Specification of a diagrammatic reasoning tool

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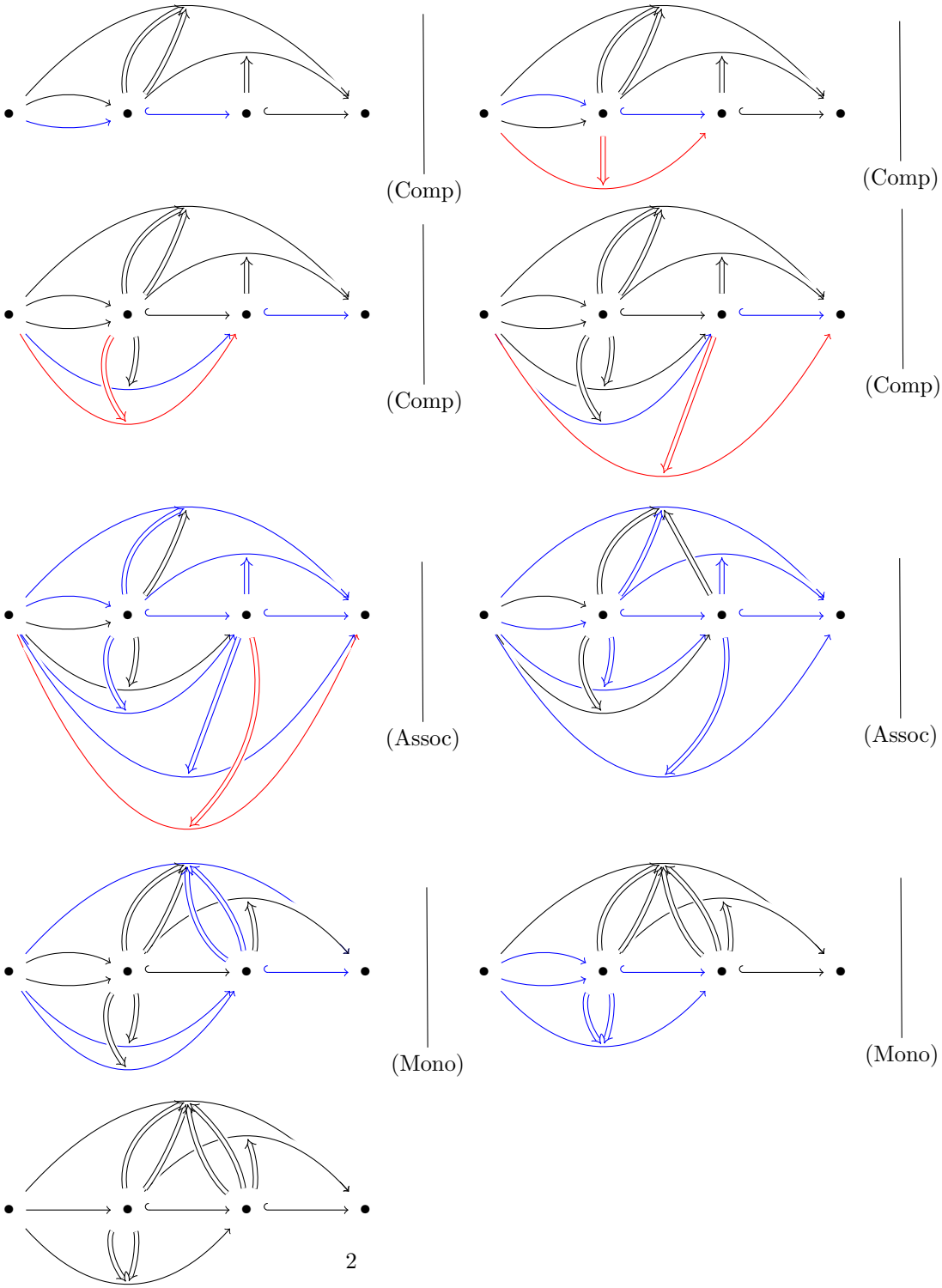
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This is a description of a tool that allows to reason graphically. The user interaction process consists of two stages:

1. an algebraic specification, that is, specification of an algebraic structure;
2. a construction mode to build elements in any model, based on some input data.

Example 0.1 (Proof that monomorphisms compose) *The algebraic structure is the structure of categories. The graphical reasoning mode would allow to construct the composite of two monomorphisms, and mark it as a monomorphism.*

Contents

1 Algebraic specification

The algebraic specification consists in two parts:

- A sort specification, consisting of a list of sort declarations that specifies what are the different sorts of elements available in the algebraic structure.
- A rewriting system specification, that is, a list of rewriting rules.

Example 1.1 *Continuing Example 0.1, the sort specification would be a list of fours sorts: objects, morphisms, commutative triangles, and monomorphisms. One rewriting rule would consist in completing any composable pair of morphism can be completed to a commutative triangle.*

From the user perspective, the algebraic specification mainly consists in drawing diagrams, where certain elements are visually "distinguished". In this document, we will use the red color, but it could be something else. Sometimes, we need to have two kinds of distinguished elements in the same diagram; for that purpose, we will use the red and blue colors.

1.1 Sort specification

The sorts are specified one after the other. Each sort is specified with a diagram, where the undistinguished elements represent the *dependencies* of the sort, while the distinguished elements determine how this new sort should be represented.

