Signatures and models for syntax and operational semantics in the presence of variable binding

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Outline

Outline

Ingredients

- Programming languages (PLs) as graphs
  - (**Syntax**) vertices = terms
  - (**Semantics**) arrows = reductions between terms
- Parallel substitution: variables $\mapsto$ terms
  - monads and modules over them
- (untyped PLs)

Example

λ-calculus with β-reduction:

- **Syntax:** $S, T \quad ::= x | S\ T | \lambda x.S$
- **Reductions:** $(\lambda x.t)\, u \xrightarrow{\beta} t[x \mapsto u] \quad + \quad$ congruences

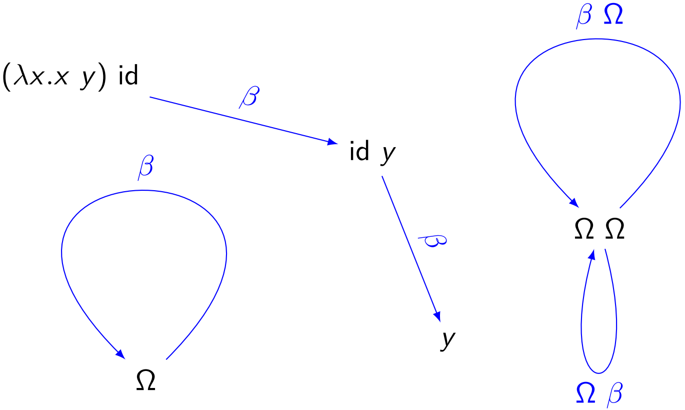
modulo α-equivalence, e.g.

$$\lambda x.x = \lambda y.y$$

Outline

PLs as graphs

Example: λ-calculus with β-reduction



- (**Syntax**) vertices = terms
- (**Semantics**) arrows = reductions (dedicated syntax: Cf labels)

Graphs

Definition

Graph $=$ a quadruple (A, V, σ, τ) where

$$A \overset{\sigma}{\underset{\tau}{\rightrightarrows}} V$$

$$A = \{\text{arrows}\} \qquad V = \{\text{vertices}\}$$

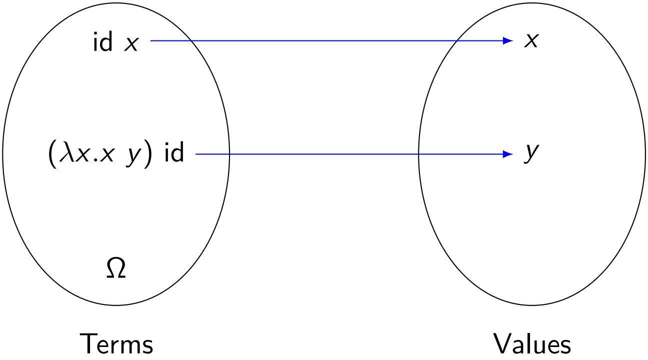
$$\begin{array}{rccl} \sigma : & A & \to & V \\ & t \xrightarrow{r} u & \mapsto & t \end{array} \qquad \begin{array}{rccl} \tau : & A & \to & V \\ & t \xrightarrow{r} u & \mapsto & u \end{array}$$

$$\sigma(r) \xrightarrow{r} \tau(r)$$

PLs as bipartite graphs

Example: λ-calculus cbv with big-step operational semantics

- term $\rightarrow$ value
- variables $=$ placeholders for values



Bipartite graphs

Definition

Bipartite graph = a quadruple (A, V_1, V_2, ∂) where

$$V_1 \xleftarrow{\sigma} A \xrightarrow{\tau} V_2$$

$$A = \{\text{arrows}\} \qquad \begin{aligned} V_1 &= \{\text{vertices in first group}\} \\ V_2 &= \{\text{vertices in second group}\} \end{aligned}$$

For simplicity, we focus on the particular case of **graphs**: $V_1 = V_2$.

