

# High-level signatures and initial semantics

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# Introduction

This work is about constructing and specifying the syntax of untyped languages with variables and well-behaved substitution (e.g. lambda-calculus).

We expect that our work generalize to simply-typed syntaxes.

# What is a syntax?

A **syntax** should enjoy a well-behaved substitution and provide a recursion principle.

Syntaxes are specified by **signatures**:

- to a signature is associated a notion of **model**. Such models form a category;
- the syntax is defined as the **initial object** in this category, if it exists.

Signatures which we care about are those whose category of models have an initial object.

# Purpose of our work

We present a notion of signatures based on the notion of module over a monad, and their associated models.

**Goal of our work:** Identify a large class of these signatures whose category of models have an initial object.

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## **1. Languages, monads and modules**

## 2. Signatures and their models

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## 4. Presentables signatures

# High-level signatures

Endofunctors may be considered as signatures, generalizing combinatorial signatures.

There is a natural **category of models** of such a signature: the category of its algebras.

The specified language (or **syntax**) is characterized as the initial object in the category of models.

This definition is motivated by the **recursion principle** induced by the initiality property.

# Monads

A monad  $\mathbf{R}$  corresponds to a language with variables as placeholders for any expression of  $\mathbf{R}$ .

$\mathbf{R}(\mathbf{X})$  denotes the set of expressions taking variables in  $\mathbf{X}$ . Intuitively, it should contain at least the set  $\mathbf{X}$  of variables.

Given any family  $(\mathbf{t}_x)_{x \in \mathbf{X}}$  of elements of  $\mathbf{R}(\mathbf{Y})$ , any expression  $\mathbf{e}$  in  $\mathbf{R}(\mathbf{X})$  can be substituted to yield an expression  $\mathbf{e}[\mathbf{x} \mapsto \mathbf{t}_x]$  in  $\mathbf{R}(\mathbf{Y})$ .

The substitution is required to satisfy some intuitive equations.

A **monad morphism** between two monads  $\mathbf{R}$  and  $\mathbf{S}$  is a family of maps  $(\mathbf{f}_x : \mathbf{R}(\mathbf{X}) \rightarrow \mathbf{S}(\mathbf{X}))_x$  preserving variables and substitution.

# Operations as module morphisms

In the lambda-calculus,

$$\text{app}(t, u)[x \mapsto v_x] = \text{app}(t[x \mapsto v_x], u[x \mapsto v_x])$$

i.e.

**application commutes with substitution**

Let us rewrite the right hand side:

$$\text{app}(t, u)[x \mapsto v_x] = \text{app}((t, u)[x \mapsto v_x])$$

considering the obvious substitution on pairs of lambda terms.

We abstract this situation as follows:

- pairs of lambda-terms form a **module** over the lambda-calculus monad,
- application is a **module morphism**



# Module over a monad

A module **M** over a monad **R** corresponds to expressions with variables as placeholders for any expression in the language **R**.

Given a module **M**, the set **M(X)** is the set of expressions taking variables in **X** (but contrary to monads, a variable may not immediately yield a generalized expression).

Given any family  $(t_x)_{x \in X}$  of expressions in **R(Y)**, any expression **e** in **M(X)** can be substituted to yield an expression **e**[**x**  $\mapsto$  **t<sub>x</sub>**] in **M(Y)**.

As for monads, the substitution is required to satisfy some intuitive equations.

# Examples of modules

## Modules over a monad:

Some examples of modules over a monad  $\mathbf{R}$ :

- $\mathbf{R}$  itself
- $\mathbf{R} \times \mathbf{R}$  (i.e. pairs of expressions of  $\mathbf{R}$ )
- $\mathbf{M} \times \mathbf{N}$  for any module  $\mathbf{M}$  and  $\mathbf{N}$

## Important example : Derivative of a module

- $\mathbf{R}'$  is the module defined by  $\mathbf{R}'(\mathbf{X}) = \mathbf{R}(\mathbf{X} + \{\mathbf{x}\})$  for any set  $\mathbf{X}$  of variables
- more generally, we similarly define  $\mathbf{M}'$  given a module  $\mathbf{M}$

The new variable  $\mathbf{x}$  is used to model an operation binding a variable (e.g. the lambda-abstraction).

# Examples of module morphisms

A module morphism between two modules  $M$  and  $N$  on the same monad  $R$  is a family of maps  $(f_x: M(X) \rightarrow N(X))_x$  commuting with substitution.

## Examples:

$$id_M : M \rightarrow M$$

the family of identity maps  $(id_{M(X)}: M(X) \rightarrow M(X))_X$  for any module  $\mathbf{M}$

$$app : L \times L \rightarrow L$$

the application operation of the lambda calculus monad  $\mathbf{L}$ .

## Binding variables:

In  $\lambda x.t$ , the term  $t$  depends on an additional free variable  $x$ :

If  $\lambda x.t \in L(Y)$ , then  $t \in L(Y + \{x\}) = \mathbf{L}'(\mathbf{Y})$

**abs:  $\mathbf{L}' \rightarrow \mathbf{L}$**  is a module morphism

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# Signatures

A **signature**  $\Sigma$  assigns (functorially) to each monad  $R$  a module  $\Sigma_R$  over it.

