Higher-order Arities, Signatures and Equations via Modules

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joint work with
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Keywords associated with syntax

Induction/Recursion

Substitution

Model

Syntax

Arity/Signature

Operation/Construction

This talk: give a mathematical account of this area

Motivation: LCD

LCD = ***differentiable λ-calculus*** [Ehrard-Regnier 2003].

- Syntax: not straightforward (equations involved).

  e.g. $\mathrm{s}+\mathrm{t} = \mathrm{t}+\mathrm{s}$

- Later articles: alternative presentations of the syntax (+/- verbose).

- No well-established *scheme* commonly used beyond BNF grammars.

Our work:

- a mathematical theory of presentations of monads,
- induces a scheme for presenting syntaxes.

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Next slides: 3 variants of the LCD syntax:

Our work:

- a mathematical theory of presentations of monads,
- induces a scheme for presenting syntaxes.

Syntax of LCD: version 1/3

A **syntax** for the ***differentiable λ-calculus*** by ***mutual induction:***

[Categorical Models for Simply Typed Resource Calculi]

Simple terms:

$$\Lambda^s : \quad s, t \quad ::= \quad x \mid \lambda x.s \mid sT \mid \mathsf{D}\, s \cdot t$$

Differential λ-terms:

$$\Lambda^d : \quad T \quad ::= \quad 0 \mid s \mid s + T$$

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modulo α-renaming of x

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neutral element for $+$

modulo commutativity

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Syntax: specified by operations and equations.

But which ones are allowed ? What is the limit ?

Syntax of LCD: version 2/3

Which operations/equations are allowed to specify a syntax ?

A stand-alone presentation of simple terms:

Simple terms:

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Differential λ-terms:

$$T \in \Lambda^{\mathsf{d}} = \mathsf{FreeCommutativeMonoid}(\Lambda^{\mathsf{s}})$$

Syntax of LCD: version 2/3

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A stand-alone presentation of simple terms:

Simple terms:

$$\Lambda^s : \quad s, t \quad ::= \quad x \mid \lambda x.s \mid sT \mid \mathsf{D}\, s \cdot t$$

as an operation: $\Lambda^{\mathsf{s}} \times \mathsf{FreeCommutativeMonoid}(\Lambda^{\mathsf{s}}) \to \Lambda^{\mathsf{s}}$

Differential λ-terms:

$$T \in \Lambda^{\mathsf{d}} = \mathsf{FreeCommutativeMonoid}(\Lambda^{\mathsf{s}})$$

Syntax of LCD: version 3/3

Which operations/equations are allowed to specify a syntax ?

A stand-alone presentation of differential λ-terms:

Allow summands everywhere (not only in the right arg of application)

Differential λ-terms:

$$\Lambda^{\mathrm{d}} : \quad S, T \quad ::= \quad x \mid \lambda x.S \mid S\,T \mid \mathsf{D}S \cdot T$$

$$\mid 0 \mid S + T$$

neutral element for $+$

modulo commutativity and associativity

[Categorical Models for Simply Typed Resource Calculi]

$$\lambda x.\Sigma_i t_i = \Sigma_i \lambda x.t_i$$

$$(\Sigma_i t_i)u = \Sigma_i t_i u$$

$$D(\Sigma_i t_i) \cdot (\Sigma_j u_j) = \Sigma_i \Sigma_j D t_i \cdot u_j$$

Syntax of LCD: Conclusion

How can we compare these different versions ?

In which sense are they syntaxes ?

Which operations/equations are we allowed to specify in a syntax ?

