

# General Pattern unification

August 1, 2022

We give a general correctness proof for pattern unification in a categorical setting. Other envisioned applications of our abstract categorical proof are unification for linear or polymorphic syntax (Section 8). In Section 1, we propose a general categorical setting in which pattern unification applies, with the nominal sets in mind. In this case, a second-order syntax with metavariables applied to distinct variables (according to the pattern restriction) corresponds to a free monad applied a coproduct of representable presheaves.

In Section 2, we state our main result that justifies pattern unification algorithms. Then we tackle the proof algorithm, starting with the pruning phase (Section 4), the coequalising phase (Section 5), the occur-check phase (Section 6), and finally we justify termination (Section 7).

Some benefits: we unify the operation and the variable case.

*Notation 1.* Given a natural number  $n$  we denote by  $\underline{n}$  the set  $\{0, \dots, n-1\}$ .

## 1 General setting

### 1.1 Base category

We work in a full subcategory  $\mathcal{C}$  of functors  $\mathcal{A} \rightarrow \mathbf{Set}$ , namely, those preserving finite connected limits, where  $\mathcal{A}$  is a small category in which all morphisms are regular monomorphisms and has finite connected limits.

**Example 2.** For the category of nominal sets, take  $\mathcal{A} = \mathbb{F}_m$  the category of finite cardinals and injections.

*Remark 3.* Regularity of monomorphisms (i.e., the fact that monomorphisms are equalisers) implies their effectivity (i.e., they are the equalisers of their cokernel pairs) because the category has pullbacks. In fact, this condition is not strictly needed, but it will allow us (by Property 5 below) to convert a factorisation problem (does  $g : A \rightarrow X$  factor through  $e : A \rightrightarrows B$ ?) into an equality check (does  $e$  coequalise the kernel pair of  $g$ ?), during the pruning phase (see Equation 2) making our description more effective. This happens, for instance, when checking that a variable occurs or not in some metavariable application.

The yoneda embedding  $\mathcal{A}^o \rightarrow [\mathcal{A}, \text{Set}]$  factors through  $\mathcal{C} \rightarrow [\mathcal{A}, \text{Set}]$ . We denote the fully faithful embedding as  $D \xrightarrow{K} \mathcal{C}$ . A useful lemma that we will exploit is the following:

**Lemma 4.** *Limits, coproducts, and filtered colimits in  $\mathcal{C}$  are computed pointwise.*

*Proof.* All we have to check is that limits, coproducts, and filtered colimits of functors preserving finite connected limits still preserve finite connected limits. The case of limits is clear, since limits commute with limits. The case of coproducts follows from connected limits commuting with coproducts in  $\text{Set}$ . The case of filtered colimits follows from finite limits commuting with filtered colimits in  $\text{Set}$ .  $\square$

In the rest of this section, we abstract this situation by listing a number of properties that we will use in the following to describe the unification algorithm. We will never unfold the definition of  $\mathcal{C}$  afterwards.

**Property 1.**  *$K : D \rightarrow \mathcal{C}$  is fully faithful.*

**Property 2.**  *$\mathcal{C}$  is bicomplete.*

*Proof.*  $\mathcal{C}$  is the category of models of a limit sketch, and thus is locally presentable, by [1, Proposition 1.51]. As a result, it is bicomplete [1, Remark 1.56].  $\square$

*Remark 5.* We need cocompleteness so that we can compute free monads of a finitary endofunctor as the colimit of an initial chain. Completeness ensures that this free monad is algebraically free (not sure if it is helpful).

**Definition 6.** We denote by  $D^+ \xrightarrow{K^+} \mathcal{C}$  the full subcategory of  $\mathcal{C}$  consisting of finite coproducts of objects of  $D$ .

We will be interested in coequalisers in  $D^+$ .

**Property 3.**  *$D$  is closed under finite connected colimits.*

This is to deal with the case  $M(\vec{x}) = N(\vec{y})$ .

*Proof.* Because  $K$  is fully faithful, an equivalent statement is that  $D$  is closed under finite connected colimits and  $K$  preserves them. Now, we assumed that  $\mathcal{A}$  has finite connected limits. Let us show that the yoneda embedding preserves them. We have a natural isomorphism  $[\mathcal{A}, \text{Set}](ya, JX) \simeq \mathcal{C}(Ka, X)$ , where  $y$  is the yoneda embedding in  $\mathcal{A}^o \rightarrow [\mathcal{A}, \text{Set}]$ , and  $J : \mathcal{C} \rightarrow [\mathcal{A}, \text{Set}]$  the canonical embedding. Now consider a finite connected limit  $\lim F$  in  $\mathcal{A}$ . Then,

$$\begin{aligned}
\mathcal{C}(K \lim F, X) &\cong [\mathcal{A}, \text{Set}](y \lim F, JX) \\
&\cong JX(\lim F) && \text{(By the Yoneda Lemma.)} \\
&\cong \lim(JX \circ F) && \text{(By definition of } \mathcal{C} \text{)} \\
&\cong \lim([\mathcal{A}, \text{Set}](yF-, JX)) && \text{(By the Yoneda Lemma)} \\
&\cong \lim \mathcal{C}(KF-, X) \\
&\cong \mathcal{C}(\text{colim } KF, X) && \text{(By left continuity of the hom-set bifunctor)}
\end{aligned}$$

Thus,  $K \lim F \cong \operatorname{colim} KF$ . □

**Property 4.** *Given any morphism  $f : A \rightarrow B$  in  $D$ ,  $Kf$  is epimorphic.*

*Proof.* A morphism  $f : A \rightarrow B$  is epimorphic if and only if the following square is a pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ f \downarrow & & \parallel \\ B & \xlongequal{\quad} & B \end{array}$$

We conclude by Property 3, because all morphisms in  $D = \mathcal{A}^o$  are epimorphic by assumption. □

In fact, we have the following stronger property, that we state separately because as argued in Remark 3, this is not strictly required.

**Property 5.** *Given any morphism  $f : A \rightarrow B$  in  $D$ , the epimorphism  $Kf$  is effective, i.e., the following diagram is a coequaliser*

$$\begin{array}{ccc} \operatorname{Ker}(f) & \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} & A \xrightarrow{f} B \end{array}$$

where  $\operatorname{Ker}(f) \rightrightarrows A$  is the kernel pair of  $f$  defined as the pullback of  $f$  with itself, as in

$$\begin{array}{ccc} \operatorname{Ker}(f) & \xrightarrow{p_1} & A \\ p_2 \downarrow & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

of its kernel pair

*Proof.* Since  $\mathcal{C}$  has pullbacks (Property 2), it is enough to show that  $Kf$  is regular. But since  $f$  is a regular epimorphism by assumption, we know that the  $f$  is the coequaliser of some  $u_1, u_2 : C \rightarrow A$  in  $D$ . We conclude by Property 3. □

**Property 6.** *Coproduct injections  $A_i \rightarrow \coprod_j A_j$  in  $\mathcal{C}$  are monomorphisms.*

*Proof.* This follows from Lemma 4, because a morphism  $f : A \rightarrow B$  is monomorphic if and only if the following square is a pullback

$$\begin{array}{ccc} A & \xlongequal{\quad} & A \\ \parallel & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

□

The following two properties are direct consequences of Lemma 4.