A **model** of a signature  $\Sigma$  is a monad  $R$  together with a morphism of modules  $\sigma_R : \Sigma_R \rightarrow R$ .

Models form a category (morphisms are monad morphisms commuting with  $\sigma$ ).

The **syntax generated by** a signature  $\Sigma$  is the initial object in its category of models.

This notion of signature is too general in the sense that we do not expect that this initial object always exists.

# Examples of syntax generating signatures

-  $R \mapsto R \times R$

models are monads  $R$  that comes with a module morphism  $R \times R \rightarrow R$ .  
The syntax corresponds to a language with variables and a binary operator **b**:

$\text{expr} ::= x \quad (\text{variable})$   
 $\quad \mid \mathbf{b}(t, u) \quad \text{where } t \text{ and } u \text{ are any expressions}$

-  $R \mapsto R \times R + R'$

By universal property of the disjoint sum  $+$ , models are monads  $R$  equipped with two module morphisms  $R \times R \rightarrow R$  and  $R' \rightarrow R$ .  
The syntax corresponds to lambda calculus.

# Algebraic signatures

More generally, any signature of the form  $R \mapsto R' \times R'' \times R''' + R \times R'' \times R''' \times R + \dots$  (i.e. any disjoint sum of products of finite derivatives of the monad) generates a syntax.

These **algebraic signatures** correspond to combinatorial signatures. They specify languages with  $n$ -ary operations binding a finite number of variables in their arguments.

**Our main result:** quotients of algebraic signatures also generate a syntax

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# Copie de Recursion

Question: cette section est-elle vraiment éncessaire, puisque on n'a pas vraiment de contribution là-dedans ? (ca fait partie du background initial semantics, non ?).

Benedikt suggère de ne pas faire plus d'une slide là dessus

# Recursion

Initiality of the syntax allows recursion.

## **Example: computing the set of free variables of a lambda-term**

Let **LC** be the monad of lambda-calculus.

Let  $\mathbf{t} \in \mathbf{LC}(\mathbf{X})$  be a term (whose free variables are in  $\mathbf{X}$ ). We want to compute its set of free variables  $\mathbf{fv}(\mathbf{t}) \subset \mathbf{X}$  (i.e.  $\mathbf{fv}(\mathbf{t}) \in \mathcal{P}(\mathbf{X})$ ).

### **Strategy:**

The only thing to do is to give the assignment  $\mathbf{X} \mapsto \mathcal{P}(\mathbf{X})$  the adequate structure of a monad, then of a model.

define  $\mathbf{fv} : \mathbf{LC} \rightarrow \mathcal{P}$  by initiality of LC in the category of models of its signature.

# Computing free variables

The assignement which to any set  $\mathbf{X}$  associates its power set  $\mathcal{P}(\mathbf{X})$  can be given the structure of a monad (variables are singletons, substitution is union).

$\mathbf{app}_{\mathcal{P}} : \mathcal{P}(\mathbf{X}) \times \mathcal{P}(\mathbf{X}) \rightarrow \mathcal{P}(\mathbf{X})$  and  $\mathbf{abs}_{\mathcal{P}} : \mathcal{P}(\mathbf{X} + \{\mathbf{x}\}) \rightarrow \mathcal{P}(\mathbf{X})$  should be given to yield a model. Let us study the case of  $\mathbf{app}$ :

$$fv(\mathbf{app}(t, u)) \quad \stackrel{\text{expected equation of } fv}{=} \quad fv(t) \cup fv(u)$$
  
$$\quad \quad \quad \parallel$$
  
$$\quad \quad \quad \mathbf{app}_{\mathcal{P}}(fv(t), fv(u))$$

$fv$  should be a model morphism

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$$fv(\mathbf{app}(t, u)) \quad \stackrel{\text{expected equation of } fv}{=} \quad fv(t) \cup fv(u)$$

Thus, we pose:

$$\mathbf{app}_{\mathcal{P}}(A, B) := A \cup B$$

$$\parallel$$
$$\mathbf{app}_{\mathcal{P}}(fv(t), fv(u))$$

$fv$  should be a model morphism

# Computing free variables

The case of **abs** <sub>$\mathcal{P}$</sub>  is similar.

It can be shown that **app** <sub>$\mathcal{P}$</sub>  and **abs** <sub>$\mathcal{P}$</sub>  are module morphisms, hence give the monad  $\mathcal{P}$  the structure of a model for the signature of the lambda-calculus.

By initiality of the syntax **LC**, we get a (unique) model morphism from **LC** to  $\mathcal{P}$  which satisfies:

$$\begin{aligned}fv(t\ u) &= fv(t) \cup fv(u) \\fv(\lambda x.t) &= fv(t) \setminus \{x\} \\fv(x) &= \{x\}\end{aligned}$$

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# Quotient of a signature

## Quotient of a set:

A quotient of a set  $X$  is a set  $Y$  together with a surjection  $p : X \rightarrow Y$ .

$$x \sim x' \iff p(x) = p(x')$$

## Quotient of a signature:

A quotient of a signature  $\Sigma$  is a signature  $\Psi$  together with a (natural) family of module morphisms  $(f_R : \Sigma_R \rightarrow \Psi_R)_R$  that is pointwise surjective.