Syntax of LCD: Conclusion

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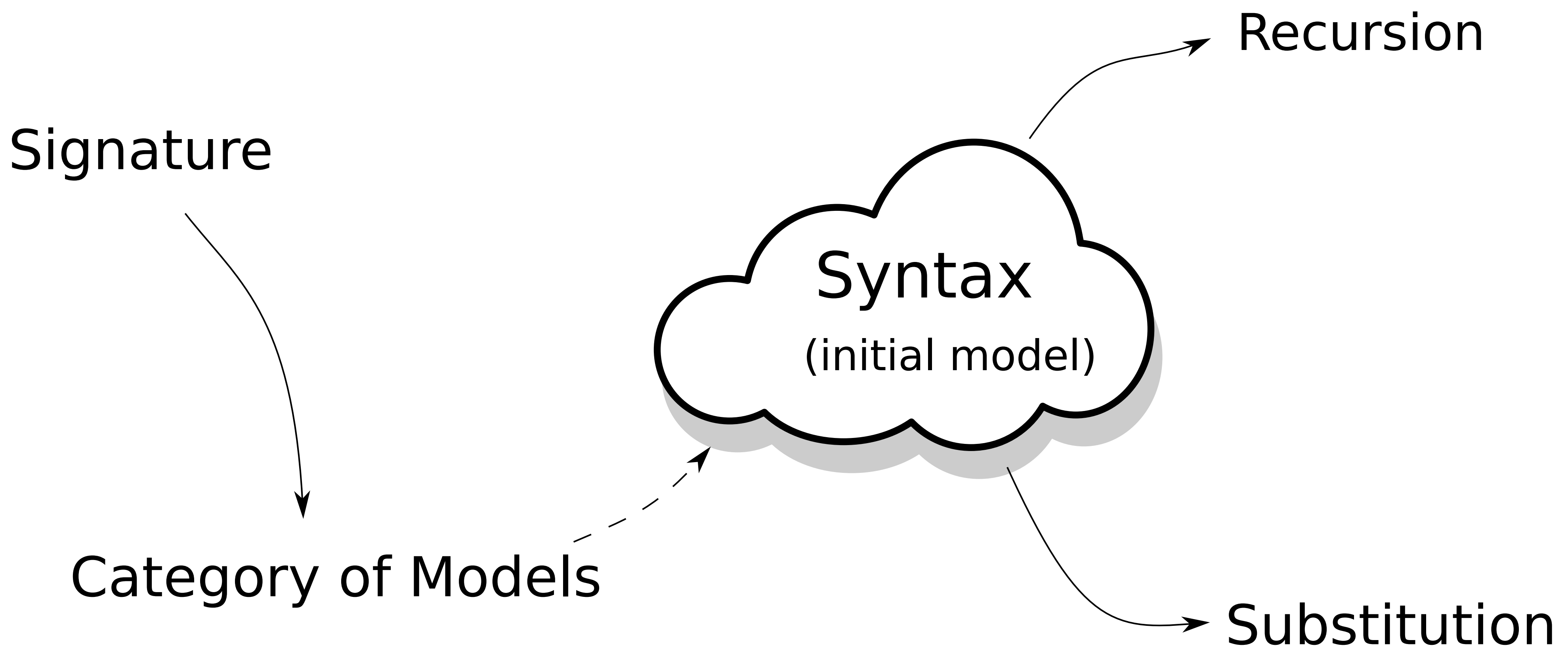
What is a syntax ?

What is a syntax?



generates a syntax = existence of the initial model

Overview

Topic: specification of untyped syntaxes (e.g. differential λ-calculus).

Our work:

1. general notion of ***1-signature*** based on ***monads*** and ***modules***.
    - *Caveat:* Not all of them do ***generate a syntax***
    - special case: classical ***algebraic 1-signatures*** generate a syntax
2. notion of **2-signature**: a pair of a 1-signature and a set of equations.
    - special case: ***algebraic 2-signatures*** generate a syntax

Related work of Fiore-Hur 2010

[Fiore-Hur 2010]: presentations of simply typed languages with

- generating *binding* operations (e.g. λ-abstraction)
- equations among them.

Our work: a variant of their approach

- for the untyped setting,
- focus on *monads* and *modules* over them

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Categorical formulation of a term language

Example: differential λ-calculus

$$\Lambda^{\mathrm{d}}: \quad S,T \quad ::= \quad x \mid \lambda x.S \mid S\,T \mid \mathrm{D}S \cdot T$$
$$\mid 0 \mid S + T$$

Free variable indexing:

$$LCD : X \mapsto \{\text{terms taking free variables in } X\}$$

$$LCD(\emptyset) = \{0, \lambda z.z, \dots\}$$
$$LCD(\{x,y\}) = \{0, \lambda z.z, \dots, x, y, x+y, \dots\}$$

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$$\begin{aligned} \mathrm{LCD}(\mathrm{f}) \ : \ \mathrm{LCD}(\mathrm{X}) &\to \quad \mathrm{LCD}(\mathrm{Y}) \\ \mathrm{t} \quad &\mapsto \quad \mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})] \end{aligned}$$

where $\mathrm{f} : \mathrm{X} \to \mathrm{Y}$

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where $\mathrm{f : X \to Y}$

