AMORO Lab: Kinematics and Dynamics of a Biglide

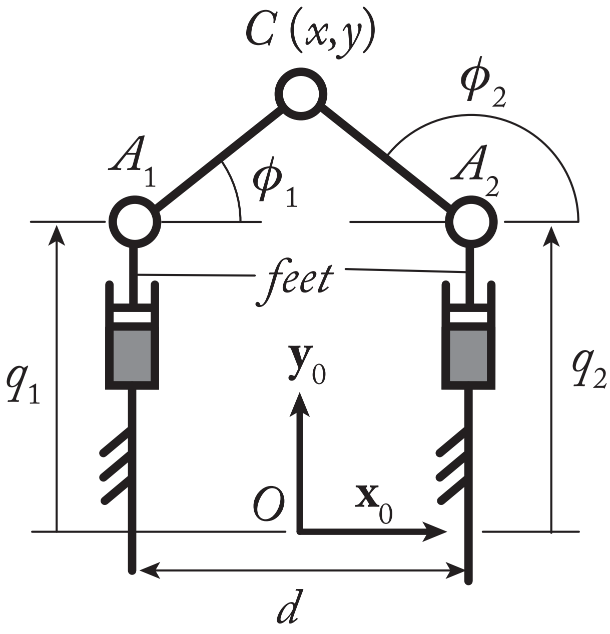

Figure 1: Kinematic model of the Biglide

1 Introduction

The main objective of the present lab is to compute the geometric, kinematic and dynamic models of a Biglide mechanism and to compare them with the results obtained with GAZEBO. Then, a controller will be designed to track a trajectory in simulation.

2 Model

The kinematic architecture of the Biglide mechanism is shown in Fig.1. For the GAZEBO model, the geometric parameters are:

- $d = 0.4$ m
- $l_{A1C} = 0.2828427$ m
- $l_{A2C} = 0.2828427$ m

The two prismatic joints are actuated. The base dynamic parameters are:

- $m_p = 3$ kg the mass of the end-effector
- $m_f = 1$ kg the mass of each foot

All other dynamic parameters are neglected.

3 Geometric models

3.1 Direct geometric model

The direct geometric model gives the position of the end-effector **C**(x,y) as a function of the active joints coordinates *(q_1,q_2)* and the assembly mode, which might be as shown in figure 1, or with the end-effector below the two revolute joints, A_1 and A_2. The objective is to obtain the vector **OC** as a function of the active joint variables only. The algorithm requires to create the vector **OC** as the sum of the segments which connect the origin to the end-effector. Using the right leg of the *Biglide*, the model of the position of the end-effector is the following one:

$$\mathbf{OC} = -\frac{d}{2} \cdot \mathbf{x_0} + q1 \cdot \mathbf{y_0} + \mathbf{A_2H} + \mathbf{HC} \tag{1}$$

being $\mathbf{A_2H}$ half of the vector connecting A_1 to A_2, and $\mathbf{HC}$ the vector connecting H to C.

$$\mathbf{A_2H} = \frac{1}{2}\mathbf{A_2A_1} = \frac{1}{2}\begin{bmatrix} -d \\ q_1 - q_2 \end{bmatrix} = \begin{bmatrix} -\frac{d}{2} \\ \frac{q_1-q_2}{2} \end{bmatrix} \tag{2}$$

Then, for the **HC** vector, there are two possible solutions, depending on the assembly mode:

$$\mathbf{HC} = \gamma\frac{h}{a}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{A_2H} \tag{3}$$

With $a = \|\mathbf{A_2H}\|$ and $h = \sqrt{l^2 - a^2}$.
Adding all these terms, we get the expression of $\mathbf{OC}$.
For this model, it is possible to demonstrate that there are two solutions for *(x,y)* from a set of known q_1 and q_2, depending on the assembly mode.