A **presentable signature** is a quotient of an algebraic signature.

**Main Theorem:** Any presentable signature generates a syntax.

# Examples of presentable signatures

Presentable signatures allow to extend a syntax generated by an algebraic (or combinatorial) signature with new kinds of operations.

## **A binary commutative operation:**

as a quotient of the signature of a binary operation  $R \mapsto R \times R$  by the action of the symmetry.

## **A syntactic closure operator:**

Such an operator allows to bind a given set of variables in an expression (thus invariant under permutation of these variables).

The signature is obtained as a quotient of the algebraic signature specifying a sequence of increasingly sequential binding operators.



# Examples of presentable signatures

## Explicit substitution:

It is possible to specify an operation  $\_ \langle \mathbf{x}_i \mapsto \mathbf{t}_i \rangle$  that mimics the behavior of the true substitution  $\_ [\mathbf{x}_i \mapsto \mathbf{t}_i]$  in the sense that it enjoys some of its coherences, for example:

- if  $\mathbf{u}$  does not depend on  $\mathbf{y}$ ,

$$u \langle x \mapsto v, y \mapsto w \rangle = u \langle x \mapsto v \rangle$$

- let  $\mathbf{u}'$  be  $\mathbf{u}$  where the variables  $\mathbf{x}$  and  $\mathbf{y}$  have been swapped,

$$u' \langle x \mapsto v, y \mapsto w \rangle = u \langle x \mapsto w, y \mapsto v \rangle$$

# Examples of presentable signatures

## A coherent fixedpoint operator:

A language with (mutual) fixedpoints comes with a construction

let rec  $\mathbf{f}_1 = \mathbf{t}_1$

and  $\mathbf{f}_2 = \mathbf{t}_2$

...

and  $\mathbf{f}_n = \mathbf{t}_n$

in  $\mathbf{f}_i$

where each  $\mathbf{f}_j$  may appear as a variable in each expression  $\mathbf{t}_i$ .

Thus, it takes  $\mathbf{n}$  expressions  $\mathbf{t}_1, \dots, \mathbf{t}_n$  depending on  $\mathbf{n}$  new variables  $\mathbf{f}_1, \dots, \mathbf{f}_n$  and produces an expression which does not depend on these variables.

As such, it can be specified by an algebraic signature.

# Coherent fixedpoint operator

But we would like to encode some of the expected behaviour of such a fixed point. For instance:

$$\begin{array}{l} \text{let rec } \mathbf{f}_1 = \mathbf{t}_1 \\ \quad \text{and } \mathbf{f}_2 = \mathbf{t}_2 \\ \text{in } \mathbf{f}_1 \end{array} = \begin{array}{l} \text{let rec } \mathbf{f}_1 = \mathbf{t}'_2 \\ \quad \text{and } \mathbf{f}_2 = \mathbf{t}'_1 \\ \text{in } \mathbf{f}_1 \end{array}$$

( $\mathbf{t}'_i$  is  $\mathbf{t}_i$  where  
 $\mathbf{f}_1$  and  $\mathbf{f}_2$  have  
been swapped)

or, if  $\mathbf{t}_1$  does not depend on  $\mathbf{f}_2$ ,

$$\begin{array}{l} \text{let rec } \mathbf{f}_1 = \mathbf{t}_1 \\ \quad \text{and } \mathbf{f}_2 = \mathbf{t}_2 \\ \text{in } \mathbf{f}_1 \end{array} = \begin{array}{l} \text{let rec } \mathbf{f}_1 = \mathbf{t}_1 \\ \text{in } \mathbf{f}_1 \end{array}$$

A construction satisfying these equations can be specified by quotienting the naive algebraic signature.

# Conclusion

We found a criterion for high-level 'monadic' signatures to specify a syntax. This criterion encompasses the classical combinatorial signatures, and allows fancier operations at the level of the syntax.

## **Future work:**

We plan to take into account more sophisticated equations in the syntax than just quotients, extend our framework to simply typed syntaxes.

# FIN PROVISOIRE

Ne pas lire les slides qui suivent (ce sont des anciennes slides que je garde au cas où).

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# Copie de Purpose of our work

The specified language (or **syntax**) is characterized as the initial object in the category of models.



The initial object may not exist and therefore the signature does not specify anything!

**Goal of our work:** Identify a large class of high-level signatures (they are not exactly endofunctors in our setting) that actually specify a language.

