A categorical diagram editor to help formalising commutation proofs

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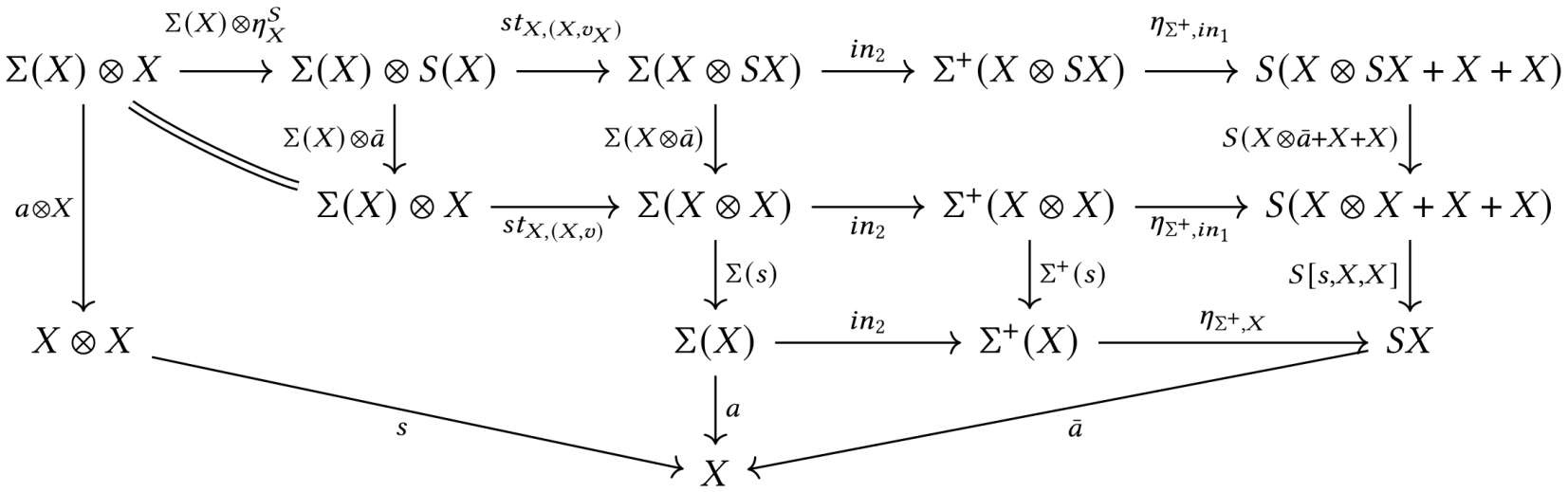
Motivation

- **Diagrammatic reasoning** is useful



× Theorem provers (e.g., Coq) are **text-based**

⇒ This talk: a tool to bridge the gap

Yet Another Diagram Editor (YADE)

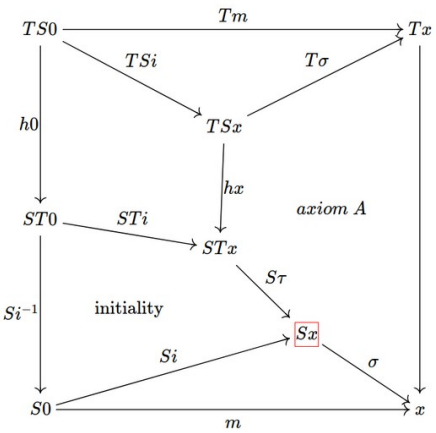
Proof assistant (Coq)

```
Lemma commutes :
  n_y ∘ F f = G f ∘ n_x.
```

Proof / statement display

Proof generation

Diagram editor (YADE)



More about YADE

- Available online[1]: run by your browser
- Written in **Elm** (~6000 LoC)