**Property 7.** *For each  $d \in D$ , the object  $Kd$  is connected, i.e., any morphism  $Kd \rightarrow \coprod_i A_i$  factors through exactly one coproduct injection  $A_j \rightarrow \coprod_i A_i$ .*

**Property 8.** *For each  $d \in D$ , the object  $Kd$  is finitely presentable, i.e.,  $\mathcal{C}(Kd, -)$  preserves filtered colimits.*

This is used to prove correctness of the occur-check by induction.

## 1.2 The endofunctor for syntax

**Assumption 7.** *We assume given a finitary endofunctor  $F$  on  $\mathcal{C}$ . We denote by  $T$  the generated free monad  $F^*$ .*

**Example 8.** Consider the endofunctor on  $Nom$  corresponding to  $\lambda$ -calculus:  $F(X) = I + X \times X + X^I$ , where  $I$  is the representable presheaf  $y1$ . Note that  $X_n^I = X_{n+1}$ .

*Remark 9.* In applications, we typically provide a finitary endofunctor on  $[\mathcal{A}, Set]$  that restricts to the category of endofunctors preserving finite connected limits. By 4, this restriction is then also finitary.

Conversely, using the assumptions below, we can also “reconstruct” the endofunctor on  $[\mathcal{A}, Set] = \hat{D}$  (see Section 6).

**Assumption 10.**  *$F(X)$  is of the shape  $\coprod_i R_i(X) \times S_i$ , where*

- $R_i$  is right adjoint to some  $L_i$ ;
- $S_i$  is an object of  $\mathcal{C}$ .
- $L_i K = \coprod_{j \in J} K L'_{i,j}$  for some finite family of endofunctors  $(L'_{i,j})_j$  on  $D$ .

**Corollary 11.**  *$T(X)$  is isomorphic to  $\coprod_i R_i T(X) \times S_i + X$ .*

*Proof.* By Lambek’s lemma [3], applied to the endofunctor  $G - + X$ . □

*Remark 12.* Typically, we want the endofunctor on syntax mapping  $X$  to  $I + F'(X)$ , where  $I$  accounts for variables and  $F'$  for recursive operations. Since  $I \cong 1 \times I$ , this first component fits into the assumed scheme by taking  $S_i = I$  and  $R_i(X) = 1$ . Indeed then, the left adjoint to  $R_i$  is the  $L_i : X \mapsto 0$ .

**Example 13.** Continuing Example 8,  $X \mapsto X \times X$  is right adjoint to  $X \mapsto X + X$ . Moreover,  $X \mapsto X^I$  is right adjoint to  $X \mapsto X \times I$ . If  $X$  is a representable presheaf  $yn$ , then  $yn \times I = yn \times y1 \simeq y(n + 1)$ . Therefore,  $L'n = n + 1$ .

## 2 Main result

Our main result is that a coequaliser diagram in  $Kl_T$  selecting objects in  $D^+$  either has no unifier, either has colimiting cocone.

Because working this disjunction is slightly inconvenient, we rephrase it a pure coequaliser by adding freely a terminal object.

**Definition 14.** Given a category  $E$ , let  $E^*$  be  $E$  extended freely with a terminal object.

*Remark 15.* Adding a terminal object results in adding a terminal cocone to all diagrams.

As a consequence, we have a following lemma.

**Lemma 16.** *Let  $J$  be a diagram in a category  $E$ . The following are equivalent:*

1.  *$J$  has a colimit has long as its category of cocones is not empty.*
2.  *$J$  has a colimit in  $E^*$ .*

Thus, we are going to work in  $Kl_T^*$  rather than  $Kl_T$ .

The following result is also useful.

**Lemma 17.** *Given a category  $E$ , the canonical functor  $E \rightarrow E^*$  creates colimits.*

This has some consequences:

1. whenever the colimit in  $Kl_T^*$  is not the terminal object, it is also a colimit in  $Kl_T$ ;
2. existing colimits in  $Kl_T$  are also colimits in  $Kl_T^*$ ;
3. in particular, coproducts in  $Kl_T$  (which are computed in  $\mathcal{C}$ ) are also coproducts in  $Kl_T^*$ .

*Notation 18.* We denote by  $\top$  the terminal object and by  $!$  any terminal morphism.

Here is our main result. TODO state it for  $D$  rather than  $D^+$  (easier to understand).

**Theorem 19.** *Let  $Kl_{D^+}^*$  be the full subcategory of  $Kl_T^*$  consisting of objects of  $D^+ \cup \{\top\}$ . Then,  $Kl_{D^+}^*$  has coequalisers and the inclusion  $Kl_{D^+}^* \rightarrow Kl_T^*$  preserves them.*

In other words, any coequaliser diagram  $A \rightrightarrows TB$  in  $Kl_T^*$  where  $A$  and  $B$  are in  $D^+$  has a colimit  $B \rightarrow TC$  where  $C \in D^+ \cup \{\top\}$ . Rephrasing it without the adjoined terminal object, this means that either such a coequaliser has a colimit which can be computed in  $Kl_{D^+}^*$ , either there is no cocone at all.

### 3 Notations

We denote the identity morphism at an object  $x$  by  $1_x$ .

If  $(g_i : A_i \rightarrow B)_{i \in I}$  is a family of arrows, we denote by  $[g_i] : \coprod_{i \in I} A_i \rightarrow B$  the induced coproduct pairing.

Coproduct injections  $A_i \rightarrow \coprod_{i \in I} A_i$  are typically denoted by  $in_i$ .

Given an adjunction  $L \dashv R$  and a morphism  $f : A \rightarrow RB$ , we denote by  $f^* : LA \rightarrow B$  its transpose, and similarly, if  $g : LA \rightarrow B$ , then  $g^* : A \rightarrow RB$ . In particular, a Kleisli morphism  $f : A \rightarrow TB$  induces a morphism  $f^* : TA \rightarrow TB$  through the adjunction between  $Kl_T$  and  $\mathcal{C}$ .

We denote the Kleisli composition of  $f : A \rightarrow TB$  and  $g : TB \rightarrow TC$  by  $f[g] = g^* \circ f$ .

### 4 Pruning phase

Here we want to compute a pushout diagram in  $Kl_T^*$ , where one branch is a coproduct of free morphisms.

$$\begin{array}{ccc} \coprod_i KA_i & \xrightarrow{\coprod_i Kf_i} & \coprod_i KB_i \xrightarrow{\eta} T \coprod_i KB_i \\ [g_i] \downarrow & & \downarrow [u_i] \\ X & \xrightarrow{\sigma} & Z \end{array}$$

where  $g_i : KA_i \rightarrow X$  and  $u_i \in \text{hom}_{Kl_T^*}(KB_i, Z)$ .

