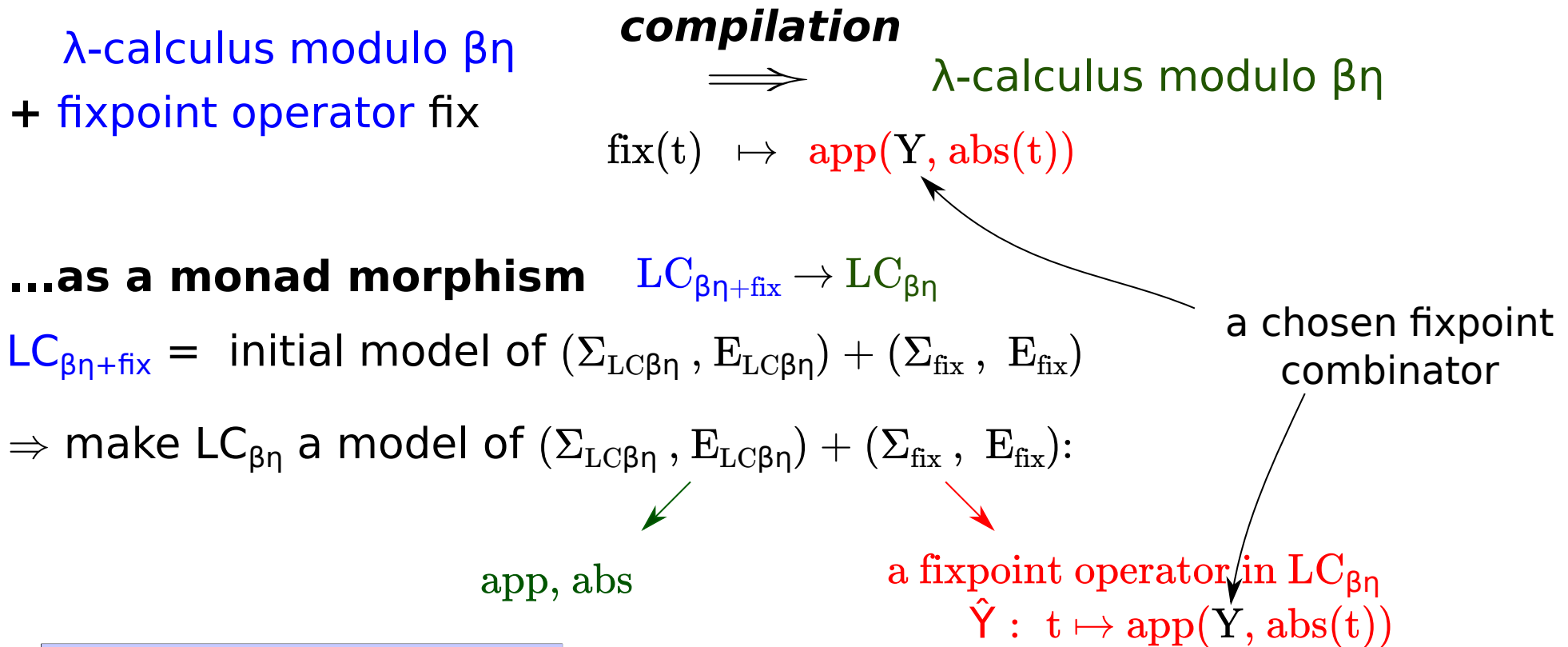
app, abs

a chosen fixpoint combinator
a fixpoint operator in $\text{LC}_{\beta\eta}$
 $\hat{\mathbf{Y}} : t \mapsto \text{app}(\mathbf{Y}, \text{abs}(t))$

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Initiality of $\text{LC}_{\beta\eta+\text{fix}} \Rightarrow$ monad morphism $\text{LC}_{\beta\eta+\text{fix}} \rightarrow \text{LC}_{\beta\eta}$

Example: Computing the size of a term

λ -calculus monad

$$\begin{array}{lcl} s_X & : & \text{LC}(\mathbf{X}) \rightarrow \mathbb{N} \\ & & t \mapsto \text{number of constructors in } t \end{array}$$

$$\begin{aligned} s(x) &= 0 \\ s(\lambda x.x) &= 1 \\ s((\lambda x.x) y) &= 2 \end{aligned}$$

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\mathbb{N} is not a monad !

Solution [CSL AHLM 2018]:

1. define $f : LC \rightarrow C$ by recursion

continuation monad $C(X) = \mathbb{N}^{\mathbb{N}^X}$

2. deduce $s : LC \rightarrow \mathbb{N}$

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\mathbb{N} is not a monad !

Solution [CSL AHLM 2018]:

1. define $f : LC \rightarrow C$ by recursion

2. deduce $s : LC \rightarrow \mathbb{N}$

continuation monad $C(X) = \mathbb{N}^{(\mathbb{N}^X)}$

assigns an arbitrary size to each variable

Intuition: $f_X : LC(X) \rightarrow \mathbb{N}^{(\mathbb{N}^X)}$

$\xRightarrow{\text{uncurry}}$

$$g : LC(X) \times \mathbb{N}^X \rightarrow \mathbb{N}$$

$$g(x, u) = u(x)$$

Example: Computing the size of a term

λ -calculus monad

$s_X : LC(X) \rightarrow \mathbb{N}$
 $t \mapsto$ number of constructors in t

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$$s(t) = g(t, (x \mapsto 0))$$

variables are of size 0

Conclusion

Summary of the talk:

- notion of 1-signature and models based on monads and modules
- 2-signature = 1-signature + set of equations
- *algebraic* 2-signatures generate a syntax, e.g. differential λ -calculus.

Main theorems formalized in Coq using the UniMath library.

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Thank you!