$\Rightarrow$ **LCD is an endofunctor on $\mathbf{Set}$**

Categorical formulation of a term language

commute with variable renaming

Operations as natural transformations:

$$\begin{aligned} + :\ & LCD \times LCD \stackrel{\cdot}{\to} LCD \\ 0 :\ & 1 \stackrel{\cdot}{\to} LCD \\ & \dots \end{aligned}$$

Variables as a natural transformation:

$$\mathrm{var} : \quad \mathrm{Id}_{\mathrm{Set}} \stackrel{\cdot}{\to} LCD$$

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model $=$ endofunctor R with natural transformations:

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Next slides: **generalize this pattern to other languages**

Binding Signatures

Definition

Binding signature = a family of lists of natural numbers.

Each list specifies one operation in the syntax:

- length of the list = number of arguments of the operation
- natural number in the list = number of bound variables in the corresponding argument

Syntax with 0, +:

$$\Big(\underbrace{()}_{0},\ \underbrace{(0,\ 0)}_{+}\Big)$$

$$0, \quad 0+x, \quad x+y$$
$$(0+x)+y, \quad 0+(x+y),$$
$$\ldots$$

Lambda calculus:

$$\Big(\underbrace{(0\ ,\ 0)}_{app}\ ,\ \underbrace{(1)}_{abs}\Big)$$

Initial semantics for binding signatures

model of (0, +) $=$ endofunctor R with a natural transformation:

$$[+, 0, \mathrm{var}] : \ (R \times R) \coprod 1 \coprod \mathsf{Id} \stackrel{.}{\to} R$$

morphism $=$ natural transformation commuting with 0, + and var.

Similarly, any binding signature gives rise to a category of models.

Well-established theorem

The initial model of a binding signature Σ always exists.

Question: Does this initial model come with a well-behaved substitution?

Answer: Yes: see e.g. [Fiore, Plotkin, Turi 1999], [Ghani & Uustalu 2003]

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Initiality still holds in the subcategory of models with a substitution.

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The Big Picture of 1-signatures and models

Binding signatures $\hookrightarrow$ Our 1-signatures

A **1-signature** Σ is a functorial assignment:

$$R \mapsto \Sigma(R)$$

Example: (0,+)

$$\Sigma_{0,+}(R) = (R \times R) \coprod 1$$

A **model of Σ** is a pair:

$$(R, \quad \rho : \Sigma(R) \to R)$$

LCD $=$ model of $\Sigma_{0,+}$

$$[+, 0] : (LCD \times LCD) \coprod 1 \to LCD$$

monad := endofunctor with substitution

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Substitution and monads

Reminder:

- $LCD(X) = \{$ differential λ-terms taking free variables in X $\}$
- Variables as a natural transformation $\mathrm{var} : \mathrm{Id}_{\mathrm{Set}} \to LCD$
- **Variable renaming** by functoriality:

$$LCD(\mathrm{f})(\mathrm{t}) = \mathrm{t}[\,\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})\,]$$ where $\mathrm{f} : \mathrm{X} \to \mathrm{Y}$ is a *renaming*

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Variable renaming $=$ special case of **substitution**:

$$\begin{array}{rcccc} \mathrm{bind}_{\mathrm{f}} & : & LCD(\mathrm{X}) & \to & LCD(\mathrm{Y}) \\ & & \mathrm{t} & \mapsto & \mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})] \end{array}$$ where $\mathrm{f} : \mathrm{X} \to LCD(\mathrm{Y})$

Substitution and monads

Reminder:

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$(\mathit{LCD}, \mathrm{var}, \mathrm{bind})$ = a **monad**.

monad morphism = mapping preserving varaiables and substitutions.

Preview: Operations are module morphisms

+ commutes with substitution

$$(t+u)[x \mapsto v_x] = t[x \mapsto v_x] + u[x \mapsto v_x]$$

Categorical formulation

$LCD \times LCD$ supports LCD-substitution $\rightsquigarrow$ $LCD \times LCD$ is a **module over $\boldsymbol{LCD}$**

+ commutes with substitution $\rightsquigarrow$ $+ : LCD \times LCD \to LCD$ is a **module morphism**

Building blocks for binding signatures

Essential constructions of **modules over a monad $\boldsymbol{R}$**:

- R itself
- M x N for any modules M and N (in particular, R x R)
- The **derivative of a module $\boldsymbol{M}$** is the module M' defined by $M'(X) = M(X \coprod \{\diamond\})$.