# Copie de Introduction

**Ultimate goal:** (untyped) languages with variables and substitution

**Classical signatures:** a combinatorial specification of a language as families of lists of natural numbers:

- each list specifies an operation in the language.
- the number of its arguments is the size of the list
- each natural number in the list indicates the number of bound variables in the corresponding argument

**Example : Lambda-calculus:**

application (2 arguments)

lambda-abstraction (1 argument  
binding 1 variable)

$((0, 0), (1))$

**This work:** a general notion of signature



# Copie de Examples of presentable signatures

## A coherent fixedpoint operator:

A unary fixed point operator for a monad is an operation (or module morphism)  $\mathbf{Y}$  turning an expression  $\mathbf{e}$  depending on an additional formal variable  $\star$  into an expression  $\mathbf{Y}(\mathbf{e})$  that doesn't depend on it, such that the fixedpoint equation holds:

$$Y(e[\star \mapsto Y(e)]) = Y(e)$$

(for the lambda-calculus monad modulo beta-eta equivalence, it corresponds to a fixedpoint combinator)

# Examples of monads (à supprimer ?)

- the syntax of arithmetic expressions
- the (untyped) syntax of lambda-calculus  $L$  (*modulo alpha equivalence*)

|                       |                      |
|-----------------------|----------------------|
| $\text{expr} ::= x$   | <i>(variable)</i>    |
| $\quad   t\ u$        | <i>(application)</i> |
| $\quad   \lambda x.t$ | <i>(abstraction)</i> |

- the (untyped) syntax of lambda-calculus modulo beta-equivalence and eta-equivalence

# 'High-level' VS classical signatures

+ Our 'high-level' signatures are more abstract and contrast with 'low-level' signatures which seem quite ad-hoc.

— Our signatures, are too general: **we don't expect that all of them specify a language** (i.e. that the initial object always exist in the category of models associated to a signature).

## Goal of our work:

Identify a large class of (high-level) signatures which actually specify a language.

# Combinatorial signatures

A **combinatorial signature** is given by a family of lists of natural numbers:

- each list specifies an operation in the language.
- the number of its arguments is the size of the list
- each natural number in the list indicates the number of bound variables in the corresponding argument

## Example : Lambda-calculus:

Two operations:

application (2 arguments)

lambda-abstraction (1 argument  
binding 1 variable)

$((0, 0), (1))$

# Copie de Languages as monads

## A monad **A** as a language with variables:

- for each set  $X$ , a set  $A(X)$  of expressions taking free variables in  $X$ .
- any variable  $x \in X$  is a valid expression that we note  $\text{var}_X(x) = \underline{x} \in A(X)$
- given a family  $(t_x)_{x \in X}$  of expressions in  $A(Y)$ , we can perform for any expression **e** in **A(X)** the substitution  $e[x \mapsto t_x]$  lying in  $A(Y)$

Three monadic laws:

$$\text{COMPOSITION OF SUBSTITUTIONS} \quad e[x \mapsto t_x][y \mapsto u_y] = e[x \mapsto t_x[y \mapsto u_y]]$$

$$\text{IDENTITY SUBSTITUTION} \quad e[x \mapsto x] = e$$

$$\text{VARIABLE SUBSTITUTION} \quad \forall x \in X \quad x[y \mapsto t_y] = t_x$$

# Copie de Overview of the methodology

1. Introduce a notion of signature.
2. Construct an associated notion of model (suitable as domain of interpretation of the syntax generated by the signature). Such models form a category.
3. Define the syntax generated by a signature as its initial model, when it exists.
4. Identify a class of signatures that generate a syntax: **presentable signatures**

# Copie de Operations as module morphisms

:



For each set  $X$ , the sum of two expressions  $e, e' \in A(X)$  take free variables in  $X$ :

$$\begin{aligned}\forall X, \text{ add}_X : A(X) \times A(X) &\rightarrow A(X) \\ (e, e') &\mapsto e + e'\end{aligned}$$

Note that (*commutation with substitution*):

$$(e + e')[x \mapsto t_x] = e[x \mapsto t_x] + e'[x \mapsto t_x]$$

We characterize this situation as follows:

$A(X) \times A(X)$  expressions are "*substitutable*"   $A \times A$  is a **module** on  $A$   
 $\text{add}$  commutes with substitution   $\text{add}$  is a **module morphism**

# Examples of monads

- the assignement  $X \mapsto \mathcal{P}(X) = \{ U \mid U \subset X \}$  yields a monad  $\mathcal{P}$ .

$$\forall X, \text{var}_X : X \rightarrow \mathcal{P}(X)$$
$$x \mapsto \{x\}$$

Let  $U \subset X$  (i.e.  $U \in \mathcal{P}(X)$ ) and  $(V_x)_{x \in X}$  a family of subsets of  $Y$ .

Substitution is defined as union:

$$U[x \mapsto V_x] = \bigcup_{x \in U} V_x \in \mathcal{P}(Y)$$



# Induction

## Example: computing the free variables of a lambda-term

We compute it by induction on the syntax:

$$fv(x) = \{x\} \quad \text{(variable)}$$

$$fv(tu) = fv(t) \cup fv(u) \quad \text{(application)}$$

$$fv(\lambda x.t) = fv(t) \setminus \{x\} \quad \text{(abstraction)}$$

This is formalized in our setting as a family of maps  $(fv_x: L(X) \rightarrow \mathcal{P}(X))_x$  which *commutes with variable and substitution*:

$$\begin{aligned} fv(var_L(x)) &= \{x\} \\ &= var_{\mathcal{P}}(x) \end{aligned} \qquad \begin{aligned} fv(u[x \mapsto t_x]_L) &= \bigcup_{y \in fv(u)} t_y \\ &= fv(u)[x \mapsto fv(t_x)]_{\mathcal{P}} \end{aligned}$$