**Question 20.** *Can we prove in general that the pushout of a free morphism, if it exists, is a free morphism?*

We denote such a situation by

$$X \vdash f_1 := g_1, f_2 := g_2, \dots \Rightarrow u_1, u_2, \dots; \sigma \dashv Z$$

abbreviated as

$$X \vdash \vec{f} := \vec{g} \Rightarrow \vec{u}; \sigma \dashv Z$$

or even

$$X \vdash (f_i)_i := g \Rightarrow u; \sigma \dashv Z$$

with  $g = [g_i]$  and  $u = [u_i]$ . **Could we use Reddy's syntax, to differentiate the input/output? We need to know what his syntax is the internal language of.**

The simplest case is when the coproduct is empty: then, the pushout is  $X$ .

$$\overline{X \vdash \vec{() := \vec{() \Rightarrow \vec{(); 1_X \dashv X}}$$

Another simple case is when  $X = \top$ . Then, the pushout is the terminal cocone. Thus we have the rule

$$\overline{\top \vdash \vec{f} := \vec{g} \Rightarrow \vec{!}; ! \dashv \top}$$

The pushout can be decomposed into smaller components.

$$\frac{X \vdash \vec{f} := \vec{g} \Rightarrow \vec{u}; \sigma \dashv Z \quad Z \vdash \vec{f}' := \vec{g}'[\sigma] \Rightarrow \vec{u}'; \sigma' \dashv Z'}{X \vdash \vec{f}, \vec{f}' := \vec{g}, \vec{g}' \Rightarrow \vec{u}[\sigma'], \vec{u}'; \sigma' \circ \sigma \dashv Z'} \quad (1)$$

This follows from the stepwise construction of coequalisers (Lemma 41).

Thanks to the previous rule, we can focus on the case where the coproduct is the singleton (since we focus on finite coproducts of elements of  $D$ ). Thus, we want to compute the pushout

$$\begin{array}{ccc} KA & \xrightarrow{Kf} & KB \xrightarrow{\eta} TKB \\ g \downarrow & & \\ TC & & \end{array}$$

Since  $TC \simeq I + C + \coprod_i R_i TC \times S_i$  (Corollary 11) and  $KA$  is connected (Property 7),  $KA \rightarrow TC$  factors through one of the following coproduct injections:

- $\eta : C \hookrightarrow TC$  (metavariable case)
- $in_i : R_i TC \times S_i \hookrightarrow TC$  (operation case)

In the next subsections, we discuss the different cases.

#### 4.1 Case $KA \hookrightarrow C$

We want to compute the pushout of free morphisms

$$\begin{array}{ccc} KA & \xrightarrow{Kf} & KB \xrightarrow{\eta} TKB \\ g \downarrow & & \\ C & & \\ \eta \downarrow & & \\ TC & & \end{array}$$

Since the functor  $\mathcal{C} \rightarrow Kl_T$  is left adjoint, the pushout can be computed in  $\mathcal{C}$ . Now,  $C$  is in  $D^+$ . Since  $KA$  is connected (Property 7),  $KA \xrightarrow{g} C$  factors as  $KA \xrightarrow{g'} KD \rightarrow KD + C' \cong C$ . Therefore, we have to compute the pushout

$$\begin{array}{ccc} KA & \xrightarrow{Kf} & KB \\ \downarrow & & \\ KD & & \\ \downarrow & & \\ KD + C' & & \end{array}$$

which is the composition of pushouts

$$\begin{array}{ccc}
KA & \longrightarrow & KB \\
\downarrow & & \downarrow \\
KD & \longrightarrow & KE \\
\downarrow & & \downarrow \\
KD + C' & \longrightarrow & KE + C'
\end{array}$$

where  $E$  is given by Property 3

. Therefore, we have the rule

$$\frac{
\begin{array}{ccc}
KA & \xrightarrow{Kf} & KB \\
g' \downarrow & & \downarrow u \\
KD & \xrightarrow{v} & KE
\end{array} \text{ is a pushout}
}{
KD + C' \vdash f := \eta \circ in_1 \circ g' \Rightarrow \eta \circ in_1 \circ u; T(v + C') \dashv KE + C'
}$$

TODO: examples.

## 4.2 Case $KA \hookrightarrow R_i TC \times S_i$

We want to compute the pushout

$$\begin{array}{ccc}
KA & \xrightarrow{Kf} & KB \xrightarrow{\eta} TKB \\
g \downarrow & & \\
R_i TC \times S_i & & \\
in_i \downarrow & & \\
TC & &
\end{array}$$

A cocone in  $Kl_T$  is given by an object  $Y$  with morphisms  $KB \rightarrow TY \leftarrow C$  such that the following diagram commutes.

$$\begin{array}{ccccc}
KA & \xrightarrow{Kf} & KB & & \\
g \downarrow & & \downarrow & & \\
R_i TC \times S_i & \longrightarrow & R_i TY \times S_i & \longrightarrow & TY
\end{array}$$

Exploiting the fact that both  $KA$  and  $KB$  are connected (Property 7) and the characterisation of  $TY$  as a coproduct  $R_i TY \times S_i + \dots$  (Corollary 11), it follows that  $KB \rightarrow TY$  factors through  $R_i TY \times S_i \hookrightarrow TY$ . Since  $R_i TY \times S_i \rightarrow TY$  is monomorphic (as a coproduct injection, again by Property 6), a cocone in  $Kl_T$



is given by an object  $Y$  with morphisms  $C \rightarrow TY$  and  $KB \rightarrow R_i TY \times S_i$  such that the following diagram commutes.

$$\begin{array}{ccc} KA & \xrightarrow{Kf} & KB \\ g \downarrow & & \downarrow \\ R_i TC \times S_i & \longrightarrow & R_i TY \times S_i \end{array}$$

which is equivalent to the commutation of the two following diagrams

$$\begin{array}{ccc} KA & \xrightarrow{Kf} & KB \\ g_2 \downarrow & & \downarrow \\ S_i & \xrightarrow{=} & S_i \end{array} \quad \begin{array}{ccc} KA & \xrightarrow{Kf} & KB \\ g_1 \downarrow & & \downarrow \\ R_i TC & \longrightarrow & R_i TY \end{array}$$

Uniqueness of  $KB \rightarrow S_i$  making the left diagram commutes is ensured by the fact that  $Kf$  is epimorphic (Assumption 4). Thus, we already have the following rule in case no such morphism exists (this happens concretely for  $M(x) := y$  when  $y \neq x$ ).

$$\frac{g_2 \text{ does not factor through } Kf}{C \vdash f := in_i \circ (g_1, g_2) \Rightarrow !; ! \dashv \top} \quad (2)$$

In fact, using Property 5, the premise can be rephrased as an inequality:

$$\frac{\text{Ker}(Kf) \xrightarrow{p_1} KA \xrightarrow{g_2} S_i \neq \text{Ker}(Kf) \xrightarrow{p_2} KA \xrightarrow{g_2} S_i}{C \vdash f := in_i \circ (g_1, g_2) \Rightarrow !; ! \dashv \top}$$

*Remark 21.* If  $S_i$  is orthogonal [1, Definition 1.32] to all morphisms (i.e., given any span  $S_i \leftarrow A \rightarrow B$ , there exists a unique  $B \rightarrow S_i$  completing the triangle), as in the case where  $S_i$  is the output type “dirac”, then this rule never applies.

