A note on (locally) presentable categories

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Contents

A *locally presentable category* (or just *presentable* category) is a category which is not too big and not too small: it is cocomplete, but is the generated by a small set of objects. The main reference for this beast is [1].

A morphism between presentable categories is defined as cocontinuous functor in Anel's and Joyal's Topo-logie.

Presentable categories can be defined as cocomplete *accessible categories*, or as complete accessible categories (after nlab, the page on accessible categories and locally presentable categories).

An accessible category is defined as a λ-accessible category (you don't need to know what it is at this stage) for some regular cardinal λ. This a notion of λ-presentable category, and a presentable categories is a λ-presentable category, for some regular cardinal λ. Any λ-presentable category is also μ-presentable for any regular cardinal $\mu > \lambda$. This is not the case for accessible categories! We only know, for them, that given any regular cardinal λ, there exists $\kappa > \lambda$ such that any λ-accessible category is also κ-accessible [1, Theorem 2.11] (such a condition is noted $\lambda \triangleleft \kappa$ and λ is said sharply smaller, and it is transitive). In particular, $\omega \triangleleft \lambda$ for any (infinite) regular cardinal λ, and a regular cardinal is always sharply smaller than its successor. For each set of regular cardinals, there is one which is sharply greater than all of them. A similar thing can be said of accessible functors.

Presentable categories satisfy nice properties.

Definition 0.1. *A functor is said* ***to have a rank*** *if it has a rank λ, for some regular cardinal λ, that is, if it preserves λ-directed colimits. A functor is said (λ-)**accessible** if it has a rank (λ) and the domain/codomain are (λ-)accessible.*

Theorem 0.2. *[1, 1.66] A functor between presentable categories is right adjoint iff it is continuous and has a rank.*

A similar version is true of left adjointness according to nlab (wihtout the accessibility condition because it is automatically satisfied). In fact, by looking at the proof of [1, Representation theorem 1.45], it seems that any cocontinous functor out of a presentable category has a right adjoint (because any presentable category is of the shape $\mathrm{Cont}_\lambda(A)$, and in the proof mentioned above, they say that the cocontinuous induced functor to a cocomplete category has a left adjoint obtained easily using yoneda. I suspect cocompleteness of the target is not necessary here).

Remark 0.3. *Anticipating the fact that any presentable category is of the shape $Cont_\lambda(A)$, a right adjoint to a presentable categorie is always a (kind of) nerve functor. This is seen, by postcomposing with the right adjoint inclusion in $\hat{A}$, and using yoneda to get the definition of this right adjoint.*

Theorem 0.4 (nlab)**.** *Let T be an accessible monad on a presentable category C. Then C^T is presentable.*

1 Regular cardinals

Definition 1.1. *A cardinal is regular if it is strictly greater than any smaller union of smaller cardinals.*

Any successor cardinal is regular. The existence of regular uncountable limit cardinal (called weakly inacessible cardinal) is not known to be consistent with ZFC. It is consistent to assume that any uncountable limit cardinal is not regular.

2 Filtered/directed colimits

A λ-filtered category is a category C in which any diagram $\mathsf{D} \to \mathsf{C}$ with D λ-small has a cocone. A λ-directed category is a poset which induces a λ-filtered category.

For any λ-filtered category C, there is a λ-directed one D with a cofinal functor $\mathsf{D} \to \mathsf{C}$. The existence of a cofinal functor means that colimits with respect to C is the same as colimits with respect to D.

That is why we can restrict ourselves to λ-directed colimits, rather than the more general λ-filtered colimits. We can also restrict to chains for ω-filtered colimits, although there is no cofinality argument (see [1, Example 1.8]).

Theorem 2.1. *[1, Corollary 1.7]: A category has ω-filtered (or ω-directed) colimits iff it has colimits of chains. For such categories K, a functor of domain K preserve ω-filtered colimits iff it preserves colimits of chains.*

3 Presentable categories

Let λ be a regular cardinal.

Definition 3.1. *A λ-presentable category is a category C which is a free λ-cocompletion $Cont_\lambda(Pres_\lambda(C))$ of some small full subcategory $Pres_\lambda(C)$ [1, Representation theorem 1.46].. Then, $Pres_\lambda(C)$ consist of* λ-presentableobjects

Here, in this free λ-cocompletion, the right adjoint would start from cocomplete (not λ-cocomplete!) categories and go to the category of categories and λ-cocontinous functors (i.e., functors preserving any λ-small colimit, not only the directed ones).

