# A note on (locally) presentable categories

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## Contents

In this document, conjectures are statements that I believe are true, although I did not have checked them or find a proper reference for them. The proofs of [1] are here reformulated, sometimes using (co)ends (see [2] for a demonstration of their usefulness).

A *locally presentable category* (or just *presentable* category) is a category which is not too big and not too small: it is cocomplete, but is the generated by a small set of objects. The main reference for this beast is [1].

A morphism between presentable categories is defined as cocontinuous functor in Anel's and Joyal's Topo-logie.

Presentable categories can be defined as cocomplete *accessible categories*, or as complete accessible categories (after nlab, the page on accessible categories and locally presentable categories).

An accessible category is defined as a $\lambda$-accessible category (you don't need to know what it is at this stage) for some regular cardinal $\lambda$. This a notion of $\lambda$-presentable category, and a presentable categories is a $\lambda$-presentable category, for some regular cardinal $\lambda$. Any $\lambda$-presentable category is also $\mu$-presentable for any regular cardinal $\mu > \lambda$. This is not the case for accessible categories! We only know, for them, that given any regular cardinal $\lambda$, there exists $\kappa > \lambda$ such that any $\lambda$-accessible category is also $\kappa$-accessible [1, Theorem 2.11] (such a condition is noted $\lambda \triangleleft \kappa$ and $\lambda$ is said sharply smaller, and it is transitive). In particular, $\omega \triangleleft \lambda$ for any (infinite) regular cardinal $\lambda$, and a regular cardinal is always sharply smaller than its successor. For each set of regular cardinals, there is one which is sharply greater than all of them. A similar thing can be said of accessible functors.

Presentable categories satisfy nice properties.

**Theorem 0.1.** *[1, 1.66] A functor between presentable categories is right adjoint iff it is continuous and preserve $\lambda$-directed colimits for some regular cardinal $\lambda$.*

A similar version is true of left adjointness according to nlab (wihtout the accessibility condition because it is automatically satisfied). In fact, by looking at the proof of [1, Representation theorem 1.45], it seems that any cocontinous functor out of a presentable category and to a copowered category has a right adjoint (because any presentable category is of the shape $\mathrm{Cont}_\lambda(A)$, and in the proof mentioned above, they say that the cocontinuous induced functor to a cocomplete category has a left adjoint obtained easily using yoneda. I suspect cocompleteness of the target is not necessary here).

**Remark 0.2.** *Anticipating the fact that any presentable category is of the shape $Cont_\lambda(A)$, a right adjoint to a presentable categorie is always a (kind of) nerve functor. This is seen, by postcomposing with the right adjoint inclusion in $\hat{A}$, and using yoneda to get the definition of this right adjoint.*

**Theorem 0.3** (nlab)**.** *Let $T$ be a monad preserving $\lambda$-directed colimits on a presentable category $C$. Then $C^T$ is presentable.*

# 1 Regular cardinals

**Definition 1.1.** *A cardinal is regular if it is strictly greater than any smaller union of smaller cardinals.*

Any successor cardinal is regular. The existence of regular uncountable limit cardinal (called weakly inaccessible cardinal) is not known to be consistent with ZFC. It is consistent to assume that any uncountable limit cardinal is not regular.

# 2 Filtered/directed colimits

A $\lambda$-filtered category is a category $\mathsf{C}$ in which any diagram $\mathsf{D} \to \mathsf{C}$ with $\mathsf{D}$ $\lambda$-small has a cocone. A $\lambda$-directed category is a poset which induces a $\lambda$-filtered category.

For any $\lambda$-filtered category $\mathsf{C}$, there is a $\lambda$-directed one $\mathsf{D}$ with a cofinal functor $\mathsf{D} \to \mathsf{C}$. The existence of a cofinal functor means that colimits with respect to $\mathsf{C}$ is the same as colimits with respect to $\mathsf{D}$.

That is why we can restrict ourselves to $\lambda$-directed colimits, rather than the more general $\lambda$-filtered colimits. We can also restrict to chains for $\omega$-filtered colimits, although there is no cofinality argument (see [1, Example 1.8]).

**Theorem 2.1.** *[1, Corollary 1.7]: A category has $\omega$-filtered (or $\omega$-directed) colimits iff it has colimits of chains. For such categories $K$, a functor of domain $K$ preserve $\omega$-filtered colimits iff it preserves colimits of chains.*

# 3 Preliminaries

By reading [3] dating back from 2014, it seems that few is known about the commutation of limits and colimits in Set, except for particular shapes.

**Notation 3.1.** *Given a presheaf $F : A \to Set$, we denote $\int F$ or $\oint^{a:A} F(a)$ (following [2]) the category of elements of $F$.*

**Notation 3.2.** *We denote the limit of a functor $F : A \to B$ by $\lim F = \int_{a:A} F(a)$. We denote the colimit of a functor $F : A \to B$ by $\mathsf{colim} F = \int^{a:A} F(a)$.*

**Notation 3.3.** *The opposite of category $A$ is denoted by $A^o$.*

**Definition 3.4.** *A **presheaf** is a functor $A \to Set$, where $A$ is a small category. We denote the category of presheaves $A^o \to Set$ by $\hat{A}$.*

# 4 Recap on coends

See [2] for a compelling view on the usefulness of coends. Here are some results (some well-known, some less).

**Lemma 4.1.** $[C, D](F, G) \simeq \int_c D(F(c), G(c))$.

**Proposition 4.2** (Weighted colimits reduce to conical colimits)**.** *Let $W : C^o \to Set$ and $F : C \to D$ be functors, where $D$ has copowers and $C$ is small. The weighted colimit $\int^c W(c) \times F(c)$ computes as the conical colimit $\int^{(c,x):\int W} F(c)$.*

**Corollary 4.3.** *Here is a list of easy consequences of Proposition 4.10*

1. *The weighted limit $\int^c F(c)^{W(c)}$ computes as the conical colimit $\int_{(c,x):\int W} F(c)$.*
2. *(coyoneda lemma) $\int^c F(c) \times C(c, c') \simeq F(c')$*
3. *(yoneda lemma) $\int_c F(c)^{C(c',c)} \simeq F(c')$*
4. *(coends as colimits) for any functor $H : C \times C^o \to D$, denoting by $y : C \times C^o \to Set$ the yoneda bifunctor, we have $\int^c H(c, c) \simeq \int^{(c_1 \xrightarrow{f} c_2):\int y} H(c_1, c_2)$.*
5. *(coend as limits) (the dual of the previous item)*

*Proof.* The first statement comes from duality. The yoneda lemma is the dual of the coyoneda lemma. The coyoneda lemma comes from the fact that $\int C(-, c') = C/c'$. Thus,

$$\int^c F(c) \times C(c, c') \simeq \int^{(c \to c'):C/c'} F(c)$$

Thus, this is a colimit over $C/c'$. But $id_{c'}$ is terminal in $C/c'$, so the colimit is trivial and results in $F(c')$.

For coends as colimits, start from the right hand side:

$$\begin{aligned}\int^{(c_1 \xrightarrow{f} c_2):\int y} H(c_1, c_2) &\simeq \int^{c_1,c_2} H(c_1, c_2) \times C(c_2, c_1) \quad \text{by Proposition 4.10.}\\ &\simeq \int^{c_1} H(c_1, c_1) \quad \text{by the coyoneda lemma.}\end{aligned}$$

□

Thus, the elimination of weighted colimits (Proposition 4.10) can be seen as a generalization of the yoneda lemma. Interestingly, we can also deduce Proposition 4.10 from the (co)yoneda lemma and the computation of coends as colimits (item 4 in the corollary above): see Lemma 4.10.

