

# Specification of a diagrammatic reasoning tool

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This is a description of a tool that allows to reason graphically. The user interaction process consists of two stages:

1. an algebraic specification, that is, specification of an algebraic structure;
2. a construction mode to build elements in any model, based on some input data.

**Example 0.1 (Proof that monomorphisms compose)** *The algebraic structure is the structure of categories. The graphical reasoning mode would allow to construct the composite of two monomorphisms, and mark it as a monomorphism.*

## 1 Algebraic specification

The algebraic specification consists in two parts:

- A sort specification, that specifies what are the different sorts of elements available in the algebraic structure.
- A rewriting system specification, that is, a list of rewriting rules.

**Example 1.1** *Continuing Example 0.1, the sort specification would be a list of four sorts: objects, morphisms, commutative triangles, and monomorphisms (which form a **subsort** of morphisms). One rewriting rule would consist in completing any composable pair of morphism can be completed to a commutative triangle.*

From the user perspective, the algebraic specification mainly consists in drawing diagrams, where certain elements are visually "distinguished" (in this document, we will use the red color, but it could be something else).

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## 1.1 Sort specification

The sorts are specified one after the other. Each sort is specified with a diagram, where the undistinguished elements represent the *dependencies* of the sort, while the distinguished elements determine how this new sort should be represented.

**Example 1.2** *Let us detail the sort specification of Example 1.1.*

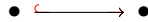
**Objects** *The sort has no dependencies, and we want to represent them as vertices, so the diagram would be:*



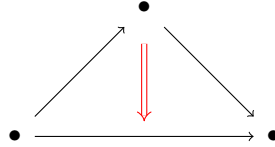
**Morphisms** *The sort has two dependencies: the source and target object. We want to represent a morphism as an arrow, so the diagram is:*



**Monomorphisms** *The sort depends on a morphism, and we want to represent it as an additional "monic" mark on the arrow.*



**Commutative triangles** *The sort has dependencies triangles. We can represent a commutative diagram as a 2-cell.*



## 1.2 Rewriting rules

### 1.2.1 First-order rules

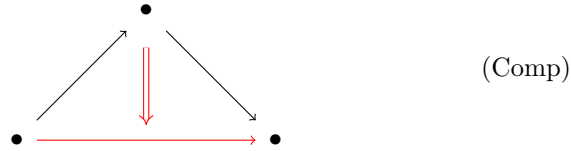
A rewriting rule is specified as a *rewriting diagram*, where the non distinguished part is the pattern. The idea is that when this pattern is found in the workspace

(we call this a *match*), the workspace can be extended with the distinguished part.

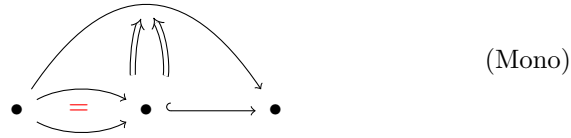
A rewriting diagram may also include equality tags between elements with the same dependencies (e.g., between two morphisms with the same source and target): those elements will be merged in the extended workspace.

**Example 1.3** *Let us detail the first-order rewriting rules of Example 1.1.*

### Composition



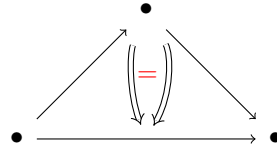
### Monomorphic property



*Note that there is an ambiguity here about the dependencies of the 2-cells. The interface should give a way to make them visually explicit, e.g., using a color convention as follows:*



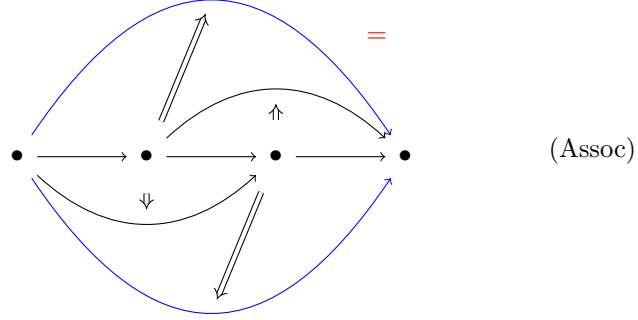
*Note that we could have added for convenience<sup>1</sup> an equality tag between 2-cells, but it would be more principled to have a separate rewrite rule to make commutative triangles irrelevant:*



*Alternatively, we could have a built-in mechanism to handle proof-irrelevant sorts.*

<sup>1</sup>This is to make the output diagram smaller.