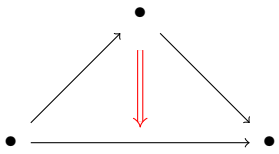
Example 1.2 *Let us detail the sort specification of Example 1.1.*

Objects *The sort has no dependencies, and we want to represent them as vertices, so the diagram would be:*

Morphisms *The sort has two dependencies: the source and target object. We want to represent a morphism as an arrow, so the diagram is:*

Monomorphisms *The sort depends on a morphism, and we want to represent it as an additional "monic" mark on the arrow.*

Commutative triangles *The sort has dependencies triangles. We can represent a commutative diagram as a 2-cell.*



We will see how to introduce proof-irrelevant sorts, such as the monomorphic tag (section 1.2.1), or functional relations, such as composition of morphisms (section 1.2.2).

1.2 First-order rewriting rules

A rewriting rule is specified as a *rewriting diagram*, where the non distinguished part is the pattern. The idea is that when this pattern is found in the workspace (we call this a *match*), the workspace can be extended with the distinguished part.

Remark 1.1 *Intuitively, the rewriting diagram is a rule that says: "if you see these (non distinguished) elements, you can add those (distinguished) elements". Logically, it is a formula* $\forall..\exists...$

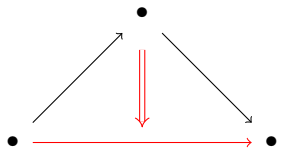
A rewriting diagram may also include equality tags between elements with the same dependencies (e.g., between two morphisms with the same source and target): those elements will be merged in the extended workspace.

Remark 1.2 *Logically, it is a formula* $\forall..\exists..P$ *where* P *is a cunjunction of equalities between variables introduced by* $\forall$*.*

Remark 1.3 *The unique existence property* $\forall\vec{x}\exists!\vec{y}..P(\vec{x})$ *is equivalent to the conjunction of* $\forall\vec{x}\exists\vec{y}P(\vec{x})$ *and* $\forall\vec{x}/P\forall\vec{y}\forall\vec{y}'\vec{y}=\vec{y'}$*, where* $\vec{x}/P$ *is a smaller vector of variables (taking into account the equations in* P*), and thus can be represented by two rewriting diagrams. The software should be able to automatically turn a unique existence rewriting diagram into two rewriting diagrams.*

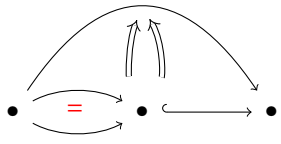
Example 1.3 *Let us detail the first-order rewriting rules of Section 1.*

Composition



(Comp)

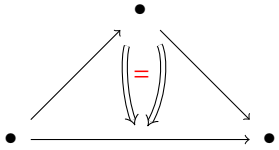
Monomorphic property



(Mono)

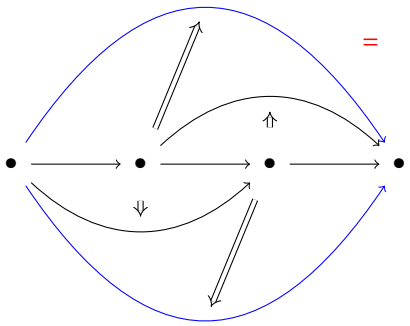
Note that there is an ambiguity here about the dependencies of the 2-cells. The interface should give a way to make them visually explicit, e.g., using a color convention as follows:

Note that we could have added for convenience[1] *an equality tag between 2-cells, but it would be more principled to have a seperate rewrite rule to make commutative triangles irrelevant:*



Alternatively, we could have a built-in mechanism to handle proof-irrelevant sorts.

Associativity of composition *We use the blue color for the specifying which elements are related by the equality tag.*



(Assoc)

Question 1.1 *What about identities? custom labelling of nodes?*

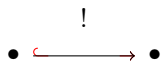
1.2.1 Proof-irrelevant sorts

Definition 1.1 *A sort is said* proof-irrelevant *if there is a rewrite rule that makes it unique, given the same dependencies.*

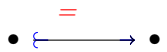
[1] This is to make the output diagram smaller.

A proof-irrelevant tag can be introduced in the sort declaration with an exclamation mark.

Example 1.4 *The monomorphism tag is proof-irrelevant.*



That is, we have the following rule, where equality relates the two monomorphism tags.

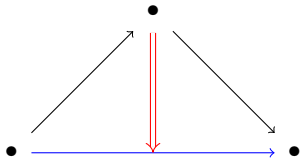


(MonoIrr)

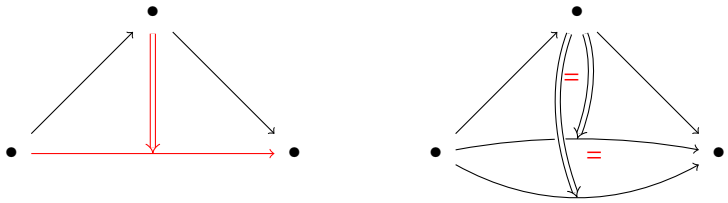
1.2.2 Functional relations

Some proof-irrelevant sorts are also used to introduce a functional relation. In that case we could compact everything in the sort declaration.

Example 1.5 *The sort witnessing composition can be declared with the following diagram.*



The red stuff is the new sort. The blue stuff is what can be constructed from the black stuff. This diagram should be understood as the combination of the following rules.