3.2 Passive joints geometric model

In the computation of the passive joint variables, the end-effector's position and of the active joint variables are exploited. The left coordinate is useful to retrieve the ϕ_1 value, in the following way:

$$\phi_1 = arctan\left(\frac{y - q_1}{x + \frac{d}{2}}\right) \tag{4}$$

In an analog way, the other passive joint value was retrieved exploiting the right leg's coordinates:

$$\phi_2 = arctan\left(\frac{y - q_2}{x - \frac{d}{2}}\right) \tag{5}$$

3.3 Inverse geometric model

The inverse geometric model of a robot with a closed chain structure gives the active joint variables as a function of the coordinates of the end-effector. In order to do so, it's required to solve the geometric constraint equations of the loop, which can be written as:

$$\|\mathbf{A_1C}\|^2 = (\mathbf{OC} - \mathbf{OA_1})^2 = l^2_{A_1C} \tag{6}$$

and

$$\|\mathbf{A_2C}\|^2 = (\mathbf{OC} - \mathbf{OA_2})^2 = l^2_{A_2C} \tag{7}$$

for the left and right legs respectively. Then, going on with computation, the resulting equations will be:

$$(x + \frac{d}{2})^2 + (y - q_1)^2 = l^2_{A_1C} \tag{8}$$

and

$$(x - \frac{d}{2})^2 + (y - q_2)^2 = l^2_{A_2C} \tag{9}$$

which lead to:

$$q_1 = y \pm \sqrt{l^2_{A_1C} - (x + \frac{d}{2})^2} \tag{10}$$

and

$$q_2 = y \pm \sqrt{l_{A_2C}^2 - (x - \frac{d}{2})^2} \tag{11}$$

The total number of solution for the inverse geometric model is 4, since, for every couple *(x,y)*, there are two solutions for q_1 and two for for q_2.

4 First order kinematic models

4.1 Forward and inverse kinematic model

To compute the first-order kinematic model, we start by determining the position of the end-effector using two vector equations, one for each leg.

$$\begin{aligned} \boldsymbol{\xi} &= -\frac{d}{2}\mathbf{x_0} + q_1\mathbf{y_0} + l\mathbf{u_1} \\ \boldsymbol{\xi} &= +\frac{d}{2}\mathbf{x_0} + q_2\mathbf{y_0} + l\mathbf{u_2} \end{aligned} \tag{12}$$

Where:

$$\mathbf{u_1} = \begin{bmatrix} \cos(\Phi_1) \\ \sin(\Phi_1) \end{bmatrix} \quad \mathbf{u_2} = \begin{bmatrix} \cos(\Phi_2) \\ \sin(\Phi_2) \end{bmatrix} \quad \boldsymbol{\xi} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{13}$$

It is known that the derivative of a unit rotating vector $\mathbf{u}$ can be addressed as:

$$\frac{d\mathbf{u}}{dt} = \dot{\Phi} \begin{bmatrix} -\sin(\phi) \\ \cos(\phi) \end{bmatrix} = \dot{\Phi}\mathbf{v} = \dot{\Phi}E\mathbf{u} \tag{14}$$

with $\mathbf{v}$ used to express the vector $\mathbf{u}$ rotated by $\frac{\pi}{2}$:

$$E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{v} = E\mathbf{u} \tag{15}$$

Taking this fact into account, we can calculate the time derivative of the $\boldsymbol{\xi}$ equation as:

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \dot{q_1}\mathbf{y_0} + \dot{\Phi}_1 l\mathbf{v_1} \\ \dot{\boldsymbol{\xi}} &= \dot{q_2}\mathbf{y_0} + \dot{\Phi}_2 l\mathbf{v_2} \end{aligned} \tag{16}$$

Since the values of the passive joint velocities $\dot{\Phi}_1$ and $\dot{\Phi}_2$ are unknown, it's required to identify the vector whose dot product with the kinematic equations of the legs will delete the terms involving these passive joint velocities. To obtain this, it's necessary to perform the dot product of equations (16) with $\mathbf{u_1}$ and $\mathbf{u_2}$, which will respectively cancel out $\mathbf{v_1}$ and $\mathbf{v_2}$, since they are orthogonal.

$$\begin{aligned} \mathbf{u_1^T}\dot{\boldsymbol{\xi}} &= \mathbf{u_1^T}\dot{q}_1\mathbf{y_0} + \mathbf{u_1^T}\dot{\Phi}_1 l\mathbf{v_1} \\ \mathbf{u_2^T}\dot{\boldsymbol{\xi}} &= \mathbf{u_2^T}\dot{q}_2\mathbf{y_0} + \mathbf{u_2^T}\dot{\Phi}_2 l\mathbf{v_2} \end{aligned} \tag{17}$$

