Unification with binding

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In this document, we show existence of the most general unifier of a family of term pairs involving binding operations and n-ary metavariables applied to distinct variables. The proof involves some categorical arguments and justifies the algorithm, detailed in Section 3.

1 Notations

In a category with coproducts, we denote by $in : A_i \hookrightarrow \coprod_i A_i$ the coproduct injection. Given a monad T on a category C, we denote the Kleisli category by Kl_T: objects are objects of C, and morphisms between c and c' are morphisms in C between c and Tc'. Note that there is a bijection between Kleisli morphisms $c \to Tc'$ and T-algebra morphisms $Tc \to Tc'$. We sometimes denote by f^* the T-algebra morphism induced by a Kleisli morphism. Composition of g and f is given by $g^* \circ f$.

2 Setting

Let Σ be a binding signature. Then, there is a free Σ-monoid monad T on $Set^{\mathbb{N}}$ [1, 2] such that $T(X)$ is the syntax of Σ extended with an n-ary operation for each $x \in X_n$. More precisely, $T(X)_n$ is the set of terms whose free variables are in $\{0, \ldots, n-1\}$.

Example 1. Consider the case of λ-calculus. Then, $T(\emptyset)_n$ is the set of λ-terms t taking free variables in $\{0, \ldots, n-1\}$.

Consider $X \in Set^{\mathbb{N}}$ such that $X_2 = \{M\}$ and $X_{n\neq 2} = \emptyset$. Then $T(X)_n$ is the set of λ-terms t involving a metavariable M of arity 2 such that $fv(t) \subset \{0, \ldots, n-1\}$.

Consider $Y \in Set^{\mathbb{N}}$ such that $Y_2 = \{M, N\}$, $Y_0 = \{C\}$ and $Y_n = \emptyset$ otherwise. Then $T(Y)_n$ is the λ-term t with metavariables M,N of arity 2 and a constant metavariable C such that $fv(t) \subset \{0, \ldots, n-1\}$.

Note that metavariables are allowed to be applied to arbitrary terms here, not only (distinct) variables.

Definition 2. A morphism $X \to T(Y)$ is said *safe* if it factors through $T'(Y) \to T(Y)$, where $T'(Y)_n$ is the subset of $T(Y)_n$ consisting of terms that where metavariables are applied to distinct variables only, rather than arbitrary terms.

Notation 3. We denote by Kl_T the Kleisli category of the monad T: objects are families in $Set^{\mathbb{N}}$ and a morphism $X \to Y$ is a family morphism $X \to T(Y)$.

Remark 4. A substitution of metavariable is a morphism in the Kleisli category. For example, a Kleisli morphism from X to Y (as in the previous example) is a morphism $X \to T(Y)$, i.e., for each $n \in \mathbb{N}$, a map $X_n \to T(Y)_n$ which assigns a term taking free variables in $\{0, \ldots, n-1\}$ for each n-ary metavariable.

Definition 5. A family $X \in Set^{\mathbb{N}}$ is said *finite* if $\coprod_n X_n$ is finite.

Our main result consists in the following

Theorem 6. *Let $V \rightrightarrows T(W)$ be a pair of safe morphisms between finite families V and W. If there is a coequalising Kleisli morphism, then there is coequaliser in $Kl(T)$.*

This theorem allows to compute the most general unifier of two terms.

Corollary 7. *Let $t, u \in T(V)_n$ for some finite family $V \in Set^{\mathbb{N}}$ (thought as a specification of metavariables). If there exists a substitution unifying t and u, then there exists a most general unifier, i.e., a substitution $V \to T(W)$ such that any other unifying substitution uniquely factors through it.*

Proof. Giving two terms $t, u \in T(V)_n$ amounts to giving two parallel morphisms $yn \rightrightarrows T(V)$, where yn denotes the family defined by $yn_p = \emptyset$ if $p \neq n$ and yn_n is a singleton set. The most general unifier, if it exists, is the coequaliser of t and u. □

More generally, we can compute any finite colimit under the same condition.