**Notation 4.4.** *If $F : \int y \to Set$, with $y : C \times C^o \to Set$, we denote its colimit by $\int^c F(c, c, id_c)$, sometimes even omitting $id_c$ in the expression. Thanks to item 4 of Corollary 4.3, it is consistent with the coend notation $\int^c G(c, c)$ because in such $G$ induces by precomposition a functor $G' : \int y \to Set$ and then $\int^c G'(c) = \int^c G(c, c)$. Faire une theorie generale des cofins fondees sur cette notation*

**Lemma 4.5.** *Let $F : C \times C^o \to Cat$ be a functor to small categories with $C$ small and $J : \int^c F(c, c) \to D$. Then*

$$\int^{(c;a):\int^c F(c,c)} J(c; a) \simeq \int^{(c_1 \xrightarrow{f} c_2):\int y} \int^{a:F(c_1,c_2)} J(c_1; f(a)).$$

**Remark 4.6.** *With notation 4.4, this becomes*

$$\int^{(c;a):\int^c A(c,c)} J(c; a) \simeq \int^c \int^{a:A(c,c)} J(c; a)$$

*Furthermore, by duality,*

$$\int_{(c;a):\int^c A(c,c)} J(c; a) \simeq \int_c \int_{a:A(c,c)} J(c; a)$$

*Proof of Lemma 4.5.* It relies on the fact (TODO: check) that $[\int^c A(c, c), D]$ is isomorphic to $\int_c [A(c, c), D]$. ☐

**Lemma 4.7.** *Let $F : A \to Set$ be a presheaf. Then*

$$\oint^a F(a) \simeq \int^a F(a) \times a/A$$

*Proof.* It comes from the coyoneda lemma, the fact that $a/A = \oint^{a'} A(a, a')$ and colimits commute with the Grothendieck construction. ☐

**Lemma 4.8.** *Let $F : C \to Set$ a presheaf and $J : \int F \to D$ a functor, where $D$ has some suitable properties. Then,*

$$\int^{(c,x):\int F} J(c, x) \simeq \int^{(c_1 \xrightarrow{f} c_2):\int y} \sum_{x \in F(c_1)} J(c_2, f(x))$$

**Remark 4.9.** *With notation 4.4, this becomes*

$$\int^{(c,x):\int F} J(c, x) \simeq \int^c \sum_{x \in F(c)} J(c, x)$$

*Furthermore, by duality,*

$$\int_{(c,x):\int F} J(c, x) \simeq \int_c \prod_{x \in F(c)} J(c, x)$$

*TODO: generalise to arbitrary coends (seems necessary in Lemma 4.14)*

*Proof of Lemma 4.8.* Let $F : C \to \mathrm{Set}$. Then (remember that the Grothendieck construction preserves colimits):

$$\int F = \oint^c F(c) \simeq \int^c F(c) \times c/A \quad \text{by Lemma 4.7}$$

Then, by Lemma 4.5,

$$\begin{aligned}\int^{(c,x):\int F} J(c,x) &\simeq \int^{(c_1 \xrightarrow{f} c_2):\int y} \int^{(x, c_2 \xrightarrow{u} c):F(c_2)\times c_1/C} J(c, u(f(x))) \\ &\simeq \int^{(c_1 \xrightarrow{f} c_2):\int y} \sum_{x \in F(c_2)} \int^{(c_1 \xrightarrow{u} c):c_1/C} J(c, u(f(x)))\end{aligned}$$

But note that $id_{c_1}$ is initial in $c_1/C$. Therefore, for any $K : (c_1/C)^o \to D$, we have that $\mathsf{colim} K \simeq K(id_{c_1})$. This concludes the argument. □

**Lemma 4.10** (Weighted colimits reduce to conical colimits)**.** *Here, we state Proposition and prove it from two of its corollaries: the yoneda lemma, and the computation of coends as colimits (item 4 of Propostion 4.3). TODO: check that the proof does not secretly relies on the conclusion.*

*Let $W : C^o \to Set$ and $F : C \to D$ be functors, where $D$ has copowers and $C$ is small. The weighted colimit $\int^c W(c) \times F(c)$ computes as the conical colimit $\int^{(c,x):\int W} F(c)$.*

*Proof.*

$$\int^{(c,x):\int W} F(c) \simeq \int^{(c_1 \xrightarrow{f} c_2):\int y} \sum_{x \in W(c_1)} F(c_2) \quad \text{by Lemma 4.8}$$

On the other hand,

$$\int^c W(c) \times F(c) \simeq \int^{(c_1 \xrightarrow{f} c_2):\int y} W(c_1) \times F(c_2) \quad \text{by item 4 of Corollary 4.3}$$

□

**Corollary 4.11.**

$$\int^{(c,x)\in \oint^c F(c)} J(c) \simeq \int^c F(c) \times J(c)$$

*Proof.*

$$\begin{aligned}\int^{(c,x)\in \oint^c F(c)} J(c) &\simeq \int^{(c_1 \xrightarrow{f} c_2):\int y} \sum_{x \in F(c_1)} J(c_2) \\ &\simeq \int^c \sum_{x \in F(c)} J(c)\end{aligned}$$

□

**Lemma 4.12** (Limits commute with dependent pairs)**.** *The canonical morphism*

$$\int_{d:D} \sum_{x \in K(d)} G(d,x) \to \sum_{\alpha \in \lim K} \int_d G(d, \alpha_d)$$

*is an isomorphism.*

**Remark 4.13.** *Recall that the mapping $H \mapsto H' = \sum_{x \in G(-)} H(c,-)$ induces an equivalence between $[\int G, Set]$ and $[C, Set]/G$. The converse functor take a functor $F \xrightarrow{\alpha} G$ and maps it to the functor $(c,g) \mapsto \alpha_c^{-1}(\{g\})$. Then, Lemma 4.14 says that a section of $H'$ is a section of $G$ together with a dependent section.*

In fact, we prove a stronger result:

**Lemma 4.14.** *Let $F, G : C \to Set$ and $H : \int G \to Set$ be a functor. Then,*

$$[C, Set](F, \sum_{x \in G(-)} H(c,-)) \simeq \sum_{\alpha : F \to G} \int_{(c,x) \in \int F} H(c, \alpha_c(x))$$

*Proof.* A natural transformation between $F$ and $H' = \sum_{x \in G(-)} H(c,-))$ is the same as a natural transformation $\alpha : F \to G$, and a morphism $F \to H'$ over $G$. Through the equivalence with the category $[\int G, \mathrm{Set}]$, this second component is equivalently a morphism $[\int G, \mathrm{Set}](\alpha^{-1}(\{-\}), H)$, that is, an element of

$$\int_{c,g} \mathrm{Set}(\alpha_c^{-1}(g), H(c,g))$$

We want it to be isomorphic to:

$$\int_{(c,x) \in \int F} H(c, \alpha_c(x)) \simeq \int_{(c,x) \in \int F} H(c, \alpha_c(x))$$

TODO: finish ☐

*Proof of Lemma 4.12.* Apply Lemma 4.14 with $F = 1$. ☐

# 5 Doctrines

Il y a un truc a faire avec les foncteurs familiaux!! qui preservent presque les limites connectees, mais pas toutes. Pourtant il y a toujours une histoire de coproduits, qui est le dual des limites connectees. avec l'histoire des sketches, va falloir reflechir a pourquoi on peut toujours se ramener a $\mathcal{S}$ l'ensembles des categories $\lambda$-petites

**Definition 5.1.** *A **doctrine** $\mathcal{S}$ (terminology borrowed from [4]) is a (possibly large) collection of small categories, called shapes. We say that a colimit is $\mathcal{S}$-shaped if the domain of the diagram lies in $\mathcal{S}$.*

**Definition 5.2.** *Let $C$ and $D$ be two small categories. The* ***pole*** *$Z(C\perp D)$* ***of*** *$C$* ***against*** *$D$ is the collection of functors $F : C \times D \to Set$ such that the canonical morphism*

$$\int^{c} \int_{d} F(c,d) \to \int_{d} \int^{c} F(c,d)$$

*is an isomorphism. If $Z(C\perp D) = [C \times D, Set]$, then we say that the pole is* ***maximal*** *and that $C$-colimits commute with $D$-limits.*

**Remark 5.3.** *$F : C \times D \to Set$ is in $Z(C\perp D)$ is equivalent to the preservation by $Lan_y 1 : [C, Set] \to Set$ of the limit of the diagram $D \to [C, Set]$ induced by $F$. Consequently, $C$-colimits commute with $D$-limits if $Lan_y 1$ is $D$-continuous.*

**Remark 5.4.**

$$Lan_y F(G) = Lan_y G(F)$$

$$Lan_{y_{\int F}} 1(G) = Lan_y F(G) \qquad \text{if } G \text{ factors through the forgetful functor } \int F \to C.$$

*The last equality is the content of elimination of weights. In fact, $Lan_y F(G)$ is the $G$-weighted colimit of $F$ (and reciprocally).*

**Definition 5.5.** *Let $F : C \to Set$ be a presheaf. A* ***dependent*** *$F$**-weighted*** ***colimit*** *is a colimit of a functor $G : (\int F)^o \to F$.*