In matrix form:

$$\begin{bmatrix} \mathbf{u_1^T} \\ \mathbf{u_2^T} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \mathbf{u_1^T y_0} & 0 \\ 0 & \mathbf{u_2^T y_0} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \tag{18}$$

Where:

$$\begin{bmatrix} \mathbf{u_1^T} \\ \mathbf{u_2^T} \end{bmatrix} = \mathbf{A} \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \dot{\boldsymbol{\xi}} \quad \begin{bmatrix} \mathbf{u_1^T y_0} & 0 \\ 0 & \mathbf{u_2^T y_0} \end{bmatrix} = \mathbf{B} \quad \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \dot{\mathbf{q}} \tag{19}$$

Which describes the kinematic model:

$$\mathbf{A}\dot{\boldsymbol{\xi}} = \mathbf{B}\dot{\mathbf{q}} \tag{20}$$

The forward and inverse kinematic models are obtained by either inverting A or B.

4.2 Passive joints kinematic model

Similar to how the velocities of the end-effector were derives as a function of the active joint velocities, it is possible to use equations (16) to compute the passive joint velocities. To express the end-effector velocities in terms of the two passive joint velocities, it is necessary to multiply by the vectors $\mathbf{v_i}$, with i=1,2, in order to get the vector of the passive joints' velocities, since its multiplication with itself gives as result 1, being a unitary length vector. The result is:

$$\begin{aligned} \mathbf{v_1^T}\dot{\boldsymbol{\xi}} &= \mathbf{v_1^T}\dot{q}_1\mathbf{y_0} + \mathbf{v_1^T}\dot{\Phi}_1 l\mathbf{v_1} \\ \mathbf{v_2^T}\dot{\boldsymbol{\xi}} &= \mathbf{v_2^T}\dot{q}_2\mathbf{y_0} + \mathbf{v_2^T}\dot{\Phi}_2 l\mathbf{v_2} \end{aligned} \tag{21}$$

In matrix form:

$$\begin{bmatrix} \mathbf{v_1^T} \\ \mathbf{v_2^T} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = l \begin{bmatrix} \mathbf{v_1^T y_0} & 0 \\ 0 & \mathbf{v_2^T y_0} \end{bmatrix} \begin{bmatrix} \dot{\Phi}_1 \\ \dot{\Phi}_2 \end{bmatrix} \tag{22}$$

Where:

$$\begin{bmatrix} \mathbf{v_1^T} \\ \mathbf{v_2^T} \end{bmatrix} = \mathbf{A} \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \dot{\boldsymbol{\xi}} \quad \begin{bmatrix} \mathbf{v_1^T y_0} & 0 \\ 0 & \mathbf{v_2^T y_0} \end{bmatrix} = \mathbf{B} \quad \begin{bmatrix} \dot{\Phi}_1 \\ \dot{\Phi}_2 \end{bmatrix} = \dot{\boldsymbol{\Phi}} \tag{23}$$

5 Second order kinematic models

5.1 Forward and inverse kinematic model

To compute the second order kinematic model, it's necessary to use the derivative of equations 16.

$$\begin{aligned} \ddot{\boldsymbol{\xi}} &= \ddot{q}_1 \mathbf{y_0} + l\ddot{\Phi}_1 \mathbf{v_1} - l\dot{\Phi}_1^2 \mathbf{u_1} \\ \ddot{\boldsymbol{\xi}} &= \ddot{q}_2 \mathbf{y_0} + l\ddot{\Phi}_2 \mathbf{v_2} - l\dot{\Phi}_2^2 \mathbf{u_2} \end{aligned} \tag{24}$$

Again, it's possible to apply the dot product of the first line of equation (24) with $\mathbf{u_1}$ and the second line with $\mathbf{u_2}$, as they will cancel respectively $\mathbf{v_1}$ and $\mathbf{v_2}$:

$$\begin{aligned} \mathbf{u_1^T} \ddot{\boldsymbol{\xi}} &= \ddot{q}_1 \mathbf{u_1^T} \mathbf{y_0} + l\ddot{\Phi}_1 \mathbf{u_1^T} \mathbf{v_1} - l\dot{\Phi}_1^2 \\ \mathbf{u_2^T} \ddot{\boldsymbol{\xi}} &= \ddot{q}_2 \mathbf{u_2^T} \mathbf{y_0} + l\ddot{\Phi}_2 \mathbf{u_2^T} \mathbf{v_2} - l\dot{\Phi}_2^2 \end{aligned} \tag{25}$$