**Associativity of composition** We use the blue color for the specifying which elements are related by the equality tag.



**Question 1.1** What about identities? custom labelling of nodes?

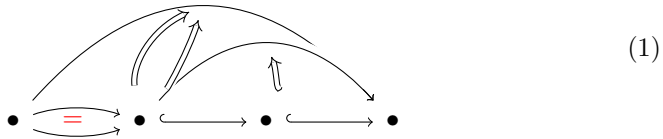
The user has two ways of creating new admissible rewriting rules, that is, rules that are valid in any model of the original set of rewriting rules.

**Chaining** Chaining rewriting rules;

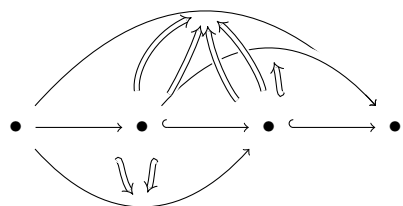
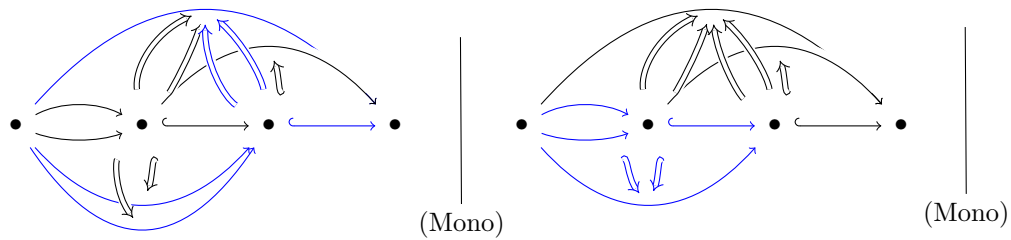
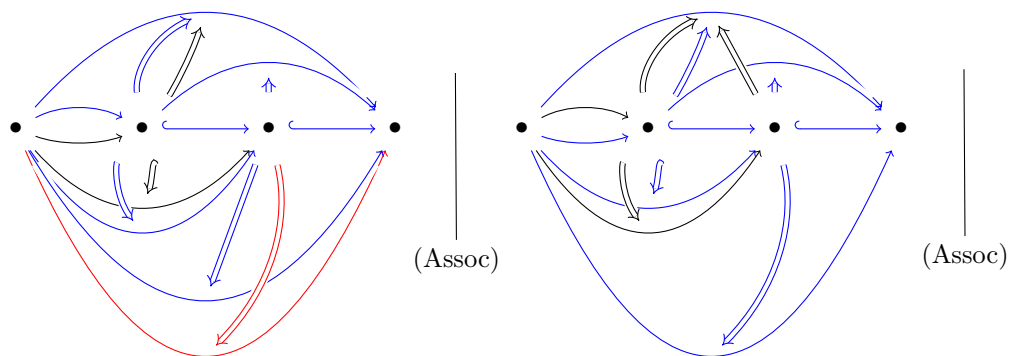
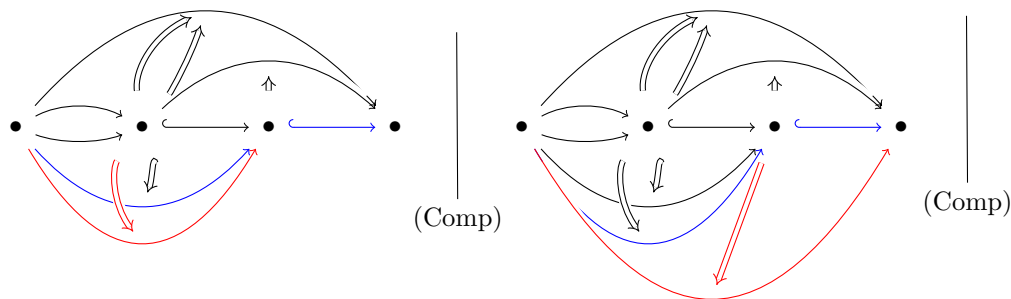
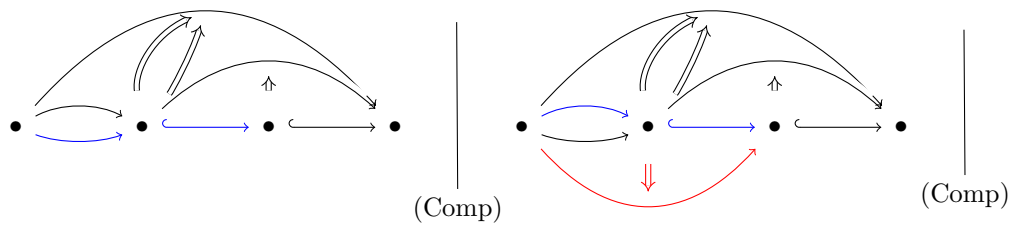
**Pruning** Removing distinguished elements from a rewriting diagram.