**Remark 5.6.** *Thanks to the reduction of weights, $F$-weighted colimits are particular cases of dependent ones.*

**Definition 5.7.** *We denote $\mathcal{S}^\perp$ the doctrine such that $\mathcal{S}^\perp$-shaped colimits commute with $\mathcal{S}$-shaped limits in Set.*

**Remark 5.8.** *Considering [4, 3], little is known about the general commutation of limits with colimits, except for specific doctrines. Typically, we consider small doctrines $\mathcal{S}$ and then $\mathcal{S}^\perp$ is large.*

**Remark 5.9.** *$\mathcal{S}^\perp$ is closed under finite products.*

**Lemma 5.10.** *Let $y : C^o \times C \to Set$ be the yoneda functor, with $C$ small. We have an easy chain of implications, for a presheaf $F : C \to Set$:*

1. *$(\int F)^o \in \mathcal{S}^\perp$, that is, dependent $F$-weighted colimits commute with $\mathcal{S}$-limits (we then say that $F$ is* ***$\mathcal{S}$-flat****) ;*
2. *$F$ is a $\mathcal{S}^\perp$-colimit of representable presheaves (because it is a $F$-weighted colimit of representable presehaves);*
3. *$Lan_y F$ preserves any $\mathcal{S}$-shaped limits ($F$-weighted colimits commute with $\mathcal{S}$-limits) ;*
4. *$Lan_y F$ preserves any $\mathcal{S}$-shaped limits of representable presheaves; ($F$-weighted colimits commute with $\mathcal{S}$-limits of representable presheaves)*

5. *$F$ preserves any $S$-shaped limits ($F$-weighted colimits commute with $\mathcal{S}$-limits of representable presheaves which already exist in $C$).*

*Moreover, if $C$ is $\mathcal{S}$-complete, then item 5 entails item 4.*

*Proof.* Straightforward using the coend formula for the left Kan extension. □

**Definition 5.11.** *Let $\mathcal{S}$ be a doctrine. It is **deligthful** if item 3 implies item 1 in Lemma 5.10, i.e. for any presheaf $F$, if $F$-weighted colimits commute with $\mathcal{S}$-limits, then so do any dependent $F$-weighted colimits. As a consequence, the first 3 items are logically equivalent.*

That is why a delightful doctrine is delightful. [4] proposes the subclass of sound doctrines.

**Definition 5.12.** *A doctrine $\mathcal{S}$ is **sound for a collection of presheaves** $F : C \to$ **Set** if item 4 in Lemma 5.10 entails item 1, so that all theses items up to item 1 are equivalent (and thus it is delightful). In other words, if $F$-weighted colimits commute with $\mathcal{S}$-limits of representable presheaves, then do so any dependent $F$-weighted colimits.*

*A doctrine is said **sound** if it is sound for the total collection of presheaves.*

**Example 5.13.** *Here are some examples [4, Example 2.3] of sound doctrines:*

- *(easy) $\mathcal{S} = \{\lambda\text{-small categories}\}$, and then $\mathcal{S}^{\perp}$ is the collection of $\lambda$-filtered categories;*
- *(non trivial) the doctrine of finite discrete categories;*
- *$\mathcal{S} = \{\text{finite connected categories}\}$, and then $\mathcal{S}^{\perp}$ is the collection of sifted categories;*
- *(non-examples) the doctrine of countable products (i.e. countable discrete categories) and the doctrine of pullbacks are not sound.*

**Remark 5.14.** *If a doctrine $\mathcal{S}$ is sound on terminal presheaves, then, given any small category $C$, if $C$-colimits commute $\mathcal{S}$-limits of representable presheaves, then $C$ commutes with $\mathcal{S}$-limits.*

*Note that by yoneda, the codomain of the canonical*

$$\int^{c:C} \int_d C(Sd, c) \to \int_d \int^c C(Sd, c)$$

*is a singleton set. As a colimit consists in taking the connected components of the category of elements, another way to state soundness is that the category of cocones of any functor $S : D^o \to C$ is connected. As a consequence [4, Proposition 2.5], any $\mathcal{S}^o$-cocomplete contegory $C$ is in $\mathcal{S}^{\perp}$ (could also be deduced directly from $\int_d C(Sd, c) \simeq C(\int^d Sd, c)$).*

In fact, any doctrine satisfying this property is sound.

**Lemma 5.15.** *If a doctrine $\mathcal{S}$ is sound on terminal presheaves, then it is sound [4, Theorem 2.4].*

*Proof.* So we are given a presheaf $F : C \to \mathrm{Set}$ such that for any functor $H : (\int F)^o \times D \to \mathrm{Set}$, the canonical morphism

$$\int^{(c,x):\int F} \int_d H(c,d) \to \int_d \int^{(c,x):\int F} H(c,d)$$

is an isomorphism. Now we would like to show that $(\int F)^o \in \mathcal{S}^\perp$. As $\mathcal{S}$ is sound, it is enough to show that given any functor $H : D \to (\int F)^o C$ with $D \in \mathcal{S}$, the canonical morphism

$$\int^{f:(\int F)^o} \int_d \textstyle\int F(f, Hd) \to \displaystyle\int_d \int^f \textstyle\int F(f, Hd)$$

is an isomorphism.

Let us rephrase this statement: by universal property of $\int F$ as the comma category $1/F$, a functor $H : D \to (\int F)^o$ is equivalently given by a functor $G : D \to C^o$ together with an element $\alpha \in \lim FG$. We then want that

$$\int^{(c,x):\int F} \int_d \sum_{f:C(c,Gd)} (f(x) = \alpha_d) \to \int_d \int^{(c,x)} \sum_{f:C(c,Gd)} (f(x) = \alpha_d)$$

is an isomorphism.

We would like to apply our hypothesis on $\int F$, but we need first to remove the the dependency on $x$. The trick is to show take the coproduct of these morphisms over all possible $\alpha \in \lim GH$ is an isomorphism.

By the fact that limits commute with dependent pairs in Set (Lemma 4.12), the domain of the resulting coproduct morphism is canonically isomorphic to

$$\int^{(c,x):\int F} \int_d \sum_{x' \in FGd} \sum_{f:C(c,Gd)} (f(x) = x') \simeq \int^{(c,x):\int F} \int_d C(c, Gd).$$

By a similar argument, the codomain of the resulting morphism is canonically isomorphic to

$$\int_d \int^{(c,x):\int F} C(c, Gd).$$

Now we can apply our hypothesis on $\int F$, which concludes the argument. $\square$

**Remark 5.16.** *In fact, Lemma* **??** *can be taken as a definition of sound doctrines: if item 4 in Lemma 5.10 entails item 1, then the doctrine is sound (consider the case of $F$ as the terminal functor).*

**Conjecture 5.17.** *Let $C$ be a small category and $\mathcal{S}$ be a doctrine. Let*

- *$U_{\mathcal{S}}$ be the forgetful 2-functor from categories with $\mathcal{S}$-shaped colimits and functors preserving them to the category of small categories.*
- *$V_{\mathcal{S}}$ be be the forgetful 2-functor from cocomplete categories and cocontinuous functors to the 2-category of small categories and functors preserving $\mathcal{S}$-shaped colimits;*
- *$Dir_{\mathcal{S}}(C)$ the full subcategory of $\hat{C}$ consisting of $\mathcal{S}$-shaped colimits of representable presheaves;*
- *$Cont_{\mathcal{S}}(C)$ the full subcategory of $\hat{C}$ consisting of presheaves preserving $\mathcal{S}$-shaped colimits.*

*Then,*

- *the yoneda embedding exhibits $Dir_{\mathcal{S}}(C)$ as the weak free object for $U_{\mathcal{S}}$ (weak in the sense that the universal morphism is only up to iso);*
- *if $\mathcal{S}$ is a small set, then the yoneda embedding exhibits $Cont_{\mathcal{S}}(C)$ as the weak free object with respect to $V_{\mathcal{S}}$ (weak in the sense that the universal morphism is only up to iso).*

This is a conjecture but I will give arguments.

