High-level signatures and initial semantics

Ambroise Lafont

joint work with Benedikt Ahrens, André Hirschowitz, Marco Maggesi

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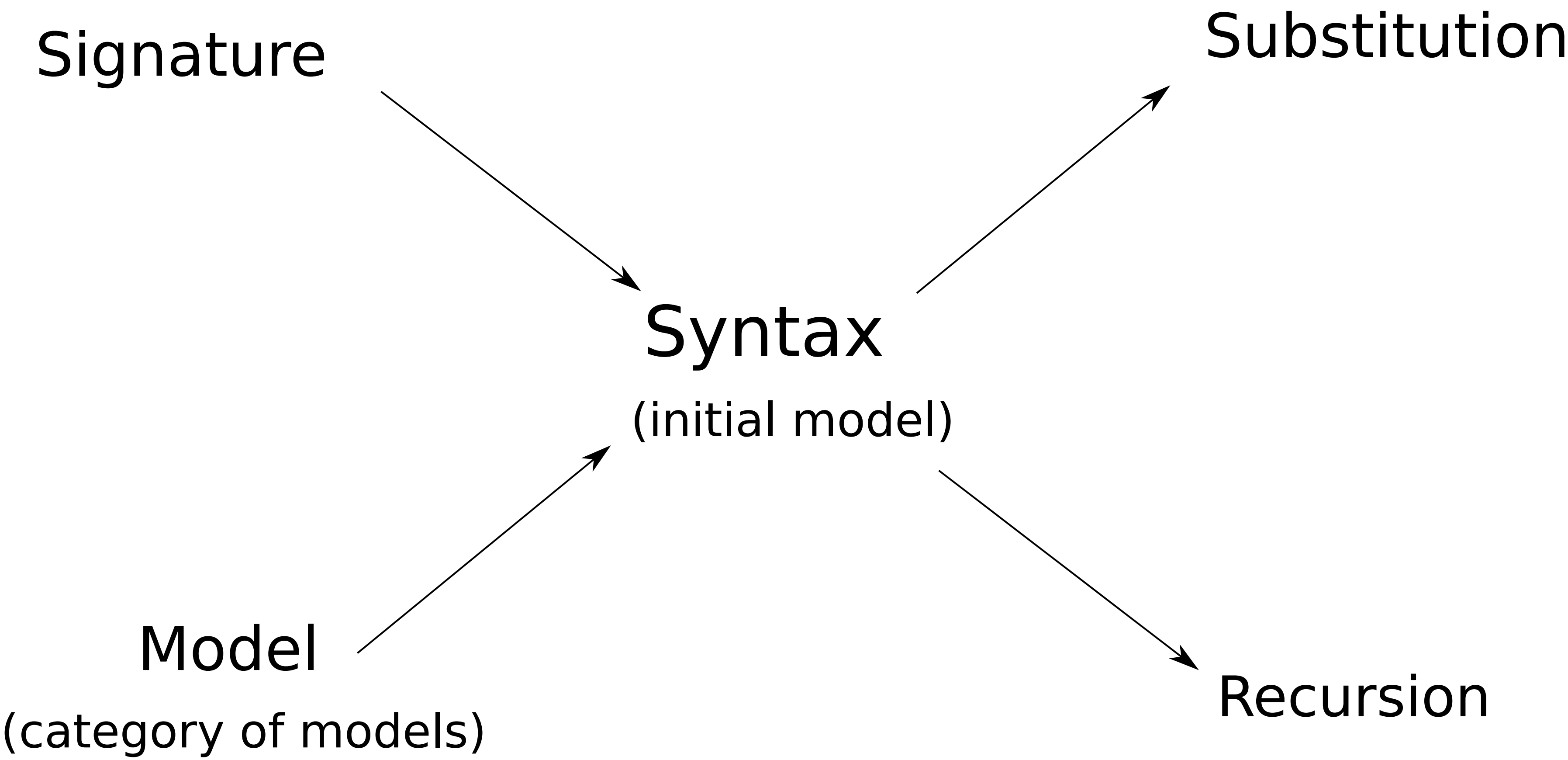
Introduction

Purpose of our work: specify and construct untyped syntaxes with variables and a well-behaved substitution (e.g. lambda-calculus).

More specifically (terms in italics will be explained):

1. we have a notion of *signature* too general (all of them do not *specify a syntax*)
2. classical *binding signatures* embed into our signatures as *algebraic signatures,* and indeed specify a syntax.
3. our main result: any *quotient* of algebraic signatures also specifies a syntax

What is a syntax?