**Example: the composition of two monomorphisms is monomorphic**

In this section, we show that the previous set of rewriting rules is enough to build the following rewriting rule, that shows that the composition of two monomorphisms is monomorphic.

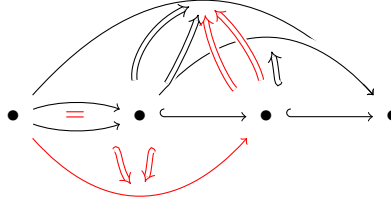


We start by chaining rewriting rules as follows. We use blue to denote a match, and the red color to show the added part, and violet for the match and added part.



**Question 1.2** Clearly, this is too complicated... How can we make it simpler? We should offer a way to focus on some subpart of the diagram (for example, elements that depends on some selected parts), hide some stuff.

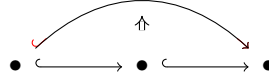
This sequence is compactified by the software into a single rewriting rule:



By removing distinguished elements, we recover the above rewriting rule (1).

### 1.3 Second-order rewriting rules

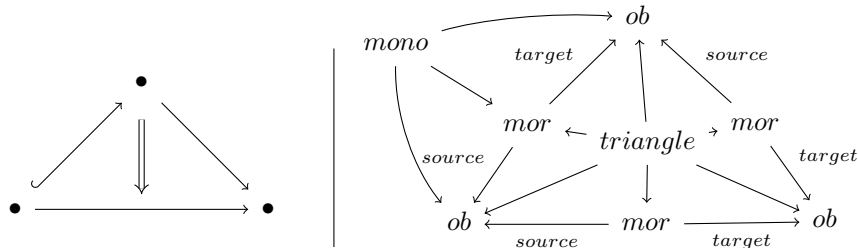
The admissible rewriting rule (1) shows that the composition of two monomorphisms is monomorphic. This makes following rewriting rule legitimate:



Unfortunately, there is no way to deduce this rule using the chaining and pruning mechanisms. This motivates what we call *second-order rewriting rules*. TODO

## 2 Internal representation

A diagram is stored as a graph: each element in the graph is a node, and each dependency is an edge. For example, the following diagram below left is represented as the graph below right:



Looking for a match can be done at the level of this graph representation.

### 3 Semantics

A sort specification corresponds to a finite direct category. A diagram with distinguished elements corresponds to a natural transformation between two functors from this category to the category of sets.