An abstract Howe theorem

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LIMD
September 2020

Motivation: generalisation of theorem statements

- Often, theorems are stated for one "typical" programming language.
- Goal: provide high-level tools for stating them for all suitable languages and models.

State of the art

- Formats: Tyft/tyxt, GSOS, PATH,...
  - Do not cover denotational models (exclusively syntactic).
  - Low-level.
- Bialgebraic semantics (Turi and Plotkin '97).
  - Deeply developed.
  - Very good at quantitative semantics.
  - Functional languages only starting to be investigated (Peressotti '17).
- Transition monads (Hirschowitz et al. '20).
  - Focus on signatures (less primitive than ours).
  - No big metatheoretical theorem (yet).
- Previous work on cellular monads (POPL '19, SOS/EXPRESS '19).
  - Does not cover higher-order languages.
  - Virtually no notion of signature, models constructed by hand.

Summary of contributions

1. General setting: Howe context.
2. Notion of signature for programming languages, in any Howe context.
3. Definition of substitution-closed bisimilarity.
   Particular case: open extension of Abramsky's applicative bisimilarity in cbn λ-calculus.
4. A semantic format for congruence of bisimilarity:

Main theorem

If the signature preserves functional bisimulations (plus mild technical hypotheses), then substitution-closed bisimilarity is a congruence.

Proof: abstract analogue of Howe's method.

This talk

Sketch main ideas on one example, big-step, cbn λ-calculus.

1. Introduction
2. Brief recap on applicative bisimilarity
3. Howe context for cbn λ
4. Models of syntax
5. Models of transition rules
6. Substitution-closed bisimilarity
7. Main result
8. Conclusion

Call-by-name λ-calculus

Slightly non-standard presentation.

$$\frac{}{\lambda x.e \Downarrow e} \qquad \frac{e_1 \Downarrow e_3 \qquad e_3[e_2] \Downarrow e_4}{e_1\ e_2 \Downarrow e_4}$$

Typing: $\Downarrow \quad \subseteq \quad$ closed terms $\quad \times \quad$ terms with 1 free variable.
Example:

Applicative bisimilarity

Definition

A relation R between terms is

- *substitution-closed* iff

$$e_1 \; R \; e_2 \qquad \text{then} \qquad e_1[\sigma] \; R \; e_2[\sigma].$$

- a *bisimulation* iff

$$\begin{array}{ccc}
e_1 - R - e_2 & & e_1 - R - e_2 \\
\Downarrow \qquad\qquad \Downarrow & \text{and} & \Downarrow \qquad\qquad \Downarrow \\
e_1' - R - e_2' & & e_1' - R - e_2'.
\end{array}$$

(for all / exists).

Standard applicative bisimilarity := largest substitution-closed bisimulation. Notation $\sim^{\otimes}$.

Congruence theorem

Theorem

Applicative bisimilarity is a congruence, in particular

- $e_1 \sim^{\otimes} e_2$ *entails* $\lambda x.e_1 \sim^{\otimes} \lambda x.e_2$,
- $e_1 \sim^{\otimes} e_2$ *and* $e_3 \sim^{\otimes} e_4$ *entails* $e_1\ e_3 \sim^{\otimes} e_2\ e_4$.

Naive proof attempt

- Let $\sim^\bullet$ denote the context closure of $\sim^\otimes$.
- Prove that it is a bisimulation.
  Indeed, if so,
  - $\sim^\otimes \subseteq \sim^\bullet \subseteq \sim^\otimes$, hence
  - $\sim^\otimes = \sim^\bullet$,
  - but $\sim^\bullet$ is context-closed.

But! Hard to prove the bisimulation property.

$$\begin{tikzcd}
e_1\ e_2 \arrow[r, dash, "\sim^\bullet"] \arrow[d] & e'_1\ e'_2 \arrow[d] \\
(\lambda x.e_3)\ e_2 \arrow[r, dash, "\sim^\bullet"] \arrow[d] & (\lambda x.e'_3)\ e'_2 \arrow[d] \\
e_3[e_2] \arrow[r, dash, "\sim^\bullet ?"] & e'_3[e'_2]
\end{tikzcd}$$

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e_3[e_2] \arrow[r, dash, dashed, "\sim^\bullet ?"] & e'_3[e'_2]
\end{tikzcd}
$$

Howe closure

Solution: context closure + $\dfrac{e \sim^{\bullet} e' \sim e''}{e \sim^{\bullet} e''}$

What now?