Signatures which we care about: those whose category of models have an *initial object*.

Our work

We present a notion of signature (and associated models) based on the notion of module over a monad.

Goal of our work: Identify a large class of these signatures whose category of models have an initial object.

Our main result: Quotients of "binding signatures" have a syntax.

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Example: 0, ★

Consider the syntax generated by a binary operation ★ and a constant **0** (and variables):

$$
\begin{aligned}
\text{expr} ::= \; & x && \textit{(variable)} \\
\mid \; & t_1 \star t_2 && \textit{(binary operation)} \\
\mid \; & 0 && \textit{(constant)}
\end{aligned}
$$

The syntax induces an endofunctor B (on Set) mapping a set of variables to the set of expressions built out of them:

$$B(\emptyset) = \{0, 0 \star 0, \ldots\}$$
$$B(\{x, y\}) = \{0, 0 \star 0, \ldots, x, y, x \star y, \ldots\}$$

Example: 0, ★

The binary operation ★ induces a natural transformation:

$$B \times B \to B$$

The constant **0** induces a natural transformation:

$$1 \to B$$

Variables induce a natural transformation

$$\mathrm{Id}_{\mathrm{Set}} \to B$$

Using disjoint union, they gather into a single natural transformation:

$$B \times B + 1 + \mathrm{Id}_{\mathrm{Set}} \to B$$

i.e. B is an algebra for the endofunctor $F \mapsto F \times F + 1 + \mathrm{Id}_{\mathrm{Set}}$ on the category $\mathrm{End}_{\mathrm{Set}}$ of endofunctors on Set.

Actually, B can be defined to be the initial algebra of $\boldsymbol{F}$.

Binding Signatures [Fiore-Plotkin-Turi 1999]

Definition

Binding signature = a family of lists of natural numbers.

Each list specifies an operation in the syntax:

- length of the list = number of arguments of the operation
- natural number in the list = number of bound variables in the corresponding argument

Syntax with 0, ★: $\big((), (0, 0)\big)$

0-aire operation **0** binary operation ★

Lambda calculus: $\big((0\ ,\ 0)\ ,\ (1)\big)$

application lambda-abstraction

Signatures and endofunctors [FPT]

In the same spirit as in the first example $(0, \star)$, any binding signature can be turned into an endofunctor Σ on the category $\mathrm{End}_{\mathrm{Set}}$.

A natural notion of model: $\Sigma + \mathrm{Id}_{\mathrm{Set}}$ -algebra

Indeed, the initial $\Sigma + \mathrm{Id}_{\mathrm{Set}}$-algebra of a binding signature Σ always exists.

What about substitution?

Does this initial algebra come with a well-behaved substitution?

Substitution following [FPT]

The endofunctor Σ induced by a binding signature comes with a *strength* which allows [FPT] to refine the notion of model:

***Σ*-monoids =** $\Sigma + \mathrm{Id}_{\mathrm{Set}}$-algebras equipped with a well-behaved substitution.

***Σ*-monoid morphisms** = algebra morphisms commuting with substitution.

Theorem [FPT]:

The initial $\Sigma + \mathrm{Id}_{\mathrm{Set}}$-algebra of a binding signature Σ comes with a well-behaved substitution that makes it initial in the category of ***Σ*-monoids**.

This suggests defining signatures to be endofunctors on $\mathrm{End}_{\mathrm{Set}}$ *with strength* (as in [Matthes-Uustalu 2004]).

Our signatures

In the next slides, we present our notion of signature.

Binding signatures $\hookrightarrow$ Endofunctors with strength $\overset{\mathcal{I}}{\hookrightarrow}$ Our signatures

Conjecture: for any endofunctor with strength Σ, our category of models is equivalent to the [FPT] one:

$$\mathrm{Models}(\mathcal{I}(\Sigma)) \cong \Sigma\text{-monoids}$$

(modulo a technical restriction, namely considering only finitary endofunctors on Set)

Our signatures and models

A **signature** Σ is a functorial assignment:

monad   module over R

$$R \mapsto \Sigma(R)$$

A **model of Σ** is a pair:

$$(R, \quad \sigma : \Sigma(R) \to R)$$

monad   module morphism

monad	:=	endofunctor with substitution
module over a monad	:=	endofunctor with substitution
module morphism	:=	natural transformation preserving substitution

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Monads

The functor B corresponds to the language (0, ★) with variables as placeholders for any expression of B.