**Question 5.18.** *What happens to $Cont_{\mathcal{S}}(C)$ if $\mathcal{S}$ is not small?*

**Remark 5.19.** *$Dir_{\mathcal{S}^\perp}(C)$ is a full subcategory of $Cont_{\mathcal{S}}(C)$.*

**Lemma 5.20.** *If $\mathcal{S}$ is sound and $C$ is $\mathcal{S}$-complete, then $Dir_{\mathcal{S}^\perp}(C)$ is equivalent to $Cont_{\mathcal{S}}(C)$.*

*Proof.* By Remark **??**. □

**Conjecture 5.21.** *If $\mathcal{S}$ is sound and $Dir_{\mathcal{S}^\perp}(C)$ is equivalent to $Cont_{\mathcal{S}}(C)$, then $C$ is $\mathcal{S}$-complete.*

The universal property of $\mathrm{Dir}_{\mathcal{S}}(C)$ should be easy to prove. We will thus focus on the universal property of $\mathrm{Cont}_{\mathcal{S}}(C)$.

**Lemma 5.22.** *The yoneda embedding $y : C \to Cont_{\mathcal{S}}(C)$ preserves $\mathcal{S}$-shaped colimits.*

*Proof.* Let $K : D \to C$ be a $\mathcal{S}$-shaped diagram with an existing colimit in $C$. Let us check that $y\mathsf{colim}K$ has the required universal property, that is, given any presheaf $F$ in $\mathrm{Cont}_{\mathcal{S}}(C)$, we have $\hat{C}(y\mathsf{colim}K, F)$ is canonically isomorphic to $[D, \hat{C}](yK, F) \simeq \lim FK$. But $\hat{C}(y\mathsf{colim}K, F)$ is canonically isomorphic to $F\mathsf{colim}K$, which is canonically isomorphic to $\lim FK$. □

First, some interesting (although not necessary lemma):

**Lemma 5.23.** *Any cocontinuous functor $L : Cont_{\mathcal{S}}(C) \to B$, with $B$ locally small, has a right adjoint.*

*Proof.* If this is true, then the right adjoint is given by $R(b)(x) = \hat{A}(yx, R(b)) = B(Lyx, b)$. Here, $R$ clearly defines a functor from $B$ to $\hat{A}$. We check that any $R(b)$ preserves $\mathcal{S}$-shaped colimits: let $\mathsf{colim} K$ be a $\mathcal{S}$-shaped colimit in $A$. Then, $R(b)(\mathsf{colim} K) = B(Ly\mathsf{colim} K, b) = B(\mathsf{colim} LyK, b) = \lim B(yK, b) = \lim R(b)K$. $\square$

**Remark 5.24.** *This also follows from the adjoint functor theorem stating that any cocontinuous functor out of a total category (such as presentable categories) is a left adjoint.*

**Remark 5.25.** *This provides examples of a left adjoint (the left Kan extension) which the composition of a right adjoint (the embedding in the total presheaf category) and a left adjoint (the left Kan extension).*

Another noticeable fact:

**Lemma 5.26.** *$Cont_{\mathcal{S}}(C)$ is complete.*

*Proof.* Because limits commute with limits. $\square$

But in fact, everything follows from the following lemma:

**Lemma 5.27.** *If $\mathcal{S}$ is small, then $Cont_{\mathcal{S}}(C)$ is a reflective subcategory of $\hat{C}$.*

*Proof.* We apply Conjecture 7.21. We only have to devise an orthogonal factorization system on $\hat{C}$ such that a presheaf is fibrant if and only it preserves $\mathcal{S}$-shaped colimits. But, observe that a presheaf $F : C^o \to \mathrm{Set}$ preserves a colimit $\int^i D(i)$ if and only if $F \to 1$ is right orthogonal to $m_D : y_{\int^i D(i)} \to \int_i y_{D(i)}$. Thus we consider the orthogonal factorization system cofibrantly generated by the collection of such morphisms $m_D$. Note that this is a small set because $C$ is small and $\mathcal{S}$ also is. $\square$

**Corollary 5.28.** *$Cont_{\mathcal{S}}(C)$ is cocomplete, by Lemma A.1.*

**Remark 5.29.** *$\mathcal{S}^{\perp}$-shaped colimits in $Cont_{\mathcal{S}}(C)$ are computed pointwise.*

**Lemma 5.30.** *If $\mathcal{S}$ is closed under binary products (for example, $\mathcal{S} = \mathcal{S}'^{\perp}$), $Dir_{\mathcal{S}}(C)$ has $\mathcal{S}$-colimits, which are computed pointwise.*

**Lemma 5.31.** *Given a functor $F : C \to B$ preserving $\mathcal{S}$-shaped colimits, with $B$ cocomplete, then the left Kan extension of $F$ along the yoneda embedding in $Cont_{\mathcal{S}}(C)$ is the unique cocontinuous functor (up to iso) yielding $F$ when precomposing with the yoneda embedding.*

*Proof.* That it is cocontinuous comes from Lemma A.3. That it is unique up to iso follows from Lemma A.4. $\square$

**Definition 5.32.** *Let $\mathcal{C}$ be a locally small category. Let $Pres_{\mathcal{S}}(\mathcal{C})$ be the full subcategory of objects $c$ of $\mathcal{C}$ such that $y_c : \mathcal{C} \to Set$ is in $Cont_{\mathcal{S}}(\mathcal{C}^o)$, that is,*

*$\mathcal{C}(c,-)$ preserves $\mathcal{S}$-shaped colimits. In other words, $Pres_{\mathcal{S}}(\mathcal{C})$ is the pullback (beware: non locally small categories involved)*

$$\begin{tikzcd}
Pres_{\mathcal{S}}(\mathcal{C}) \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
Cont_{\mathcal{S}}(\mathcal{C}^o) \arrow[r] & {[\mathcal{C}, Set]}
\end{tikzcd}$$

**Conjecture 5.33.** *$Cont_{\mathcal{S}}(\mathcal{C}) \to \hat{\mathcal{C}}$ and $Dir_{\mathcal{S}}(\mathcal{C})$ are isofibrations, and thus the pullback in Definition 5.32 is also a weak pullback.*

**Lemma 5.34.** *$Pres_{\mathcal{S}^\perp}(\mathcal{C})$ is closed under any $\mathcal{S}$-colimits that exists in $\mathcal{C}$. In particular, if $\mathcal{C}$ is cocomplete (for example $\mathcal{C} = Cont_{\mathcal{S}}(C)$ with $\mathcal{S}$ and $C$ small), then $Pres_{\mathcal{S}^\perp}(\mathcal{C})$ has $\mathcal{S}$-colimits.*

**Conjecture 5.35.** *$Pres_{\mathcal{S}^\perp}(Cont_{\mathcal{S}}(C))$ is the closure of $C$ under $\mathcal{S}$-colimits in $Cont_{\mathcal{S}}(C)$ (which may not always exist, but do if $\mathcal{S}$ and $C$ are small).*

**Question 5.36.** *What about $Pres_{\mathcal{S}^\perp}(Dir_{\mathcal{S}}(C))$?*

**Corollary 5.37.** *If $\mathcal{S}$ and $C$ are small, and $C$ is $\mathcal{S}$-cocomplete, then $Pres_{\mathcal{S}^\perp}(Cont_{\mathcal{S}}(C)) \simeq C$. Thus in this case, a presheaf in $Pres_{\mathcal{S}^\perp}(Cont_{\mathcal{S}}(C))$ preserves in fact any colimit, as it is a representable presheaf.*

**Question 5.38.** *Which kind of free cocompletion of $C$ is $Pres_{\mathcal{S}^\perp}(Cont_{\mathcal{S}}(C))$?*

**Conjecture 5.39.**
1. *$Dir_{\mathcal{S}} \circ Pres_{\mathcal{S}} \circ Dir_{\mathcal{S}} \simeq Dir_{\mathcal{S}}$*
2. *$Cont_{\mathcal{S}} \circ Pres_{\mathcal{S}^\perp} \circ Cont_{\mathcal{S}} \simeq Cont_{\mathcal{S}}$*

*Proof.* □

**Definition 5.40.** *A $\mathcal{S}$-presentable category is a category $\mathcal{C}$ such that any object is a $\mathcal{S}^\perp$-colimit of objects of $Pres_{\mathcal{S}^\perp}(\mathcal{C})$, and $Pres_{\mathcal{S}^\perp}(\mathcal{C})$ is essentially small.*

**Conjecture 5.41.** *Any category is $\mathcal{S}$-presentable if and only if it is equivalent to some $Cont_{\mathcal{S}}(C)$ with $C$ small.*