- Categorical point of view on
  - syntax,
  - dynamics,
  - substitution-closed bisimulation,
  - Howe's method.
- Amenable to generalisation: see paper.

Main question for us

What is a model of call-by-name λ-calculus?

Models of syntax

The category $\mathbb{F} := \mathbf{Set}_f^{op}$

$$\begin{tikzcd} 0 & 1 \arrow[l] \arrow[r] & 2 \arrow[l, bend right] \arrow[l, bend left] & \ldots & n & \ldots \end{tikzcd}$$

Let $X \in \widehat{\mathbb{F}}$, presheaf on $\mathbb{F}$, i.e., $X \colon \mathbf{Set}_f \to \mathbf{Set}$.

- $X(n)$: "terms" with potential free variables in $\{x_1, \ldots, x_n\}$.
- $X(n) \xrightarrow{X(f)} X(m)$: "renaming".

Example

- $L(n) =$ actual terms over n.
- $$\begin{array}{rcl} L(2) & \xrightarrow{L(\mathsf{swap})} & L(2) \\ \lambda x.(x_1\ x_2\ x) & \mapsto & \lambda x.(x_2\ x_1\ x). \end{array}$$

Models of syntax

Model of syntax: $X \in \widehat{\mathbb{F}}$ equipped with

Operations $\lambda_n \colon X(n+1) \to X(n) \qquad app_n \colon X(n)^2 \to X(n)$.

Substitution
- Let $(Y \otimes Z)(n) = \sum_p Y(p) \times Z(n)^p$ (modulo std eqs.).
- Elements $y(\!|\zeta|\!)$ are like formal substitutions.
- Substitution: $\begin{array}{rcl} m_X \colon X \otimes X & \to & X \\ x(\!|\chi|\!) & \mapsto & x[\chi]. \end{array}$

Variables $e_X \colon I \to X$, where $I(n) = n$ (notation for $\{1, \ldots, n\}$).

Model of syntax = Σ_0-monoid :=

$X \in \widehat{\mathbb{F}}$ + app, λ, substitution, variables + compatibility conditions.

Free Σ_0-transition monoids

Definition

Category $\Sigma_0\,\textbf{-mon}$ of Σ_0-monoids.

$$\begin{tikzcd} \widehat{\mathbb{F}} \arrow[r, bend left, "\mathscr{L}_0"] \arrow[r, phantom, "\perp"] & \Sigma_0\,\textbf{-mon} \arrow[l, bend left, "\mathscr{U}_0"] \end{tikzcd}$$

Syntax $:= \mathscr{L}_0(\varnothing)$.

Remark

"High-level" definition of syntax: no fuss about α-equivalence.

Transition systems

The category $\mathbb{F}^{\Downarrow}$

$$\begin{tikzcd}
 & \Downarrow & & & \\
0 \arrow[ur, "s"] & & 1 \arrow[ul, "t"'] \arrow[ll] \arrow[r] & 2 \arrow[l, bend right] \arrow[l, bend left] & \ldots & n
\end{tikzcd}$$

Let $X \in \widehat{\mathbb{F}^{\Downarrow}}$, presheaf on $\mathbb{F}^{\Downarrow}$, i.e., $X \colon (\mathbb{F}^{\Downarrow})^{op} \to \mathbf{Set}$.

- $X(n)$, $X(f)$: "terms" and "renaming" as before.
- $X(\Downarrow)$: "evaluation witnesses".
- $X(s) \colon X(\Downarrow) \to X(0)$: source/input.
- $X(t) \colon X(\Downarrow) \to X(1)$: body of value.

Transition system with syntactic structure on states

Consider the following pullback in **CAT**.

$$\begin{tikzcd} \Sigma_0\text{-}\mathbf{Mon} \arrow[r, "\mathscr{D}"'] \arrow[d, "\mathscr{U}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma_0\text{-}\mathbf{mon} \arrow[d, "\mathscr{U}_0"] \\ \widehat{\mathbb{F}^{\Downarrow}} \arrow[r, "\Delta_{\mathbb{F}}"'] & \widehat{\mathbb{F}} \end{tikzcd}$$

- Objects: $X \in \widehat{\mathbb{F}^{\Downarrow}}$ with Σ_0-monoid structure on the restriction $\Delta_{\mathbb{F}}(X) \in \widehat{\mathbb{F}}$.
- Name: transition Σ_0-monoids.