$B(X)$ denotes the set of expressions taking variables in X.

Substitution (required to satisfy some intuitive equations):

$$B(X) \to (X \to B(Y)) \to B(Y)$$

Such a functor is called a **monad**.

A **monad morphism** between two monads R and S is a family of maps $(f_X : R(X) \to S(X))_X$ preserving variables and substitution.

Operations as module morphisms

In the (0, ★) language,

$$(t \star u)[x \mapsto v_x] = t[x \mapsto v_x] \star u[x \mapsto v_x]$$

★ commutes with substitution

In the right hand side, substitution acts on a pair of expressions.

We abstract this situation as follows:

- pairs of expressions form a **module** $B \times B$ over the monad B,

- ★ yields **module morphism** from $B \times B$ to B

Module over a monad

The endofunctor $B \times B$ corresponds to expressions with variables as placeholders for any expression in the language B.

Substitution with B-expressions (required to satisfy some intuitive equations):

$$(B \times B)(X) \to (X \to B(Y)) \to (B \times B)(Y)$$

Such a functor is called a **module over the monad $\boldsymbol{B}$**.

Examples of modules

Modules over a monad:

Some examples of modules over a monad $\boldsymbol{R}$:

- R itself

- M x N for any modules M and N (in particular, R x R)

- M' is the module defined by M'(X) = M(X + {x}) for any set X of variables given a module M. We call it the **derivative of $\boldsymbol{M}$**.

  The new variable x is used to model an operation binding a variable (e.g. the lambda-abstraction).

Examples of module morphisms

A **module morphism** between two modules M and N on the same monad R is a family of maps $(f_X{:}M(X) \to N(X))_X$ commuting with substitution.

$id_M : M \to M$

the family of identity maps $(id_{M(X)}{:}M(X) \to M(X))_X$ for any module M

$\star : B \times B \to B$

$app : L \times L \to L$

the application operation of the lambda calculus monad L.

$abs : L' \to L$

Indeed, in $\lambda x.t$, the term t depends on an additional free variable x:

If $\lambda x.t \in L(Y)$, then $t \in L(Y + \{x\})$ **= $\boldsymbol{L'(Y)}$**

Signatures and models

A **signature** Σ is a functorial assignment:

monad module over R

$$R \mapsto \Sigma(R)$$

A **model of Σ** is a pair:

$$(R, \quad \sigma : \Sigma(R) \to R)$$

monad module morphism

A **model morphism** $m : R \to S$ is a monad morphism commuting with σ:

$$\begin{tikzcd}
\Sigma(R) \arrow[r, "\sigma"] \arrow[d, "\Sigma(m)"'] & R \arrow[d, "m"] \\
\Sigma(S) \arrow[r, "\sigma"'] & S
\end{tikzcd}$$

Existence of syntax = initial model?

Notion of signature too general: existence of the syntax (**= initial model**) ?

Counter-example: the signature $R \mapsto \mathscr{P} \circ R$

powerset endofunctor on $\mathbb{S}\mathrm{et}$

Examples of signatures with syntax

- $R \mapsto 1 + R \times R$

  By universal property of the disjoint sum, models are monads R equipped with module morphisms $1 \to R$ and $R \times R \to R$. The syntax corresponds to our example with **0** and ★.

- $R \mapsto R \times R + R'$

  Models are monads R equipped with two modules morphisms: $R \times R \to R$ and $R' \to R$. The syntax corresponds to lambda calculus.

Algebraic signatures

More generally, the syntax always exists for any signature induced by a disjoint sum of products of finite derivatives of the monad ($R \mapsto R' \times R'' \times R''' + R \times R'' \times R''' \times R + ...$).

We call such a signature an **algebraic signature**. They correspond to binding signatures through the inclusion:

$$\text{Binding signatures} \hookrightarrow \text{Endofunctors with strength} \overset{\mathcal{I}}{\hookrightarrow} \text{Our signatures}$$

Our main result: Quotients of algebraic signatures have a syntax.

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Quotient of a signature

Quotient of a set:

A quotient of a set X is a set Y together with a surjection $p : X \to Y$.

$$x \sim x' \qquad \Longleftrightarrow \qquad p(x) = p(x')$$

Quotient of a signature:

A quotient of a signature Σ is a signature Ψ together with a (natural) family of module morphisms $(f_R : \Sigma(R) \to \Psi(R))_R$ that is pointwise surjective.

$$R \mapsto \begin{tikzcd} \Sigma(R) \arrow[d, two heads, "f_R"] \\ \Psi(R) \end{tikzcd}$$

Presentable signatures

A **presentable signature** is a quotient of a binding signature.

