Unification with binding

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1 Setting

Let Σ be a binding signature. Then, there is a free Σ-monoid monad T on $Set^{\mathbb{N}}$ such that $T(X)$ is the syntax of Σ extended with an n-ary operation for each $x \in X_n$. More precisely, $T(X)_n$ is the set of terms whose free variables are in $\{0, \dots, n-1\}$.

Example 1. λ-calculus. Then, $T(0)_n = \{\lambda\text{-term t}|fv(t) \subset \{0, \dots, n-1\}\}$

Now take $X \in Set^{\mathbb{N}}$ such that $X_2 = \{M\}$ and $X_{n\neq 2} = \emptyset$. Then $T(X)_n = \{\lambda\text{-term t with a metavariable } M \text{ of arity } 2|fv(t) \subset \{0, \dots, n-1\}\}$

Another example: $Y_2 = \{M, N\}$, $Y_0 = \{C\}$ and $Y_n = \emptyset$ otherwise. Then $T(Y)_n = \{\lambda\text{-term t with metavariables } M\text{,}N \text{ of arity 2 and a constant metavariable } C|fv(t) \subset \{0, \dots, n-1\}\}$

Thanks to the monoid structure, T-algebras inherit a relative monad structure along $I \in Set^{\mathbb{N}}$, which induce a (finitary) monad structure on sets (see [1])

Notation 2. If $n \in \mathbb{N}$, we designate the n^{th} cardinal set $\{0, \dots, n-1\}$ by n.

If $n \in \mathbb{N}$, we designate by yn the coyoneda embedding into $Set^{\mathbb{N}}$, i.e., $yn(p)$ is empty if $n \neq p$ and a singleton set otherwise. Note that morphisms from yn to X are in one-to-one correspondence with elements of X_n.

$I \in Set^{\mathbb{N}}$ designates the family $I_n = n$

Remark 3. A substitution of metavariable is a morphism in the Kleisli category. For example, a Kleisli morphism from X to Y (as in the previous example) is a morphism $X \to T(Y)$, i.e., for each $n \in \mathbb{N}$, a map $X_n \to T(Y)_n$.

Problem 4. Let $t, u : yn \to T(V)$ for some family $V \in Set^{\mathbb{N}}$ (thought as a specification of metavariables). We are looking for a weak coequaliser of t and u in the Kleisli category of T. That is, we are looking for some W together with a morphim $V \to T(W)$ coequalising t and u and factors any other coequalising morphism $V \to T(Z)$.

We can even try to look for an authentic coequaliser, meaning the fatorisation is unique. We prove that such a coequaliser exists as long as there exists at least one unifier.

Definition 5. A coequaliser in a category C of two morphisms $t, u : c \to c'$ is a morphism $g : c' \to d$ s.t. $g \circ t = g \circ u$ such that given any morphism $f : c' \to e$ s.t., $f \circ t = f \circ u$, there exists a unique morphism $h : d \to e$ that factors f through g.

Example 6. If $V \to T(W)$ is a weak coequaliser, then $V \to T(W + y1)$ also is.

Remark 7. A (weak) coequaliser in Kleisli may not be a (weak) coequaliser in the larger category of T-algebras. For example, consider the empty signature. Unifying $M(x, y) = M(y, x)$ results in the substitution $M(x, y) \mapsto N$ where N is a constant metavariable. Consider a T-algebra A equipped with a binary commutative operation op and no closed terms. Then, there is no factorising morphism for the substitution $M \mapsto op$

Question 8. *What about the subcategory of T-algebras that preserves connected limits, as monads on sets? Such monads are equivalent to those equipped with a (cartesian) monad morphism to the list monad, which intuitively simply lists the free variables in the order they appear. They are also equivalent to non-symmetric operads.*

2 Unifying $M(\vec{x}) = M(\vec{y})$

Consider the case $M(\vec{x}) = M(\vec{y})$, where $\vec{x}$ (resp. $\vec{y}$) are distinct variables.

If M is a metavariable of arity m, then this situation is modelled by two injective maps $m \underset{y}{\overset{x}{\rightrightarrows}} N$. Let $p \xrightarrow{e} m$ denote its equaliser: p is the number of indices i such that $x_i = y_i$.