1.2.3 Creation of new admissible rewriting rules

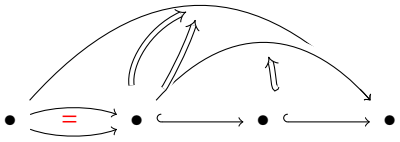
The user has two ways of creating new admissible rewriting rules, that is, rules that are valid in any model of the original set of rewrting rules.

Chaining Chaining rewriting rules;

Pruning Removing distinguished elements from a rewriting diagram.

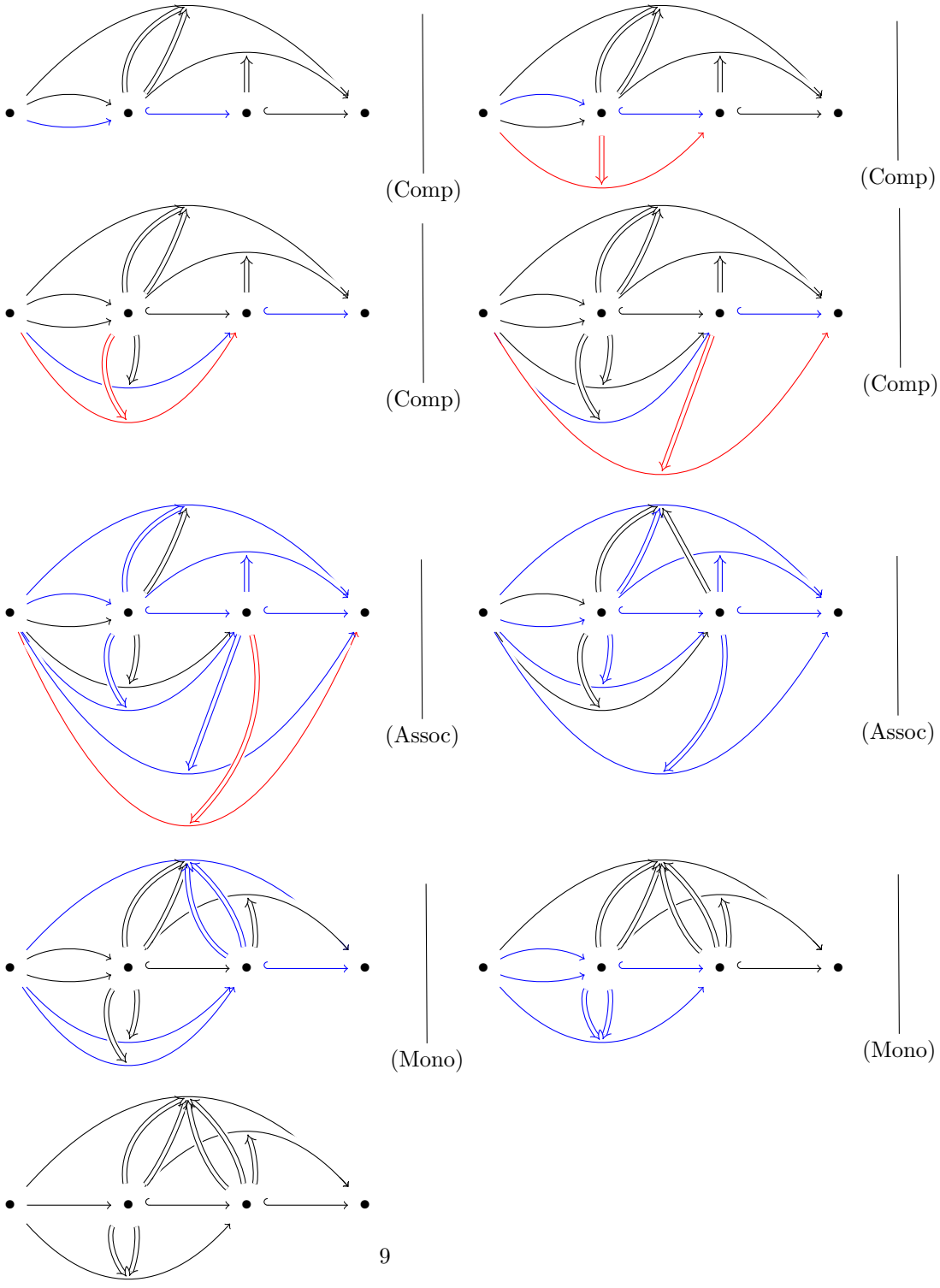
1.2.4 Example: composition of monomorphisms

Monomorphisms compose. In this section, we show that the previous set of rewriting rules is enough to build the following rewriting rule, that shows that the composition of two monomorphisms is monomorphic.



(1)

We start by chaining rewriting rules as follows. We use blue to denote a match, and the red color to show the added part, and violet (or orange) for the match and added part.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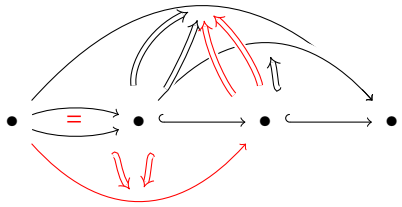
Question 1.2 *Clearly, this is too complicated... How can we make it simpler? See Appendix B for some attempts. We should offer a way to focus on some subpart of the diagram (for example, elements that depends on some selected parts), hide some stuff.*

This sequence is compactified by the software into a single rewriting rule:



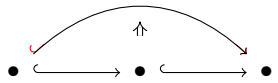
(2)

By removing distinguished elements, we recover the above rewriting rule (1).

