Signatures and models for syntax and operational semantics in the presence of variable binding

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Outline

1. Operational monads
   - Syntax as monads
   - Semantics as modules
2. General signatures
   - Main Results
   - Basic Ideas for Proofs/Implementations
3. Syntax
4. Semantics

Substitution for syntax and semantics

(e.g. λ-calculus with β-reduction).

Syntax

Substitution of terms: $t[x \mapsto u_x]$ replaces each free variable x with the term u_x, in the term t.

$\Rightarrow$ Terms form a **monad** T (on sets)

Operational Semantics

Substitution of reductions:

$$\frac{t \xrightarrow{r} u}{t[x \mapsto u_x] \xrightarrow{r[x \mapsto u_x]} u[x \mapsto u_x]}$$

$\Rightarrow$ Reductions between terms form a **module over** T.

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Substitution and monads

Example: λ-calculus

$$S,T \quad ::= \quad x \mid \lambda x.\, S \mid S\; T$$

Free variable indexing:

$$LC : X \mapsto \{\text{terms taking free variables in } X\}$$

$$LC(\emptyset) = \{0, \lambda z.z, \dots\}$$
$$LC(\{x,y\}) = \{0, \lambda z.z, \dots, x, y, x\,y, \dots\}$$

Parallel substitution:

$$\begin{aligned} \mathrm{bind}_f \ : \ LC(X) & \to LC(Y) \\ t & \mapsto t[x \mapsto f(x)] \end{aligned} \qquad \text{where} \quad f : X \to LC(Y)$$

$\Rightarrow$ (LC, $\mathrm{var}_X : X \subset LC(X)$, bind) = **monad on Set** [Altenkirch-Reus 99]

monad morphism = mapping preserving variables and substitutions.

Preview: Operations as module morphisms

+ commutes with substitution

$$(t+u)[x \mapsto v_x] = t[x \mapsto v_x] + u[x \mapsto v_x]$$

Categorical formulation

$LC \times LC$ supports LC-substitution	$\rightsquigarrow$	$LC \times LC$ is a **module over** LC
+ commutes with substitution	$\rightsquigarrow$	$+ : LC \times LC \to LC$ is a **module morphism**

[Hirschowitz-Maggesi 2007 : Modules over Monads and Linearity]

Examples of modules

module over a monad R: supports the R-monadic substitution

- R itself
- $M \times N$ for any modules M and N

  e.g. $\mathrm{R} \times \mathrm{R}$: $\qquad f : X \to R(Y)$

  $$(\mathrm{t},\mathrm{u})[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})] := (\mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})], \mathrm{u}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})])$$

- M' = **derivative of a module** M: $M'(X) = M(\underbrace{X \coprod}_{\text{disjoint union}} \underbrace{\{ \diamond \}}_{\text{fresh variable}})$.

  used to model an operation binding a variable (Cf next slide).

Operations as module morphisms

operations = **module morphisms** = maps commuting with substitution.

$$\mathrm{app} : \mathrm{LC} \times \mathrm{LC} \to \mathrm{LC}$$

$$\mathrm{abs} : \mathrm{LC'} \to \mathrm{LC}$$

$$\begin{aligned} \mathrm{abs}_X : \mathrm{LC}(X \coprod \{\diamond\}) &\to \mathrm{LC}(X) \\ t &\mapsto \lambda\diamond.t \end{aligned}$$

Combining operations into a single one using disjoint union

$$[\mathrm{app}, \mathrm{abs}] : (\mathrm{LC} \times \mathrm{LC}) \coprod \mathrm{LC'} \to \mathrm{LC}$$

Make Titles Informative.

- You can also use overlay specifications to create overlays.
- This allows you to present things in any order.
- This is shown second.

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- Shown on all slides.

Some Example Block Title

- $e^{i\pi} = -1.$
- $e^{i\pi/2} = i.$

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Theorem

On first slide.

Corollary

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Corollary

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Make Titles Informative.

Theorem

In left column.

Corollary

In right column.
New line

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Theorem

In left column.

Corollary

In right column.
New line

Outline

Summary

- The first main message of your talk in one or two lines.
- The second main message of your talk in one or two lines.
- Perhaps a third message, but not more than that.

- Outlook
  - What we have not done yet.
  - Even more stuff.

For Further Reading I

A. Author.
Handbook of Everything.
Some Press, 1990.

S. Someone.
On this and that.
Journal on This and That. 2(1):50–100, 2000.