(This is a definition of a monad morphism)

# Induction

## Example: computing the free variables of a lambda-term

*fv* also commutes with 'application' and 'abstraction'

$$\begin{aligned} app_{\mathcal{P}} : \mathcal{P} \times \mathcal{P} &\rightarrow \mathcal{P} \\ (V, V') &\mapsto V \cup V' \end{aligned}$$

$$\begin{aligned} abs_{\mathcal{P}, X} : \overbrace{\mathcal{P}'(X)}^{\mathcal{P}(X + \{n\})} &\rightarrow \mathcal{P} \\ V &\mapsto V \setminus \{n\} \end{aligned}$$

Actually, these commutations **define** *fv* uniquely by induction:

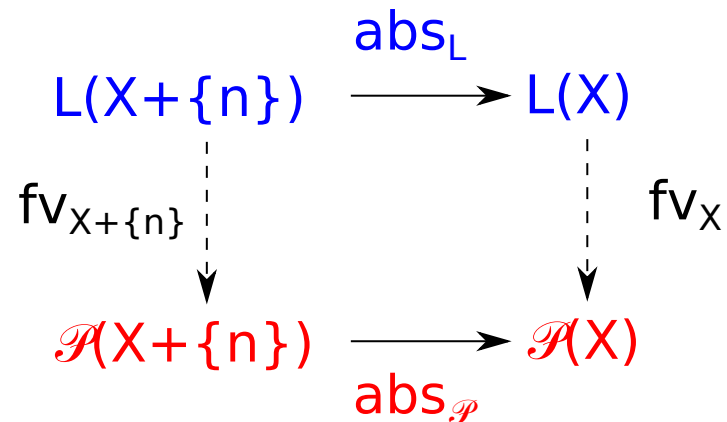
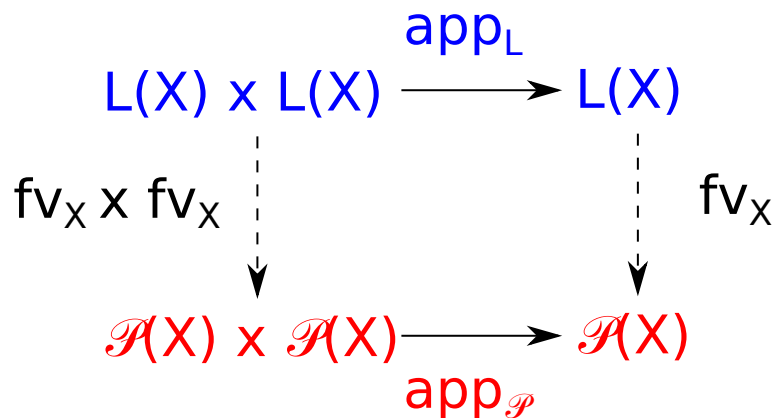
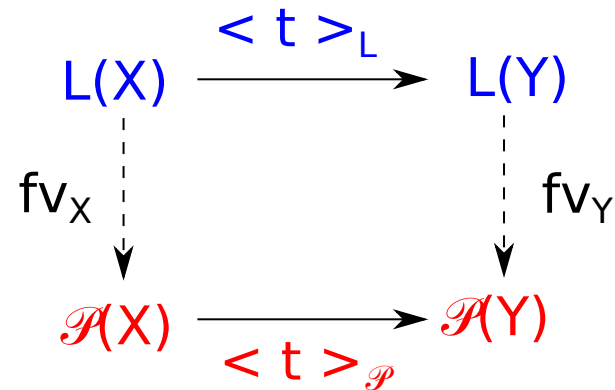
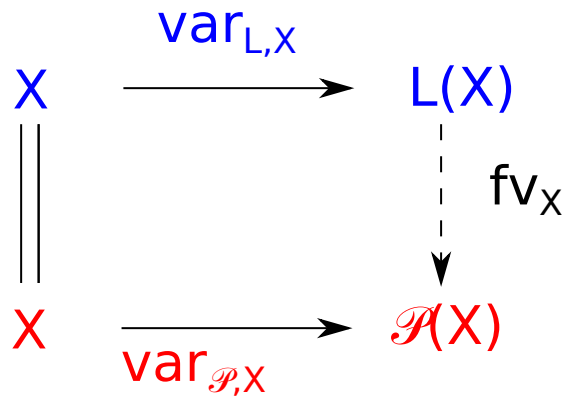
$$fv(x) = \{x\} \quad \text{(commutation with variable)}$$

$$fv(tu) = fv(t) \cup fv(u) \quad \text{(commutation with application)}$$

$$fv(\lambda x.t) = fv(t) \setminus \{x\} \quad \text{(commutation with abstraction)}$$

# Induction and initiality

$fv$  is the unique family of maps that makes the following diagrams commute:



# Induction and initiality

More generally, let  $R$  be a monad with application and abstraction.