Question 1.3 *We could have a way to automatically apply some rules when possible, like the associativity rule?*

1.3 Second-order rewriting rules

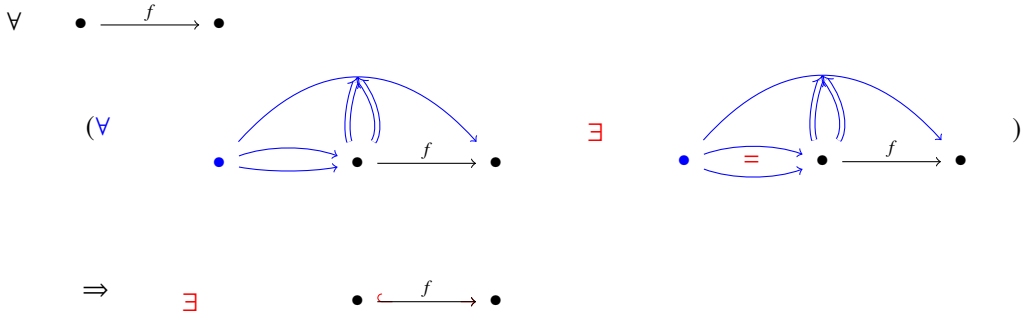
The admissible rewriting rule (1) shows that the composition of two monomorphisms is monomorphic. This makes following rewriting rule legitimate:



(MarkMono)

Unfortunately, there is no way to deduce this rule using the chaining and pruning mechanisms.

The formula that we would like to account is of the following shape:



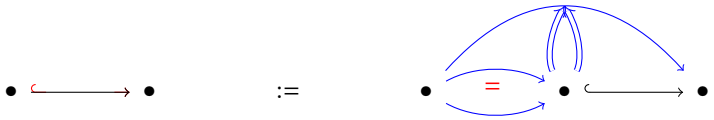
It does not fit the format $\forall..\exists..P$ of our previous rewriting rules (see Section 1.2).

This motivates what we call *second-order rewriting rules*, which handles formulae of the shape above, where the second line can occur many time. That is, we handle formulae of the shape $\forall(\forall\exists..)^*\exists...$

1.3.1 Sort definition

Second-order rules are typically useful to characterise proof-irrelevant sorts by their defining properties.

This motivates a new scheme: on the left, we have a sort declaration as in 1.1, and on the right, we have a list of rewriting rules that characterise this sort. For example we can define the sort of monomorphisms as follows:

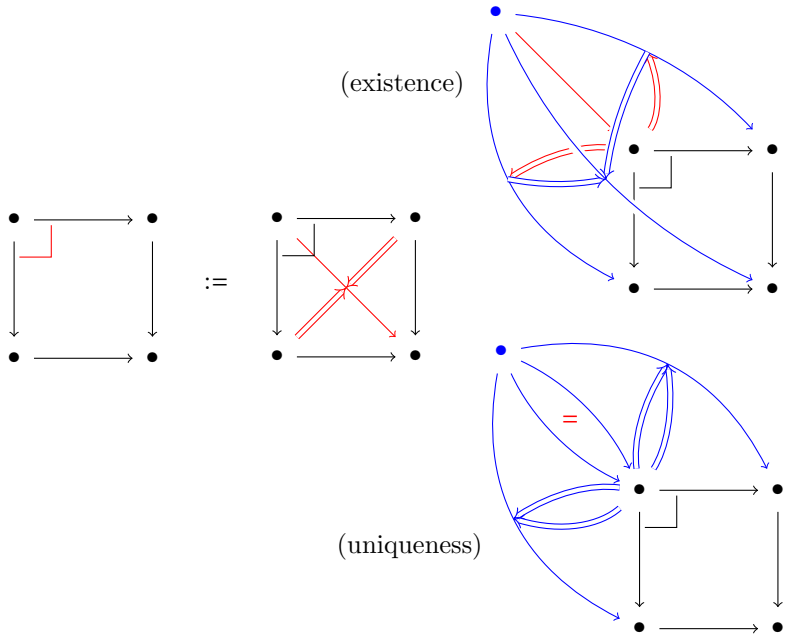


In this case we have only one rule on the right. This rule has two kinds of distinguished elements: in blue, the universally quantified, in red, the existentially quantified ones. The non-distinguished elements must be exactly the same as the left diagram.

This scheme actually defines three rules at once:

1. proof-irrelevance for monicity (rule (MonoIrr));
2. the first-order rule (Mono) that can be used when given a morphism marked as monic;
3. the second-order rule (MarkMono) that can be used to mark a morphism as monic.

