Signatures and models for syntax and operational semantics in the presence of variable binding

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Motivation

How do we formally specify a **programming language**?

In the literature: no common well-established discipline.

Differential λ-calculus [Ehrhrad-Regnier 2003]

$\sim$10 pages (section 2 $\rightarrow$ beginning of section 3) describing the programming language[a] and proving some properties.

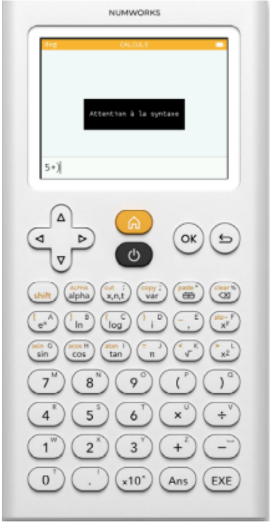
[a] not yet satisfyingly addressed by this PhD.

This PhD: a discipline for presenting programming languages

- from small elementary data
- automatically ensuring some properties

What is a programming language?

Example: arithmetic expressions in a calculator



Syntax (of expressions) = formal language

- *vocabulary* : available symbols/keys
- *grammar rules* : what is a valid expression.

e.g. + is a *binary operation*.

Syntax and variables

Focus of this PhD

Variables in expressions

$$(x + 5) \times y$$

x, y = **variables** = placeholders for other expressions
Substitution: variables $\mapsto$ expressions:

$$\begin{cases} \text{replace } x \text{ with } 3 \\ \text{replace } y \text{ with } z \times z \end{cases} \quad \rightsquigarrow \quad (3 + 5) \times (z \times z)$$

Bound variables and α-equivalence

α-**equivalence** for arithmetic propositions with quantifiers.

$$\exists x.x > 100 \quad \text{should be identified with} \quad \exists y.y > 100$$

"x is bound by $\exists$ in $\exists x.x > 100$"

Syntax and recursion

Recursion (for syntax) = principle for investigating a piece of valid syntactic data.

Examples of use of recursion

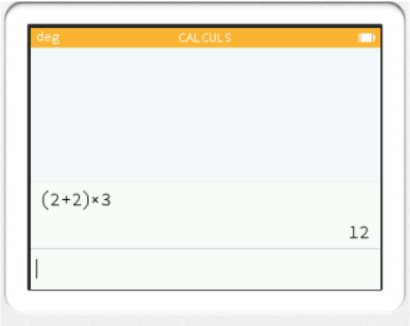
- count the number of operations in an arithmetic expression
- compute an arithmetic expression

What is a programming language?

Program execution

Program = valid *syntactic* text

Execution = modification of the program:



$$(2+2)\times 3 \xrightarrow{\text{1 step of execution}} 4 \times 3 \xrightarrow{\text{1 step of execution}} 12$$

Operational semantics = description of how programs execute.

What is a programming language?

Finally

Programming language (PL) = syntax + operational semantics.

Specification of a PL = features uniquely characterizing a PL.

In 2 steps:

1. syntax

Example: specification of the syntax of arithmetic expressions

- numbers = constants
- $+$ and $\times$: operations expecting two expressions.

2. semantics

Caveat

There are **ineffective specifications**: no PL satisfies them.

Stupid example

Syntax with two constants 0 and 1 s.t. $0 = 1$ and $0 \neq 1$.

Initial Semantics

Specification through initial semantics for justifying **recursion**.

This PhD:

1. Proposes a mathematical definition of PLs as **reduction monads**;
2. Proposes a notion of **signature** for specifying the syntax and semantics of a reduction monad:
   - **category of models** of a signature
   - initial model = specified object
   - initiality ⇒ recursion

   Initial Semantics
3. Rules out **ineffective** signatures: identifies a criterion ensuring existence of the initial model.

Outline

1. Reduction monads
   - Graphs
   - Substitution
2. Syntax
   - Operations
   - Equations
3. Semantics
   - Reduction rules
   - Reduction signatures

Outline

Ingredients

- Programming languages (PLs) as graphs
  - (**Syntax**) vertices = terms
  - (**Semantics**) arrows = reductions between terms
- Simultaneous substitution: variables $\mapsto$ terms
  - monads and modules over them
- (untyped PLs)

Example

λ-calculus with β-reduction:

- **Syntax:** $S, T \quad ::= x | S\ T | \lambda x.S$
- Modulo α-**equivalence**, e.g.