Remark 3.2. *$Cont_\lambda$ is obtained by restricting the category of presheaves to those preserving λ-small limits (as the yonedas do). Beware that colimits do not compute as in the total presheaf category (I can't find an argument for its cocompleteness in [1] by the way). Is any functor still a canonical colimit of yonedas in $Cont_\lambda$? The induced cocomplete functor is obtained by restricting the universal functor from $\hat{A}$ (because of the universal property of $\hat{A}$. Note: we get a cocomplete functor from $\hat{A}$ to $Cont_\lambda$ by universal property of $\hat{A}$. This is obviously the left adjoint to the inclusion functor (indeed, any cocontinuous functor out of $\hat{A}$ has a left adjoint, using yoneda to get the value of it)*

Compare with the characterization of λ-accessible category:

Definition 3.3. *A λ-accessible category is a cocompletion of some small full subcategory with respect to λ-directed colimits [1, Representation theorem 2.26].*

Here, we think of an adjunction between categories and categories with λ-directed colimits and functors preserving them. The free stuff is obtained by restricting the category of presheaves to λ-directed colimits of yoneds.

Question 3.4. *Is it clear that a λ-presentable category is λ-accessible with these characterizations? Not really...*

Alternative and (not)

Definition 3.5.

Definition 3.6. *A λ-**presentable object** in a category C is an object such that its coyoneda embedding preserves λ-directed colimits.*

It says no more that any morphism from such an object to a λ-directed colimit factors (essentially uniquely) through some object of the colimiting cocone.

Lemma 1. *A λ-presentable object is also κ-presentable, for $\kappa > \lambda$.*

Proof. Obvious, because a κ-filtered category is also λ-filtered. □

Note that for accessible categories, we have a similar statement [1, 2.26] where the adjunction happens between the category of λ-directed cocomplete categories and the usual category of categories.

Question 3.7. *Can you show with these characterizations that a presentable category is accessible?*

Definition 3.8 (Alternative definition)**.** *A λ-accessible category is defined as a category closed under λ-directed colimit and there is a small set of λ-presentable objects such that every object is a λ-directed colimit of objects of A.*

Definition 3.9. *(Another def [1, 1.20]) A locally λ-presentable category is a cocomplete category with a strong generator formed of λ-presentable objects.*

To go from the strong generator to the set of λ-presentable objects above: take λ-small colimits of objects in the strong generator.

Question 3.10. *How do you show the previous characterization of accessible categories?*

Lemma 2. *A locally λ-presentable category is also locally κ-presentable, for $\kappa > \lambda$.*

Proof. We consider the alternative definition of locally presentable category. The set of "presenting" κ-presentable objects is induced by noticing that λ-presentable objects are also κ-presentable, and form a strong generator. Then, we take κ-small colimits of them. □

3.1 Cauchy completion

(let us recall that the free split idempotency is called cauchy completion and gives a criterion to compare presheaf categories)

Theorem 3.11. *[1, 2.4] Each accessible category has split idempotents.*

Theorem 3.12. *Every small category with split idempotents is accessible.*

4 Finitely presentable categories

Any presheaf category is finitely presentable [1]: the representable functors form the strong generator, and thus the presentable objects are the finite colimits of these representable functors.

5 Gabriel-Ulmer

This section is based on Exercise 1.s and Remark 1.46 of [1].

Definition 5.1.

Theorem 5.2. *For each regular cardinal λ, we denote*

- *$c_\lambda CAT$ the category of small categories λ-complete and functor preserving these limits;*
- *$lp_\lambda CAT$ the category of λ-presentable categories and continuous and λ-accessible functors.*

Then, the following functors are equivalences:

$$\begin{aligned} R : c_\lambda CAT &\to lp_\lambda CAT \\ C &\mapsto Cont_\lambda C^o \\ L : lp_\lambda CAT &\to c_\lambda CAT \\ C &\mapsto Pres_\lambda(C)^o \end{aligned}$$

Actually, they even induce a biequivalence ($c_\lambda CAT$ and $lp_\lambda CAT$ can be equipped with a 2-categorical structure).

Question 5.3. *Is it a strict 2-equivalence? I guess no, otherwise it would have been said.*

This theorem induces two (equivalent) contravariant functors from the poset of regular cardinals (seen as a category) to the category of small categories. Performing the Grothendieck construction yield an equivalence between some category of presentable categories and some category of small categories (fibered over the category of regular cardinals).

