A categorical diagram editor to help formalising commutation proofs

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- **Diagrammatic reasoning** is useful

$$
\begin{tikzcd}
\Sigma(X) \otimes X \arrow[r, "\Sigma(X)\otimes\eta^S_X"] \arrow[dr, equal] \arrow[dd, "a\otimes X"'] & \Sigma(X) \otimes S(X) \arrow[r, "st_{X,(X,v_X)}"] \arrow[d, "\Sigma(X)\otimes\bar{a}"'] & \Sigma(X \otimes SX) \arrow[r, "in_2"] \arrow[d, "\Sigma(X\otimes\bar{a})"'] & \Sigma^+(X \otimes SX) \arrow[r, "\eta_{\Sigma^+,in_1}"] & S(X \otimes SX + X + X) \arrow[d, "S(X\otimes\bar{a}+X+X)"'] \\
 & \Sigma(X) \otimes X \arrow[r, "st_{X,(X,v)}"'] & \Sigma(X \otimes X) \arrow[r, "in_2"'] \arrow[d, "\Sigma(s)"] & \Sigma^+(X \otimes X) \arrow[r, "\eta_{\Sigma^+,in_1}"'] \arrow[d, "\Sigma^+(s)"] & S(X \otimes X + X + X) \arrow[d, "S[s,X,X]"'] \\
X \otimes X \arrow[dr, "s"'] & & \Sigma(X) \arrow[r, "in_2"] \arrow[d, "a"] & \Sigma^+(X) \arrow[r, "\eta_{\Sigma^+,X}"] & SX \arrow[dll, "\bar{a}"] \\
 & & X & &
\end{tikzcd}
$$

× Theorem provers (e.g., Coq) are **text-based**

⇒ This talk: a tool to bridge the gap

Yet Another Diagram Editor (YADE)

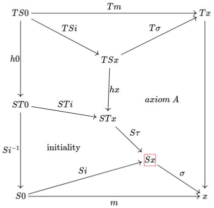
Proof assistant (Coq)

```
Lemma commutes :
  n_y ∘ F f = G f ∘ n_x.
```

Proof / statement display

Proof generation

Diagram editor (YADE)



More about YADE

- Available online[1]: run by your browser
- Written in **Elm** (~6000 LoC)

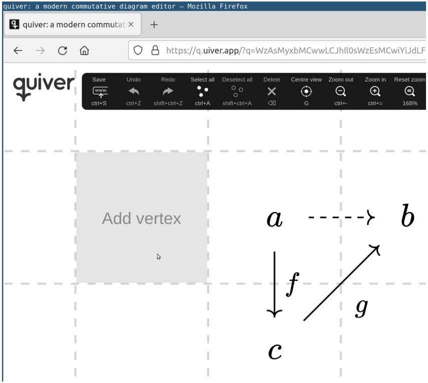
"A delightful language for reliable web applications."

- Functional PL
- Compiles to js

[1] https://amblafont.github.io/graph-editor/index.html

Comparison with Quiver

"quiver[1] *is a modern, graphical editor for commutative and pasting diagrams, capable of rendering high-quality diagrams for screen viewing, and exporting to LaTeX via tikz-cd."*



Quiver features:

- More display options (colors, ...)
- Rigid grid layout
- Export to LaTeX
- × No proof generation

Note: YADE exports to quiver

[1]https://q.uiver.app/

This talk

- Short presentation of YADE's features that help formalisation
- Demo

Plan

1. Parse equations
2. Forward reasoning
3. Backward reasoning
4. Demo

Plan

A simple format for equations

```
a -- f -> b -- g -> c  =  a -- h -> d -- k -> c
```

$$\Downarrow$$

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h"'] & b \arrow[d, "g"] \\ d \arrow[r, "k"'] & c \end{tikzcd}$$

Load an equation from Coq

Specific Coq notations to comply to the above format

Plan

Diagrams are proofs

$$
\begin{tikzcd}
X \arrow[dr, dashed, "m"] \arrow[drr, bend left, "\boxed{u}"] \arrow[ddr, bend right, "\boxed{v}"'] & & \\
& A \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "(3)"] & B \arrow[d, "\boxed{f}"] \\
& C \arrow[r, "\boxed{g}"'] & D
\end{tikzcd}
$$

(1) (2)

Generated formal proof

If inner subdiagrams (1)-(3) commute,
then the outer diagram commutes.

Algorithm for proof generation

$$
\begin{tikzcd}
X \arrow[dr, dashed, "m"] \arrow[drr, bend left, "\boxed{u}"] \arrow[ddr, bend right, "\boxed{v}"'] & & \\
& A \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[ur, phantom, "(1)"] \arrow[dl, phantom, "(2)"] \arrow[dr, phantom, "(3)"] & B \arrow[d, "\boxed{f}"] \\
& C \arrow[r, "\boxed{g}"'] & D
\end{tikzcd}
$$

1. Identify all subdiagrams
2. Start from one branch of the outer diagram
3. Repeatedly “apply” subdiagrams to progress, until reaching the other branch.