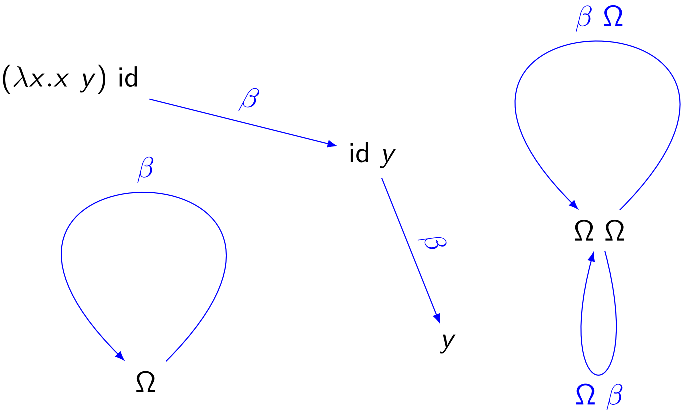
$$\lambda x.x = \lambda y.y$$

- **Reductions:** $(\lambda x.t)\, u \xrightarrow{\beta} t[x \mapsto u] \qquad + \quad$ congruences

Outline

PLs as graphs

Example: λ-calculus with β-reduction



- (**Syntax**) vertices = terms e.g. $\Omega = (\lambda x.x\,x)\,(\lambda x.x\,x)$
- (**Semantics**) arrows = reductions

Graphs

Definition

Graph $=$ a quadruple (A, V, σ, τ) where

$$A = \{\text{arrows}\} \qquad\qquad V = \{\text{vertices}\}$$

$$A \underset{\tau}{\overset{\sigma}{\rightrightarrows}} V$$

$$\sigma : \begin{array}{ccc} A & \to & V \\ t \xrightarrow{r} u & \mapsto & t \end{array} \qquad\qquad \tau : \begin{array}{ccc} A & \to & V \\ t \xrightarrow{r} u & \mapsto & u \end{array}$$

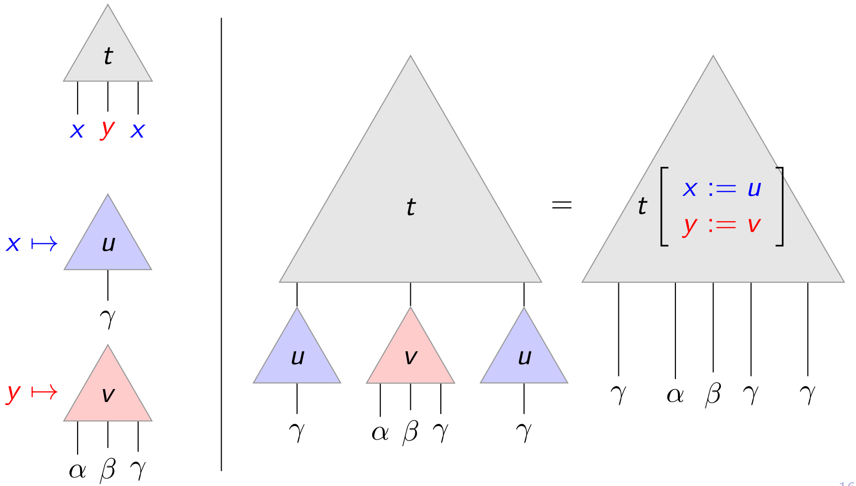
$$\sigma(r) \xrightarrow{r} \tau(r)$$

Outline

Simultaneous substitution

Syntax comes with substitution

terms (e.g. λ-terms) = trees with free variables as (distinguished) leaves.