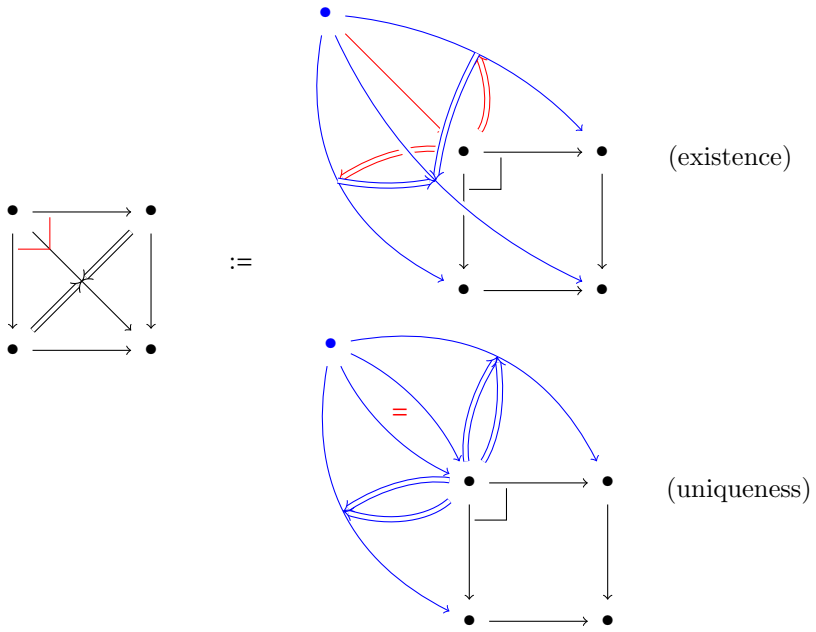
Let us do the example of pullbacks.



So here we have three rules. Note that the fact a pullback square commutes is not enforced by the dependency relation, but by the first rule.

The alternative is of course possible; then we don't need the first of the three diagrams. Let us present the other version below. Let us allow that in the rewriting rules, we can omit some non-distinguished part (here, the "inside" of

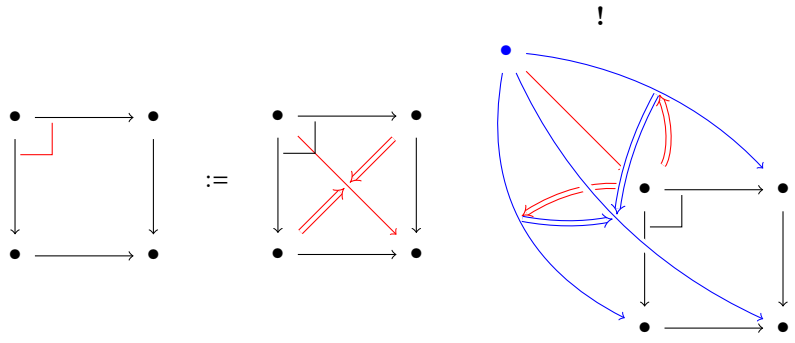
the pullback square) for legibility.



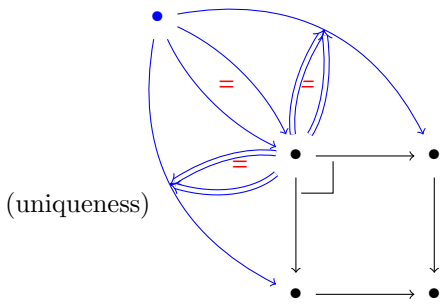
1.3.2 Uniqueness property

In a sort definition as above, we allow some of the rules to be marked as "unique" using an exclamation mark. The meaning is that it adds another rule whose underlying diagram consists of the red stuff and its dependencies, but the red stuff is duplicated. Now, turn all the red stuff into blue, and adds red equalities between pairs of duplicated blue stuff.

For example, pullbacks can be defined as follows.



The second rule marked with "!" will then generate the following new rule.



It has more equalities than the one in the original sort definition, since it adds spurious equalities between composition witnesses (two-cells), which are already proof-irrelevant.

1.3.3 Beyond sort definitions

All the examples of second-order rewriting rules we've seen so far were sort definitions. But there are also cases were it would be a bit artificial to introduce a sort, where we would like to use second-order rewriting rules, see Equation (5)

1.4 Third-order order rules and beyond

One may worry that second-order rules are not strong enough to capture all the possible rewriting rules that we want. What if we want additional nesting of quantifiers, like the third-order rewriting rule below?

$$\forall x.\, (\forall y.(\forall z \exists w.P) \Rightarrow \exists..Q) \Rightarrow \exists..R \tag{3}$$

In fact, we can "emulate" them with second-order rewriting rules with a trick: we can introduce a new sort, and use it to represent the inner quantifiers[2].

1.4.1 Reduction to second-order rewriting rules

Let us illustrate this trick with Equation (3). We introduce a new sort S depending on x, y such that

$$\forall xy\, (\exists s : S(x,y) \;\Leftrightarrow\; \forall z \exists w P) \tag{4}$$

so that Equation (3) is equivalent to the following second-order rule:

$$\forall x.\, (\forall y (s : S(x,y)) \exists..Q) \Rightarrow \exists..R$$

[2]Intriguingly enough, github copilot correctly guessed the end of this sentence, after I wrote "emulate them".