**Conjecture 5.42** (Gabriel-Ulmer)**.** *There is a 2-equivalence between $\mathcal{S}$-complete categories and $\mathcal{S}$-presentable category (through $Pres_{\mathcal{S}^\perp}$ and $Cont_{\mathcal{S}}$).*

**Conjecture 5.43.** *If $\mathcal{C}$ is $\mathcal{S}$-presentable and $\mathcal{S}$ is sound and closed under opposite, then $Pres_{\mathcal{S}^\perp}(\mathcal{C})$ is $\mathcal{S}^o$-cocomplete, and thus is in $\mathcal{S}^\perp$, by Remark 5.14.*

# 6 Stuff about presentable categories

**Lemma 6.1.** *Given a $\lambda$-presentable category $\mathcal{C}$, there exists $\mu > \lambda$ such that $Pres_\mu(\mathcal{C})$ is stable per subobjects*

*Proof.* Take the closure of $\mathrm{Pres}_\lambda(\mathcal{C})$ under subobjects (which is still a set). As any object is compact in a presentable category, we can take $\mu$ to be such that any object in this closure is $\mu$-compact. $\square$

**Example 6.2.** *To find an example of a $Pres_\lambda(\mathcal{C})$ not closed under subobjects, it is enough to find a small category $C$ such that some subobject of a representable presheaf (that is, some sieve) is not a finite colimit of representable presheaves. For example, consider the chain category with a terminal object adjoined, and consider the functor to set mapping any object to 1, except the terminal object which is mapped to 0. This is a sieve, but, it is not a finite colimit of representable presheaves.*

**Conjecture 6.3.** *If $C$ is a small category and $\mathcal{C}$ is $\lambda$-presentable, then so is $[C, \mathcal{C}]$.*

# 7 Presentable categories (old new attempt)

Let $\lambda$ be a regular cardinal.

**Definition 7.1.** *A **$\lambda$-presentable object** in a category $C$ is an object such that its yoneda embedding preserves $\lambda$-directed colimits.*

It says no more that any morphism from such an object to a $\lambda$-directed colimit factors (essentially uniquely) through some object of the colimiting cocone.

**Lemma 7.2.** *A $\lambda$-presentable object is also $\kappa$-presentable, for $\kappa > \lambda$.*

*Proof.* Obvious, because a $\kappa$-filtered category is also $\lambda$-filtered. $\square$

**Definition 7.3.** *A $\lambda$-accessible category is a category closed under $\lambda$-directed colimits and there is a small set $A$ of $\lambda$-presentable objects such that every object is a $\lambda$-directed colimit of objects of $A$.*

**Lemma 7.4.** *Consider the full subcategory $Pres_\lambda(C)$ of $\lambda$-presentable objects of a category $C$. If $C$ is locally $\lambda$-accessible, $Pres_\lambda(C)$ is essentially small (in the following we will reason as if it were small), and every object in $C$ is a $\lambda$-directed colimits of objects of $Pres_\lambda(C)$.*

**Definition 7.5.** *A $\lambda$-presentable category is a $\lambda$-accessible category which is cocomplete.*

**Proposition 7.6.** *A category is $\lambda$-presentable category if and only if it is a cocomplete category with a strong generator formed of $\lambda$-presentable objects.*

To go from the strong generator to a proper set of generating $\lambda$-presentable objects above: take $\lambda$-small colimits of objects in the strong generator.

**Conjecture 7.7.** *Given a $\lambda$-presentable $C$ category with a strong generator $A$ of $\lambda$-presentable objects,*

- *$Pres_\lambda(C)$ is closed under $\lambda$-small colimits;*
- *$Pres_\lambda(C)$ is equivalent to the full subcategory of $C$ consisting of $\lambda$-small colimits of elements $A$.*

*Proof.* I haven't tried to prove it. $\square$

**Example 7.8.** *Any presheaf category is $\omega$-presentable: consider the strong generator consisting of representable presheaves.*

**Lemma 7.9.** *A locally $\lambda$-presentable category is also locally $\kappa$-presentable, for $\kappa > \lambda$.*

*Proof.* We consider the alternative characterisation of locally presentable categories of Proposition 7.6. The set of "presenting" $\kappa$-presentable objects is induced by noticing that $\lambda$-presentable objects are also $\kappa$-presentable, and form a strong generator. Then, we take $\kappa$-small colimits of them. $\square$

There is an alternative definition of presentable categories as free cocompletions.

**Definition 7.10.** *Let $U_{Cont_\lambda}$ denotes the forgetful functor from cocomplete categories and cocontinuous functors to small categories and $\lambda$-cocontinous functors between them (that is, functors preserving $\lambda$-small colimits). In particular, the codomain of $U_{Cont_0}$ is the usual category of small categories. Let $\hat{A}$ denotes the presheaf category $[A^o, Set]$, for a small category $A$.*

In the following, $A$ denotes a small category.

**Lemma 7.11.** *The yoneda embedding $y : A \to \hat{A}$ exhibits the free object with respect to $U_{Cont_0}$. The unique cocontinuous functor induced by the universal property is computed as a left Kan extension.*

*Proof.* Let $G : A \to B$ where $B$ is a cocomplete category. This induces a functor $Lan_y(G) : \hat{A} \to B$ which has a right adjoint (as explained later, I think this is true even if $B$ were not cocomplete). It is cocontinuous and unique because any presheaf is a colimit of representables ones, whose image is fixed by $G$. $\square$

**Lemma 7.12.** *([1, Proposition 1.45]) Let $Cont_\lambda(A)$ be the full subcategory of $\hat{A}$ consisting of presheaves preserving $\lambda$-small colimits of $A$. The yoneda embedding $y : A \to Cont_\lambda A$ is the free object of $A$ with respect to $U_{Cont_\lambda}$.*

See the reference for the proof. The only thing which is not obvious is the fact that $\mathrm{Cont}_\lambda A$ is cocomplete. This relies on the orthogonal reflection section of [1]. We recast this section in §7.1 in terms of orthogonal factorization systems which look more standard to us. Hopefully, the results involved in §7.1 could be inserted at the same place than the original orthogonal reflection section (that is, they do not rely on results proven later in [1], in which case the argument could be circular).

Interestingly enough, the proof in [1] entails the following:

**Conjecture 7.13.** *Any cocontinous functor $Cont_\lambda(A) \to B$ has a right adjoint, which is roughly a nerve functor. Thus, anticipating Proposition 7.14,* ***any cocontinuous functor out of a presentable category has a right adjoint****.*

I write it as a conjecture because [1] assumes $B$ is cocomplete, but I don't think it is important.

**Proposition 7.14.** *([1, Representation Theorem, 1.46]) A category $C$ is $\lambda$-presentable if and only if there exists a small category $A$ such that $C$ is equivalent to $Cont_\lambda(A)$.*

**Conjecture 7.15.** *In Proposition 7.14 above, the categories $A$ that work are exactly the full subcategories generated by a strong generator of $\lambda$-presentable objects.*

I should check that $C$ is equivalent to $\mathrm{Cont}_\lambda(A)$ for a strong generator $A$ of $C$ consisting of $\lambda$-presenting objects. Anyway, we have the following lemma:

**Lemma 7.16.** *Representable functors in $Cont_\lambda(A)$ form a strong generator consisting of $\lambda$-presentable objects.*

The only thing missing for $\lambda$-presentability of $\mathrm{Cont}_\lambda(A)$ is that it is cocomplete, and this is proven in §7.1.

Regarding the proof, it is straightforward to check that representable presheaves form a strong generator. That they are $\lambda$-presentable relies on the following lemma:

**Lemma 7.17.** *$Cont_\lambda(A)$ is closed under $\lambda$-directed colimits, which are computed pointwise.*

In fact, $\mathrm{Cont}_\lambda(A)$ is even cocomplete as we will see in §7.1. Colimits in $\mathrm{Cont}_\lambda(A)$ are computed pointwise exactly when the pointwise colimit is in $\mathrm{Cont}_\lambda(A)$ (this is a general argument working for any full subcategory). Then, this lemma results from the fact that $\lambda$-directed colimits commute with $\lambda$-small limits in Set.

**Proposition 7.18.** *([1, Representation Theorem 1.45]) A category $C$ is $\lambda$-presentable if and only if it is a reflective subcategory of a presheaf category, closed under (pointwise) $\lambda$-directed colimits.*

This follows from Proposition 7.14 and the following result:

**Proposition 7.19.** *([1, Theorem 1.39]) Let $C$ be a reflective subcategory of a $\lambda$-presentable category $K$ such that the embedding preserves $\lambda$-directed colimits. Then $C$ is $\lambda$-presentable.*

*Proof.* Here we provide a different proof than [1], not relying on orthogonal reflection.