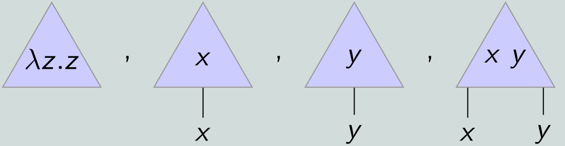
Simultaneous substitution made formal

Free variables indexing

$$X \mapsto \{\text{terms taking free variables in } X\}$$

Example: λ-calculus

$L(\{x,y\}) = \{$



$, \dots \}$

Simultaneous substitution

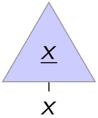
$$\forall f : X \to L(Y), \quad \boxed{\begin{array}{rl} L(X) & \to L(Y) \\ t & \mapsto t[x \mapsto f(x)] \quad (\text{or } t[f]) \end{array}}$$

Monads capture simultaneous substitution

λ-calculus as a monad $(L, \_[\_], \eta)$

1. Simultaneous substitution $(L, \_[\_])$
2. Variables are terms

$$\begin{array}{cccc} \eta_X : & X & \to & L(X) \\ & x & \mapsto & \end{array}$$



3. Monadic laws:

$$\underline{x}[f] = f(x) \qquad\qquad t[x \mapsto \underline{x}] = t$$

$+$ associativity:

$$t[f][g] = t\,[x \mapsto f(x)[g]]$$

Substitution for semantics

We saw that syntax is expected to support substitution. This is also true of semantics.

Our notion of PL:

- **Syntax**: a monad $(L, \_[\_], \eta)$
- **Semantics**:
  - graphs $R(X) \underset{\tau}{\overset{\sigma}{\rightrightarrows}} L(X)$ for each X

$$R(X) = \begin{array}{c}\text{total set of reductions between}\\ \text{terms taking free variables in } X\end{array}$$

  - substitution of reduction: variables $\mapsto$ L-**terms**.

$$\frac{t \xrightarrow{r} u}{t[f] \xrightarrow{r[f]} u[f]}$$

Substitution for semantics made formal

R as a **module** over L

R supports L-monadic substitution:

$$\forall f : X \to \boldsymbol{L}(Y), \qquad \boxed{\begin{array}{rl} R(X) & \to R(Y) \\ r & \mapsto r[x \mapsto f(x)] \quad (\text{or } t[f]) \end{array}}$$

$$r[x \mapsto \underline{x}] = r \qquad\qquad r[f][g] = r\,[x \mapsto f(x)[g]]$$

Remark: any monad T is a module over itself.

σ and τ as L-**module morphisms**

By definition of σ and τ, $\quad \sigma(r[f]) \xrightarrow{r[f]} \tau(r[f])$

Then, $\quad \dfrac{\sigma(r) \xrightarrow{r} \tau(r)}{\sigma(r)[f] \xrightarrow{r[f]} \tau(r)[f]}$ enforces $\quad \begin{array}{l} \sigma(r[f]) = \sigma(r)[f] \\ \tau(r[f]) = \sigma(r)[f] \end{array}$

Commutation with substitution $\Leftrightarrow$ Module morphisms $\sigma, \tau : R \to L$.

Reduction monads

Summary: graphs + substitution.

Definition

A **reduction monad** $R \overset{\sigma}{\underset{\tau}{\rightrightarrows}} T$ consists of

- T = monad (= module over itself)
- R = module over T
- $\sigma, \tau : R \to T$ are T-module morphisms.

Example

λ-calculus with β-reduction.

How can we specify a reduction monad?

1. signature for the (syntactic) operations for the monad;
2. reduction rules, involving some specified syntactic operations.

Use of a general notion of **signature** managing this dependency.

Outline

Overview

- Syntax = monad L
- Operations = module morphisms $\Sigma(L) \to L$
- 1-signatures specify operations
- 2-signatures specify operations + equations.

Outline

Operations as module morphisms

Application commutes with substitution

$$(t\ u)[x \mapsto v_x] = t[x \mapsto v_x]\ u[x \mapsto v_x]$$

Categorical formulation

$L \times L$ supports L-substitution	$\rightsquigarrow$	$L \times L$ is a **module over** L
application commutes with substitution	$\rightsquigarrow$	$\mathrm{app} : L \times L \to L$ is a **module morphism**

[Hirschowitz-Maggesi 2007 : Modules over Monads and Linearity]

Examples of modules

We argued that syntactic operations are **module morphisms**. Basic examples of modules?