2.1 Restriction to the case $V = ym$

Isolating the metavariable M in V, we can write $V = V' + ym$, and $t, u : yN \to T(V)$ rewrites as $yN \simeq 0 + yN \underset{i+u'}{\overset{i+t'}{\rightrightarrows}} T(V') \oplus T(ym) \simeq T(V' + ym)$, where $\oplus$ denotes the coproduct in the Kleisli category. Thus, by usual colimit construction, the coequaliser of t and u is the coproduct of $T(V')$ with the coequaliser of $t', u' : yN \to T(ym)$.

2.2 Category of coequalising morphisms

We want to construct the coequaliser, that is, the initial object in the category of coequalising morphisms of t', u'.

This category is equivalent to the category whose objects consists of:

- a free T-algebra O
- an element $s : 1 \to O(m)$

- s.t., s equalises $O(m) \overset{O(x)}{\underset{O(y)}{\rightrightarrows}} O(N)$

Lemma 9. *Free T-algebras preserves connected limits*

Proof sketch. Monads on sets preserving connected limits are equivalent to (cartesian?) monads equipped with a (cartesian) monad morphism to the list monad (this morphism simply lists the free variables in the order they appear). Clearly, this is the case of free T-algebras (see the description at the beginning). □

As a consequence, the category of coequalising morphisms is equivalent to the category of free T-algebras O equipped with an element $1 \to O(p)$

2.3 Coequaliser arrow

Clearly, the initial object is $T(yp)$, equipped with its canonical element $1 \to T(yp)(p)$ corresponding by yoneda to $yp \to T(yp)$. In particular, the unification results in a fresh metavariable whose arity is p.

3 Unifying $M_1(\vec{x}) = M_2(\vec{y})$

Consider the case $M_1(\vec{x}) = M_2(\vec{y})$, where $\vec{x}$ (resp. $\vec{y}$) are distinct variables.

If M_1 and M_2 are metavariables of arities m_1 and m_2, then this situation is modelled by two injective maps

$$\begin{tikzcd} & m_1 \arrow[d, "x"] \\ m_2 \arrow[r, "y"'] & N \end{tikzcd}$$

Let $m_1 \xleftarrow{e_1} p \xrightarrow{e_2} m_2$ denote its pullback: p is the number of common values between $\vec{x}$ and $\vec{y}$.

$p = \sum_{i \in m_1, j \in m_2}, x_i = y_j$

3.1 Restriction to the case $V = ym_1 + ym_2$

Isolating the metavariables M_1 and M_2 in V, we can write $V = V' + ym_1 + ym_2$, and $t, u : yN \to T(V)$ rewrites as $yN \simeq 0 + yN \overset{i+t'}{\underset{i+u'}{\rightrightarrows}} T(V') \oplus T(ym_1) \oplus T(ym_2) \simeq T(V' + ym_1 + ym_2)$, where $\oplus$ denotes the coproduct in the Kleisli category, $t', u' : yN \to T(y_{m_1} + y_{m_2})$ and $i : 0 \to T(V')$.

Lemma 10. *The coequaliser of $a + b \overset{f+g}{\underset{f'+g'}{\rightrightarrows}} c + d$ is the coproduct of the coequaliser of $f, f' : a \to c$ and $g, g' : b \to d$.*

In this case, we need to compute the coequaliser of $i, i : 0 \to T(V')$ which is just $T(V')$ and the coequaliser of $t', u' : yN \to T(y_{m_1} + y_{m_2})$.

3.2 Category of coequalising morphisms

We want to construct the coequalisers (in Kleisli), that is, the initial object in the category of cocones over $yN \overset{t'}{\underset{u'}{\rightrightarrows}} T(ym_1 + ym_2)$.

This category is equivalent to the category whose objects consists of:

- a free T-algebra O
- maps $y_{m_1} + y_{m_2} \to O$, or equivalently elements $s_1 : 1 \to O(m_1)$ and $s_2 : 1 \to O(m_2)$
- s.t.,

$$\begin{tikzcd} 1 \arrow[r, "s_1"] \arrow[d, "s_2"'] & O(m_1) \arrow[d, "O(x)"] \\ O(m_2) \arrow[r, "O(y)"'] & O(N) \end{tikzcd}$$

Again, as a consequence of Lemma 9, this category is equivalent to the category of free T-algebras O equipped with an element $1 \to O(p)$.

3.3 Coequaliser arrow

Clearly, the initial object is $T(yp)$, equipped with its canonical element $1 \to T(yp)(p)$ corresponding by yoneda to $yp \to T(yp)$.