or, as a matrix expression:

$$\begin{bmatrix} \mathbf{u_1^T} \\ \mathbf{u_2^T} \end{bmatrix} \ddot{\boldsymbol{\xi}} = \begin{bmatrix} \mathbf{u_1^T} \mathbf{y_0} & 0 \\ 0 & \mathbf{u_2^T} \mathbf{y_0} \end{bmatrix} \ddot{\mathbf{q}} - l \begin{bmatrix} \dot{\mathbf{\Phi}}_\mathbf{1}^\mathbf{2} \\ \dot{\mathbf{\Phi}}_\mathbf{2}^\mathbf{2} \end{bmatrix} \tag{26}$$

This second order kinematic model is under the form:

$$\mathbf{A} = \mathbf{B}\ddot{\mathbf{q}} - \mathbf{d} \tag{27}$$

being $\mathbf{d}$ the vector, without considering the minus sign, on the right hand side of the equation.

5.2 Passive joints kinematic model

For the second-order passive joint kinematic model, the procedure closely resembles the one of the active model. Just as in the first-order case, the vector $\mathbf{v_1^T}$ and $\mathbf{v_2^T}$ will pre-multiply both sides of the relative equation, leading to:

$$\begin{aligned} \mathbf{v_1^T} \ddot{\boldsymbol{\xi}} &= \ddot{q}_1 \mathbf{v_1^T} \mathbf{y_0} + l\ddot{\Phi}_1 \mathbf{v_1^T} \mathbf{v_1} - l\dot{\Phi}_1^2 \mathbf{v_1^T} \mathbf{u_1} \\ \mathbf{v_2^T} \ddot{\boldsymbol{\xi}} &= \ddot{q}_2 \mathbf{v_2^T} \mathbf{y_0} + l\ddot{\Phi}_2 \mathbf{v_2^T} \mathbf{v_2} - l\dot{\Phi}_2^2 \mathbf{v_2^T} \mathbf{u_2} \end{aligned} \tag{28}$$

Or, under the matrix form:

$$\begin{bmatrix} \mathbf{v_1^T} \\ \mathbf{v_2^T} \end{bmatrix} \ddot{\boldsymbol{\xi}} = l\mathbf{I}_{2x2} \ddot{\mathbf{\Phi}} - l \begin{bmatrix} \dot{\mathbf{\Phi}}_\mathbf{1}^\mathbf{2} \\ \dot{\mathbf{\Phi}}_\mathbf{2}^\mathbf{2} \end{bmatrix} \tag{29}$$

6 Dynamic model

The dynamic model of the *Biglide* is based on the following dynamic parameters:

- m_p= 3 kg, the mass of the end-effector;
- m_f= 1 kg, the mass of each foot.

Since no other information is given regarding the dynamic parameters of the structure, assume that the mass of the diagonal links is null, as well as their inertial moment $I_{O_{i2}}$. Due to this, their kinetic energy will not contribute to the calculation of the energy of the entire structure. It is possible to virtually cut it at the end effector joint and deal with a virtual tree structure plus a virtually free-moving platform.
For the computation of the dynamic model of the structure, it's necessary to start by finding the kinetic and potential energies, to build the **Lagrangian equation** of the whole system. Afterwards, the Lagrangian's derivatives will be calculated to retrieve the components of the vector of generalized forces and moments τ. The Lagrangian equation is defined as follows:

$$L = E - U \tag{30}$$

Where, E is the *kinetic energy* of the system and U the *potential energy*. Since it is a planar mechanism, the gravity effect will be neglected. The potential energy of the system will then be null. The Lagrange's equation will then be equal to the kinetic energy of the system.
The total kinetic energy of the system will be expressed as the sum of all energies of the bodies' components. In the current case the only bodies with kinetic energy different from zero are the two feet and the end effector. Therefore, the total energy will be equal to:

$$E_{tot} = E_{foot_1} + E_{foot_2} + E_p \tag{31}$$

The dynamic model of the *Biglide* is obtained by computing first the dynamic model of the tree structure without the end effector, considering all the joints in the single opened chains as actuated, even if they are passive. The formulation of the kinetic energy for a generic rigid body is the following one:

$$E_i = \frac{1}{2}(m_i v_i^T v_i + \omega_i^T I_{Oi} \omega_i + 2^i m s_i^T (v_i \times \omega_i)) \tag{32}$$

