Signatures and models for syntax and operational semantics in the presence of variable binding

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PhD, 2019

Outline

- Operational monads
 - Syntax as monads
 - Semantics as modules
- 2 General signatures
 - Main Results
 - Basic Ideas for Proofs/Implementations
- Syntax
- 4 Semantics

(e.g. λ -calculus with β -reduction).

Syntax

Substitution of terms: $t[x \mapsto u_x]$ replaces each free variable x with the term u_x , in the term t.

 \Rightarrow Terms form a **monad** T (on sets)

Operational Semantics

Substitution of reductions:

$$\frac{t \xrightarrow{r} u}{t[x \mapsto u_x] \xrightarrow{r[x \mapsto u_x]} u[x \mapsto u_x]}$$

 \Rightarrow Reductions between terms form a **module over** T.

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Substitution and monads

Example: λ-calculus

$$S,T$$
 ::= $x \mid \lambda x.S \mid ST$

Free variable indexing:

$$LC: X \mapsto \{\text{terms taking free variables in } X\}$$

$$LC(\emptyset) = \{0, \lambda z. z, \dots\}$$

$$LC(\{x, y\}) = \{0, \lambda z. z, \dots, x, y, x y, \dots\}$$

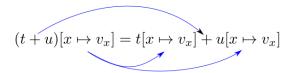
Parallel substitution:

$$egin{array}{lll} \mathrm{bind_f} &: \mathit{LC}(\mathrm{X}) & o & \mathit{LC}(\mathrm{Y}) \ & \mathrm{t} & \mapsto & \mathrm{t}[\mathrm{x} \mapsto \mathrm{f}(\mathrm{x})] \end{array} \qquad \qquad \mathsf{where} \quad \mathrm{f}: \mathrm{X} o \mathit{\underline{LC}}(\mathrm{Y})$$

 \Rightarrow (LC, var_X : X \subset LC(X) , bind) = **monad on Set** [Altenkirch-Reus 99] **monad morphism** = mapping preserving variables and substitutions.

Preview: Operations as module morphisms

+ commutes with substitution



Categorical formulation

$$LC \times LC$$
 supports LC -substitution

 \sim

 $LC \times LC$ is a module over LC

+ commutes with substitution



 $+:LC\times LC\to LC$ is a

module morphism

[Hirschowitz-Maggesi 2007 : Modules over Monads and Linearity]



Examples of modules

module over a monad R: supports the R-monadic substitution

- R itself
- $M \times N$ for any modules M and N

e.g.
$$R \times R$$
: $f: X \to R(Y)$

$$(t,\!u)[x\mapsto f(x)]:=(t[x\!\mapsto\! f(x)],\,u[x\mapsto f(x)])$$

disjoint union fresh variable

• M' = derivative of a module M: $M'(X) = M(X | \{ \{ \} \} \})$.

used to model an operation binding a variable (Cf next slide).

Operations as module morphisms

operations = **module morphisms** = maps commuting with substitution.

$$\begin{aligned} \operatorname{app}: \operatorname{LC} \times \operatorname{LC} &\to \operatorname{LC} \\ \operatorname{abs}: \operatorname{LC}' &\to \operatorname{LC} \\ \operatorname{abs}_X: \operatorname{LC}(\operatorname{X} \coprod \{\diamond\}) \to \operatorname{LC}(X) \\ t &\mapsto \lambda \diamond. t \end{aligned}$$

Combining operations into a single one using disjoint union

$$[app, abs] : (LC \times LC) \coprod LC' \rightarrow \underline{LC}$$

- You can also use overlay specifications to create overlays.
- This allows you to present things in any order.
- This is shown second.

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- Shown on all slides.

Some Example Block Title

- $e^{i\pi} = -1$.
- $e^{i\pi/2} = i$

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Example

On first slide.

Example

On second slide

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Theorem

On first slide.

Corollary

On second slide

Theorem

On first slide.

Corollary

On second slide.

Theorem

In left column.

Corollary

In right column.

Theorem

In left column.

Corollary

In right column. New line

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Summary

- The first main message of your talk in one or two lines.
- The second main message of your talk in one or two lines.
- Perhaps a third message, but not more than that.
- Outlook
 - What we have not done yet.
 - Even more stuff.

For Further Reading I



A. Author.

Handbook of Everything.

Some Press, 1990.



S. Someone.

On this and that.

Journal on This and That. 2(1):50–100, 2000.