Outline

1. Reduction monads
   - Graphs
   - Substitution

2. General signatures

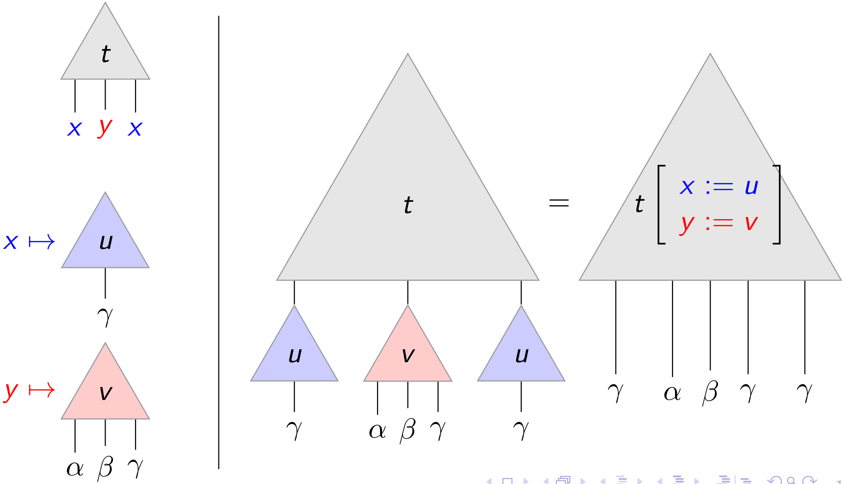
3. Syntax
   - Operations
   - Equations

4. Semantics

Parallel substitution

Syntax comes with substitution

terms (e.g. λ-terms) $=$ trees with free variables as (distinguished) leaves.

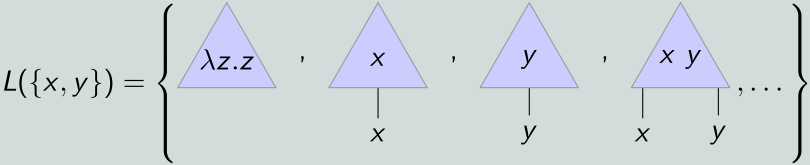


Parallel substitution made formal

Free variables indexing

$$X \mapsto \{\text{terms taking free variables in } X\}$$

Example: λ-calculus



Parallel substitution

For any $f : X \to L(Y)$,

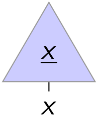
$$\begin{array}{rccc} \mathrm{bind}_f : & L(X) & \to & L(Y) \\ & t & \mapsto & t[x \mapsto f(x)] \qquad (\text{or } t[f]) \end{array}$$

Monads

λ-calculus as a monad (L, bind, η)

1. Parallel substitution (L, bind)
2. Variables are terms

$$\begin{array}{rccc} \eta_X : & X & \to & L(X) \\ & x & \mapsto & \end{array}$$



3. Monadics laws:

$$\underline{x}[f] = f(x) \qquad\qquad t[x \mapsto \underline{x}] = t$$

$+$ associativity:

$$t[f][g] = t\,[x \mapsto f(x)[g]]$$

Substitution for semantics

Our notion of PL:

- **Syntax**: a monad (L, bind, η)
- **Semantics**:
  - graphs $R(X) \underset{\tau}{\overset{\sigma}{\rightrightarrows}} L(X)$ for each X

    $$R(X) = \begin{array}{c}\text{total set of reductions between}\\ \text{terms taking free variables in } X\end{array}$$

  - substitution of reduction: variables $\mapsto$ L-**terms**.

    $$\frac{t \xrightarrow{r} u}{t[f] \xrightarrow{r[f]} u[f]}$$

Substitution for semantics made formal

R as a **module** over L

For any $f : X \to \boldsymbol{L}(Y)$,

$$\begin{aligned} \mathrm{bind}_f : \quad R(X) &\to R(Y) \\ r &\mapsto r[x \mapsto f(x)] \quad (\text{or } r[f]) \end{aligned}$$

s.t.

$$r[x \mapsto \underline{x}] = r \qquad\qquad r[f][g] = r\,[x \mapsto f(x)[g]]$$

σ and τ as L-**module morphisms**

$$\sigma(r[f]) \xrightarrow{r[f]} \tau(r[f])$$

Then, $\dfrac{\sigma(r) \xrightarrow{r} \tau(r)}{\sigma(r)[f] \xrightarrow{r[f]} \tau(r)[f]}$ enforces $\begin{array}{l} \sigma(r[f]) = \sigma(r)[f] \\ \tau(r[f]) = \sigma(r)[f] \end{array}$

Commutation with substitution $\Leftrightarrow$ Module morphisms $\sigma, \tau : R \to L$.

Reduction monads

Definition

Reduction monad: a quadruple (L, R, σ, τ) s.t.