$$X \xrightarrow{\text{var}_{R,X}} R(X)$$

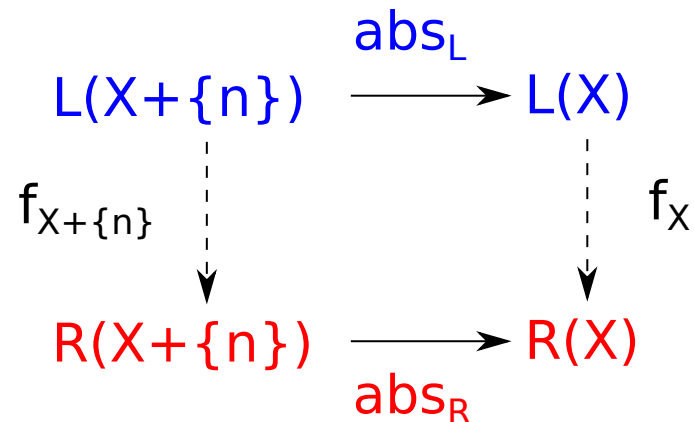
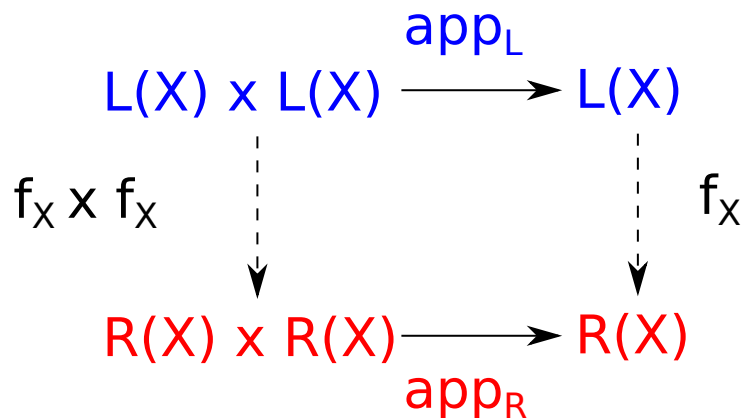
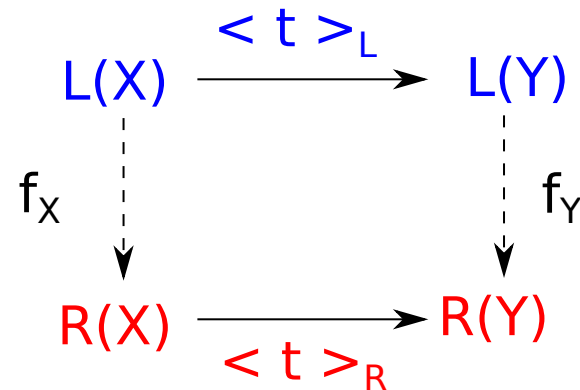
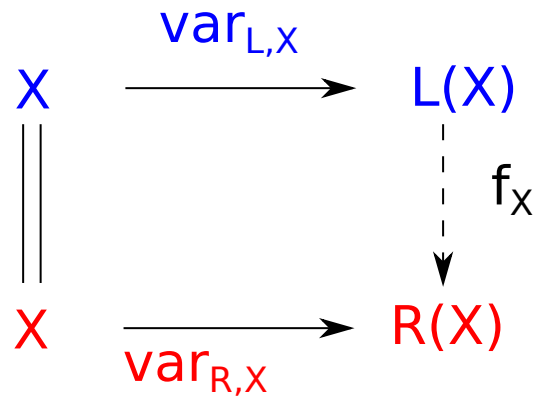
$$R(X) \xrightarrow{\langle t \rangle_R} R(Y)$$