Now, let us explain how we can enforce the equivalence of Equation (4). For the left-to-right implication, a first-order rewriting rule is enough:

$$\forall xy(s : S(x, y))z \ \exists w.P$$

For the reverse implication, we can write a second order rewriting rule:

$$\forall xy(\forall z \exists w.P) \ \Rightarrow \exists s : S(x, y)$$

1.4.2 The second-order thesis

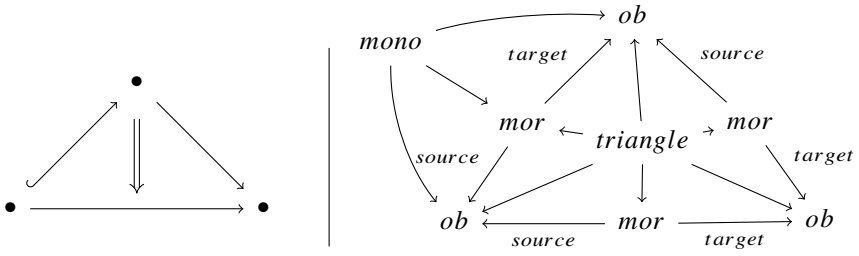
Not only can third-order rewriting rules be emulated with second-order rules and new sorts, as shown by the previous section, but we claim that this is what mathematicians actually do in practice: they never work with third-order statements; they would rather introduce an intermediate definition to flatten any statement. This claim is what we call the *second-order thesis*. Happy to hear if you have any counter-example.

2 Semantics

A sort specification corresponds to a finite direct category. A diagram with distinguished elements corresponds to a natural transformation between two functors from this category to the category of sets.

3 Internal representation

A diagram is stored as a graph: each element in the graph is a node, and each dependency is an edge[3]. For example, the following diagram below left is represented as the graph below right:



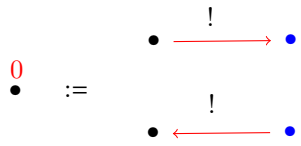
Looking for a match can be done at the level of this graph representation.

A Abelian categories

Objects and morphisms, pullbacks, epimorphisms.

[3]Semantically, this is the graph underlying the category of elements of the presheaf.

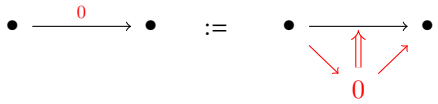
A.1 Zero



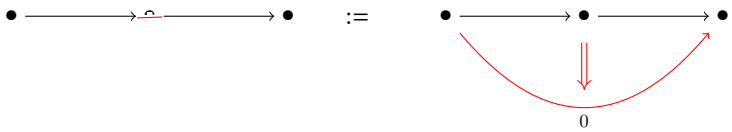
Note that this sort definition enforces both existence and uniqueness of the initial/terminal morphism.

To simplify the notations, we will mark a zero object with a label "0" in the following diagrams. The software should provide a way to ensure this.

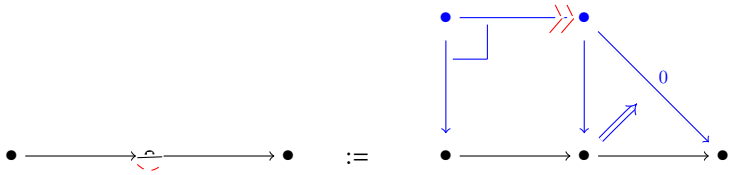
Let us define zero morphisms.



A.2 Chain complex



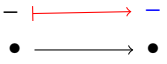
A.3 Exact



A.4 Diagram chasing rules

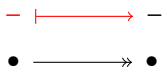
The following rules provide some rules for diagram chasing (it is by no mean exhaustive).

A.4.1 Sort specification

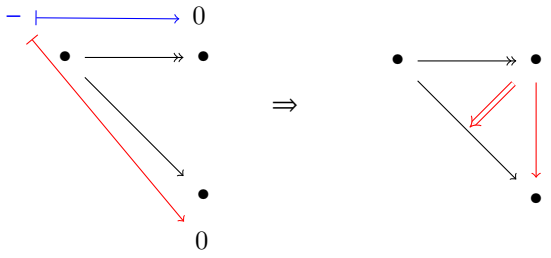


(functional relation)

A.4.2 Rewriting rules



And now this is a second-order rewriting rule.

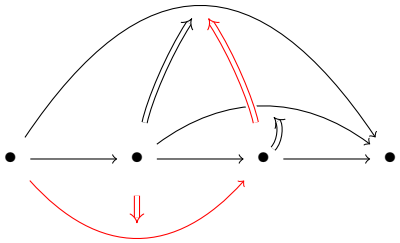


(5)

B Attempts to simplify the proof that monomorphisms compose

The last attempt is pretty convincing; it we require tactics.

Simplifying the above example First we can construct the following associativity rule.