Corollary 8. *Let $J : D \to Kl_T$ be a finite diagram selecting safe Kleisli morphisms and finite families. If there is a cocone, then J has a colimit.*

Proof. This follows from the computation of colimits as coequalisers and coproducts [3, Theorem V.2.2]. □

Let us finish this section by introducing some useful definitions, notations, and lemmas.

Lemma 9. *Free T-algebras extend to functors $\mathbb{F} \to Set$ preserving pullbacks, where $\mathbb{F}$ is the full subcategory of sets consisting in finite cardinals. Action on morphisms is given by renaming. Moreover, Kleisli morphisms are compatible with renaming, so that there is a functor $\mathcal{U} : Kl_T \to \mathcal{P}$, where $\mathcal{P}$ is the category of pullback preserving functors $\mathbb{F} \to Set$.*

Notation 10. If $n \in \mathbb{N}$, we designate the n^{th} cardinal set $\{0, \ldots, n-1\}$ by n.

If $n \in \mathbb{N}$, we designate by yn the yoneda embedding into $Set^{\mathbb{N}}$, i.e., $yn(p)$ is empty if $n \neq p$ and a singleton set otherwise. We call *representable* any family which is isomorphic to some yn.

$I \in Set^{\mathbb{N}}$ designates the family $I_n = n$.

Lemma 11. *Any family $X \in Set^{\mathbb{N}}$ is isomorphic to a coproduct of representable families. A family X is finite if and only if such a coproduct is finite.*

Proof. Clearly, any family X is isomorphic to $\coprod_n X_n yn \simeq \coprod_n \coprod_{x \in X_n} yn$. □

3 Algorithm

The algorithm takes as input

- a (finite) metavariable context $\Gamma \in Set^{\mathbb{N}}$, that we denote by a list $M_1 : m_1, \ldots M_n : m_n$ of metavariable symbols M_i with their associated arities m_i;
- a list of term pairs $t_i =_{n_i} u_i$, where n_i that t_i and u_i takes free variables in $\{0, \ldots, n_i - 1\}$

It outputs:

- a metavariable context Δ
- a Kleisli map $\sigma : \Gamma \to T(\Delta)$, that is a substitution assigning to each metavariable $M : m$ declared in Γ a term with m free variables involving metavariables Δ.

In this section we inductively specify the algorithm through the judgement

$$\Gamma \vdash t_1 =_{n_1} u_1, \ldots t_p =_{n_p} u_p \Rightarrow \sigma \dashv \Delta$$

that we sometimes abbreviate as

$$\Gamma \vdash \vec{t} =_{\vec{n}} \vec{u} \Rightarrow \sigma \dashv \Delta$$

The completeness statement (proven by induction) is the following.

Proposition 12. *Given a list of safe term pairs $\vec{t} = \vec{u}$ taking metavariables in Γ, if there exists a (finite) derivation tree of $\Gamma \vdash \vec{t} =_{\vec{n}} \vec{u} \Rightarrow \sigma \dashv \Delta$, then σ is the most general unifier. Moreover such a derivation tree exists as long as there exists at least one unifier.*

We will justify this proposition by showing that each introduced rule is sound, in the sense that that it supports the induction step.

Note that at most one rule is applyable given an input $\vec{t} = \vec{u}$.

Proposition 13. *There is at most one derivation tree of $\Gamma \vdash \vec{t} =_{\vec{n}} \vec{u} \Rightarrow \sigma \dashv \Delta$.*

Let us list indeed the rules according to the input $\vec{t} = \vec{u}$ in the conclusion:

- Empty list $\vec{t} = \vec{u} = ()$, in Section 3.1;
- Non empty, non singleton lists, $t_0 = u_0, \vec{t} = \vec{u}$, in Section 3.2;
- $M(\vec{x}) = N(\vec{y})$, in Section 3.3;
- $o(\vec{t}) = o(\vec{u})$, in Section 3.4;
- $M(\vec{x}) = o(\vec{t})$, in Section 3.5;
- $M(\vec{x}) = x_i$, in Section 3.6.

Let us mention the missing cases, which can never be unified anyway.