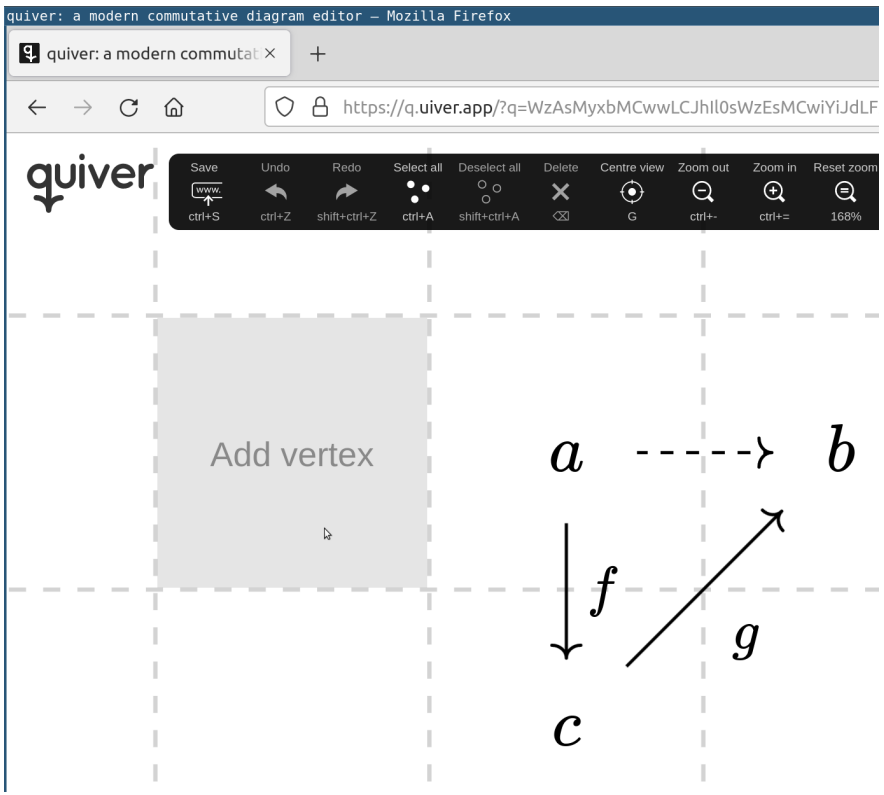
"A delightful language for reliable web applications."

- Functional PL
- Compiles to js

[1] `https://amblafont.github.io/graph-editor/index.html`

Comparison with Quiver

"quiver[1] is a modern, graphical editor for commutative and pasting diagrams, capable of rendering high-quality diagrams for screen viewing, and exporting to LaTeX via tikz-cd."



Quiver features:

- More display options (colors, ...)
- Rigid grid layout
- Export to LaTeX
- × No proof generation

Note: YADE exports to quiver

[1] https://q.uiver.app/

This talk

- Short presentation of YADE's features that help formalisation
- Demo

Plan

1. Parse equations
2. Forward reasoning
3. Backward reasoning
4. Demo

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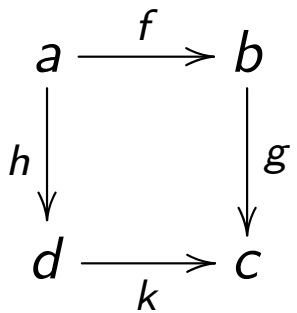
A simple format for equations

```
a -- f -> b -- g -> c  =  a -- h -> d -- k -> c
```