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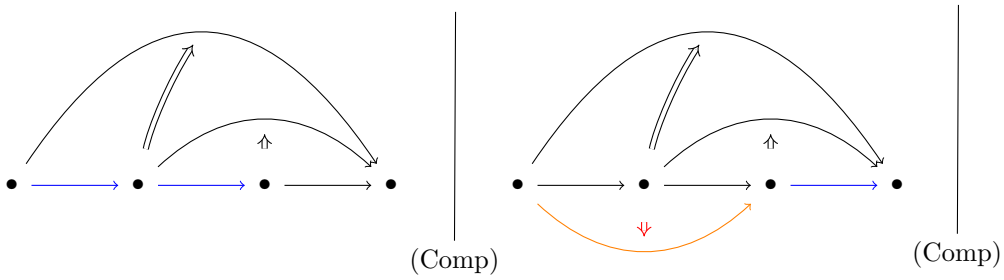
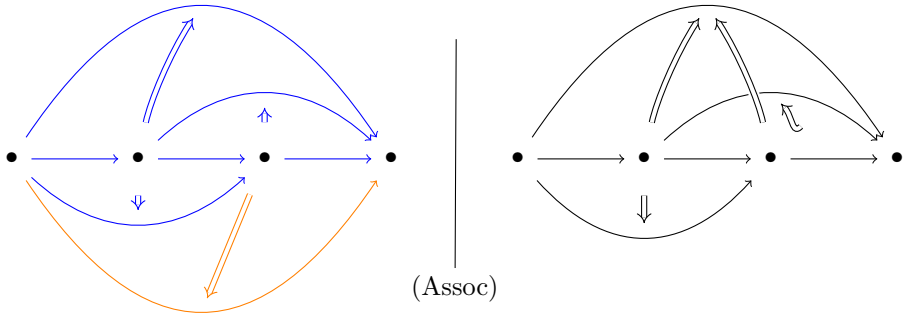
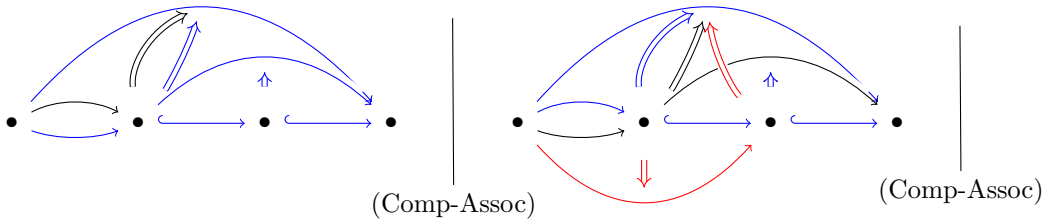
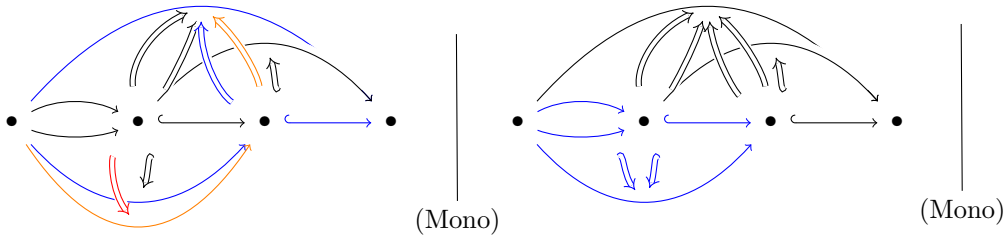
Let us derive it more carefully.

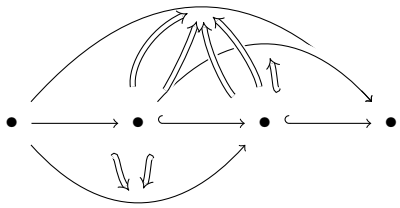




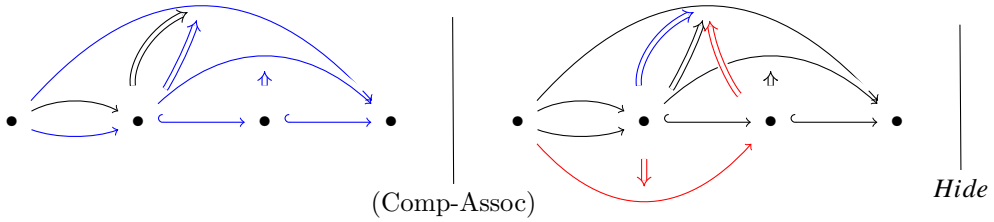
We present a second version of the long proof of (2).

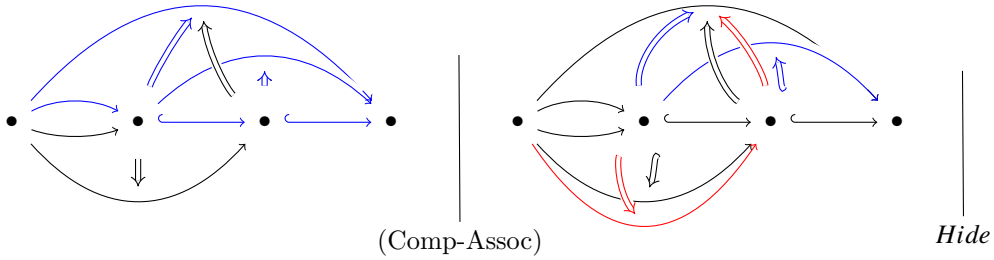

(Comp-Assoc)
(Comp-Assoc)

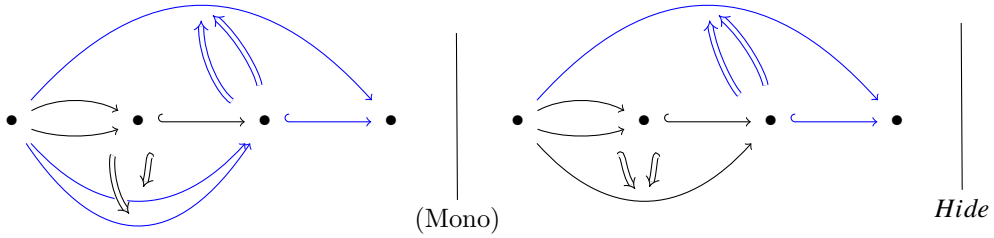

(Mono)
(Mono)