- $o(\vec{t}) = o'(\vec{u})$ when $o \neq o'$;
- $M(\vec{x}) = y$ when $y \notin \vec{x}$

From this list it follows

3.1 Empty list

$$\frac{}{\Gamma \vdash () \Rightarrow id \dashv \Gamma}$$

Soundness is straightforward.

3.2 Stepwise construction

We can restrict to the case where of a single coequaliser.

$$\frac{\Gamma \vdash t_0 =_{n_0} u_0 \Rightarrow \sigma_0 \dashv \Delta_1 \qquad \Delta_1 \vdash \vec{t}[\sigma_0] =_{\vec{n}} \vec{u}[\sigma_0] \Rightarrow \sigma \dashv \Delta_2 \qquad \vec{t} \text{ is not empty}}{\Gamma \vdash t_0 =_{n_0} u_0, \vec{t} =_{\vec{n}} \vec{u} \Rightarrow \sigma \circ \sigma_0 \dashv \Delta_2}$$

Soundness of this rule follows from the following general result categorical result.

Lemma 14. *Let $f_1, g_1 : A_1 \to B$ and $f_2, g_2 : A_2 \to B$ be morphisms in some category. Then the coequaliser of the induced parallel morphism $A_1 \coprod A_2 \rightrightarrows B$ is the morphism $B \to C \to D$ defined as follows (assuming the involved coequalisers exist):*

1. *$B \to C$ is the coequaliser of $A_1 \rightrightarrows B$;*
2. *$C \to D$ is the coequaliser of $A_2 \rightrightarrows B \to C$.*

Corollary 15. *The above rule is sound.*

Proof. Assume there is a common unifier for $t_0 = u_0, \vec{t} = \vec{u}$. By induction hypothesis, the premises are derivable and thus the rule can be applied. Moreover, again by induction hypothesis, the premises produce most general unifiers. The output substitution of the conclusion is also the most general unifier, thanks to the previous lemma by specialising it to the Kleisli category Kl_T, taking $A_1 = yn_0$ and $A_2 = \coprod_i yn_i$. □

3.3 Case $M(x_1, \ldots, x_m) =_q N(y_1, \ldots, y_n)$

We need to make the distinction whether $M = N$ or not.

$$\frac{(i_1, \ldots, i_p) \text{ is the family of common positions: } x_{\vec{i}} = y_{\vec{i}}}{\Gamma \vdash M(\vec{x}) =_q M(\vec{y}) \Rightarrow M(1, \ldots, m) \mapsto N(i_1, \ldots i_p) \dashv \Gamma \backslash \{M\}, N : p}$$

The premise states that $i : p \to m$ is the equaliser of $\vec{x}, \vec{y} : m \to q$.

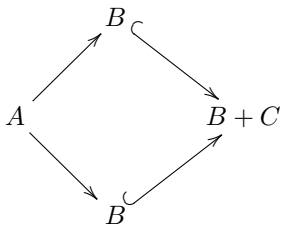
$$\frac{M \neq N \qquad (i_1, \ldots, i_p) \text{ and } (j_1, \ldots, j_p) \text{ are maximal such that } x_{\vec{i}} = y_{\vec{j}}}{\Gamma \vdash M(\vec{x}) =_q N(\vec{y}) \Rightarrow M(1, \ldots, m) \mapsto O(i_1, \ldots i_p), N(1, \ldots, n) \mapsto O(j_1, \ldots, j_p) \dashv \Gamma \backslash \{M, N\}, O : p}$$

The premise states that $m \xleftarrow{i} p \xrightarrow{j} n$ is a pullback of $m \xrightarrow{\vec{x}} q \xleftarrow{\vec{y}} n$.

Proposition 16. *The above rules are sound.*

Proof. We shall prove that the output substitution is the most general unifier.