Load an equation from Coq

Specific Coq notations to comply to the above format

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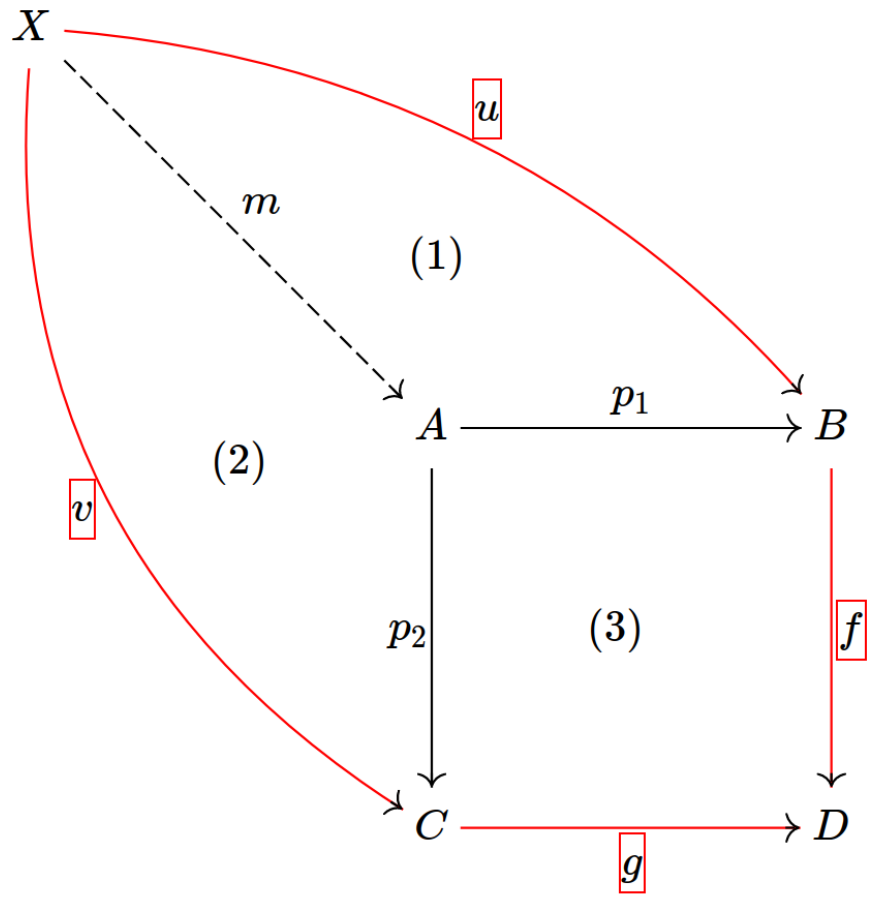
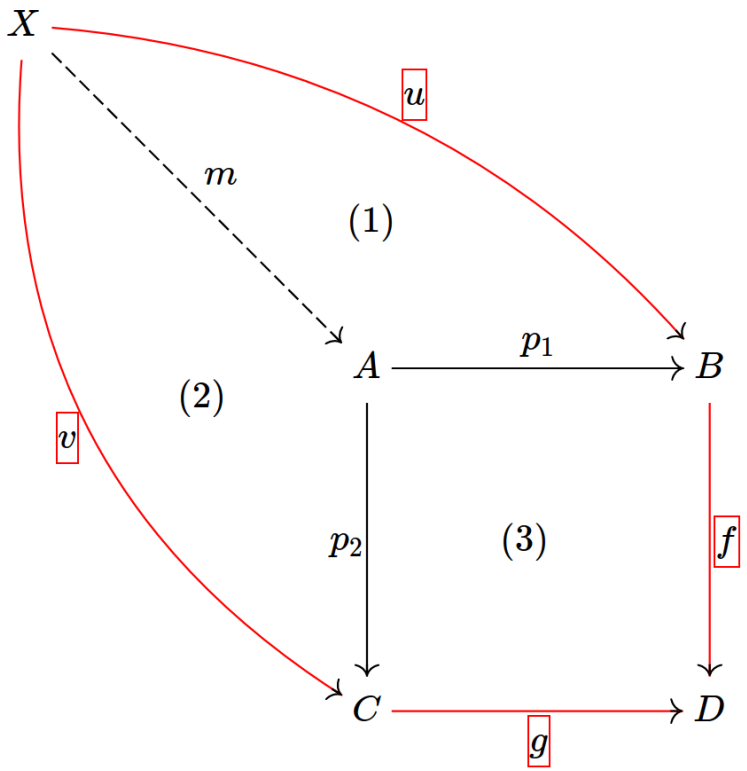
Diagrams are proofs



Generated formal proof

If inner subdiagrams (1)-(3) commute,
then the outer diagram commutes.

Algorithm for proof generation



1. Identify all subdiagrams
2. Start from one branch of the outer diagram
3. Repeatedly "apply" subdiagrams to progress, until reaching the other branch.

Target: the UniMath mathematical library

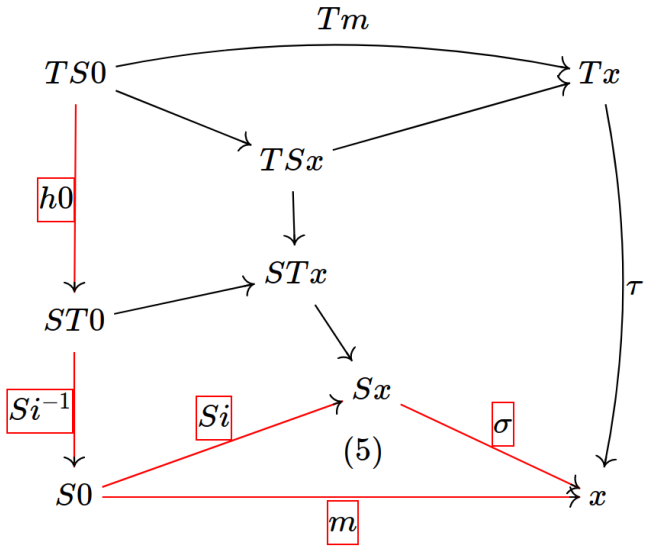
History

- started in 2014 with Voevodsky's repository Foundations
- Large Coq mathematical library (~300 000 lines)
  - ~ two thirds on (bi)category theory
  - Verbose style
- Inconsistent: universe level checks are disabled
  ($\Rightarrow$ impredicative encoding of quotients)

Why targeting this library?