$$R(X) \times R(X) \xrightarrow{\text{app}_R} R(X)$$

$$R(X + \{n\}) \xrightarrow{\text{abs}_R} R(X)$$

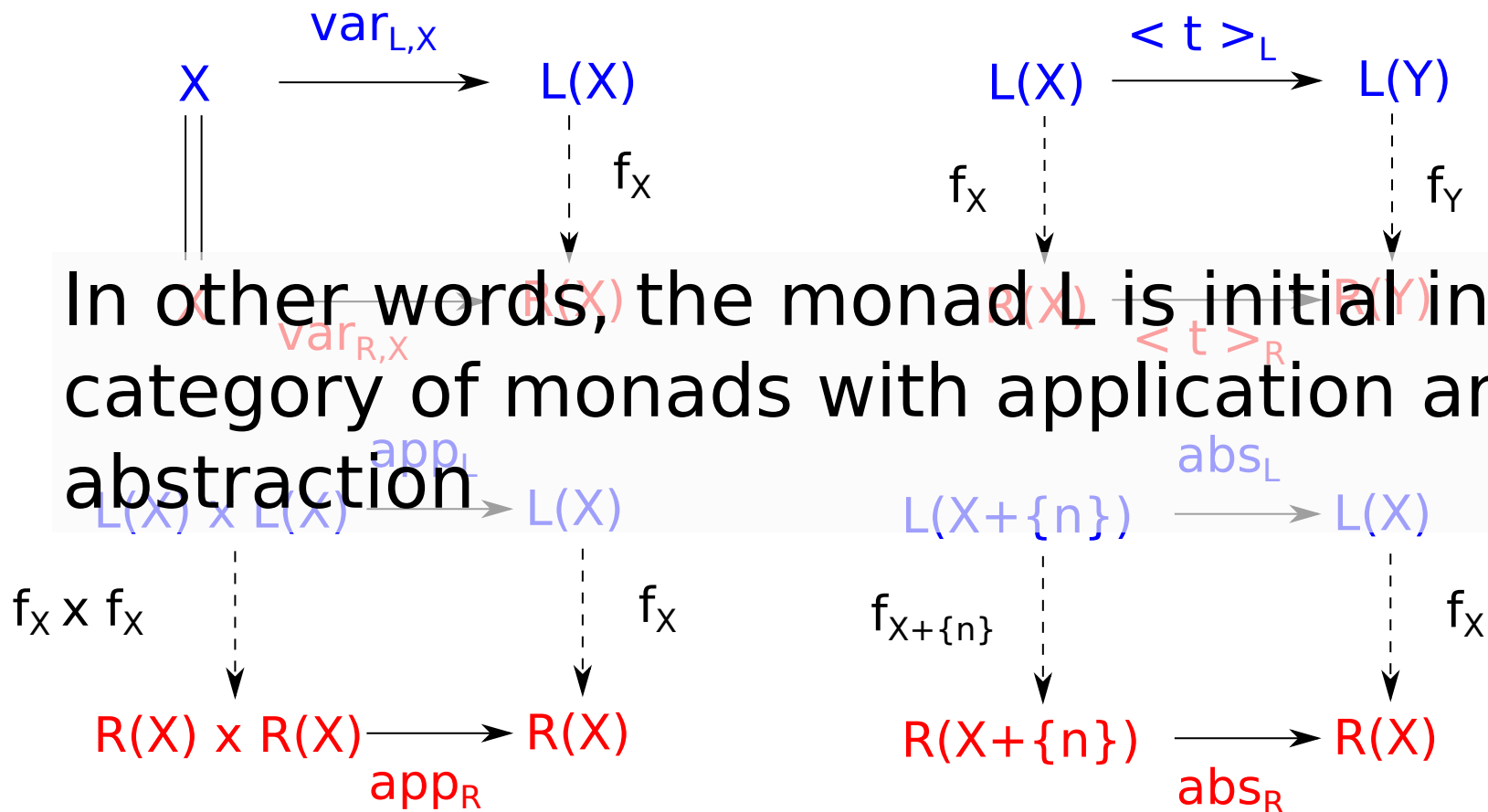
# Induction and initiality

More generally, let  $R$  be a monad with application and abstraction. Then there is a unique family  $(f_x)_x$  of maps (defined by induction) that makes the following diagrams commute:



# Induction and initiality

More generally, let  $R$  be a monad with application and abstraction. Then there is a unique family  $(\mathbf{f}_X)_X$  of maps (defined by induction) that makes the following diagrams commute:



# Syntax and initiality

## A definition of a syntax:

A **syntax** is a monad that comes with an *induction principle*, i.e. which is initial in a suitable category of *monads + operations that it implements*.

## Example:

The monad L of lambda calculus is initial in the category of *monads + application and abstraction*.

We say that L is the **syntax generated by the signature of application and abstraction**.

We will now present a general definition of **signatures**.

# Signatures

## What a signature should be:

$L$  is initial among the monads  $R$  that model the signature  $\Sigma_L$  of application and abstraction, i.e. monads  $R$  that come with module morphisms:

$$app_R : R \times R \rightarrow R$$

$$abs_R : R' \rightarrow R$$

or  $[app_R, abs_R] : R \times R + \underbrace{R'}_{\Sigma_L(R)} \rightarrow R$



A syntax  $S$  is initial among the monads  $R$  that model its associated signature  $\Sigma$ , i.e. monads  $R$  that come with a module morphism:

$$\sigma_R : \Sigma_R \rightarrow R$$

Thus, a signature  $\Sigma$  should assign to any monad  $R$  a module  $\Sigma_R$  over it.



# Signatures

Let  $\mathbf{R}$  be a monad that models the signature of application and abstraction. Then there exists a unique monad morphism  $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{R}$  which commutes with abstraction and application:

$$\begin{array}{ccc}
 L(X) \times L(X) & \xrightarrow{\text{app}_L} & L(X) \\
 \downarrow \mathbf{f}_X \times \mathbf{f}_X & & \downarrow \mathbf{f}_X \\
 R(X) \times R(X) & \xrightarrow{\text{app}_R} & R(X)
 \end{array}$$

$$f_X(\text{app}_L(t, u)) = \text{app}_R(f_X(t), f_X(u))$$



(and similarly for abs)

$$f_X(\text{abs}_L(t)) = \text{abs}_R(f_{X+\{n\}}(t))$$

Let  $\mathbf{R}$  be a monad that models a signature  $\Sigma$  (there is a module morphism  $\sigma_R : \Sigma_R \rightarrow \mathbf{R}$ ). Then there exists a unique monad morphism  $\mathbf{f} : \mathbf{S} \rightarrow \mathbf{R}$  which commutes with  $\sigma$ :

$$\begin{array}{ccc}
 \Sigma_L(X) & \xrightarrow{\sigma_L} & L(X) \\
 \downarrow \text{??} & & \downarrow \mathbf{f}_X \\
 \Sigma_R(X) & \xrightarrow{\sigma_R} & R(X)
 \end{array}$$