Transition system with syntactic structure on states

Consider the following pullback in $\mathbf{CAT}$.

$$\begin{tikzcd}
\Sigma_0\,\text{-}\mathbf{Mon} \arrow[r, "\mathscr{D}"'] \arrow[d, "\mathscr{U}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma_0\,\text{-}\mathbf{mon} \arrow[l, bend right, "\mathscr{M}"'] \arrow[d, "\mathscr{U}_0"] \\
\widehat{\mathbb{F}^{\Downarrow}} \arrow[u, bend left, "\mathscr{L}"] \arrow[r, "\Delta_{\mathbb{F}}"'] & \widehat{\mathbb{F}}
\end{tikzcd}$$

- Objects: $X \in \widehat{\mathbb{F}^{\Downarrow}}$ with Σ_0-monoid structure on the restriction $\Delta_{\mathbb{F}}(X) \in \widehat{\mathbb{F}}$.
- Name: transition Σ_0-monoids.

Both projections have left adjoints!

Models of transition rules

Remember our variant of cbn λ-calculus:

$$\frac{}{\lambda x.e \Downarrow e} \qquad\qquad \frac{e_1 \Downarrow e_3 \qquad e_3[e_2] \Downarrow e_4}{e_1\ e_2 \Downarrow e_4}$$

Model of rules: transition Σ_0-monoid $X \in \widehat{\mathbb{F}^{\Downarrow}}$ equipped with

$$X(1) \to X(\Downarrow) \qquad \text{and} \qquad A_\beta(X) \to X(\Downarrow)$$

where $A_\beta(X) = \{(r_1, e_2, r_2) \mid r_2 \cdot s = (r_1 \cdot t)[e_2]\}$,
+ compatibility conditions for source and target.

Models of rules as algebras

Lemma

Models of rules are vertical algebras for a suitable endofunctor

$$\begin{tikzcd}
\Sigma_0\text{-}\mathbf{Mon} \arrow[rr, "\check{\Sigma}_1"] \arrow[dr] & & \Sigma_0\text{-}\mathbf{Mon} \arrow[dl] \\
& \Sigma_0\text{-}\mathbf{mon} &
\end{tikzcd}$$

Proof: $\check{\Sigma}_1(X)(\Downarrow) = X(1) + A_\beta(X)$...

Syntactic transition system

Theorem

$$
\begin{tikzcd}
\widehat{\mathbb{F}^{\Downarrow}} \arrow[r, bend left, "\mathscr{L}"] \arrow[r, phantom, "\perp"] & \Sigma_0\text{-}\mathbf{Mon} \arrow[l, bend left, "\mathscr{U}"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \textit{models of rules} \arrow[l, bend left]
\end{tikzcd}
$$

Syntactic transition system $\mathbf{Z}$:= initial model.

Yoneda

Some presheaves in $\widehat{\mathbb{F}^{\Downarrow}}$:

- $\mathbf{y}_0$: just one element over 0.
- $\mathbf{y}_1$: just one element over 1.
- $\mathbf{y}_{\Downarrow}$: just one element over $\Downarrow$ + its source and target.

Functional bisimulation

A morphism f in $\widehat{\mathbb{F}^{\Downarrow}}$ is a functional bisimulation iff

$$\begin{tikzcd} \mathbf{y}_0 \arrow[r, "x"] \arrow[d, "\mathbf{y}_s"'] & X \arrow[d, "f"] \\ \mathbf{y}_{\Downarrow} \arrow[r, "e'"'] \arrow[ur, dashed, "e"] & Y \end{tikzcd}$$

i.e., concretely

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "e"'] & f(x) \arrow[d, "e'"] \\ x' \arrow[r, mapsto] & y. \end{tikzcd}$$

Notation

$\mathbf{y}_s \boxslash f$, $f \in \{\mathbf{y}_s\}^{\boxslash}$.