  The derivative is used to model an operation binding a variable

  (Cf next slide).

Syntactic operations are module morphisms

module morphism = maps commuting with substitution.

$$id_M : M \to M$$

$$0 : 1 \to LCD$$

$$+ : LCD \times LCD \to LCD$$

$$app : LCD \times LCD \to LCD$$

$$abs : LCD' \to LCD$$

The Big Picture again

A **1-signature** Σ is a functorial assignment:

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monad module over R

A **model of Σ** is a pair:

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monad module morphism

A **model morphism** $m : (R,\rho) \to (S,\sigma)$ is a monad morphism commuting with the module morphism:

$$\begin{tikzcd} \Sigma(R) \arrow[r, "\rho"] \arrow[d, "\Sigma(m)"'] & R \arrow[d, "m"] \\ \Sigma(S) \arrow[r, "\sigma"'] & S \end{tikzcd}$$

Syntax

Definition

Given a 1-signature Σ, its **syntax** is an initial object in its category of models.

Question: Does the syntax exist for every 1-signature?

Answer: No.

Counter-example: the 1-signature $R \mapsto \mathscr{P} \circ R$

powerset endofunctor on $\mathbb{S}\mathrm{et}$

Examples of 1-signatures generating syntax

- **(0,+) language**:

  Signature: $R \mapsto 1 \coprod (R \times R)$

  Model: $(R\ ,\quad 0 : 1 \to R\ ,\quad + : R \times R \to R)$

  Syntax: $(B\ ,\quad 0 : 1 \to B\ ,\quad + : B \times B \to B)$

- **lambda calculus**:

  Signature: $R \mapsto R' \coprod (R \times R)$

  Model: $(R\ ,\quad abs : R' \to R\ ,\quad app : R \times R \to R)$

  Syntax: $(\Lambda\ ,\quad abs : \Lambda' \to \Lambda\ ,\quad app : \Lambda \times \Lambda \to \Lambda)$

Can we generalize this pattern?

Initial semantics for algebraic 1-signatures

Theorem [Hirschowitz & Maggesi 2007]

Syntax exists for any **algebraic 1-signature**, i.e. 1-signature built out of derivatives, products, disjoint unions, and the 1-signature $R \mapsto R$.

Algebraic 1-signatures correspond to binding signatures through the embedding:

$$\text{Binding signatures} \hookrightarrow \text{Our 1-signatures}$$

Question: Can we enforce some equations in the syntax ?
For example: commutativity of $+$ for the differential λ-calculus.

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Example: a commutative binary operation

Specification of a binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R\ ,\ + : R \times R \to R)$

What is an appropriate notion of model for a commutative binary operation ?

Example: a commutative binary operation

Specification of a commutative binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R\ ,\ + : R \times R \to R)$ s.t. $t + u = u + t$ (1)

What is an appropriate notion of model for a commutative binary operation ?

Answer: a monad equipped with a commutative binary operation

Example: a commutative binary operation

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What is an appropriate notion of model for a commutative binary operation ?

Answer: a monad equipped with a commutative binary operation

Equation (1) states an equality between R-module morphisms:

$$\begin{tikzcd} \mathrm{R} \times \mathrm{R} \arrow[rr, bend left, "+"] \arrow[r, bend right, "\text{swap}"'] & \mathrm{R} \times \mathrm{R} \arrow[r, bend right, "+"'] & \mathrm{R} \end{tikzcd}$$

Review: Signatures with equations

- [Fiore-Hur 2010]: inductively defined set of possible equations.

- [AHLM CSL 2018]: "quotients" of algebraic 1-signatures

  Examples:

  - a binary commutative operation

  - application of the simple terms of differential λ-calculus (2^{nd} variant)

  $$\text{app} : \text{LCD} \times \text{FreeCommutativeMonoid(LCD)} \to \text{LCD}$$

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$$\text{app} : \text{LCD} \times \text{FreeCommutativeMonoid(LCD)} \rightarrow \text{LCD}$$

This work: more general equations (e.g. associativity of a binary op).