4 Unifying $M(\vec{x}) = u$

Consider the case $M(\vec{x}) = u$, where $\vec{x}$ are distinct variables and u is not a metavariable application.

If M appears in u, then there is no unifier. Same if the free variables of u are not in $\vec{x}$. We now assume the contrary. If M is a metavariable of arity m, then this situation is modelled by an injective map $m \xrightarrow{x} N$ and a morphism $u : ym \to T(V')$, where $V = V' + ym$.

4.1 Category of coequalising morphisms

The category of coequalising morphisms is equivalent to the category whose objects consists of

- a free T-algebra O
- a morphism $f : V' \to O$
- an element $s : 1 \to O(m)$

such that

$$\begin{tikzcd}
1 \arrow[rr, "s"] \arrow[d, "u'"'] & & O(m) \arrow[dd, "O(x)"] \\
T(V')(m) \arrow[d, "f^*"'] & & \\
O(m) \arrow[rr, "O(x)"'] & & O(N)
\end{tikzcd}$$

Moreover, since x is monomorphic, $O(x)$ also is. Indeed,

Lemma 11. *Free T-algebras preserve monomorphisms, as monads on sets.*

Proof. This is true of any pullback-preserving functor, and thus follows from Lemma 9 □

Thus, the above required commutation amounts to commutation of the following triangle

$$\begin{tikzcd}
1 \arrow[rr, "s"] \arrow[dr, "u'"'] & & O(m) \\
& T(V')(m) \arrow[ur, "f"'] &
\end{tikzcd}$$

As a consequence, the category of coequalising morphisms is isomorphic to the coslice category under V'.

5 Unifying $o(\vec{t}) = o(\vec{u})$

Consider the case $o(\vec{t}) = o(\vec{u})$, where o is an operation of binding arity $(k_1, \ldots, k_m) \in \mathbb{N}^m$.

So, t and u are morphisms $yn \to T^{(\vec{k})}(V) = T^{(k_1)}(V) \times \cdots \times T^{(k_m)}(V) \xrightarrow{o} T(V)$ where $F^{(k)}(n) = F(n+k)$. Coequalising morphisms consists of

- a free T-algebra O
- a morphism $f : V \to O$
- such that the following diagram commute

$$\begin{tikzcd}
yn \arrow[r] \arrow[d] & T^{(\vec{k})}(V) \arrow[r, "o"] & T(V) \arrow[dd, "f^*"] \\
T^{(\vec{k})}(V) \arrow[d, "o"'] & & \\
T(V) \arrow[rr, "f^*"'] & & O
\end{tikzcd}$$

or equivalently (since f^* is a T-algebra morphism)

$$\begin{tikzcd}
yn \arrow[r] \arrow[d] & T^{(\vec{k})}(V) \arrow[r, "f^{(\vec{k})}"] & O^{(\vec{k})} \arrow[dd, "o"] \\
T^{(\vec{k})}(V) \arrow[d, "f^{(\vec{k})}"'] & & \\
O^{(\vec{k})} \arrow[rr, "o"'] & & O
\end{tikzcd}$$

But since o is monomorphic, this is again equivalent to making the following diagram commute

$$\begin{tikzcd}
yn \arrow[r] \arrow[d] & T^{(\vec{k})}(V) \arrow[d, "f^{(\vec{k})}"] \\
T^{(\vec{k})}(V) \arrow[r, "f^{(\vec{k})}"'] & O^{(\vec{k})}
\end{tikzcd}$$

Now, morphisms $yn \to T^{(\vec{k})}(V)$ are in one-to-one correspondence with morphisms $y(n + \vec{k}) = y(n + k_1) + \cdots + y(n + k_m) \to T(V)$ and thus it means that $f : V \to O$ coequalises the morphisms $\coprod_i y(n + k_i) \rightrightarrows T(V)$.

So, we are looking for a coequaliser for the above morphism. Equivalently we are looking for the colimit over the following diagram

$$\begin{tikzcd}
& T(V) & \\
y(n + k_1) \arrow[ur, shift left] \arrow[ur, shift right] & \dots & y(n + k_m) \arrow[ul, shift left] \arrow[ul, shift right]
\end{tikzcd}$$

We can construct such a colimit (reference?) by performing the subsequent coequalisers in order.

References

[1] Thorsten Altenkirch, James Chapman, and Tarmo Uustalu. Monads need not be endofunctors. *Logical Methods in Computer Science*, 11(1), 2015.