Bisimulation

Definition

In $\widehat{\mathbb{F}^{\Downarrow}}$, a span $X \leftarrow R \rightarrow Y$ is a bisimulation iff both legs are functional bisimulations.

Substitution-closed spans

For $X \in \Sigma_0\text{-}\mathbf{Mon}$, a span $R \to X \times X$ is substitution-closed iff (omitting $\mathscr{D}$ for readability):

$$\begin{tikzcd} R \otimes X \arrow[rr, dashed] \arrow[d] & & R \arrow[d] \\ X^2 \otimes X \arrow[r] & (X \otimes X)^2 \arrow[r] & X^2. \end{tikzcd}$$

Essentially:

$$x_1 \; R \; x_2 \qquad \text{entails} \qquad x_1[\sigma] \; R \; x_2[\sigma].$$

Substitution-closed bisimulation relation = applicative bisimulation.

Substitution-closed bisimilarity

Proposition

For any $X \in \Sigma_0\text{-}\mathbf{Mon}$, there is a terminal substitution-closed bisimulation, $\sim_X^{\otimes}$, called substitution-closed bisimilarity.

Relevance:

- Recall $\mathbf{Z}$, the initial model.
- One can prove that $\sim_{\mathbf{Z}}^{\otimes}$ coincides with the relation originally considered by Howe: open extension of applicative bisimilarity.

Generalisation

cbn λ	general case
$\mathbb{F} \hookrightarrow \mathbb{F}^{\Downarrow}$	"two-level" category $\mathbb{C}_0 \hookrightarrow \mathbb{C}$
$\otimes$	monoidal structure on $\widehat{\mathbb{C}_0}$
$\mathbf{y}_0 \to \mathbf{y}_{\Downarrow} \leftarrow \mathbf{y}_1$	"border inclusions" from level 0
Σ_0	any "pointed strong" endofunctor
...	...

Main result

Theorem

For any suitable signature (Σ_0, Σ_1), substitution-closed bisimilarity on the initial model $(\sim^{\otimes}_{\mathbf{Z}})$ is a congruence.

$$\begin{tikzcd}
\Sigma_0(\sim^{\otimes}_{\mathbf{Z}}) \arrow[rr, dashed] \arrow[d] & & \sim^{\otimes}_{\mathbf{Z}} \arrow[d] \\
\Sigma_0(\mathbf{Z}^2) \arrow[r] & \Sigma_0(\mathbf{Z})^2 \arrow[r] & \mathbf{Z}^2
\end{tikzcd}$$

Essentially:

$$e_1 \sim^{\otimes}_{\mathbf{Z}} e'_1, \ldots, e_n \sim^{\otimes}_{\mathbf{Z}} e'_n \qquad \text{entails} \qquad op(e_1, \ldots, e_n) \sim^{\otimes}_{\mathbf{Z}} op(e'_1, \ldots, e'_n).$$

What's suitable?

Lemma

$\widehat{\mathbb{F}^{\Downarrow}}$ *is isomorphic to the category of triples* $(X_0 \in \widehat{\mathbb{F}}, X_1 \in \mathbf{Set}, \partial_X)$, *where*

$$\begin{array}{c} X_{\Downarrow} \\ \big\downarrow{\scriptstyle \partial_X} \\ X_0(0) \times X_0(1). \end{array}$$

Σ_0 -$\mathbf{Mon}$*: same with* Σ_0*-monoid structure on* X_0.

What's suitable?

Definition

The functor $\check{\Sigma}_1$ is suitable iff it may be decomposed as

$$\begin{tikzcd}
X_{\Downarrow} \arrow[d] & \overset{\Sigma_1}{\longmapsto} & \Sigma_1(X)_{\Downarrow} \arrow[d] & \longmapsto & \Sigma_1(X)_{\Downarrow} \arrow[d] \\
X_0(0) \times X_0(1) & & \Sigma_0(X_0)(0) \times X_0(1) & & \Sigma_0(X_0)(0) \times X_0(1) \arrow[d] \\
& & & & X_0(0) \times X_0(1)
\end{tikzcd}$$

such that Σ_1 preserves functional bisimulations.