# Signatures

Let  $\mathbf{R}$  be a monad that models the signature of application and abstraction. Then there exists a unique monad morphism  $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{R}$  which commutes with abstraction and application:

$$\begin{array}{ccc}
 L(X) \times L(X) & \xrightarrow{\text{app}_L} & L(X) \\
 \downarrow \mathbf{f}_X \times \mathbf{f}_X & & \downarrow \mathbf{f}_X \\
 R(X) \times R(X) & \xrightarrow{\text{app}_R} & R(X)
 \end{array}$$

$$f_X(\text{app}_L(t, u)) = \text{app}_R(f_X(t), f_X(u))$$



(and similarly for abs)

$$f_X(\text{abs}_L(t)) = \text{abs}_R(f_{X+\{n\}}(t))$$

Let  $\mathbf{R}$  be a monad that models a signature  $\Sigma$  (there is a module morphism  $\sigma_R : \Sigma_R \rightarrow \mathbf{R}$ ). Then there exists a unique monad morphism  $\mathbf{f} : \mathbf{S} \rightarrow \mathbf{R}$  which commutes with  $\sigma$ :

$$\begin{array}{ccc}
 \Sigma_L(X) & \xrightarrow{\sigma_L} & L(X) \\
 \downarrow \Sigma(\mathbf{f})_X & & \downarrow \mathbf{f}_X \\
 \Sigma_R(X) & \xrightarrow{\sigma_R} & R(X)
 \end{array}$$

# Signatures

Let  $\mathbf{R}$  be a monad that models the signature of application and abstraction. Then there exists a unique monad morphism  $\mathbf{f} : \mathbf{L} \rightarrow \mathbf{R}$  which commutes with abstraction and application. Thus, a signature  $\Sigma$  assigns to any monad morphism  $\mathbf{f} : \mathbf{R} \rightarrow \mathbf{R}'$  a family of maps  $(\Sigma(\mathbf{f})_X : \Sigma_R(X) \rightarrow \Sigma_{R'}(X))_X$ .

As for module morphisms, we require that this family commutes with substitution:

$$\Sigma(\mathbf{f})_Y(e[x \mapsto t_x]_{\Sigma_R}) = \Sigma(\mathbf{f})_X(e)[x \mapsto f_X(t_x)]_{\Sigma'_R}$$

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$$\begin{array}{ccc} \Sigma_L(X) & \xrightarrow{\sigma_L} & L(X) \\ \Sigma(\mathbf{f})_X \downarrow & & \downarrow f_X \\ \Sigma_R(X) & \xrightarrow{\sigma_R} & R(X) \end{array}$$

## PLAN

1. Languages, monads and modules
2. Induction and Initiality
- 3. Signatures**

# Definition of signatures

A **signature**  $\Sigma$  is given by:

- for each monad  $R$ , a module  $\Sigma_R$  over it
- for each monad morphism  $f : R \rightarrow S$ , a family  $\Sigma(f) : \Sigma_R \rightarrow \Sigma_S$  of morphisms which commutes with substitution:

$$\Sigma(f)_Y(e[x \mapsto t_x]_{\Sigma_R}) = \Sigma(f)_X(e)[x \mapsto f_X(t_x)]_{\Sigma'_R}$$

- such that (functoriality)

$$\Sigma(f \circ g) = \Sigma(f) \circ \Sigma(g) \quad \text{and} \quad \Sigma(id_R) = id_{\Sigma_R}$$

A **model** of a signature  $\Sigma$  is a monad  $R$  together with a morphism of modules  $\sigma_R : \Sigma_R \rightarrow R$

A **model morphism** of a signature  $\Sigma$  between two models  $R$  and  $R'$  is a monad morphism  $f : R \rightarrow S$  which commutes with  $\sigma$ :  $\sigma_R \circ f = \Sigma_f \circ \sigma_{R'}$

The **syntax generated by** a signature  $\Sigma$  is its initial model.

# Syntax generated by a signature

This notion of signature is very general so that we do not expect that all of them generate a syntax.

## Examples of syntax generating signatures:

- $R \mapsto R \times R$ :

models are monads  $R$  that comes with a module morphism  $R \times R \rightarrow R$ .

The syntax corresponds to a language with variables and a binary

operator  $b$ :      $\text{expr} ::= x$             (*variable*)  
                        |  $b(t, u)$     *where  $t$  and  $u$  are any expressions*

- $R \mapsto R \times R + R'$ :

By universal property of the disjoint sum  $+$ , models are monads  $R$  equipped with two modules morphisms  $R \times R \rightarrow R$  and  $R' \rightarrow R$ .

## The syntax corresponds to lambda calculus

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