Target: the UniMath mathematical library

History

- started in 2014 with Voevodsky's repository Foundations
- Large Coq mathematical library ($\sim$300 000 lines)
  - $\sim$ two thirds on (bi)category theory
  - Verbose style
- Inconsistent: universe level checks are disabled ($\Rightarrow$ impredicative encoding of quotients)

Why targeting this library?

- I use UniMath in my formalisation projects.
- The implementation could be easily adapted to another target.

Commutation proofs in UniMath

$$
\begin{tikzcd}
TS0 \arrow[rrrr, bend left, "Tm"] \arrow[drr] \arrow[dd, "\boxed{h0}"'] & & & & Tx \arrow[dddd, bend left, "\tau"] \\
& & TSx \arrow[urr] \arrow[d] & & \\
ST0 \arrow[rr] \arrow[dd, "\boxed{Si^{-1}}"'] & & STx \arrow[dr] & & \\
& & & Sx \arrow[dr, "\boxed{\sigma}"] \arrow[d, phantom, "(5)"] & \\
S0 \arrow[urrr, "\boxed{Si}"] \arrow[rrrr, "\boxed{m}"'] & & & & x
\end{tikzcd}
$$

Applying[1] (5) to the bottom left branch

$$
\begin{aligned}
& (h_0 \cdot Si^{-1}) \cdot m \\
& \quad \big\downarrow \ \textit{rewrite assoc'} \\
& h_0 \cdot (Si^{-1} \cdot m) \\
& \quad \big\downarrow \ \textit{apply cancel\_precomposition} \\
& Si^{-1} \cdot m \\
& \quad \big\downarrow \ \textit{apply cancel\_precomposition} \\
& m \\
& \quad \big\downarrow \ \textit{apply (5)} \\
& Si \cdot \sigma
\end{aligned}
$$

[1] rewrite (5) sometimes works but is hard to maintain.

Generated Coq script

$$
\begin{tikzcd}
X \arrow[dr, dashed, "m"] \arrow[drr, bend left, "\boxed{u}"] \arrow[ddr, bend right, "\boxed{v}"'] \arrow[drr, phantom, "(1)"] \arrow[ddr, phantom, "(2)"] & & \\
& A \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "(3)"] & B \arrow[d, "\boxed{f}"] \\
& C \arrow[r, "\boxed{g}"'] & D
\end{tikzcd}
\Longrightarrow
$$

```
Goal { u · f = v · g }.

assert(eq : { u = m · p_1 }).
{ admit. }
etrans.
{
  apply cancel_postcomposition.
  apply eq.
}
clear eq.
assert(eq : { p_1 · f = p_2 · g }).
{ admit. }
etrans.
{
  repeat rewrite assoc'.
  apply cancel_precomposition.
  repeat rewrite assoc.
  apply eq.
}
repeat rewrite assoc.
clear eq.
assert(eq : { m · p_2 = v }).
{ admit. }
etrans.
{
  apply cancel_postcomposition.
  apply eq.
}
clear eq.
apply idpath.
Qed.
```

Commutation of subdiagrams are explicitly asserted and admitted.

Plan

1. Parse equations
2. Forward reasoning
3. **Backward reasoning**
4. Demo

Forward vs Backward reasoning in Coq

$n : G \Rightarrow F$ natural.

How to rewrite $n_x \cdot Ff$ into $Gf \cdot n_y$?

Forward reasoning in Coq

1. State naturality specialised to $f : x \to y$

$$n_x \cdot Ff = Gf \cdot n_y$$

2. Prove the above equation by naturality
3. Apply the equation

Backward reasoning in Coq

Directly apply naturality of n

$\Rightarrow$ No need to provide the r.h.s $Gf \cdot n_y$

Backward reasoning with YADE

1. Generate Coq script that focuses on the subterm
2. Manually apply the lemma, in Coq
3. Copy and paste the result in YADE

$\Rightarrow$ The diagram is completed automatically

$$
\begin{tikzcd}
a \arrow[r, "u"] \arrow[ddr, bend right, "\dots"'] & Gx \arrow[r, "\boxed{n_x}"] \arrow[dr] & Fx \arrow[d, "\boxed{Ff}"] \\
& & Fy \arrow[dl, "v"] \\
& b &
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
a \arrow[r, "u"] \arrow[dddrr, bend right, "\dots"'] & Gx \arrow[rr, "\boxed{n_x}"] \arrow[dr, "\boxed{Gf}"'] & & Fx \arrow[dd, "\boxed{Ff}"] \\
& & Gy \arrow[dr, "\boxed{n_y}"'] & \\
& & & Fy \arrow[dl, "v"] \\
& & b &
\end{tikzcd}
$$

Plan

Let

- $\delta : ST \to TS$ be a monadic distributive law.
- $R := TS$
- $\mu' : RR \to R$ defined in the only sensible way.

Then,

$$\begin{tikzcd} RRR \arrow[r, "R\mu'"] \arrow[d, "\mu' R"'] & RR \arrow[d, "\mu'"] \\ RR \arrow[r, "\mu'"'] & R \end{tikzcd}$$