Rigorous definition

The category $\mathbb{F}_{\mathbf{s}}$

$$\begin{tikzcd}
 & \Downarrow & & & & \\
0 \arrow[ur, "s"] & & 1 \arrow[ll] \arrow[r] & 2 \arrow[l, bend right] \arrow[l, bend left] & \dots & n
\end{tikzcd}$$

Dynamic signatures $\check{\Sigma}_1$ in fact defined to induce

$$\begin{tikzcd}
\Sigma_0\text{-}\mathbf{Mon} \arrow[r, "\Sigma_1"] \arrow[d] & \widehat{\mathbb{F}_{\mathbf{s}}} \arrow[d] \\
\widehat{\mathbb{F}} \arrow[r, "\Sigma_0"'] & \widehat{\mathbb{F}}
\end{tikzcd}$$

Definition

Functional bisimulation in $\widehat{\mathbb{F}_{\mathbf{s}}}$: $\{s\}^{\boxslash}$.

Why is cbn λ suitable?

Lemma

If Σ_1 is familial then

$$\textit{suitable} \iff \textit{cellular}.$$

- Cellular $\approx$ input arities of rules are in ${}^{\boxslash}(\{s\}^{\boxslash})$
  $=$ input arities of rules are functional cobisimulations.
- Let us see why cbn λ input arities are cellular.

Input arity

By example: $\dfrac{e_1 \Downarrow e_3 \qquad e_3[e_2] \Downarrow e_4}{e_1\ e_2 \Downarrow e_4}$

Goal

Find E_β such that $\Sigma_0\text{-}\mathbf{Mon}(E_\beta, X) \cong A_\beta(X)$, naturally in X.

$$
\begin{tikzcd}
\mathscr{L}(\mathbf{y}_0) \arrow[r, "\mathscr{L}(\mathbf{y}_s)"] \arrow[d, "(k_\Downarrow \cdot t)[k_0]"'] & \mathscr{L}(\mathbf{y}_\Downarrow) \arrow[d] \\
\mathscr{L}(\mathbf{y}_\Downarrow + \mathbf{y}_0) \arrow[r] & E_\beta \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
A_\beta(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X(\Downarrow) \arrow[d, "x_\Downarrow \mapsto x_\Downarrow \cdot s"] \\
X(\Downarrow) \times X(0) \arrow[r, "x_\Downarrow, x_0 \mapsto (x_\Downarrow \cdot t)[x_0]"'] & X(0)
\end{tikzcd}
$$

Cellularity

The composite $\mathscr{L}(\mathbf{y}_0 + \mathbf{y}_0) \to \mathscr{L}(\mathbf{y}_\Downarrow + \mathbf{y}_0) \to E_\beta$ is in ${}^{\boxslash}(\{s\}^{\boxslash})$.

Cellularity for cbn λ-calculus

Lemma

Stability properties for functional cobisimulations.

- *Contain s and all isomorphisms.*
- *Closed under (transfinite) composition.*
- *Closed under pushouts.*
- *Closed under retracts.*

$$\begin{tikzcd}
\mathscr{L}(\mathbf{y}_0) \arrow[r] \arrow[d] & \mathscr{L}(\mathbf{y}_{\Downarrow}) \arrow[d] & \\
\mathscr{L}(\mathbf{y}_0 + \mathbf{y}_0) \arrow[r] & \mathscr{L}(\mathbf{y}_{\Downarrow} + \mathbf{y}_0) \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & E_\beta
\end{tikzcd}$$

Cellularity for cbn λ-calculus

Lemma

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$$\begin{tikzcd}
 & \mathscr{L}(\mathbf{y}_0) \arrow[r] \arrow[d] & \mathscr{L}(\mathbf{y}_{\Downarrow}) \arrow[d] \\
\mathscr{L}(\mathbf{y}_0 + \mathbf{y}_0) \arrow[r] & \mathscr{L}(\mathbf{y}_{\Downarrow} + \mathbf{y}_0) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & E_\beta
\end{tikzcd}$$

Summary

Semantic format for congruence of substitution-closed bisimilarity

Input arities should be functional cobisimulations.

- Shown here: example of cbn λ.
- In the paper: cbv λ.

Perspectives

- More examples!
- Other kinds of bisimilarity: normal form, environmental, contextual,... (+ weak variants)
- Other kinds of results: type soundness, compiler correctness...