Main Theorem: For any presentable signature, there is a syntax.

We now give examples of new kinds of operations specified by presentable signatures (more can be found in the article).

Example 1: Symmetric operations

Binary commutative operation +:

$$t + u = u + t$$

As a quotient of an algebraic signature:

$$R \mapsto \begin{array}{c} R \times R \\ \big\downarrow\!\!\downarrow \\ R \times R / \{(x,y) \sim (y,x)\} \end{array}$$

This generalizes to **n-ary permutation invariant operations**.

Example 2: Explicit substitution

An operation $\_\langle \mathbf{x_i} \mapsto \mathbf{t_i} \rangle$ that mimics the behavior of the meta-substitution $\_[\mathbf{x_i} \mapsto \mathbf{t_i}]$ in the sense that it enjoys some of its coherences:

- invariance under **permutation**

$$F(x, y)\langle x \mapsto t, y \mapsto u\rangle = F(y, x)\langle x \mapsto u, y \mapsto t\rangle$$

- invariance under **weakening**

$$F(x)\langle x \mapsto t, y \mapsto u\rangle = F(x)\langle x \mapsto u\rangle$$

- invariance under **contraction**

$$F(x, y)\langle x, y \mapsto t\rangle = F(x, x)\langle x \mapsto t\rangle$$

Example 2: Explicit substitution

Explicit substitution as a quotient of the algebraic signature:

$$\Sigma(R) := R' \times R \;+\; R'' \times R \times R \;+\; R''' \times R \times R \times R \;+\; \ldots$$

$$t\langle x \mapsto u \rangle \qquad t\langle x \mapsto u, y \mapsto v \rangle \qquad t\langle x \mapsto u, y \mapsto v, z \mapsto w \rangle$$

Quotiented by the following relation:

- **permutation**: $t\langle x \mapsto u, y \mapsto v \rangle \sim t[x \rightleftarrows y]\langle x \mapsto v, y \mapsto u \rangle$
- **weakening**: $t\langle x \mapsto u \rangle \sim t\langle x \mapsto v, y \mapsto u \rangle$
- **contraction**: $t\langle x \mapsto u, y \mapsto u \rangle \sim t[y{:=}x]\langle x \mapsto u \rangle$

Conclusion

We have given a criterion for signatures to specify a syntax. This criterion encompasses the classical binding signatures, and allows new operations in the syntax.

Our main theorem have been formalized using the Coq library UniMath.

Future work:

- take into account more sophisticated equations in the syntax than just quotients (e.g. associative binary operation, lambda-calculus modulo beta/eta equivalence);

- extend our framework to simply typed syntaxes.

FIN PROVISOIRE

Ne pas lire les slides qui suivent (ce sont des anciennes slides que je garde au cas où).

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Fixpoint operator with coherences

But we would like to encode some of the expected behaviour of such a fixed point:

- invariance under permutation
- invariance under weakening
- invariance under contraction. Roughly:

```
let rec f₁ = F(f₁,f₂)
    and f₂ = F(f₁,f₂)
in f₁
```

=

```
let rec f = F(f,f)
in f
```

A construction satisfying these invariances can be specified by quotienting the naive algebraic signature.

Fixpoint operator

A fixpoint operator:

A language with (mutual) fixpoints comes with a construction

$$\begin{aligned}&\texttt{let rec } \mathbf{f}_1 = \mathbf{t}_1\\ &\qquad \texttt{and } \mathbf{f}_2 = \mathbf{t}_2\\ &\qquad\qquad \cdots\\ &\qquad \texttt{and } \mathbf{f}_n = \mathbf{t}_n\\ &\texttt{in } \mathbf{f}_i\end{aligned}$$

where each $\mathbf{f}_j$ may appear as a variable in each expression $\mathbf{t}_i$.

Thus, it takes **n** expressions $\mathbf{t}_1,..,\mathbf{t}_n$ depending on **n** fresh variables $\mathbf{f}_1,..,\mathbf{f}_n$ and produces an expression which no longer depend on them.

As such, it can be specified by a binding signature.

Example 2: Syntactic closure operator

Syntactic closure operator $\forall$

$\forall xyz.t$ binds the variables x, y and z in the term t

Example of an operation invariant under **permutation** and **weakening**:

- permutation: $\forall xy.t = \forall yx.t$

- weakening: $\forall x.t = \forall xy.t$ if t does not depend on y