Equations

Given a 1-signature Σ, (e.g. binary operation: $\Sigma(\mathrm{R}) = \mathrm{R} \times \mathrm{R}$)

a Σ-**equation** A $\rightrightarrows$ B is a functorial assignment:

$$R \mapsto \left(A(R) \rightrightarrows B(R) \right)$$

model of Σ

parallel pair of module morphisms over R

e.g. commutativity:

$$R \mapsto \left(R \times R \underset{+\circ swap}{\overset{+}{\rightrightarrows}} R \right)$$

A **2-signature** is a pair

$$(\Sigma, \mathrm{E})$$

1-signature

set of Σ-equations

***model* of a 2-signature** $(\boldsymbol{\Sigma}, \mathbf{E})$:

- a model R of Σ
- s.t. ∀ (A $\rightrightarrows$ B) $\in$ E, the two morphisms $A(R) \rightrightarrows B(R)$ are equal

Initial semantics for algebraic 2-signatures

Algebraic 2-signature:

$$(\Sigma, \mathrm{E})$$

algebraic 1-signature

set of **elementary** Σ-equations

Theorem

Syntax exists for any algebraic 2-signature

Given a 1-signature Σ, a Σ-equation A $\rightrightarrows$ B is **elementary** if:

1. A "preserves pointwise epimorphisms"

(e.g., any "algebraic 1-signature", such as $\mathrm{R} \mapsto \mathrm{R}{\times}\mathrm{R}$)

2. B is of the form $\mathrm{R} \mapsto \mathrm{R}$'...' (e.g. $\mathrm{R} \mapsto \mathrm{R}$)

Example: λ-calculus modulo βη

The algebraic 2-signature $(\Sigma_{\mathrm{LC}\beta\eta}, E_{\mathrm{LC}\beta\eta})$ of λ-calculus modulo βη:

$$\boldsymbol{\Sigma}_{\mathbf{LC}\boldsymbol{\beta\eta}}\,(R) := \Sigma_{\mathrm{LC}}(R) = (R \times R) \coprod R'$$

model of $\boldsymbol{\Sigma}_{\mathbf{LC}}$ = monad R with module morphisms:

$$\mathrm{app} : R \times R \to R \qquad \mathrm{abs} : R' \to R$$

β-equation: $(\lambda x.t)\;u = \underbrace{t[x \mapsto u]}_{\sigma_R(t,u)}$ **η-equation**: $t = \lambda x.(t\;x)$

$$\mathbf{E}_{\mathbf{LC}\boldsymbol{\beta\eta}} = \{\ \beta\text{-equation},\ \eta\text{-equation}\ \}$$

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η-equation: $t = \lambda x.(t\ x)$

$$\begin{tikzcd} R' \times R \arrow[rr, bend left, "\sigma_R"] \arrow[r, bend right, "\mathrm{abs} \times R"'] & R \times R \arrow[r, bend right, "\mathrm{app}"'] & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[rr, bend left, "\mathrm{id}_R"] \arrow[r, bend right, "R\iota_1"'] & R' \arrow[r, bend right, "\mathrm{abs}"'] & R \end{tikzcd}$$

$$\mathbf{E_{LC\beta\eta}} = \{\ \beta\text{-equation},\ \eta\text{-equation}\ \}$$

Example: fixpoint operator

Definition [AHLM CSL 2018]

A **fixpoint operator** in a monad R is a module morphism $f : R' \to R$ s.t.

for any term $t \in R(X \coprod \{ \diamond \})$, $f(t) = t[\diamond \mapsto f(t)]$,

i.e. (1)
$$\begin{tikzcd} R' \arrow[rr, bend left, "f"] \arrow[r, "(id_{R'}, f)"'] & R' \times R \arrow[r, "\sigma_R"'] & R \end{tikzcd}$$
commutes.

The algebraic 2-signature (Σ_{fix}, E_{fix}) of a fixpoint operator:

$$\Sigma_{fix}(R) := R' \qquad E_{fix} = \{ (1) \}$$

Proposition [AHLM CSL 2018]

Fixpoint operators in $LC_{\beta\eta}$ are in one to one correspondance with fixpoint combinators (i.e. λ-terms Y s.t. $t\,(Y\,t) = Y\,t$ for any t).