$C$ is cocomplete by Lemma A.1. Now we have to give a set of $\lambda$-presenting objects generating $C$ under $\lambda$-directed colimits. We take the image of $\mathrm{Pres}_\lambda(K)$ by the left adjoint. $\square$

## 7.1 Orthogonal factorization systems

This section is somehow the counterpart of orthogonal reflection section of [1]. It relies on the observation that orthogonality classes there correspond to fibrant objects for a cofibrantly generated orthogonal factorization system. Thus, the transfinite induction that they do correspond to the usual small object argument (although usually performed for weak factorization systems).

The goal here is to show that $\mathrm{Cont}_\lambda(A)$ is cocomplete (and even $\lambda$-presentable): this is the missing part of Lemma 7.12. More precisely, we show that $\mathrm{Cont}_\lambda$ is a reflective subcategory of $\hat{A}$. We don't recall what orthogonal factorization systems are.

**Proposition 7.20.** *Given a small set $I$ of morphisms in a presentable category, Then $({}^\perp(I^\perp), I^\perp)$ is an orthogonal factorization system.*

*Proof.* The main point of the proof is the construction of the factorization. This is done by a transfinite construction.

This is claimed by the nlab page on orthogonal factorization systems, although I haven't found a proper reference for it (maybe Kelly's transfinite paper, as this result is also mentionned on the related nlab page). □

**Conjecture 7.21.** *Given an orthogonal factorization system, the full subcategory of fibrant objects is reflective (the left adjoint simply factorizes the terminal morphism).*

*Proof.* I did not find a proper reference for this. I remember reading stuff about idempotent monads related to orthogonal factorization systems. □

**Corollary 7.22.** *$Cont_\lambda(A)$ is a reflective subcategory of $\hat{A}$ (and hence is cocomplete, by Lemma A.1).*

This relies on the following observation:

**Remark 7.23.** *A presheaf $F : A^o \to Set$ preserves a colimit $\int^i D(i)$ if and only if $F \to 1$ is right orthogonal to $y_{\int^i D(i)} \to \int_i y_{D(i)}$.*

**Remark 7.24.** *Anticipating the sketch section, we can see why models of limit-sketches gather into a locally presentable category. For colimit sketches, we would consider left orthogonality rather than right orthogonality.*

**Corollary 7.25.** *$Cont_\lambda(A)$ is $\lambda$-presentable.*

*Proof.* It is cocomplete (Corollary 7.22) and has a strong generator consisting of representables functors (Lemma 7.16). Thus, it is $\lambda$-presentable according to Definition 7.6. □

### 7.2 Fun facts about orthogonal factorization systems

In fact, orthogonal factorization systems are particular weak factorization systems (at least if the category has pushouts)! They are such that the cofibrations are closed under a certain pushout construction (and there is probably a dual statement for fibrations).

**Lemma 7.26.** *Let $f : A \to B$ be a morphism. Consider the universal morphism $B +_A B \to B$ (through the cocone consisting of identities), and a morphism $g : C \to D$. A commutative square*

$$\begin{tikzcd} B +_A B \arrow[r] \arrow[d] & C \arrow[d, "g"] \\ B \arrow[r] & D \end{tikzcd}$$

*is equivalently given by (where $g : C \to D$ is considered as given) two fillings of a commutative square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] & D \end{tikzcd}$$

*A filling is then a witness that the two fillings are equal.*

Thus, an factorization system which is closed under these codiagonals is an orthogonal factorization system.

# 8 Accessible categories

We have already introduced accessible categories in Definition 7.3.

**Question 8.1.** *How do you show the equivalence between different definitions of accessible categories? and the fact that any presentable category is accessible?*

**Definition 8.2.** *According to the nlab (page on sketch), an accessible category is a full subcategory of a presheaf category that's closed under $\kappa$-filtered colimits for some $\kappa$.*

**Remark 8.3.** *Contrary to what happens with presentability, a $\lambda$-accessible category may not be $\kappa$-presentable, for $\kappa > \lambda$. But there always exist such a $\kappa$ for which it is true. See the section "Raising the index of accessibility" in [1]. They introduce in [1, 2.12] a transitive relation "sharply smaller" between two cardinals, for which it is true:*

- *a cardinal is always sharply smaller than its successor;*
- *$\omega$ is sharply smaller than any other regular cardinal;*
- *if $\lambda \leq \mu$, then $\lambda$ is sharply smaller than the successor of $2^\mu$;*

- *as a consequence (not sure why) of the last item, for any set of cardinals, there exists a cardinal which is sharply greater than any cardian in this set.*

As a consequence of this remark:

**Proposition 8.4.** *For any family of accessible categories, there exists arbitrary large regular cardinals $\mu$ such that they are $\mu$-accessible.*

It often happens that small categories are accessible:

**Lemma 8.5.** *If a category has equalizers or coequalizers, then it has split idempotents.*

*Proof.* The splitting, according to nlab, consists in taking the (co)equalizer of the idempotent with the identity morphism. □

**Lemma 8.6.** *[1, 2.4] Each accessible category has split idempotents.*

*Proof.* The diagram consisting of one object, the identity and the idempotent is $\lambda$-filtered. The colimit provides a splitting. □

**Proposition 8.7.** *[1, 2.6] Every small category with split idempotents is accessible.*

**Question 8.8.** *When $C$ is $\lambda$-presentable, $Pres_\lambda(C)$ has small $\lambda$-colimits and thus is accessible. What is $Pres(Pres_\lambda(C))$ then?*

**Remark 8.9.** *The Cauchy completion is an idempotent monad which freely adds split idempotents. I don't know if it will be relevant at some point, but it is known that two presheaves categories are equivalent if and only if the base categories have equivalent Cauchy completions.*

**Proposition 8.10.** *([1, Remarks 2.2, (3)]) Any object $c$ of a $\lambda$-accessible category is presentable (although not necessarily $\lambda$-presentable).*

A bound on the involved cardinal is the cardinal of the set of morphisms from any element set of the generating set of presentable objects to $c$. The proposition indeed follows from the following lemma, and the fact that $\lambda$-presentable objects are also $\kappa$-presentable, for $\kappa > \mu$

**Lemma 8.11.** *([1, 1.16]) A $\lambda$-small colimit of $\lambda$-presentable objects is $\lambda$-presentable.*

There is a not obvious converse to this fact:

**Proposition 8.12.** *([5, Proposition 2.3.11]) $\mu$-presentable objects are $\mu$-small $\lambda$-directed colimits of $\lambda$-presentable objects.*

**Remark 8.13.** *The fact that a full and faithful embedding between two accessible categories preserves $\lambda$-directed colimits for some $\lambda$ is undecidable [1, 2.17.(3)].*

**Proposition 8.14.** *([1, 2.24]) Let $F : A \to Set$ be a functor. Then the opposite of its category of elements $\int F$ is $\lambda$-filtered (that is, every $\lambda$-small diagram has a cone in $\int F$) if and only if it left Kan extension $\hat{A} \to Set$ along yoneda preserve $\lambda$-small limits.*

*In this case, as $F$ is a colimit over $(\int F)^o$, $F$ is a $\lambda$-filtered colimit of representable presheaves.*

**Remark 8.15.** *Such a functor is called* flat*. We can restrict to $\lambda$-small limits of representable presheaves in the statement above. We also know from Lemma 7.11 that this left Kan extension is cocontinuous.*

Recall that $\mathrm{Lan}_y F(K) = \int^a F(a) \times K(a)$. We are sloppy and use equality where isomorphism would be more appropriate.

Let $K : D \to A$. Then, $\lim FK = \int_d F(K(d)) = \int_d \int^a F(a) \times A(a, K(d)) = \int_d \mathrm{Lan}_y F(A(-, K(d)))$ which by hypothesis is $\mathrm{Lan}_y F(\int_d A(-, K(d))) = \int^a F(a) \times \int_d A(a, K(d))$. To conclude, $\int_d A(a, K(d))$ is the set of natural transformations between $a$ (as a constant functor) and $K$: that is, it is the set of cones.