6 Sketches

Definition 6.1. *A* ***sketch*** *is a small category C with a family of (small) diagrams $F_i : D_i \to C$ together with a choice of a cone or a cocone for each of this diagram.*

Definition 6.2. *The* ***category of models of a sketch*** $(C, (F_i, c_i)_i)$ *is the full subcategory of $[C, Set]$ mapping any mapping any chosen cones and cocones to limits and colimits.*

Theorem 6.3. *Any accessible category is equivalent to the category of models of a sketch.*

Theorem 6.4. *Any presentable category is equivalent to the category of models of a* ***limit sketch****, that is, a sketch with only cones (and no cocones).*

Question 6.5. *How do you retrieve the regular cardinal from this description? Probably, it is some cardinal bound over the size of the diagrams?*

Theorem 6.6. *(nlab) A category is λ-accessible iff it is equivalent to a full subcategory of a presheaf category closed under λ-filtered colimits.*

Question 6.7. *How does the raising of the indexing regular cardinal translates into this definition?*

Theorem 6.8. *A category is presentable if it is equivalent to a reflective subcategory of a presheaf category.*

Note that a accessible category is presentable precisely if the embedding into the presheaf category has a left adjoint, indeed:

Theorem 6.9. *[2, Proposition 2.4.8] or [1, 2.23] Any right adjoint functor between accessible categories is accessible.*

Theorem 6.10. *[1, Corollary 2.62] For any sketch, there exists another one whose cones/cocones are all limiting/colimiting and has an equivalent category of models.*

By Gabriel-Ulmer duality, we already know that for limit sketches: you can take the λ-complete category corresponding to the presentable category and take all the λ-small limits as cones.

Question 6.11. *What does exactly this construction on a sketch S do?*

7 GATs and EATs

generalized algebraic theories (or gats) are defined in [3], and also more formally (and recently) in [4]

7.1 QIITs are (particular) GATs

It is not exactly clear what are GATs introduced by [3], in particular what is the status of equations of sorts. GATs are like QIITs but with a possible infinite context. More concretely, a GAT consists of a possibly infinite set of axiomatic judgments, and each judgement must be wellformed (with respect to a finite subset of axioms). Infinitary QIITs allow to capture some of them, but not all.

Question 7.1. *Can we find a counter example?*

Also, in QIITs, we can have equality constructors taking equalities in arguments, that we cannot do directly with GATs. But fortunately, such constructors can be encoded by introducing new types in the universe which are the limits of other ones (such incorporating the equations).

Question 7.2. *[3] claims that GATs are equivalent to EATs, but how to deal with equations of sorts? How are they converted in the EAT version ?*

Cartmell also proves that they are equivalent to contextual categories, and for this he needs these sorts equations (see (iii) at the top of p107 of his PhD manuscript).

Taichi sees two difficulties when trying to replace sort equations with isomorphisms:

- If we replace a sort equation $A = B$ with a sort isomorphism $e : A \simeq B$, we also have to replace occurences of coercion along $A = B$ with applications of e.
- A naive replacement can create a different GAT. For example, consider the GAT consisting of a constant sort A and equation $A = A$ (although this equation is redundant, it is a valid GAT). A model of $(A, A = A)$ is just a set, but a model of $(A, e : A \simeq A)$ is a set equipped with an automorphism. They do not seem to be equivalent.

I expect that 1. is possible, although I confess I don't know how to do it in general. For the second point, I would suggest the following: given any sort equation $A = A$ that you can prove in the theory induced by a GAT with sort equations, you get (I hope) an automorphism on the translation of A (in the translated GAT which turns sort equations into isomorphisms). Then, for each such sort equation, add in the translated GAT that this automorphism should be the identity.

7.2 QIITs are QITs (as GATs are EATs)

Example: $Con : \mathcal{U}, Ty : Con \to \mathcal{U}$ is translated as $Con : \mathcal{U}, \int Ty : \mathcal{U}, con : \int Ty \to Con$ (this is the idea of translating GATs into EATS, as in [3]).

Question 7.3. *Maybe this generalizes to HIITs and HITs?*

8 Equational systems?

References

[1] J. Adamek and J. Rosicky. *Locally Presentable and Accessible Categories*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1994.

[2] Andreas Blass. Michael makkai and robert paré. accessible categories: the foundations of categorical model theory. contemporary mathematics, vol. 104. american mathematical society, providence1989, viii 176 pp. *Journal of Symbolic Logic*, 58(1):355–357, 1993.

[3] John Cartmell. Generalised algebraic theories and contextual categories. *Annals of Pure and Applied Logic*, 32:209 – 243, 1986.

[4] Richard Garner. Combinatorial structure of type dependency. *Journal of Pure and Applied Algebra*, 219(6):1885 – 1914, 2015.