Combining algebraic 2-signatures

Algebraic 2-signatures can be combined:

fixpoint operator

λ-calculus modulo βη

$$(\Sigma_{\mathrm{fix}}, E_{\mathrm{fix}}) \quad + \quad (\Sigma_{\mathrm{LC}\beta\eta}, E_{\mathrm{LC}\beta\eta})$$

$$=$$

$$(\Sigma_{\mathrm{fix}} \coprod \Sigma_{\mathrm{LC}\beta\eta} \;,\; E_{\mathrm{fix}} \cup E_{\mathrm{LC}\beta\eta})$$

λ-calculus modulo βη with an explicit fixpoint operator

Example: free commutative monoid

An algebraic 2-signature $(\Sigma_{\mathrm{mon}}, E_{\mathrm{mon}})$ for the free commutative monoid monad:

$$\mathbf{\Sigma}_{\mathrm{mon}}(R) := 1 \coprod (R \times R)$$

model of Σ_{mon} = monad R with module morphisms:

$$0 : 1 \to R \qquad + : R \times R \to R$$

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An algebraic 2-signature $(\Sigma_{\text{mon}}, \mathrm{E}_{\text{mon}})$ for the free commutative monoid monad:

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model of Σ_{mon} = monad R with module morphisms:

$$0 : 1 \to \mathrm{R} \qquad + : \mathrm{R} \times \mathrm{R} \to \mathrm{R}$$

4 elementary Σ-equations:

$$\begin{tikzcd} \underset{t}{R} \arrow[r, bend left, "0+t"] \arrow[r, bend right, "t"'] & R \end{tikzcd} \qquad \begin{tikzcd} \underset{s,t,u}{R \times R \times R} \arrow[r, bend left, "(s+t)+u"] \arrow[r, bend right, "s+(t+u)"'] & R \end{tikzcd}$$

$$\begin{tikzcd} \underset{t}{R} \arrow[r, bend left, "t+0"] \arrow[r, bend right, "t"'] & R \end{tikzcd} \qquad \begin{tikzcd} \underset{s,t}{R \times R} \arrow[r, bend left, "s+t"] \arrow[r, bend right, "t+s"'] & R \end{tikzcd}$$

Our target: LCD

Syntax of the *differentiable λ-calculus*:

Differential λ-terms

$$
\begin{array}{rcll}
s,t & ::= & \mathrm{x} & \\
& \mid & \lambda \mathrm{x.t} & \left.\right\} \text{λ-calculus} \\
& \mid & \mathrm{s\ t} & \\
& \mid & \mathrm{Ds} \cdot \mathrm{t} & \\
& \mid & \mathrm{s} + \mathrm{t} & \left.\right\} \text{free commutative monoid} \\
& \mid & 0 &
\end{array}
$$

and (bi)linearity of constructors with respect to +:

$$\lambda \mathrm{x.(s+t)} = \lambda \mathrm{x.s} + \lambda \mathrm{x.t} \qquad \ldots$$

Algebraic 1-signature for LCD

Syntax of the *differentiable λ-calculus*:

Differential λ-terms

Corresponding 1-signature

$$
\begin{array}{rcll}
s,t & ::= & \mathrm{x} & \\
& | & \lambda \mathrm{x.t} & \multirow{2}{*}{$\left.\right\}\ \Sigma_{\mathrm{LC}}(\mathrm{R}) = \mathrm{R}' \coprod (\mathrm{R} \times \mathrm{R})$} \\
& | & \mathrm{s\ t} & \\
& | & \mathrm{Ds} \cdot \mathrm{t} & \mathrm{R} \mapsto \mathrm{R} \times \mathrm{R} \\
& | & \mathrm{s} + \mathrm{t} & \multirow{2}{*}{$\left.\right\}\ \Sigma_{\mathrm{mon}}(\mathrm{R}) = 1 \coprod (\mathrm{R} \times \mathrm{R})$} \\
& | & 0 &
\end{array}
$$

Algebraic 1-signature for LCD

Syntax of the *differentiable λ-calculus*:

Differential λ-terms

Corresponding 1-signature

$$
\begin{array}{rcll}
s,t & ::= & x & \\
& | & \lambda x.t & \multirow{2}{*}{$\Big\}\quad \Sigma_{LC}(R) = R' \coprod (R \times R)$} \\
& | & s\ t & \\
& | & Ds \cdot t & R \mapsto R \times R \\
& | & s + t & \multirow{2}{*}{$\Big\}\quad \Sigma_{mon}(R) = 1 \coprod (R \times R)$} \\
& | & 0 &
\end{array}
$$

Resulting algebraic 1-signature: $\Sigma_{LCD}(R) = \Sigma_{LC}(R) \coprod (R \times R) \coprod \Sigma_{mon}(R)$

Elementary equations for LCD

Commutative monoidal structure:

$$
E_{\mathrm{mon}} \begin{cases} s + t = t + s & R \times R \rightrightarrows R \\ s + (t + u) = (s + t) + u & R \times R \times R \rightrightarrows R \\ 0 + t = t & R \rightrightarrows R \\ t + 0 = t & R \rightrightarrows R \end{cases}
$$

Linearity:

$$\lambda x.(s+t) = \lambda x.s + \lambda x.t \qquad R \times R \rightrightarrows R$$

$$D(s+t) \cdot u = Ds \cdot u + Dt \cdot u \qquad R \times R \times R \rightrightarrows R$$

$$Ds \cdot (t+u) = Ds \cdot t + Ds \cdot u \qquad R \times R \times R \rightrightarrows R$$

$$\ldots$$

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Initiality of R $\Rightarrow$ model morphism $\mathrm{R} \to \mathrm{S}$ $\Rightarrow$ monad morphism $\mathrm{R} \to \mathrm{S}$

Example: Computing the set of free variables

LC = initial model of $(\Sigma_{\mathrm{LC}}\,, \varnothing)$ $\Sigma_{\mathrm{LC}}(\mathrm{R}) = (\mathrm{R} \times \mathrm{R}) \coprod \mathrm{R}'$

$\mathcal{P}$ = power set monad

Definition of a (monad) morphism $\mathrm{fv} : \mathrm{LC} \to \mathcal{P}$ **s.t.**

$$\mathrm{fv}(\mathrm{app}(\mathrm{t},\mathrm{u})) = \mathrm{fv}(\mathrm{t}) \cup \mathrm{fv}(\mathrm{u}) \qquad \mathrm{fv}(\mathrm{abs}(\mathrm{t})) = \mathrm{fv}(\mathrm{t}) \setminus \{\diamond\}$$

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$\Rightarrow$ make $\mathcal{P}$ a model of Σ_{LC}:

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Initiality of $\mathrm{LC} \Rightarrow$ $\mathrm{fv} : \mathrm{LC} \to \mathcal{P}$ satisfying the above equations (as a model morphism).

Example: Translating λ-calculus with fixpoint

$\mathrm{LC}_{\beta\eta\mathrm{fix}}$ = initial model of $(\Sigma_{\mathrm{LC}\beta\eta}, \mathrm{E}_{\mathrm{LC}\beta\eta}) + (\Sigma_{\mathrm{fix}}, \mathrm{E}_{\mathrm{fix}})$

λ-calculus modulo βη with a fixpoint operator $\mathrm{fix} : \mathrm{LC}_{\beta\eta\mathrm{fix}}' \to \mathrm{LC}_{\beta\eta\mathrm{fix}}$

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λ-calculus modulo βη

monad morphism

Definition of a translation $\mathrm{f} : \mathrm{LC}_{\beta\eta\mathrm{fix}} \to \mathrm{LC}_{\beta\eta}$ **s.t.**

$$\mathrm{f}(\mathrm{u}) = \text{"}\mathrm{u}[\,\mathrm{fix}(\mathrm{t}) \mapsto \mathrm{app}(\mathrm{Y}, \mathrm{abs}(\mathrm{t}))\,]\text{"}$$

a chosen fixpoint combinator

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$\mathrm{app, abs}$

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Solution [CSL AHLM 2010]:

continuation monad C(X) = $\mathbb{N}^{(\mathbb{N}^{X})}$

1. define $\mathrm{f} : \mathrm{LC} \to \mathrm{C}$ by recursion
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affects an arbitrary size to each variable

Intuition: uncurrying $f_X : LC(X) \to \mathbb{N}^{(\mathbb{N}^X)}$ yields $g : LC(X) \times \mathbb{N}^X \to \mathbb{N}$

Example: Computing the size of a term

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$$\mathrm{s}(\mathrm{t}) = \mathrm{g}(\mathrm{t}, (\mathrm{x} \mapsto 0))$$

variables are of size 0

Conclusion

Summary of the talk:

- presented a notion of 1-signature and models
- defined a 2-signature as a 1-signature and a set of equations
- identified a class of 2-signatures that generate a syntax

The main theorem has been formalized in Coq using the UniMath library.