*Proof of Proposition 8.14.* It is clear that if $(\int F)^o$ is $\lambda$-filtered then the left Kan extension preserves $\lambda$-small limits, because $F$ is a colimit over $(\int F)^o$, and $\lambda$-small limits commute with $\lambda$-directed colimits in Set. Let us prove the converse and suppose that the left Kan extension preserve $\lambda$-small limits of representable presheaves. We show that any $\lambda$-small diagram in $\int F$ has a cone.

By universal property of $\int F$ as the comma category $1/F$, for any category $D$, a functor $D \to \int F$ is equivalently given by:

- a functor $K : D \to A$,
- and an object $\alpha \in \hom(1, FK))$.

Note that $\hom(1, FK) = \lim(FK)$. We are going to show that if the left Kan extension of $F$ preserves $D$-shaped limits, then $\lim(FK)$ is canonically isomorphic to $\int^a F(a) \times Cone(a, K)$, where $Cone(a, K)$ is the set of cones for $K$ with tip $a$. Then, any antecedent of $\alpha$ through the surjection $\amalg_a F(a) \times Cone(a, K) \to \int^a F(a) \times Cone(a, K)$ provides a suitable cone.

Recall that $\mathrm{Lan}_y F(K) = \int^a F(a) \times K(a)$. We are sloppy and use equality where isomorphism would be more appropriate. Now, $\lim FK = \int_d F(K(d)) = \int_d \int^a F(a) \times A(a, K(d)) = \int_d \mathrm{Lan}_y F(A(-, K(d)))$ which by hypothesis is

$$\mathrm{Lan}_y F(\int_d A(-, K(d))) = \int^a F(a) \times \int_d A(a, K(d)).$$

To conclude, $\int_d A(a, K(d))$ is the set of natural transformations between $a$ (as a constant functor) and $K$: that is, it is the set of cones.

$\square$

**Remark 8.16.** *The proof of Proposition 8.14 does not constructively provide a cone, as it requires to choose an antecedent of some element by a surjective function.*

There is an alternative definition of accessible categories as free cocompletions. In the following, $A$ denotes a small category.

**Definition 8.17.** *Let $U_{Dir_\lambda}$ denotes the forgetful functor from categories with $\lambda$-directed colimits and functors preserving them to the category of small categories. We denote $Dir_\lambda(A)$ the full subcategory of $\hat{A}$ consisting of $\lambda$-directed colimits of representables presheaves.*

## 8.1 Things to know

**Theorem 8.18.** *[1, 2.47] An accessible category is complete if and only if it is cocomplete (in this case, it is locally presentable).*

# 9 Gabriel-Ulmer

What is the category of algebras for the monad $\mathrm{Pres}_\lambda \circ \mathrm{Cont}_\lambda$?

# 10 Presentable categories (old version)

Let $\lambda$ be a regular cardinal.

**Definition 10.1.** *A $\lambda$-presentable category is a category $C$ which is a free $\lambda$-cocompletion $Cont_\lambda(Pres_\lambda(C))$ of some small full subcategory $Pres_\lambda(C)$ [1, Representation theorem 1.46]. Then, $Pres_\lambda(C)$ consist of* $\lambda$-presentableobjects

Here, in this free $\lambda$-cocompletion, the right adjoint would start from cocomplete (not $\lambda$-cocomplete!) categories and go to the category of categories and $\lambda$-cocontinous functors (i.e., functors preserving any $\lambda$-small colimit, not only the directed ones).

**Remark 10.2.** *$Cont_\lambda$ is obtained by restricting the category of presheaves to those preserving $\lambda$-small limits (as the yonedas do). Beware that colimits do not compute as in the total presheaf category (I can't find an argument for its cocompleteness in [1] by the way). Is any functor still a canonical colimit of yonedas in $Cont_\lambda$? The induced cocomplete functor is obtained by restricting the universal functor from $\hat{A}$ (because of the universal property of $\hat{A}$. Note: we get a cocomplete functor from $\hat{A}$ to $Cont_\lambda$ by universal property of $\hat{A}$ . This is obviously the left adjoint to the inclusion functor (indeed, any cocontinuous functor out of $\hat{A}$ has a left adjoint, using yoneda to get the value of it)*

**Remark 10.3.** *The remark after [1, Representation theorem 1.46] says that a $\lambda$-presentable category $K$ is equivalent to $Cont_\lambda(Pres_\lambda(K))$.*

**Remark 10.4.** *I suspect that $A$ consists of presentable objects in $Cont_\lambda(A)$, but it can be less that all of them.*

Compare with the characterization of $\lambda$-accessible category:

**Definition 10.5.** *A $\lambda$-accessible category is a cocompletion $Dir_\lambda(A)$ of some small full subcategory $A$ with respect to $\lambda$-directed colimits [1, Representation theorem 2.26].*

Here, we think of the forgetful functor from categories with $\lambda$-directed colimits and functors preserving them to the category of categories as the right adjoint functor. The free stuff is obtained by restricting the category of presheaves to $\lambda$-directed colimits of yonedas.

**Question 10.6.** *Is it clear that a $\lambda$-presentable category is $\lambda$-accessible with these characterizations? Not really... I think the key point is to identify the requirements on $A$ to have $Dir_\lambda(A) = Cont_\lambda(A)$.*

**Remark 10.7.** *It is clear that $\lambda$-directed colimits of yonedas preserve $\lambda$-small colimits (because $\lambda$-small limits commute with $\lambda$-directed colimits in Set), so $Dir_\lambda(A) \subset Cont_\lambda(A)$. So I guess, we can always extend a $\lambda$-accessible category with the other presheaves that preserves small limits, and then we get a $\lambda$-presentable category.*

**Remark 10.8.** *The first remark following [1, Representation theorem 2.26] says that given a $\lambda$-accessible category $K$, we have $K = Dir_\lambda(Pres_\lambda(K))$.*

**Question 10.9.** *As $\lambda$-accessibility differs from $\lambda$-presentability, it is not always the case that $(K =)Cont_\lambda(A) = Dir_\lambda(A)$. But as any presentable category is accessible, there exists $B$ such that $K = Dir_\lambda(B)$. And after the previous remarks, we know that we can choose $B = Pres_\lambda(K)$. And then, $Cont_\lambda(B) = Dir_\lambda(B)$. In other words, $Cont_\lambda = Dir_\lambda \circ Pres_\lambda \circ Cont_\lambda$. We also know that $Cont_\lambda \circ Pres_\lambda Cont_\lambda = Cont_\lambda$ and similarly for $Dir_\lambda$.*

*What is this $B$ with respect to $A$? More precisely, what does $Pres_\lambda(Cont_\lambda(A))$ do ? Maybe a $\lambda$-small completion?? In fact, what can we say about $Pres_\lambda(K)$, for $K$ $\lambda$-accessible or presentable? For presentability, thinking of the example of presheaves, where presentable objects are finite colimits of yonedas, we might guess that $Pres_\lambda(Cont_\lambda(A))$ or $Pres_\lambda(Dir_\lambda(A))$ is the category of $\lambda$-small colimits of elements of $A$ (one should be this one, and the other is then different otherwise every accessible category would be presentable). We may guess that $Pres_\lambda \circ Dir_\lambda(A)$ is smaller that $Pres_\lambda \circ Cont_\lambda(A)$, because $Dir_\lambda(A) \subset Cont_\lambda(A)$, but I am not sure. Is $Pres_\lambda$ monotonous? I think this relates to Gabriel-Ulmer duality!! In fact $Pres_\lambda(K)$ is always small-complete when $K$ is $\lambda$-presentable. But how does it fit with finite colimits of yonedas in the case of presheaves? Oh yes, it makes sense now.. Is there Gabriel-Ulmer duality for accessible categories?*

Alternative and (not)

**Definition 10.10.** *A* ***$\lambda$-presentable object*** *in a category $C$ is an object such that its coyoneda embedding preserves $\lambda$-directed colimits.*

It says no more that any morphism from such an object to a $\lambda$-directed colimit factors (essentially uniquely) through some object of the colimiting cocone.

**Lemma 10.11.** *A $\lambda$-presentable object is also $\kappa$-presentable, for $\kappa > \lambda$.*

*Proof.* Obvious, because a $\kappa$-filtered category is also $\lambda$-filtered. □

Note that for accessible categories, we have a similar statement [1, 2.26] where the adjunction happens between the category of $\lambda$-directed cocomplete categories and the usual category of categories.