As said before, the feet action is only translational and not rotational, so the equation for both bodies simplifies to:

$$E_1 = \frac{1}{2} m_1 v_1^T v_1 \tag{33}$$

$$E_2 = \frac{1}{2} m_2 v_2^T v_2 \tag{34}$$

being v_1 and v_2 equal to $\dot{\mathbf{q_1}}$ and $\dot{\mathbf{q_2}}$ respectively. The computation of the Lagrangian of the end effector can be simply computed as the kinetic energy of a point mass moving on the plane. Since it's possible to easily compute the velocity of point C with the first-order kinematic model, the Lagrangian turns out to be:

$$L_p = E_p = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)m_p \tag{35}$$

By adding these three contributions of the kinetic energy, the Lagrangian of the whole system is obtained, since the potential energy is null:

$$L = \frac{1}{2}(m_1 \dot{q_1}^2 + m_2 \dot{q_2}^2 + (\dot{x}^2 + \dot{y}^2)m_p) \tag{36}$$

Then, using the Lagrange formalism, it's possible to calculate τ_a, the virtual input torques of the actuated robot's joints, τ_d, the virtual input torques of the passive robot's joints, w_P, force and moment applied by the end effector, namely:

$$\tau_{a_1} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q_1}}\right] - \frac{\partial L}{\partial q_1} = m_1 \ddot{q_1} \tag{37}$$

$$\tau_{a_2} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q_2}}\right] - \frac{\partial L}{\partial q_2} = m_2 \ddot{q_2} \tag{38}$$

$$\tau_{d_1} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\phi_1}}\right] - \frac{\partial L}{\partial \phi_1} = 0 \tag{39}$$

$$\tau_{d_2} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\phi_2}}\right] - \frac{\partial L}{\partial \phi_2} = 0 \tag{40}$$

$$w_{P_x} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] - \frac{\partial L}{\partial x} = m_p \ddot{x} \tag{41}$$

$$w_{P_x} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}}\right] - \frac{\partial L}{\partial y} = m_p \ddot{y} \tag{42}$$

So, grouping these data in vectors, it is obtained:

$$\tau_{\mathbf{a}} = \begin{bmatrix} m_1\ddot{q}_1 \\ m_2\ddot{q}_2 \end{bmatrix} \tag{43}$$

$$\tau_{\mathbf{d}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{44}$$

$$\mathbf{w_p} = \begin{bmatrix} m_P\ddot{x} \\ m_P\ddot{y} \end{bmatrix} \tag{45}$$

which, using Lagrangian multipliers, give the following formula of the vector of generalized forces:

$$\tau_t = \tau_a + J^T w_p \tag{46}$$

where $J = A^{-1}B$ is the Jacobian matrix of the system. Since the relation between $\ddot{\xi}$ and $\ddot{q}_a$, namely:

$$\ddot{\xi} = A^{-1}(B\ddot{q}_a - d) \tag{47}$$

it's possible to write the equation 46 in order to group the the inertial components and the Coriolis ones:

$$\begin{aligned}
\tau_{\text{tot}} &= \begin{bmatrix} m_1\ddot{q}_1 \\ m_2\ddot{q}_2 \end{bmatrix} + m_p(A^{-1}B)^T\ddot{\xi} = \\
&= \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + m_p(A^{-1}B)^T A^{-1}(B\ddot{q}_a - d) = \\
&= \left(\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + m_p(A^{-1}B)^T A^{-1}B \right) \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} - m_p(A^{-1}B)^T A^{-1}d = \\
&= \left(\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + m_p J^T J \right) \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} - m_p J^T A^{-1}d
\end{aligned}$$

where the parenthesis multiplied by the vector of $\ddot{q}_a$ will be the *inertial matrix* M, positive definite, while the other term will be the *Coriolis* and *centrifugal effect*.

$$\tau_{tot} = M\ddot{q} + C \tag{48}$$

with

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + J^T m_p J \tag{49}$$

and

$$C = J^T m_p A^{-1} d \tag{50}$$