Since  $R_i$  has a left adjoint  $L_i$  such that  $L_i K = \int^j K L'_{i,j}$  (Assumption 10), the right diagram is equivalent to making the following diagram commute.

$$\begin{array}{ccc} \coprod_j K L'_{i,j} A & \xrightarrow{\coprod_j K L'_{i,j} f} & \coprod_j K L'_{i,j} B \\ g_1^* \downarrow & & \downarrow u \\ TC & \xrightarrow{\sigma} & TY \end{array}$$

Thus, this justifies the following rule

$$\frac{C \vdash (L'_{i,j} f)_j := g_1^* \Rightarrow u; \sigma \dashv Y \quad g_2 = h \circ Kf}{C \vdash f := in_i \circ (g_1, g_2) \Rightarrow in_i \circ u^*; \sigma \dashv Y} \quad (3)$$

## 5 Coequalising phase

Here we want to compute a coequalising diagram in  $Kl_T^*$ , where the domain is in  $D^+$ .

$$\coprod_i KA_i \xrightarrow[t_i]{[u_i]} \Gamma \xrightarrow{\sigma} \Delta$$

where  $t_i, u_i \in \text{hom}_{Kl_T^*}(KB_i, \Gamma)$ . We denote such a situation by

$$\Gamma \vdash t_1 =_{A_1} u_1, \dots, t_p =_{A_p} u_p \Rightarrow \sigma \dashv \Delta$$

that we sometimes abbreviate as

$$\Gamma \vdash \vec{t} =_{\vec{A}} \vec{u} \Rightarrow \sigma \dashv \Delta$$

The simplest case is when the coproduct is empty: then, the coequaliser is  $\Gamma$

$$\overline{\Gamma \vdash () \Rightarrow 1_\Gamma \dashv \Gamma}$$

Another simple case is when  $\Gamma = \top$ . Then, the pushout is the terminal cocone. Thus we have the rule

$$\overline{\top \vdash \vec{t} =_{\vec{A}} \vec{u} \Rightarrow ! \dashv \top}$$

Again, thanks to Lemma 41, such a coequaliser can be decomposed into smaller components.

$$\frac{\Gamma \vdash t_0 =_{n_0} u_0 \Rightarrow \sigma_0 \dashv \Delta_1 \quad \Delta_1 \vdash \vec{t}[\sigma_0] =_{\vec{n}} \vec{u}[\sigma_0] \Rightarrow \sigma \dashv \Delta_2}{\Gamma \vdash t_0 =_{n_0} u_0, \vec{t} =_{\vec{n}} \vec{u} \Rightarrow \sigma_0[\sigma] \dashv \Delta_2} \quad (4)$$

Thanks to the previous rules, we can focus on the case where the coproduct is a singleton (since we focus on finite coproducts of elements of  $D$ ), and  $\Gamma = TC$ . Thus, we want to compute the coequaliser

$$KA \xrightarrow[t]{u} TC$$

Since  $TC \simeq I + C + \coprod_i R_i TC$  (Assumption 10) and  $KA$  is connected (Property 7),  $t, u : KA \rightarrow TC$  factor through one of the following coproduct injections:

- $\eta : C \hookrightarrow TC$  (metavariables)
- $in_i : R_i TC \times S_i \hookrightarrow TC$  (variables / operations)

In the next subsections, we discuss the different cases.

- $\eta = \dots$ , or  $M(\vec{x}) = \dots$ , in case of a successful occur-check, in Section 5.1;

- $\eta = \eta$ , or  $M(\vec{x}) = N(\vec{y})$ , in Section 5.2, which is partly redundant with the previous case
- $in_i = in_i$ , or  $o(\vec{t}) = o(\vec{u})$ , in Section 5.3;

Let us mention schematically some other cases, which can never be unified in  $Kl_T$ , in most case by Property 7, and thus are solved using  $\top$ .

- $in_i = in_{i'}$  with  $i \neq i'$  (in the examples,  $o(\vec{t}) = o'(\vec{u})$  when  $o \neq o'$ )
- $\eta = \dots$ , or  $M(\vec{x}) = u$ , with failing occur-check (Section 6), i.e., when  $M$  appears deep in  $u$ .

### 5.1 Successful occur-check

The occur-check phase is described in more details in Section 6. Here, we assume that

1.  $C \simeq KB + C'$  for some  $B, C'$ ;
2.  $t : KA \rightarrow TC$  factors as  $KA \xrightarrow{f} KB \xrightarrow{in_B} TC \simeq KB + \dots$ ;
3.  $u : KA \rightarrow TC$  factors as  $KA \xrightarrow{g} TC' \xrightarrow{in_{TC'}} TC$ .

In the examples, the second condition means that  $t$  is a metavariable, and the last condition means that this metavariable does not occur in  $u$ , i.e., the occur-check is successful.

Thus, the coequaliser

$$KA \begin{matrix} \xrightarrow{t} \\ \xrightarrow{u} \end{matrix} TC$$

is a coequaliser (in  $Kl_T^*$ ) of the shape

$$\begin{array}{ccc} & Y & \\ X \swarrow & & \searrow \\ & Z & \\ X \searrow & & \swarrow \\ & Y + Z & \end{array},$$

with  $X = KA$ ,  $Y = TKB$  and  $Z = TC'$

*Remark 22.* There is a canonical isomorphism between the category of cocones over such a coequaliser diagram and the category of cocones over the pushout diagram  $Y \leftarrow X \rightarrow Z$  exists.

Therefore, computing this coequaliser amounts to computing the pushout. We are thus in the situation of the pruning phase, and we can justify the rule

$$\frac{C' \vdash f := g \Rightarrow v; \sigma \dashv Z}{KB + C' \vdash in_B \circ f = in_{TC'} \circ g \Rightarrow [v, \sigma] \dashv Z}$$

## 5.2 Case $M(x_1, \dots, x_m) =_q N(y_1, \dots, y_n)$

Here we are in the situation where  $t, u : KA \rightarrow TC$  factor as  $t', u' : KA \rightarrow C$  through  $\eta : C \rightarrow TC$ . Note that since postcomposition with  $\eta$  is precisely the left adjoint (and thus cocontinuous) functor from  $\mathcal{C}$  to  $Kl_T$ , it is enough to compute the coequaliser in  $\mathcal{C}$  and then precompose it with  $\eta$ .