**Question 10.12.** *Can you show with these characterizations that a presentable category is accessible?*

**Definition 10.13** (Alternative definition)**.** *A $\lambda$-accessible category is defined as a category closed under $\lambda$-directed colimit and there is a small set of $\lambda$-presentable objects such that every object is a $\lambda$-directed colimit of objects of $A$.*

**Definition 10.14.** *(Another def [1, 1.20]) A locally $\lambda$-presentable category is a cocomplete category with a strong generator formed of $\lambda$-presentable objects.*

To go from the strong generator to the set of $\lambda$-presentable objects above: take $\lambda$-small colimits of objects in the strong generator.

**Question 10.15.** *How do you show the previous characterization of accessible categories?*

**Lemma 10.16.** *A locally $\lambda$-presentable category is also locally $\kappa$-presentable, for $\kappa > \lambda$.*

*Proof.* We consider the alternative definition of locally presentable category. The set of "presenting" $\kappa$-presentable objects is induced by noticing that $\lambda$-presentable objects are also $\kappa$-presentable, and form a strong generator. Then, we take $\kappa$-small colimits of them. □

## 10.1 Cauchy completion

(let us recall that the free split idempotency is called cauchy completion and gives a criterion to compare presheaf categories)

**Theorem 10.17.** *[1, 2.4] Each accessible category has split idempotents.*

**Theorem 10.18.** *Every small category with split idempotents is accessible.*

# 11 Finitely presentable categories

Any presheaf category is finitely presentable [1]: the representable functors form the strong generator, and thus the presentable objects are the finite colimits of these representable functors.

## 12 Gabriel-Ulmer

This section is based on Exercise 1.s and Remark 1.46 of [1].

**Definition 12.1.**

**Theorem 12.2.** *For each regular cardinal $\lambda$, we denote*

- *$c_\lambda CAT$ the category of small categories $\lambda$-complete and functor preserving these limits;*
- *$lp_\lambda CAT$ the category of $\lambda$-presentable categories and continuous and $\lambda$-accessible functors.*

*Then, the following functors are equivalences:*

$$\begin{aligned} R : c_\lambda CAT &\to lp_\lambda CAT \\ C &\mapsto Cont_\lambda C^o \\ L : lp_\lambda CAT &\to c_\lambda CAT \\ C &\mapsto Pres_\lambda(C)^o \end{aligned}$$

Actually, they even induce a biequivalence ($c_\lambda CAT$ and $lp_\lambda CAT$ can be equipped with a 2-categorical structure).

**Question 12.3.** *Is it a strict 2-equivalence? I guess no, otherwise it would have been said.*

This theorem induces two (equivalent) contravariant functors from the poset of regular cardinals (seen as a category) to the category of small categories. Performing the Grothendieck construction yield an equivalence between some category of presentable categories and some category of small categories (fibered over the category of regular cardinals).

## 13 Sketches

**Definition 13.1.** *A **sketch** is a small category $C$ with a family of (small) diagrams $F_i : D_i \to C$ together with a choice of a cone or a cocone for each of this diagram.*

**Definition 13.2.** *The **category of models of a sketch** $(C, (F_i, c_i)_i)$ is the full subcategory of $[C, Set]$ mapping any mapping any chosen cones and cocones to limits and colimits.*

**Theorem 13.3.** *Any accessible category is equivalent to the category of models of a sketch.*

**Theorem 13.4.** *Any presentable category is equivalent to the category of models of a **limit sketch**, that is, a sketch with only cones (and no cocones).*

**Question 13.5.** *How do you retrieve the regular cardinal from this description? Probably, it is some cardinal bound over the size of the diagrams?*

**Theorem 13.6.** *(nlab) A category is $\lambda$-accessible iff it is equivalent to a full subcategory of a presheaf category closed under $\lambda$-filtered colimits.*

**Question 13.7.** *How does the raising of the indexing regular cardinal translates into this definition?*

**Theorem 13.8.** *A category is presentable if it is equivalent to a reflective subcategory of a presheaf category.*

Note that a accessible category is presentable precisely if the embedding into the presheaf category has a left adjoint, indeed:

**Theorem 13.9.** *[5, Proposition 2.4.8] or [1, 2.23] Any right adjoint functor between accessible categories is accessible.*

**Theorem 13.10.** *[1, Corollary 2.62] For any sketch, there exists another one whose cones/cocones are all limiting/colimiting and has an equivalent category of models.*

By Gabriel-Ulmer duality, we already know that for limit sketches: you can take the $\lambda$-complete category corresponding to the presentable category and take all the $\lambda$-small limits as cones.

**Question 13.11.** *What does exactly this construction on a sketch $S$ do?*

# 14 GATs and EATs

generalized algebraic theories (or gats) are defined in [6], and also more formally (and recently) in [7]

## 14.1 QIITs are (particular) GATs

It is not exactly clear what are GATs introduced by [6], in particular what is the status of equations of sorts. GATs are like QIITs but with a possible infinite context. More concretely, a GAT consists of a possibly infinite set of axiomatic judgments, and each judgement must be wellformed (with respect to a finite subset of axioms). Infinitary QIITs allow to capture some of them, but not all.

**Question 14.1.** *Can we find a counter example?*

Also, in QIITs, we can have equality constructors taking equalities in arguments, that we cannot do directly with GATs. But fortunately, such constructors can be encoded by introducing new types in the universe which are the limits of other ones (such incorporating the equations).

**Question 14.2.** *[6] claims that GATs are equivalent to EATs, but how to deal with equations of sorts? How are they converted in the EAT version ?*

Cartmell also proves that they are equivalent to contextual categories, and for this he needs these sorts equations (see (iii) at the top of p107 of his PhD manuscript).

Taichi sees two difficulties when trying to replace sort equations with isomorphisms:

- If we replace a sort equation $A = B$ with a sort isomorphism $e : A \simeq B$, we also have to replace occurences of coercion along $A = B$ with applications of $e$.
- A naive replacement can create a different GAT. For example, consider the GAT consisting of a constant sort A and equation $A = A$ (although this equation is redundant, it is a valid GAT). A model of $(A, A = A)$ is just a set, but a model of $(A, e : A \simeq A)$ is a set equipped with an automorphism. They do not seem to be equivalent.

I expect that 1. is possible, although I confess I don't know how to do it in general. For the second point, I would suggest the following: given any sort equation $A = A$ that you can prove in the theory induced by a GAT with sort equations, you get (I hope) an automorphism on the translation of $A$ (in the translated GAT which turns sort equations into isomorphisms). Then, for each such sort equation, add in the translated GAT that this automorphism should be the identity.

### 14.2 QIITs are QITs (as GATs are EATs)?

No! As noticed by Ambrus, EATs allow equations in the arguments (i.e., partial constructors), whereas QITs do not.

## 15 Equational systems?

## A Some useful categorical results

**Lemma A.1.** *A reflective subcategory of a cocomplete category is cocomplete.*

*Proof.* Take the colimit in the cocomplete category, and apply the left adjoint. □

As already mentioned, a colimit in the full subcategory does not coincide with the one computed in the total category unless the latter is essentially in the subcategory.

**Proposition A.2.** *In Set, $\lambda$-filtered colimits are exactly those commuting with any $\lambda$-small limit.*

**Lemma A.3.** *Let $A$ be a small category, $C$ be full subcategory of $\hat{A}$ containing representable presheaves, and $B$ be a cocomplete category. Then the left Kan extension of any functor $F : A \to B$ along the yoneda embedding into $C$ maps a presheaf $G$ to the coend $\int^a F(a) \times G(a)$. In particular, it is cocontinuous.*

*Proof.* Obvious when unfolding the coend formula of left Kan extension. $\square$

**Lemma A.4.** *Let $A$ be a small category, $C$ be a reflective subcategory of $\hat{A}$ containing representable presheaves, and $B$ be a cocomplete category. Then the left Kan extension of any functor $F : A \to B$ along the yoneda embedding into $C$ is the unique cocontinuous functor which postcomposed with the yoneda embedding yields $F$.*

*Proof.* Suppose you have two such functors $F$ and $F'$ and let us denote $L$ the left adjoint to the embedding of $C$ in $\hat{A}$. As $\hat{A}$ is the free cocompletion, $F \circ L = F' \circ L$. But then, by precomposing with the full and faithful embedding $R$ and exploiting the fact that $L \circ R$ is isomorphic to the identity, we get the desired result. $\square$

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