- L = monad
- R = module over L
- $\sigma, \tau : R \to L$ are L-module morphisms.

Example

λ-calculus with β-reduction.

How can we specify a reduction monad?

1. signature for the (syntactic) operations for the monad;
2. reduction rules, involving some specified syntactic operations.

Use of a general notion of **signature** managing this dependency.

Outline

Specify reduction monads

Overview

- A signature is a sequence of arities $A_1, \ldots, A_n$

Outline

Overview

- Syntax $=$ monad L
- Operations $=$ module morphisms $\Sigma(L) \to L$
- 1-signatures specify operations
- 2-signatures specify operations $+$ equations.

Outline

Operations as module morphisms

Application commutes with substitution

$$(t\ u)[x \mapsto v_x] = t[x \mapsto v_x]\ u[x \mapsto v_x]$$

Categorical formulation

$LC \times LC$ supports LC-substitution $\rightsquigarrow$ $LC \times LC$ is a **module over** LC

application commutes with substitution $\rightsquigarrow$ $\mathrm{app} : LC \times LC \to LC$ is a **module morphism**

[Hirschowitz-Maggesi 2007 : Modules over Monads and Linearity]

Examples of modules

module over a monad R: supports the R-monadic substitution

- R itself
- $M \times N$ for any modules M and N

  e.g. $\mathrm{R} \times \mathrm{R}$: $f : X \to R(Y)$

  $(\mathrm{t},\mathrm{u})[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})] := (\mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})], \mathrm{u}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})])$

- M' = **derivative of a module** M: $M'(X) = M(\underbrace{X \coprod}_{\text{disjoint union}} \{ \underbrace{\diamond}_{\text{fresh variable}} \}).$

  used to model an operation binding a variable (Cf next slide).

Operations as module morphisms

operations = **module morphisms** = maps commuting with substitution.

$$\mathrm{app} : \mathrm{LC} \times \mathrm{LC} \to \mathrm{LC}$$

$$\mathrm{abs} : \mathrm{LC}' \to \mathrm{LC}$$

$$\begin{aligned} \mathrm{abs}_X : \mathrm{LC}(\mathrm{X} \coprod \{\diamond\}) &\to \mathrm{LC}(X) \\ t &\mapsto \lambda\diamond.t \end{aligned}$$

Combining operations into a single one using disjoint union

$$[\mathrm{app}, \mathrm{abs}] : (\mathrm{LC} \times \mathrm{LC}) \coprod \mathrm{LC}' \to \mathrm{LC}$$

1-signatures and their models

A **1-signature** Σ = functorial assignment:

$$\underbrace{R}_{\text{monad}} \mapsto \underbrace{\Sigma(R)}_{\text{module over } R}$$

A **model of** Σ is a pair:

$$(\underbrace{R}_{\text{monad}}, \quad \underbrace{\rho : \Sigma(R) \to R}_{\text{module morphism}})$$

Example: (app,abs)

$$\Sigma_{\text{app,abs}}(R) = (R \times R) \coprod R'$$

LC = model of $\Sigma_{\text{app,abs}}$

$$[\text{app}, \text{abs}] : (LC \times LC) \coprod LC' \to LC$$

+ suitable notion of model morphism [Hirschowitz-Maggesi 2012]

Syntax

Definition

Given a 1-signature Σ, its **syntax** is an initial object in its category of models.

Question: Does the syntax exist for every 1-signature?

Answer: No.

Counter-example: the 1-signature $R \mapsto \mathscr{P} \circ R$.

powerset endofunctor on Set

Examples of 1-signatures generating syntax

- **(0,+) language**: a constant 0 and a binary operation +

  Signature: $R \mapsto 1 \coprod (R \times R)$

  Model: $(R\ ,\ \ 0 : 1 \to R\ ,\ \ + : R \times R \to R)$

  Syntax: initial model

- **lambda calculus**:

  Signature: $R \mapsto R' \coprod (R \times R)$

  Model: $(R\ ,\ \ abs : R' \to R\ ,\ \ app : R \times R \to R)$

  Syntax: initial model

Can we generalize this pattern?

Initial semantics for algebraic 1-signatures

Theorem [Hirschowitz & Maggesi 2007]

Syntax exists for any **algebraic 1-signature**, i.e. 1-signature built out of derivatives, products, disjoint unions, and the 1-signature $R \mapsto R$.