We assume the following

- $C \simeq \coprod_{i \in \{a, b\}} KB_i + C'$ , where  $a, b$  may not be distinct (in practice  $C$  is in  $D^+$ ),
- $t', u' : KA \rightarrow C$  factor respectively as  $KB_a \rightarrow C$  and  $KB_b \rightarrow C$ .

Note that if  $a$  and  $b$  are distinct, then the case we describe is redundant with some specific case of the pruning phase (Section 4.1).

We want to compute the coequaliser of the free algebra morphisms.

$$\begin{array}{ccc}
 & KB_a & \\
 Kf \nearrow & & \searrow \\
 KA & & \coprod_{i \in \{a, b\}} KB_i + C' \\
 Kg \searrow & & \nearrow \\
 & KB_b &
 \end{array}$$

**Lemma 23.** *The coequaliser of a diagram of this shape, in any category which has the involved colimits, is  $X + C'$ , where  $X$  is the coequaliser of  $Kf$  and  $Kg$  if  $a = b$ , or the pushout of  $Kf$  and  $Kg$ , otherwise.*

Thanks to Property 3, we know that such a colimit exists in  $D^+$ .

All the arguments in this section justify the following rule, whose premise can always be satisfied.

$$\begin{array}{ccc}
 & KB_a & \\
 Kf \nearrow & & \searrow \\
 KA & & \coprod_{i \in \{a, b\}} KB_i + C' \xrightarrow{\sigma} X \text{ is a coequaliser} \\
 Kg \searrow & & \nearrow \\
 & KB_b &
 \end{array}
 \quad \frac{}{\coprod_{i \in \{a, b\}} KB_i + C' \vdash \eta \circ in_a \circ Kf = \eta \circ in_b \circ Kf \Rightarrow \eta \circ \sigma \dashv X}$$

## 5.3 Case $o(\vec{t}) = o(\vec{u})$

Here we assume that  $t, u : KA \rightarrow TC$  factors as  $(t', s_1), (u', s_2) : KA \rightarrow R_i TC \times S_i$  through  $R_i TC \times S_i \hookrightarrow TC$ . A cocone in  $Kl_T$  is given by an object  $Y$  with a

morphism  $C \xrightarrow{\sigma} TY$  such that the following diagram commutes.

$$\begin{array}{ccccc} KA & \xrightarrow{t', s_1} & R_i TC \times S_i & \xrightarrow{R_i \sigma^* \times S_i} & R_i TY \times S_i \\ u', s_2 \downarrow & & & & \downarrow in_i \\ R_i TC \times S_i & \xrightarrow{R_i \sigma^* \times S_i} & R_i TY \times S_i & \xrightarrow{in_i} & TY \end{array}$$

Since  $R_i TY \times S_i \rightarrow TY$  is monomorphic (as a coproduct injection, by Property 6), the above commutation is equivalent to commutation of the following diagram.

$$\begin{array}{ccc} KA & \xrightarrow{t', s_1} & R_i TC \times S_i \\ u', s_2 \downarrow & & \downarrow R_i \sigma^* \times S_i \\ R_i TC \times S_i & \xrightarrow{R_i \sigma^* \times S_i} & R_i TY \times S_i \end{array}$$

which is equivalent to the commutation of the two following diagrams

$$\begin{array}{ccc} KA & \xrightarrow{s_1} & S_i \\ s_2 \downarrow & & \downarrow = \\ S_i & \xrightarrow{=} & S_i \end{array} \quad \begin{array}{ccc} KA & \xrightarrow{t'} & R_i TC \\ u' \downarrow & & \downarrow R_i \sigma^* \\ R_i TC & \xrightarrow{R_i \sigma^*} & R_i TY \end{array}$$

Clearly, if  $s_1 \neq s_2$  there is no unifier so that we have the rule

$$\frac{s_1 \neq s_2}{C \vdash in_i \circ (t', s_1) = in_i \circ (u', s_2) \Rightarrow ! \dashv \top}$$

. We assume the contrary:  $s_1 = s_2 (= s)$ . Since  $R_i$  has a left adjoint  $L_i$  such that  $L_i K = \int^j KL'_{i,j}$  (Assumption 10), the right one is equivalent to making the following diagram commute.

$$\begin{array}{ccc} \coprod_j KL'_{i,j} A & \xrightarrow{t'^*} & TC \\ u'^* \downarrow & & \downarrow \sigma^* \\ TC & \xrightarrow{\sigma^*} & TY \end{array}$$

Thus, this justifies the following rule

$$\frac{C \vdash t'^* = u'^* \Rightarrow \sigma \dashv Y}{C \vdash in_i \circ (t', s) = in_i \circ (u', s) \Rightarrow \sigma \dashv Y} \quad (5)$$

## 6 Occur-check

The occur-check allows to jump from the coequalising phase (Section 5) to the pruning phase (Section 4), whenever the metavariable appearing at the toplevel

of the l.h.s does not appear in the r.h.s. On the other hand, if it appears on the r.h.s and is not top-level, then there is no unifier: this section is devoted to a proof of this fact, see Corollary 30.

We prove hits by working in the category of presheaves on  $D$ .

Let  $J : \mathcal{C} \rightarrow \hat{D}$  mapping  $c$  to the nerve functor  $\mathcal{C}(K-, c)$ . Moreover, let  $G : \hat{D} \rightarrow \hat{D}$  mapping  $P$  to  $\coprod_i JS_i \times \prod_j P_{L_{i,j}-}$ . Then,  $GJ$  is isomorphic to  $JF$ , i.e., the following square commutes up to isomorphism.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C} \\ J \downarrow & & \downarrow J \\ \hat{D} & \xrightarrow{G} & \hat{D} \end{array}$$

Note that  $J$  has a left adjoint (since it is continuous), but more importantly, it preserves coproducts (Property 7) and filtered colimits (Property 8). As a consequence, the following square commutes up to isomorphism.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{T} & \mathcal{C} \\ J \downarrow & & \downarrow J \\ \hat{D} & \xrightarrow{G^*} & \hat{D} \end{array}$$

Then, we can define the size of a morphism as a universal  $G$ -algebra morphism to the constant presheaf  $\mathbb{N}$ , or define the occur-check by induction as a  $G$ -algebra morphism from  $\text{hom}_{\mathcal{C}}(K-, T(B + C))$  to  $\text{hom}_{\mathcal{C}}(K-, TC) + 1$ .