Module over a monad T: supports the T-monadic substitution

Examples

- T itself
- $M \times N$ for any modules M and N:

$$\forall (t, u) \in M(X) \times N(X), \qquad X \xrightarrow{f} T(Y),$$

$$\boxed{(t, u)[f] = (t[f], u[f])} \in M(Y) \times N(Y)$$

- M'= **derivative** of a module M:

$$M'(X) = M(\overbrace{X \coprod \{\diamond\}}^{X \text{ extended with a fresh variable } \diamond})$$

used to model an operation binding a variable (Cf next slide).

Operations as module morphisms

Operations can be combined into a single one.

Operations = module morphisms = maps commuting with substitution:

Example: λ-calculus

$$\begin{aligned} \mathrm{app}: \quad & L \times L \to L \\ \mathrm{abs}: \quad & L' \to L \end{aligned} \qquad \left\{ \begin{aligned} \mathrm{abs}_X : L(X \coprod \{\diamond\}) &\to L(X) \\ t &\mapsto \lambda \diamond . t \end{aligned} \right.$$

Combine operations into a single one:

$$\boxed{[\mathrm{app}, \mathrm{abs}] : (L \times L) \coprod L' \to L}$$

where (*coproducts* of modules M and N)

$$(M \coprod N)(X) = M(X) \coprod N(X)$$

1-signatures specify operations

Definition

A **1-signature** Σ is a (functorial) assignment

$$\boxed{T \mapsto \Sigma(T)}$$

monad module over T

Definition (model of a 1-signature Σ)

A **model** of Σ is a pair (T, m) denoted by $\Sigma(T) \xrightarrow{m} T$ s.t.

- T is a monad
- $\Sigma(T) \xrightarrow{m} T$ is a T-module morphism

Example: λ-calculus

$$[\mathrm{app}, \mathrm{abs}] : \Sigma_{LC}(L) \to L \qquad \text{where } \Sigma_{LC}(L) = (L \times L) \amalg L'$$

Syntax

We defined 1-signatures and their models. When is a signature effective?

(suitable notion of model morphism [Hirschowitz-Maggesi 2012])

Definition

The **syntax** specified by a 1-signature Σ is the initial object in its category of models.

Question: Does the syntax exist for every 1-signature?

Answer: No.

Counter-example: $\Sigma(R) = \mathcal{P} \circ R$

Powerset endofunctor on *Set*.

Initial semantics for algebraic 1-signatures

We gave examples of effective 1-signatures. They were all **algebraic**.

Definition

Algebraic 1-signatures = 1-signatures built out of derivatives, finite products, disjoint unions, and the 1-signature $\Theta : T \mapsto T$.

Algebraic 1-signatures $\simeq$ binding signatures [Fiore-Plotkin-Turi 1999]
$\Rightarrow$ specification of n-ary operations, possibly binding variables.

Theorem (Fiore-Plotkin-Turi 1999)

Syntax exists for any algebraic 1-signature.

Example

λ-calculus

Question: Specify syntactic operations subject to some equations?

e.g. a *commutative* or *associative* binary operation.

Quotient of algebraic signatures

We saw that algebraic signatures are effective. Can we specify effectively operations subject to equations?

Theorem

Syntax exists for any "quotient" of algebraic 1-signatures.

Example

a *commutative* binary operation $+$:

$$\forall a, b, \quad a + b = b + a$$


What about an **associative** operation?

Outline

Example: a commutative binary operation

Specification of a binary operation

1-signature	$T \mapsto T \times T$
model	$T \times T \xrightarrow{+} T$

Question What is an appropriate notion of model for a **commutative** binary operation?

- a monad T
- with a binary operation

$\Big\}$ a model $T \times T \xrightarrow{+} T$ of $\Theta \times \Theta$

- s.t.

$$\begin{tikzcd} T \times T \arrow[rr, bend left, "+"] \arrow[dr, "\mathrm{swap}"'] & & T \\ & T \times T \arrow[ur, "+"'] & \end{tikzcd}$$

where $\mathrm{swap}(t, u) = (u, t)$

Equations

Σ = 1-signature (e.g. binary operation $\Sigma(T) = T \times T$)

Definition

A **Σ-equation** $A \overset{u}{\underset{v}{\rightrightarrows}} B$ is a (functorial) assignment

$$\boxed{M = (\Sigma(T) \to T) \quad \mapsto \quad \left(A(M) \overset{u_M}{\underset{v_M}{\rightrightarrows}} B(M) \right)}$$

model of Σ

parallel pair of T-module morphisms

Example (Binary commutative operation)

$$\Sigma(T) = T \times T$$

$$\begin{tikzcd} T \times T \arrow[d, "+"] \\ T \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} T \times T \arrow[rr, bend left, "+"] \arrow[dr, "\mathrm{swap}"'] & & T \\ & T \times T \arrow[ur, "+"'] & \end{tikzcd}$$

2-signatures and their models

We defined equations. A set of equations yields a 2-signature.