Let us consider the first rule. We decompose Γ as $M : m, \Gamma'$ the coproduct of ym and Γ'. The term $M(\vec{x})$ corresponds to the composition of $yp \xrightarrow{M(\vec{x})} T(ym) \hookrightarrow T(\Gamma)$. More abstractly, we want to compute the coequaliser, in Kl_T, of



for $A = yp$, $B = ym$, $C = \Gamma'$. It is enough to compute the coequaliser of $f : B \to D$ of $A \rightrightarrows B$, for the desired coequaliser is then $B + C \xrightarrow{f+C} D + C$. So we need to compute the coequaliser of $yp \rightrightarrows T(ym)$. Thanks to Lemma 26, the coequaliser is $ym \xrightarrow{\mathbf{y}i} T(yp)$, where $i : p \to m$ is the equaliser of $\vec{x}$ and $\vec{y}$.

The second rule can be justified similarly. □

3.4 Case $o(\vec{t}) = o(\vec{u})$

$$\frac{o \text{ has binding arity } \vec{n} \qquad \Gamma \vdash \vec{t} =_{q+\vec{n}} \vec{u} \Rightarrow \sigma \dashv \Delta}{\Gamma \vdash o(\vec{t}) =_q o(\vec{u}) \Rightarrow \sigma \dashv \Delta}$$

Lemma 17. *This rule is sound.*

Proof. The result follows easily from the fact that a substitution unifies $o(\vec{t}) = o(\vec{u})$ if and only if it unifies $\vec{t}$ with $\vec{u}$. □

3.5 Case $M(\vec{x}) = o(\vec{u})$

We need to make the distinction whether it is a *simple* term pair or not, in order to avoid the algorithm to loop.

Definition 18. A term pair $t = u$ is called *simple* it is of the shape $M(\vec{x}) = o(N_1(\vec{y_1}), \dots, N_n(\vec{y_n}))$, where

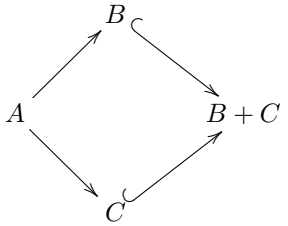
- $\vec{x}$, $\vec{y_1}$, $\dots$, $\vec{y_n}$ are lists of variables and $\vec{x}$ are distinct;
- $\forall i, N_i \neq M$;
- o is a binding operation of the signature;
- the free variables in u are included in $\vec{x}$.

$$\frac{o \text{ has binding arity } (n_1, \dots, n_p) \qquad M(\vec{x}) = o(\vec{N}) \text{ is a simple term pair}}{\Gamma \vdash M(\vec{x}) =_q o(\vec{N}) \Rightarrow M(1, \dots, m) \mapsto o(\vec{N})[x_i \mapsto i] \dashv \Gamma \backslash \{M\}}$$

$$\frac{\begin{array}{c} o \text{ has binding arity } (n_1, \dots, n_p) \qquad M(\vec{x}) = o(\vec{u}) \text{ is not a simple term pair} \qquad \forall i, u_i \neq M(\dots) \\ \Gamma \backslash \{M\}, \vec{N} : m + \vec{n} \vdash N_1(1, \dots, m + n_1) =_{q+n_1} u_1[\Uparrow^{n_1} (x_i \mapsto i)][M(1, \dots m) \mapsto o(\vec{N})], \cdots \Rightarrow \sigma \dashv \Delta \end{array}}{\Gamma \vdash M(\vec{x}) =_q o(\vec{u}) \Rightarrow \sigma, M(1, \dots m) \mapsto o(\vec{N}) \dashv \Delta}$$

Lemma 19. *The above rules are sound.*

Proof. If there is a unifier, then M can't appear in $\vec{u}$, so the condition $u_i \neq M(\dots)$ is safe in both rules. Let us write Γ as $M : m, \Gamma'$, the coproduct of ym and Γ'. The term $M(\vec{x})$ corresponds to the composition of $yq \to T(ym)$ selecting $M(\vec{x})$ with the coproduct injection $T(ym) \hookrightarrow T(\Gamma)$. Since M does not appear in the $o(\vec{u})$, it means that the corresponding morphism $yq \to T(\Gamma)$ factors through the coproduct injection $T(\Gamma') \hookrightarrow T(\Gamma)$. In other words, we want to compute the coequaliser (in Kl_T) of



with $A = yq$, $B = ym$ and $C = \Gamma'$. This is equivalent to the pushout of $B \leftarrow A \to C$