## 6.1 Preliminary lemma

There is a  $M$ -algebra structure  $M\mathbb{N} \rightarrow \mathbb{N}$  on the constant presheaf  $\mathbb{N}$  that adds one to the maximum of the arguments. For any  $o : X \rightarrow \mathbb{N}$ , this induces an algebra morphism  $MX \xrightarrow{|\cdot|_o} \mathbb{N}$ , which basically computes the maximal depth of an element of  $MX$ , with custom depth for leaves in  $X$  as specified by  $o$ . In practice we will have  $X$  as the initial presheaf (then flatness does not depend on  $o$ ), or the terminal presheaf.

**Definition 24.** Let  $o : X \rightarrow \mathbb{N}$ . A natural transformation  $n : A \rightarrow MX$  is said *o-flat* if  $A \rightarrow MX \xrightarrow{|\cdot|_o} \mathbb{N}$  is constant. It is said *flat* if there exists  $o$  that makes it *o-flat*.

**Example 25.** Any natural transformation out of a representable presheaf is 0-flat.

The crucial lemma is the following.

**Proposition 26.** *Let  $A \xrightarrow{n} M\emptyset$  be a flat morphism. Then, the following diagram is a pullback*

$$\begin{array}{ccc} A + A \times_{M\emptyset} A & \longrightarrow & MA \\ \downarrow & & \downarrow n^* \\ A & \xrightarrow{n} & M\emptyset \end{array}$$

*Proof.* See Section B. □

As an example, if  $D$  is discrete, and if  $n$  is  $yd \rightarrow M\emptyset$ , choosing an element  $u \in M\emptyset_d$ , then what it says is that given any element  $t \in MA_d$  such that  $t[u] = u$ , either  $t = \eta(id_d)$ , i.e.,  $t$  is a metavariable, either  $t = u$ .

**Question 27.** *Can we prove this more categorically by simply assuming that  $A$  is tiny?*

**Corollary 28.** *Let  $A \xrightarrow{n} M1$  be a flat morphism. Then, the following diagram is a pullback*

$$\begin{array}{ccc} A + A \times_{M1} A & \longrightarrow & M(A + 1) \\ \downarrow & & \downarrow n^* \\ A & \xrightarrow{n} & M1 \end{array}$$

*Proof.* Consider the functor  $G' = G + 1$ . By universal property, it precisely generates the monad  $M' = M(1 + -)$ , so we apply Proposition 26 for  $M'$  (any  $M'$ -flat morphism becomes  $M$ -flat by assigning the unit weight to the leaf in 1). □

**Corollary 29.** *Let  $A \xrightarrow{n} M1$  be a flat morphism and  $\sigma : B \rightarrow M1$  be another one. Then, the following diagram is a pullback*

$$\begin{array}{ccc} A \times_{M1} MB + A \times_{M1} A & \longrightarrow & M(A + B) \\ \downarrow & & \downarrow [u, \sigma]^* \\ A & \xrightarrow{u} & M1 \end{array}$$

*Proof.* Note that the right morphism decomposes as

$$\begin{array}{ccc} & & M(A + B) \\ & & \downarrow \\ & & MM(A + 1) \\ & & \downarrow \mu \\ A + A \times_{M1} A & \longrightarrow & M(A + 1) \\ \downarrow & & \downarrow \\ A & \xrightarrow{u} & M1 \end{array}$$

where we inserted the known pullback at the bottom. Since coproducts are pullback stable, it is enough to show that  $(A \times_{M1} A) \times_{M(A+1)} M(A+B) = A \times_{M1} A$  and  $A \times_{M1} MB = A \times_{M(A+1)} M(A+B)$ , which we do in the following diagrams (all squares are pullbacks).

$$\begin{array}{ccccc}
A \times_{M1} A & \longrightarrow & A & \xrightarrow{\quad\quad\quad} & M(A+B) \\
\parallel & & \parallel & & \downarrow \\
A \times_{M1} A & \longrightarrow & A & \xrightarrow{\quad\quad\quad} & MM(A+1) \\
\downarrow & & \downarrow & & \downarrow \mu \\
A \times_{M1} A & \longrightarrow & A & \xrightarrow{\quad\quad\quad} & M(A+1) \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{\quad\quad\quad u \quad\quad\quad} & M1 & & 
\end{array}$$
  

$$\begin{array}{ccccccc}
A \times_{M1} MB & \longrightarrow & MB & \xrightarrow{\quad\quad\quad} & M(A+B) & \xrightarrow{\quad\quad\quad} & 0+B & \longrightarrow & (A+B) & \xrightarrow{\quad\quad\quad} & B & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{\quad\quad\quad u \quad\quad\quad} & M1 & \xrightarrow{\quad\quad\quad} & M(A+1) & & M1 & \xrightarrow{\quad\quad\quad} & M(A+1) & & M1 & \xrightarrow{\quad\quad\quad} & M(A+1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{\quad\quad\quad u \quad\quad\quad} & M1 & \xrightarrow{\quad\quad\quad} & M(A+1) & & M1 & \xrightarrow{\quad\quad\quad} & M(A+1) & & M1 & \xrightarrow{\quad\quad\quad} & M(A+1)
\end{array}$$

□

**Corollary 30.** *Given  $\sigma : \Gamma \rightarrow T1$ ,  $u : Kd \rightarrow T1$ ,  $f : Kc \rightarrow Kd$ , if the following square commutes, then the top morphism factors through either  $T\Gamma \rightarrow T(Kd+\Gamma)$  or  $Kd \rightarrow TKd$*

$$\begin{array}{ccc}
Kc & \longrightarrow & T(Kd + \Gamma) \\
f \downarrow & & \downarrow [u, \sigma]^* \\
Kd & \xrightarrow{\quad\quad\quad u \quad\quad\quad} & T1
\end{array}$$

*Proof.* Applying  $J$  to the above diagram, we get

$$\begin{array}{ccc}
yc & \longrightarrow & M(yd + J\Gamma) \\
f \downarrow & & \downarrow [u, \sigma]^* \\
yd & \xrightarrow{\quad\quad\quad u \quad\quad\quad} & M1
\end{array}$$



Now,  $u$  is flat (Example 25), so we by the previous corollary, the top morphism factors through the pullback

$$yd \times_{M1} M(J\Gamma) + yd \times_{M1} yd \rightarrow M(yd + \Gamma). \text{ The conclusion easily follows. } \square$$

**Question 31.** *Can we do the whole algorithm by translating it in the presheaf world?*

## 7 Termination

In this section, we adapt the usual termination argument. Indeed, because objects of  $D$  are connected (Property 7), we can define without ambiguity the size  $|\Gamma|$  of a *context*  $\Gamma$  (i.e., an element of  $D^+$ ) as the length of the coproduct:

**Lemma 32.** *Let  $\coprod_{i \in I} A_i$  and  $\coprod_{j \in J} B_j$  be isomorphic finite coproducts of connected objects. Then,  $I$  and  $J$  are isomorphic sets.*

*Proof.* Assume an isomorphism  $\coprod_{i \in I} A_i \cong \coprod_{j \in J} B_j$ . By connectedness, each  $A_k \rightarrow \coprod_{j \in J} B_j$  factors (uniquely) through a coproduct injection  $B_{\alpha(k)} \rightarrow \coprod_{j \in J} B_j$ . This induces a map  $\alpha : I \rightarrow J$ . Conversely, we get a map  $\beta : J \rightarrow I$ . Now, the composition  $A_k \rightarrow \coprod_{j \in J} B_j \cong \coprod_{i \in I} A_i$  which equals  $A_k \rightarrow B_{\alpha(k)} \rightarrow A_{\beta(\alpha(k))} \rightarrow \coprod_{i \in I} A_i$  must be the coproduct injection  $A_k \rightarrow \coprod_{i \in I} A_i$ . By connectedness, it follows that  $\beta(\alpha(k)) = k$ . Therefore,  $\beta \circ \alpha = id$ , and by a symmetrical argument,  $\alpha \circ \beta = id$ .