Algebraic 1-signatures correspond to the binding signatures described in [Fiore-Plotkin-Turi 1999]

(binding signature $=$ lists of natural numbers specify n-ary operations, possibly binding variables)

Question: Can we enforce some equations in the syntax ?

e.g. commutativity and associativity of a binary operation.

Quotient of algebraic signatures

Theorem [AHLM CSL 2018]

Syntax exists for any "*quotient*" of algebraic 1-signature.

Example: a commutative binary operation

... what about an associative binary operation?

Outline

Example: a commutative binary operation

Specification of a binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R\ ,\ + : R \times R \to R)$

What is an appropriate notion of model for a commutative binary operation ?

Example: a commutative binary operation

Specification of a commutative binary operation

1-Signature: $R \mapsto R \times R$

Model: $(R\,,\ + : R \times R \to R)$ s.t. $t + u = u + t$ (1)

What is an appropriate notion of model for a commutative binary operation ?

Answer: a monad equipped with a commutative binary operation

Equation (1) states an equality between R-module morphisms:

$$\begin{tikzcd} \mathrm{R} \times \mathrm{R} \arrow[rr, bend left, "+"] \arrow[r, "\mathrm{swap}"'] & \mathrm{R} \times \mathrm{R} \arrow[r, "+"'] & \mathrm{R} \end{tikzcd}$$

Equations

Given a 1-signature Σ, (e.g. binary operation: $\Sigma(\mathrm{R}) = \mathrm{R} \times \mathrm{R}$)

a Σ-**equation** A ⇉ B is a functorial assignment:

$$R \mapsto \left(A(R) \rightrightarrows B(R) \right)$$

model of Σ

parallel pair of module morphisms over R

e.g. commutativity:

$$R \mapsto \left(R \times R \underset{+\circ swap}{\overset{+}{\rightrightarrows}} R \right)$$

A **2-signature** is a pair

$$(\Sigma, \mathrm{E})$$

1-signature

set of Σ-equations

model **of a 2-signature** $(\boldsymbol{\Sigma}, \mathbf{E})$:

- a model R of Σ
- s.t. ∀ (A ⇉ B) ∈ E, the two morphisms $A(R) \rightrightarrows B(R)$ are equal

Initial semantics for algebraic 2-signatures

Our main theorem

Syntax exists for any algebraic 2-signature.

Algebraic 2-signature: (Σ, E)

algebraic 1-signature

set of **elementary** Σ-equations

*a Σ-equation A ⇉ B is **elementary** if A maps pointwise epis to pointwise epis, and B(R) = R'..'*

Main instances of **elementary** Σ-equations A ⇉ B:

- $A =$ algebraic 1-signature e.g. $\mathrm{A(R)} = \mathrm{R} \times \mathrm{R}$
- $B(R) = R$

Example: fixpoint operator

Definition [AHLM CSL 2018]

A **fixpoint operator** in a monad R is a module morphism $\mathrm{fix}\colon \mathrm{R}' \to \mathrm{R}$ s.t. for any term $\mathrm{t} \in \mathrm{R}(\mathrm{X} \coprod \{\diamond\})$, $\mathrm{fix}(\mathrm{t}) = \mathrm{t}[\diamond \mapsto \mathrm{fix}(\mathrm{t})]$

Intuition:

$\mathtt{fix}(t) \quad := \quad$ `let rec ◇ = t in t`

Algebraic 2-signature $(\Sigma_{\mathrm{fix}}, \mathrm{E}_{\mathrm{fix}})$ of a fixpoint operator:

$$\Sigma_{\mathrm{fix}}(\mathrm{R}) := \mathrm{R}' \qquad\qquad E_{\mathtt{fix}} = \left\{ \begin{tikzcd} \begin{matrix} R' \\ t \end{matrix} \arrow[r, bend left, "\mathtt{fix}(t)"] \arrow[r, bend right, "t[\diamond\mapsto\mathtt{fix}(t)]"'] & R \end{tikzcd} \right\}$$

Outline

Summary

- The first main message of your talk in one or two lines.
- The second main message of your talk in one or two lines.
- Perhaps a third message, but not more than that.

- Outlook
  - What we have not done yet.
  - Even more stuff.

For Further Reading I

A. Author.
Handbook of Everything.
Some Press, 1990.

S. Someone.
On this and that.
Journal on This and That. 2(1):50–100, 2000.