- I use UniMath in my formalisation projects.
- The implementation could be easily adapted to another target.

Commutation proofs in UniMath



Applying[1] (5) to the bottom left branch

$$(h_0 \cdot Si^{-1}) \cdot m$$

rewrite assoc'

$$h_0 \cdot (Si^{-1} \cdot m)$$

apply cancel_precomposition

$$Si^{-1} \cdot m$$

apply cancel_precomposition

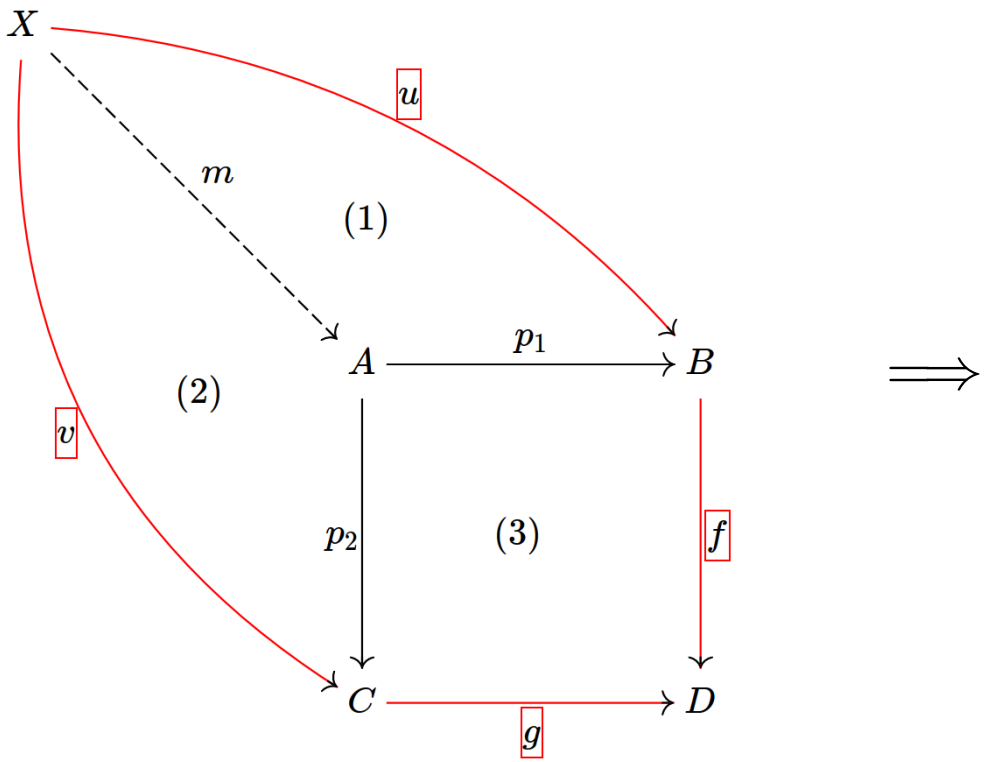
$$m$$

apply (5)

$$Si \cdot \sigma$$

[1] rewrite (5) sometimes works but is hard to maintain.

Generated Coq script




```
Goal { u · f = v · g }.

assert(eq : { u = m · p_1 }).
{ admit. }
etrans.
{
  apply cancel_postcomposition.
  apply eq.
}
clear eq.
assert(eq : { p_1 · f = p_2 · g }).
{ admit. }
etrans.
{
  repeat rewrite assoc'.
  apply cancel_precomposition.
  repeat rewrite assoc.
  apply eq.
}
repeat rewrite assoc.
clear eq.
assert(eq : { m · p_2 = v }).
{ admit. }
etrans.
{
  apply cancel_postcomposition.
  apply eq.
}
clear eq.
apply idpath.
Qed.
```

Commutation of subdiagrams are explicitly asserted and admitted.