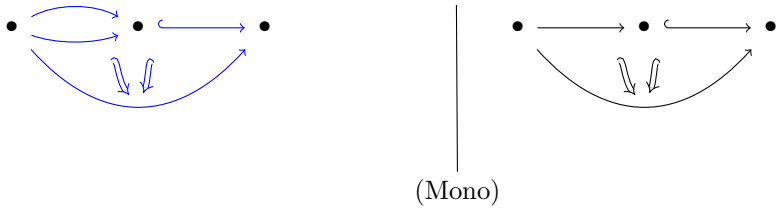


A version with a hide primitive

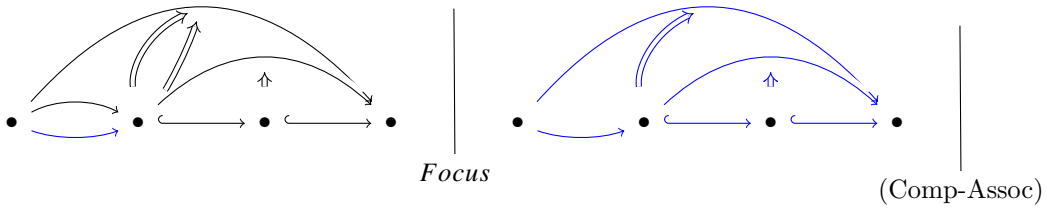


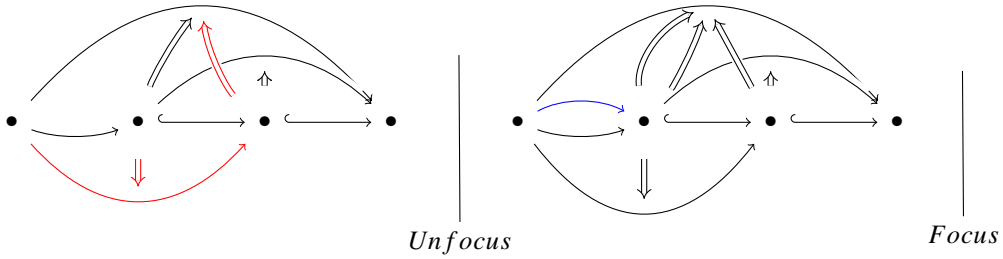


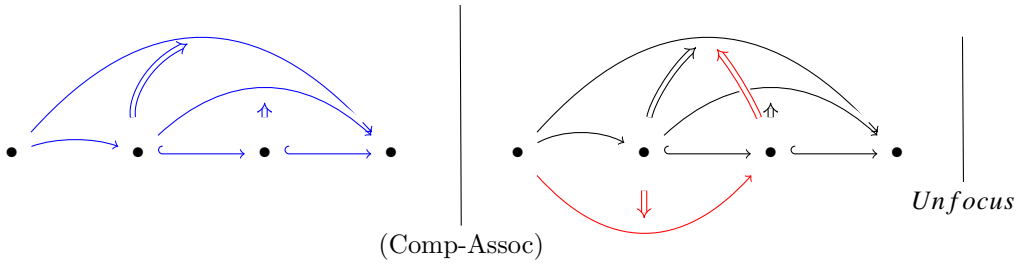


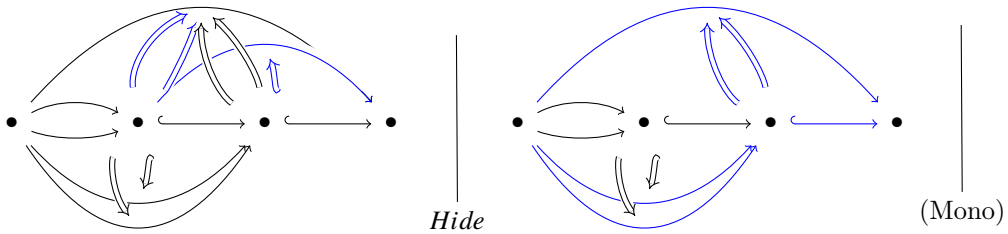


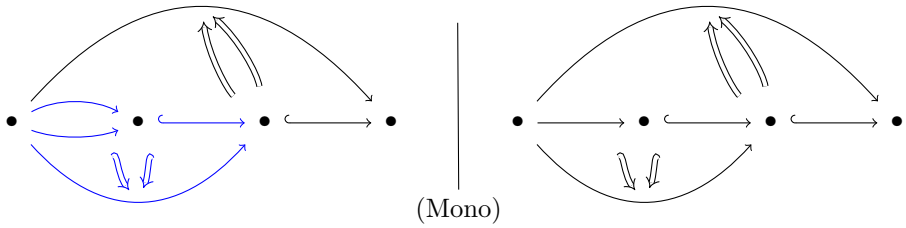
A version with hide and focus primitives The idea is that we keep all the elements that depend on the selected elements, and hide the rest.


Focus
(Comp-Assoc)


Unfocus
Focus


(Comp-Assoc)
Unfocus


Hide
(Mono)


(Mono)

Maybe names could help, to remember stuff...

A version with hide tactical We assume that we have a tactic Apply-Hide that takes as input a rewriting rule, and a set of elements to hide. It will apply the rewriting rules greedily on the selected elements, and then hide the objects.