In case of a simple term, $fv(o(\vec{u})) \subset \vec{x}$ implies that there exists $u' \in T(\Gamma')_m$ such that $u'[\vec{x}] = o(\vec{u})$, by choosing $u' = o(\vec{u})[x_i \mapsto i]$. Thus, $o(\vec{u})$ can be

decomposed as the composition $yq \xrightarrow{M(\vec{x})} T(ym) \xrightarrow{u'} T(\Gamma') \hookrightarrow T(\Gamma)$, and we want to compute the pushout

$$\begin{array}{ccc} yq & \xrightarrow{M(\vec{x})} & T(ym) \\ {\scriptstyle M(\vec{x})}\downarrow & & \\ T(ym) & & \\ {\scriptstyle u'}\downarrow & & \\ T(\Gamma') & & \end{array}$$

As we prove in the appendix (Lemma 24), since $\vec{x}$ is injective, the induced morphism $yq \xrightarrow{M(\vec{x})} T(ym)$ is epimorphic and thus we can eliminate it in the diagram, and the above pushout is simply given by $T(ym) \xrightarrow{u'} T(\Gamma') \xleftarrow{id} T(\Gamma')$, so that the coequaliser, as expected, substitute $M : m$ with u' and preserve the other metavariables.

Let us now tackle the other rule. The morphism $yq \to T(\Gamma')$ corresponding to $o(\vec{u})$ factors through $T(\Gamma')^{(\vec{n})} \xrightarrow{o} T(\Gamma')$, where $T(\Gamma')^{(\vec{n})}_k = \prod_i T(\Gamma')_{k+n_i}$. A unifier $ym \to T(\Delta) \leftarrow \Gamma'$ must maps M to some $o(\dots)$, so that $ym \to T(\Delta)$ factors as $ym \to T(\Delta)^{(\vec{n})} \xrightarrow{o} T(\Delta)$. In other words, a unifier is a family Δ with morphisms $ym \xrightarrow{g} T(\Delta)^{(\vec{n})}$ and $f : \Gamma' \to T(\Delta)$ such that the following diagram commutes.

$$\begin{array}{ccccc} yq & \xrightarrow{M(\vec{x})} & T(ym) & \xrightarrow{g^*} & T(\Delta)^{(\vec{n})} \\ {\scriptstyle u}\downarrow & & & & \\ T(\Gamma')^{(\vec{n})} & & & & \downarrow{\scriptstyle o} \\ {\scriptstyle o}\downarrow & & & & \\ T(\Gamma') & & \xrightarrow[f^*]{} & & T(\Delta) \end{array}$$

Now, since f^* is a T-algebra morphism, commutation of the previous diagram is equivalent to commutation of the following one:

$$\begin{array}{ccccc} yq & \xrightarrow{M(\vec{x})} & T(ym) & \xrightarrow{g^*} & T(\Delta)^{(\vec{n})} \\ {\scriptstyle u}\downarrow & & & & \\ T(\Gamma')^{(\vec{n})} & & & & \downarrow{\scriptstyle o} \\ {\scriptstyle f^{*(\vec{n})}}\downarrow & & & & \\ T(\Delta)^{(\vec{n})} & & \xrightarrow[o]{} & & T(\Delta) \end{array}$$

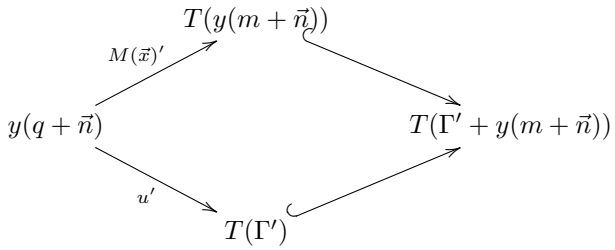
Since o is injective, this is equivalent to commutation of the following diagram.

$$\begin{array}{ccc} yq & \xrightarrow{M(\vec{x})} & T(ym) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle g^*} \\ T(\Gamma')^{(\vec{n})} & \xrightarrow[f^{*(\vec{n})}]{} & T(\Delta)^{(\vec{n})} \end{array}$$