Future work:

- add the notion of reductions;
- extend our work to simply typed syntaxes.

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Thank you!

Monads

1. $\mathrm{LCD} : \mathrm{Set} \to \mathrm{Set}$

2. A collection of functions $(\mathrm{var}_{\mathrm{X}} : X \to LCD(X))_X$

   Variables are expressions

3. For each function $\mathrm{u} : \mathrm{X} \to \mathrm{LCD(Y)}$, a function $\mathrm{bind_u} : \mathrm{LCD(X)} \to \mathrm{LCD(Y)}$

   Parallel substitution

   Notation: $\mathbf{bind_u(t)} = \mathbf{t[x \mapsto u(x)]}$

4. Monadic laws:

$$\mathrm{var(y)[x \mapsto u(x)] = u(y)}$$

$$\mathrm{t[x \mapsto var(x)] = t}$$

$$\mathrm{t[x \mapsto f(x)][y \mapsto g(y)] = t[x \mapsto f(x)[y \mapsto g(y)]\ ]}$$

Modules VS Monads

Monad

1. $\mathrm{R} : \mathrm{Set} \to \mathrm{Set}$

2. A collection of functions $(\mathrm{var}_X : X \to R(X))_X$

   Variables are expressions

3. For each function $\mathrm{u} : \mathrm{X} \to \mathrm{R}(\mathrm{Y})$, a function $\mathrm{bind}_{\mathrm{u}} : \mathrm{R}(\mathrm{X}) \to \mathrm{R}(\mathrm{Y})$

   Parallel substitution

   Notation: $\mathbf{bind_u(t) = t[x \mapsto u(x)]^R}$

4. Substitution laws:

$$\mathrm{var}(\mathrm{y})[\mathrm{x} \mapsto \mathrm{u}(\mathrm{x})]^{\mathrm{R}} = \mathrm{u}(\mathrm{y})$$
$$\mathrm{t}[\mathrm{x} \mapsto \mathrm{var}(\mathrm{x})]^{\mathrm{R}} = \mathrm{t}$$
$$\mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})]^{\mathrm{R}}[\mathrm{y} \mapsto \mathrm{g}(\mathrm{y})]^{\mathrm{R}} = \mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})[\mathrm{y} \mapsto \mathrm{g}(\mathrm{y})]^{\mathrm{R}}\ ]^{\mathrm{R}}$$

Modules VS Monads

~~**Monad**~~ **Module over a monad** R (e.g. $\mathrm{R}, \mathrm{R} \times \mathrm{R}, 2, \ldots$)

1. $\mathrm{M} : \mathrm{Set} \to \mathrm{Set}$

   $M(X)$ = expressions taking variables in X

2. ~~A collection of functions $(\mathrm{var}_X : X \to M(X))_X$~~

3. For each function $\mathrm{u} : \mathrm{X} \to \mathrm{R}(\mathrm{Y})$, a function $\mathrm{bind}_\mathrm{u} : \mathrm{M}(\mathrm{X}) \to \mathrm{M}(\mathrm{Y})$

   Parallel substitution

   Notation: $\mathbf{bind_u(t) = t[x \mapsto u(x)]^M}$

4. Substitution laws:

$$\cancel{\mathrm{var}(\mathrm{y})[\mathrm{x} \mapsto \mathrm{u}(\mathrm{x})]^{\mathrm{M}} = \mathrm{u}(\mathrm{y})}$$

$$\mathrm{t}[\mathrm{x} \mapsto \mathrm{var}(\mathrm{x})]^{\mathrm{M}} = \mathrm{t}$$

$$\mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})]^{\mathrm{M}}[\mathrm{y} \mapsto \mathrm{g}(\mathrm{y})]^{\mathrm{M}} = \mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})[\mathrm{y} \mapsto \mathrm{g}(\mathrm{y})]^{\mathrm{R}}\,]^{\mathrm{M}}$$