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Forward vs Backward reasoning in Coq

$n : G \Rightarrow F$ natural.

How to rewrite $n_x \cdot Ff$ into $Gf \cdot n_y$?

Forward reasoning in Coq

1. State naturality specialised to $f : x \to y$

$$n_x \cdot Ff = Gf \cdot n_y$$

2. Prove the above equation by naturality
3. Apply the equation

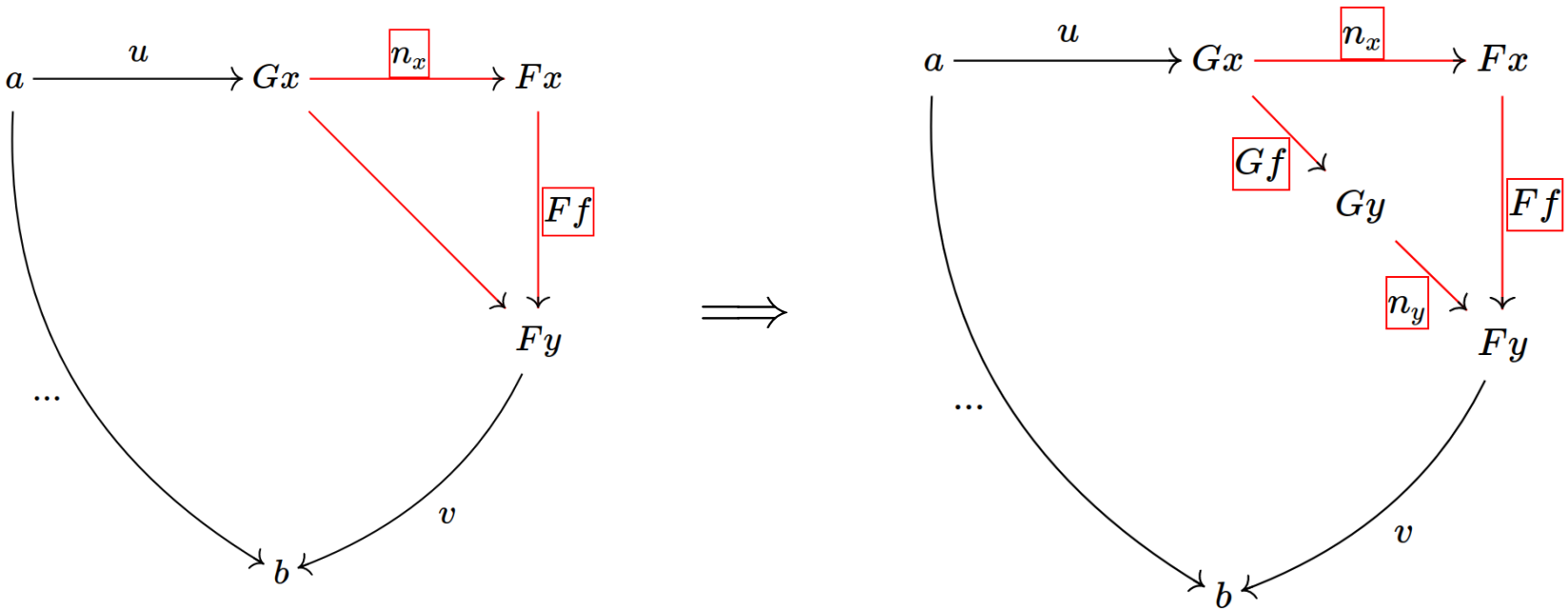
Backward reasoning in Coq

Directly apply naturality of n
$\Rightarrow$ No need to provide the r.h.s $Gf \cdot n_y$

Backward reasoning with YADE

1. Generate Coq script that focuses on the subterm
2. Manually apply the lemma, in Coq
3. Copy and paste the result in YADE

$\Rightarrow$ The diagram is completed automatically



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Demo: Associativity of the "composite" multiplication

Let

- $\delta : ST \to TS$ be a monadic distributive law.
- $R := TS$
- $\mu' : RR \to R$ defined in the only sensible way.

Then,

$$\begin{array}{ccc} RRR & \xrightarrow{R\mu'} & RR \\ {\scriptstyle \mu' R}\downarrow & & \downarrow{\scriptstyle \mu'} \\ RR & \xrightarrow[\mu']{} & R \end{array}$$