We also need to define the size  $||t||$  of a *term*  $t : Kd \rightarrow \Delta$  as  $|t|_0$ , as defined in Section 5.1.  $\square$

Let us recall the usual argument.

First, we show termination of the pruning phase 4, and then termination of the coequalising phase 5.

### 7.1 Pruning phase

There are two recursive rules: (1) and (3).

In fact, they can be straightforwardly recast so that the input  $\vec{f}$  is structurally decreasing in  $\Gamma \vdash \vec{f} := \vec{g} \Rightarrow \vec{u}; \sigma \dashv \Delta$ .

**Question 33.** *Can we define it truly recursively?*

### 7.2 Coequalising phase

There are two recursive rules: (4) and (5). Contrary to the pruning phase, the first one is not structurally recursive since a substitution is applied to the argument.

**Lemma 34.** *If there is a finite derivation tree of  $\Gamma \vdash \vec{t} = \vec{u} \Rightarrow \sigma \dashv \Delta$ , and  $\Delta \neq \perp$ , then*

- $|\Gamma| \geq |\Delta|$

- if  $|\Gamma| = |\Delta|$ , then  $\sigma$  is a renaming, in the sense that it instantiates metavariables with metavariables.

This relies on the following lemma about the pruning phase, easily proved by induction.

**Lemma 35.** *If there is a finite derivation tree of*

$$\Gamma \vdash \vec{f} := \vec{g} \Rightarrow \vec{u}; \sigma \dashv \Delta$$

*and  $\Delta \neq \top$ , then  $|\Gamma| = |\Delta|$  and  $\sigma$  is a renaming, i.e., a free morphism: it factors through  $\Delta \rightarrow T\Delta$ .*

Next, the crucial point, in the stepwise rule (4), is the following lemma.

**Lemma 36.** *If  $\sigma$  is a renaming, then  $||t[\sigma]|| = ||t||$ .*

From these

## 8 Applications

### 8.1 Nominal sets (untyped)

### 8.2 Nominal sets, simply-typed

Let  $T$  be the set of simple types. Here we consider the category of functors  $(Set_f^T \times T)_m \rightarrow Set$ , where  $Set_f^T$  is the category of finite families indexed by  $T$  (it is equivalent to the comma category  $J/T$ , where  $J : \mathbb{F} \rightarrow Set$  is the canonical inclusion), and the indice  $-_m$  means that we restrict to monomorphisms.

Note that any morphism  $\sigma : \Gamma \vdash \tau \rightarrow \Gamma + \Delta \vdash \tau$  is the equaliser of the two obvious morphisms  $\Gamma + \Delta \vdash \tau \rightrightarrows \Gamma + \Delta + \Delta \vdash \tau$ .

**Example 37.** Consider the functor for  $\lambda$ -calculus:

$$\begin{aligned} F(X)_{\Gamma \vdash \sigma} &= y_{\sigma \vdash \sigma}(\Gamma \vdash \sigma) && ((\text{variables})) \\ &+ \coprod_{\tau', \sigma'} X(\Gamma, \tau' \vdash \sigma') \times y_{\vdash \tau' \Rightarrow \sigma'}(\Gamma \vdash \sigma) && ((\text{abstraction})) \\ &+ \coprod_{\tau'} X(\Gamma \vdash \tau' \Rightarrow \sigma) \times X(\Gamma \vdash \tau') && ((\text{application})) \end{aligned}$$

. In other words,

$$\begin{aligned} F(X) &= \coprod_{\sigma} y_{\sigma \vdash \sigma} \\ &+ \coprod_{\tau', \sigma'} X(-, \tau' \vdash \sigma') \times y_{\vdash \tau' \Rightarrow \sigma'} \\ &+ \coprod_{\tau', \sigma} X(- \vdash \tau' \Rightarrow \sigma) \times X(- \vdash \tau') \times y_{\vdash \sigma} \end{aligned}$$

### 8.3 Linear syntax

Should work with the new condition on  $S_i$

Take  $\mathcal{C} = \text{Set}^{\mathbb{N}}$  and  $D$  the full subcategory of representable presheaves. Intuitively, given  $X \in \text{Set}^{\mathbb{N}}$ , the set  $X_n$  is the set of expressions with exactly  $n$  (distinct) variables. Then, we can consider the linear lambda-calculus, as an endofunctor on  $\text{Set}^{\mathbb{N}}$  mapping  $X$  to  $F(X)$  where  $F(X)_n = y1 + \coprod_{p+q=n} X_p \times X_q + (n+1) \times X_{n+1}$ . Note that we could also specify a non-linear binder by replacing  $(n+1) \times X_{n+1}$  with  $\coprod_{p>n} \binom{p}{n} X_p$ . We could also have a non linear application by replacing  $\coprod_{p+q=n} X_p \times X_q$  with  $X_n \times X_n$ .

Then,  $F^*(0)$  is the linear lambda-calculus.  $F^*(yn)$  is the syntax of linear  $\lambda$ -calculus extended with one  $n$ -ary metavariable applied to  $n$  (distinct) variables.

Note that  $F(X)$  is of the shape  $I + \coprod_i X_{p_{i,1}} \times \cdots \times X_{p_{i,m_i}} \times yn_i$  and each  $X \mapsto X_{p_{i,1}} \times \cdots \times X_{p_{i,m_i}} \times yn_i$  is left adjoint to  $X \mapsto X_{n_i} \times (yp_{i,1} + \cdots + yp_{i,m_i})$ . **No! But almost, i.e., if there exists a morphism  $A \rightarrow X_p \times y_n$ , then in fact  $A = A_n yn$  and there exists a morphism  $A_n \times yp \rightarrow X$ , but the converse is false.**

*Remark 38.* We could have done the non-linear version in this setting as well, but the abstract syntax is more convoluted (see the binomial coefficient) and metavariables must still be linear.

**Example 39.** linear lambda calculus, quantum lambda calculus

### 8.4 Polymorphic syntax

WIP. Let  $J^+ : \mathbb{F} \rightarrow \text{Set}$  denotes the functor mapping  $n$  to the  $n + 1^{th}$  cardinal  $\{0, \dots, n\}$ .