Definition

A **2-signature** is a pair (Σ, E) where

- Σ is a 1-signature for monads
- E is a set of Σ-equations

Definition

A **model** of a 2-signature (Σ, E) consists of:

- a model $M = \begin{pmatrix} \Sigma(T) \\ \downarrow \\ T \end{pmatrix}$ of Σ s.t.

$$\forall\ A \underset{v}{\overset{u}{\rightrightarrows}} B \in E, \quad \boxed{u_M = v_M} : A(M) \to B(M)$$

morphism of models = morphisms as models of Σ.

Initial semantics for algebraic 2-signatures

We defined 2-signatures and their models. When is a 2-signature effective?

Theorem

*Any **algebraic** 2-signature has an initial model.*

Definition

A 2-signature (Σ, E) is **algebraic** if:

- Σ is algebraic
- E consists of **elementary** Σ-equations

Main instances of elementary Σ-equations

$$A \rightrightarrows B \text{ s.t. } \qquad A\left(\begin{array}{c}\Sigma(T)\\ \downarrow\\ T\end{array}\right) = \Phi(T) \qquad B\left(\begin{array}{c}\Sigma(T)\\ \downarrow\\ T\end{array}\right) = T$$

for some *algebraic* 1-signature Φ.

(e.g. $\Phi(T) = T \times T$ for commutativity)

Examples of elementary equations

We saw that elementary Σ-equations yield effective 2-signatures. Examples of them?

- *associativity* of a binary operation
- fixpoint equation

$$(\mathrm{fix}\,\mathrm{f} := \mathrm{t}) \quad = \quad \mathrm{t}[\mathrm{f} := (\mathrm{fix}\, f := t)]$$

- β-reduction as an equation:

$$(\lambda x.t)\, u = t[x := u]$$

What if we want β-reduction as a *reduction* rather than an *equation*?

Outline

Specifying reduction monads

λ-calculus with (small-step) β-reduction as a reduction monad:

$$R \underset{\tau}{\overset{\sigma}{\rightrightarrows}} L$$

module over L

module morphisms

monad

- vertices $= L =$ initial model of the signature of λ-calculus.
- arrows $= R, \sigma, \tau = ?$
  - specified through *reduction rules* (to be made formal):

$$(\lambda x.t)\, u \to t[x := u] \qquad \frac{t \to t'}{t\, u \to t'\, u} \qquad \ldots$$

Outline

Analysis of a reduction rule

Example: binary congruence for application.

hypotheses

metavariables: $\underbrace{\text{as a } L\text{-module } L^4}_{t,\, t',\, u,\, u'} \mapsto \boxed{\dfrac{t \to t' \qquad u \to u'}{t\, u \to t'\, u'}}$

conclusion

Hypothesis/conclusion = pair of λ-terms using metavariables

- as parallel module morphisms $L^4 \rightrightarrows L$

$$\text{e.g.} \quad t\,u \to t'\,u' : \qquad \begin{aligned}(t, t', u, u') &\mapsto t\,u \\ (t, t', u, u') &\mapsto t'\,u'\end{aligned}$$

- **Generalization**: $L \rightsquigarrow$ any model $\Sigma_{LC}(T) \to T$ of Σ_{LC}:

(application denoted by app : $T \times T \to T$)

$$\text{e.g.} \quad t\,u \to t'\,u' : \qquad \begin{aligned}(t, t', u, u') &\mapsto \mathrm{app}(t, u) \\ (t, t', u, u') &\mapsto \mathrm{app}(t', u')\end{aligned}$$

Reduction rules

Definition

Let $\Sigma =$ signature for monads (e.g. Σ_{LC} for congruence for application).