Now, a morphism $u : yk \to Y^{(\vec{n})}$ is equivalently given by a morphism $u' : y(k+\vec{n}) \to Y$, where $y(k+\vec{n})$ denotes the coproduct $\coprod_i y(k+n_i)$. Thus, a unifier is given by a family Δ together with Kleisli morphisms $g : y(m+\vec{n}) \to T(\Delta)$ and $f : \Gamma' \to T(\Delta)$ such that the following diagram commutes.

$$\begin{array}{ccc} y(q+\vec{n}) & \xrightarrow{M(\vec{x})'} & T(y(m+\vec{n})) \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle g^*} \\ T(\Gamma') & \xrightarrow[f^*]{} & T(\Delta) \end{array}$$

Finally, a unifier is equivalently given as a cocone over the coequaliser diagram



This coequaliser is exactly what the premise of the rule is computing, by induction hypothesis. The equivalence between cocones over this diagram and the original one justifies the output substitution. □

3.6 Case $M(\vec{x}) = x_i$

$$\overline{\Gamma \vdash M(\vec{x}) =_q x_i : M(1,\ldots,m) \mapsto i \dashv \Gamma \backslash \{M\}}$$

A symmetric rule must be introduced as well.
Soundness is straightforward.

3.7 Termination

Proposition 20. *There is no infinite derivation tree of* $\Gamma \vdash \vec{t} =_{\vec{n}} \vec{u} \Rightarrow \sigma \dashv \Delta$.

Proof. TODO (Not sure how). □

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A Free T-algebras preserve pullbacks

WIP.

More formally, we prove that T can be lifted to the category $\mathcal{P}$ of functors $\mathbb{F} \to Set$ preserving pullbacks. Then, it remains to show that the functorial action on a free T-algebra is precisely the one considered.

Let us first note that a functor from a complete category to a locally small category preserves pullbacks if and only if it preserves any finite connected limit.

Definition 21. Let $\mathcal{P}$ be the category of functors $\mathbb{F} \to Set$ preserving pullbacks.

Lemma 22. *The forgetful functor $p : \mathcal{P} \to Set^{\mathbb{F}}$ has a right adjoint.*

Proof. This functor is cocontinuous, and morover $\mathcal{P}$ is accessible since it is a limit sketch. □

Proposition 23. *T lifts to $T' : \mathcal{P} \to \mathcal{P}$ yielding a natural isomorphism*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{T'} & \mathcal{P} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ Set^{\mathbb{N}} & \xrightarrow[T]{} & Set^{\mathbb{N}} \end{array}$$

TODO: finish the proof.

The rest of this section is devoted to the proof of the following statement.

Lemma 24. *Let $f : p \to q$ be an injective map. Then, the induced map $yq \to T(yp)$ is epimorphic.*

Remark 25. Any free T-algebra $T(X)$ induces a functor $Kl_T(\mathbf{y}-, X) : \mathbb{F} \to Set$ that we still denote by $T(X)$, extending $T(X) \simeq \hom(y-, TX) : \mathbb{N} \to Set$. Given a $f : p \to q$, the map $T(X)_f : T(X)_p \to T(X)_q$ is the renaming of free variables.

Lemma 26. $\mathbf{y} : \mathbb{F}^o \to Kl_T$ *preserves finite connected limits.*

Proof. Let us denote by Con the full subcategory of functors $\mathbb{F} \to Set$ preserving finite connected limits. We have a natural isomorphism $Kl_T(\mathbf{y}m, X) \simeq T(X)_m \simeq Con(Jm, TX)$, where J is the yoneda embedding in $\mathbb{F}^o \to Set^{\mathbb{F}}$: $Jn_m = m^n$. It follows that $\mathbf{y}$ preserves any colimit that J preserves. Now, since J is the coyoneda embedding, J preserves finite connected colimits, as we restrict to the category of presheaves preserving finite connected limits. □

Proof of Lemma 24. The proof follows from Lemma 26 by noting that $f : A \to B$ is monomorphic if and only if the following square is a pullback.

$$\begin{array}{ccc} A & \xrightarrow{id} & A \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

□