Let  $S$  be a nominal set with  $Sn$  the set of types taking free type variables in  $X$ .

Now, contexts are of the shape  $n; \sigma_1, \dots, \sigma_p \vdash \tau$ , where  $\sigma_i, \tau \in Sn$ . Categorically,

We first consider the category of functors  $\int^{n \in \mathbb{F}_m} \text{Set}_f^{Sn} \times Tn \rightarrow \text{Set}$  (this is the oplax colimit), i.e., a morphism between  $n, \Gamma \vdash \tau$  and  $n', \Gamma' \vdash \tau'$  is a monomorphism  $\sigma : n \rightarrow n'$  and renamings  $\Gamma[\sigma] \rightarrow \Gamma'$  such that  $\tau[\sigma] = \tau'$ .

Note that a morphism  $\sigma : n|\Gamma \vdash \tau \rightarrow n+k|\Gamma[\sigma] + \Delta \vdash \tau[\sigma]$  is the equaliser of the two obvious morphisms  $n+k|\Gamma[\sigma] + \Delta \vdash \tau[\sigma] \rightrightarrows n+k+k|\Gamma[\sigma] + \Delta + \Delta \vdash \tau[\sigma]$ .

**Example 40.** Following [2], we specify System  $F$  with the functor

$$\begin{aligned}
F(X)_{n;\Gamma \vdash \sigma} &= V_{n;\Gamma \vdash \sigma} && \text{(variables)} \\
&+ \coprod_{\tau} X(n+1; \Gamma \vdash \tau) \delta_{\sigma, \forall \tau} && \text{(type abstraction)} \\
&+ \coprod_{\tau'} X(n; \Gamma \vdash \forall \tau') \times (\sigma' : Sn) \delta_{\sigma, \tau'[\sigma']} && \text{(type application)} \\
&+ \coprod_{\tau} X(n; \Gamma, \tau \vdash \sigma') \delta_{\sigma, \tau \Rightarrow \sigma'} && \text{(abstraction)} \\
&+ \coprod_{\tau'} X(n; \Gamma \vdash \tau' \Rightarrow \sigma) \times X(n; \Gamma \vdash \tau') && \text{(application)}
\end{aligned}$$

Here,  $V$  is defined as  $V(n; \Gamma \vdash \sigma) = \#\{\sigma \in \Gamma\}$ .

## 9 Conclusion and future work

It would be nice to allow metavariables with mixed linearity constraints.

## References

- [1] J. Adámek and J. Rosický. *Locally Presentable and Accessible Categories*. Cambridge University Press, 1994.
- [2] Makoto Hamana. Polymorphic abstract syntax via grothendieck construction. 2011.
- [3] Joachim Lambek. A fixpoint theorem for complete categories. *Mathematische Zeitschrift*, 103:151–161, 1968.

## A Some lemmas

**Lemma 41.** *Let  $f_1, g_1 : A_1 \rightarrow B$  and  $f_2, g_2 : A_2 \rightarrow B$  be morphisms in some category. Then the coequaliser of the induced parallel morphism  $A_1 \amalg A_2 \rightrightarrows B$  is the morphism  $B \rightarrow C \rightarrow D$  defined as follows (assuming the involved coequalisers exist):*

1.  $B \rightarrow C$  is the coequaliser of  $A_1 \rightrightarrows B$ ;
2.  $C \rightarrow D$  is the coequaliser of  $A_2 \rightrightarrows B \rightarrow C$ .

**Lemma 42.** *Let  $M$  be an algebraically free monad on an extensive category. Then, for any Kleisli morphism  $B \rightarrow M(B')$ , for any object  $A$ , the following square is a pullback (TODO: refine the assumptions)*

$$\begin{array}{ccc}
A & \longrightarrow & M(A + B) \\
\downarrow & & \downarrow \\
A & \longrightarrow & M(A + B')
\end{array}$$

*Proof.* By Lambek's lemma,  $M(X) \cong X + GMX$  where  $M = G^*$ . Through this isomorphism, this square becomes

$$\begin{array}{ccc}
A & \longrightarrow & A + B + GM(A + B) \\
\downarrow & & \downarrow \\
A & \longrightarrow & A + B' + GM(A + B')
\end{array}$$

where horizontal morphisms are left coproduct injections. TODO: conclude by extensivity.  $\square$

**Lemma 43.** *Let  $M$  be an algebraically free cartesian monad on an extensive category. Then, for any Kleisli morphism  $B \rightarrow MB'$ , the following square is a pullback (TODO: refine the assumptions)*

$$\begin{array}{ccc}
MB & \longrightarrow & M(A + B) \\
\downarrow & & \downarrow \\
MB' & \longrightarrow & M(A + B')
\end{array}$$

*Proof.* This commuting square expands as

$$\begin{array}{ccc}
MB & \xrightarrow{Min_2} & M(A + B) \\
\downarrow & & \downarrow \\
MMB' & \longrightarrow & M(A + M(A + B')) \\
= \downarrow & & \downarrow \\
MMB' & \xrightarrow{MMin_2} & MM(A + B') \\
\downarrow \mu & & \downarrow \mu \\
MB' & \xrightarrow{Min_2} & M(A + B')
\end{array}$$

where each square is a pullback square.  $\square$

**Lemma 44.** *Grothendieck topoi are stable under products.*

*Proof.* Let  $(A_i)_{i \in I}$  be a family of Grothendieck topoi. Then, there exists a family of small categories  $(B_i)_{i \in I}$  such that each  $A_i$  is reflective subcategory of  $\hat{B}_i$  and moreover, the reflector is left exact. Then,  $\prod_i A_i \rightarrow \prod_i \hat{B}_i \cong \widehat{\prod_i B_i}$  is also coreflective and the reflector is left exact, concluding the proof.  $\square$

## B Proof of Proposition 26

The results of this section are formalised in agda (see `occurcheck-ind.agda`). The crucial point that keeps the agda formalisation simple is that we can work on  $[D_0, Set]$  rather than in  $[D, Set]$ , where  $D_0$  is the discrete category consisting of objects of  $D$ .

Now, the basic idea is that we can define by recursion a morphism  $MX \rightarrow M0 + \mathbb{N}$  such that the following is a pullback

$$\begin{array}{ccc} M0 & \longrightarrow & MX \\ \downarrow & & \downarrow \\ M0 & \longrightarrow & M0 + \mathbb{N} \end{array}$$

This morphism either returns the closed term is the given input  $t \in MX$  is a closed term, or it returns the depth of some leaf. Assuming it indeed returns  $n$ , then the height of  $t[u]$  must be bigger than the height of  $u$  raised by  $n$ .

Now, the basic idea for the proof is that if  $t[u] = u$  then it must be that  $n = 0$ , so  $t$  is a metavariable (otherwise, it is a closed term).