Definition of Σ-reduction rules

A Σ-**reduction rule** $(\vec{\sigma}, \vec{\tau})$

$$\boxed{\dfrac{\sigma_1 \longrightarrow \tau_1 \quad \ldots \quad \sigma_n \longrightarrow \tau_n}{\sigma_0 \longrightarrow \tau_0}}$$

assigns (functorially) to each model $\Sigma(T) \to T$:

- $V(T) = T$-module of metavariables (e.g. $V(T) = T^4$)
- parallel T-module morphisms $V(T) \underset{\tau_{i,T}}{\overset{\sigma_{i,T}}{\rightrightarrows}} T^{\prime\cdots\prime}$

We write

$$\sigma_i, \tau_i : V \to \Theta^{(n_i)} \qquad n_i = \text{number of derivatives}$$

Outline

Reduction signatures

Definition

A **reduction signature** is a pair $(\Sigma, \mathfrak{R})$ where

- Σ is a signature for monads
- $\mathfrak{R}$ is a family of Σ-reduction rules

Example: λ-calculus with β-reduction

- $\Sigma = \Sigma_{LC} = \Theta \times \Theta \coprod \Theta'$ for app and abs.
- Σ-reduction rules:
  - β-reduction
  - congruence for application and **abstraction**:

$$\frac{u \to u'}{\lambda x.u \to \lambda x.u'} \rightsquigarrow \frac{\pi_1 \to \pi_2}{\mathrm{abs} \circ \pi_1 \to \mathrm{abs} \circ \pi_2} \qquad \frac{T' \times T' \overset{\pi_{1,T}}{\underset{\pi_{2,T}}{\rightrightarrows}} T'}{T' \times T' \overset{\mathrm{abs}\circ\pi_{1,T}}{\underset{\mathrm{abs}\circ\pi_{2,T}}{\rightrightarrows}} T}$$

Models

We defined **reduction signatures**. What are their models?

A **model** of a signature $(\Sigma, \mathfrak{R})$ consists of:

- a reduction monad $R \overset{\sigma}{\underset{\tau}{\rightrightarrows}} T$ with a Σ-model structure on T
- for each reduction rule

$$\boxed{\dfrac{\sigma_1 \to \tau_1 \quad \dots \quad \sigma_n \to \tau_n}{\sigma_0 \to \tau_0}} \qquad V \overset{\sigma_i}{\underset{\tau_i}{\rightrightarrows}} \Theta^{(n_i)} \quad \text{in } \mathfrak{R},$$

  - a mapping, for each $v \in V(T)(X)$,

$$\begin{pmatrix} \sigma_1(v) \xrightarrow{r_1} \tau_1(v) \\ \dots \\ \sigma_n(v) \xrightarrow{r_n} \tau_n(v) \end{pmatrix} \quad \mapsto \quad \sigma_0(v) \xrightarrow{op(r_1, \dots r_n)} \tau_0(v)$$

  - compatible with substitution:

$$op(r_1, \dots r_n)[f] = op(r_1[f], \dots, r_n[f])$$

Initiality

We defined **models** of a **reduction signature**. When is a signature effective?

(suitable notion of model morphisms)

Theorem

Σ has an initial model (e.g. Σ is algebraic) $\Rightarrow$ $(\Sigma, \mathfrak{R})$ has an initial model.

Examples

- λ-calculus with small-step β-reduction
- λ-calculus with explicit substitution [Kesner 2009].

A Theory of Explicit Substitutions with Safe and Full Composition

Generalizing from *graphs* to *bipartite graphs* yields more examples:

Examples

- (small-step or big-step) cbv λ-calculus.
- π-calculus

Conclusion

Summary

- PLs as reduction monads
- Signatures for reduction monads with initiality theorem

Articles

- AHLM CSL 2018 about *quotient of algebraic 1-signatures*
- AHLM FSCD 2019 about *algebraic 2-signatures*
- AHLM POPL 2020 about *reduction monads*
- HHL (on HAL): extension to bipartite graphs and simply typed PLs

AHLM = Ahrens, A. Hirschowitz, *Lafont*, Maggesi
HHL = A. Hirschowitz, T. Hirschowitz, *Lafont*