

# High-Precision Quadratic Programming by Iterative Refinement

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# Motivation

#### Areas of Application:

- Least squares
- Computational geometry<sup>1</sup>
- Support vector machines
- SQP<sup>2</sup>

<sup>&</sup>lt;sup>2</sup> Johnson, Travis C., Christian Kirches, and Andreas Wächter. An active-set method for quadratic programming based on sequential hot-starts. SIAM Journal on Optimization 25.2 (2015): 967-994.





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#### Standard description:

$$\min_{x} \frac{1}{2}x^{T}Qx + c^{T}x$$
s. t.:  $Ax = b$ 

$$x \ge l$$

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# First Order Optimality

The Karush-Kuhn-Tucker (KKT) conditions read:

$$Ax^* = b$$

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Usually a solver provides solutions  $(\tilde{x}^*, \tilde{y}^*, \tilde{z}^*)$  of the form e.g.:

$$\tilde{z_i}^* \ge -\varepsilon \ (\varepsilon > 0)$$



$$Qx^* + c - A^Ty^* - z^* = 0$$
  $x^* - l, z^* \ge 0$   
 $Ax^* = b$   $(x^* - l)^Tz^* = 0$ 

$$Qx^* + c - A^T y^* - z^* = 0 x^* - l, z^* \ge 0$$

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$$\{1, ..., n\} = \mathcal{B} \cup \mathcal{N}$$

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$$Ax^* = b \qquad x_{\mathcal{N}}^* = l_{\mathcal{N}}, z_{\mathcal{B}}^* = 0$$

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#### Basic solution:

$$\begin{pmatrix} -Q_{\mathcal{B}\mathcal{B}} & A_{\cdot\mathcal{B}}^T \\ A_{\cdot\mathcal{B}} & 0 \end{pmatrix} \begin{pmatrix} x_{\mathcal{B}} \\ y \end{pmatrix} = \begin{pmatrix} c_{\mathcal{B}} + Q_{\mathcal{B}\mathcal{N}}l_{\mathcal{N}} \\ b - A_{\cdot\mathcal{N}}l_{\mathcal{N}} \end{pmatrix}$$



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- Solve  $A\tilde{d}=\tilde{r}$
- Correct solution

$$\tilde{x}^{i+1} = \tilde{x}^i + \tilde{d}$$



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#### Error:

$$Ax^{i+1} - b = A(x^i + \tilde{d}/\Delta) - b = \frac{\Delta Ax^i + A\tilde{d} - \Delta b}{\Delta} = \frac{A\tilde{d} - r}{\Delta}$$





## Refine Basic Solutions

For some approximate point  $(x^*, y^*)$  we refine:

$$\begin{pmatrix} -Q & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix}$$

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and get:

$$\begin{pmatrix} -\frac{\Delta_P}{\Delta_D}Q & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \hat{c}\Delta_D \\ \hat{b}\Delta_P \end{pmatrix} = \begin{pmatrix} (Qx^* + c - A^Ty^*)\Delta_D \\ (b - Ax^*)\Delta_P \end{pmatrix}$$

## **QP** Refinement

# Theorem (Gleixner, 2015)

Suppose we are given a QP in form

$$\min\{\frac{1}{2}x^{T}Qx + c^{T}x \mid Ax = b, x \ge l\},\tag{1}$$

then for  $x^* \in \mathbb{R}^n$ ,  $y^* \in \mathbb{R}^m$ , and scaling factors  $\Delta_P$ ,  $\Delta_D > 0$ , consider the transformed problem

$$\min\{\frac{\Delta_P}{2\Delta_D}x^TQx + \Delta_D\hat{c}^Tx \mid Ax = \Delta_P\hat{b}, x \ge \Delta_P\hat{l}\},\tag{2}$$

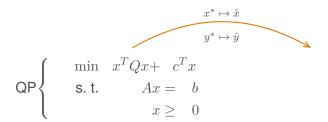
where  $\hat{c}=Qx^*+c-A^Ty^*$ ,  $\hat{b}=b-Ax^*$ , and  $\hat{l}=l-x^*$ . Then for any  $\hat{x}\in\mathbb{R}^n$ ,  $\hat{y}\in\mathbb{R}^m$  the following holds:

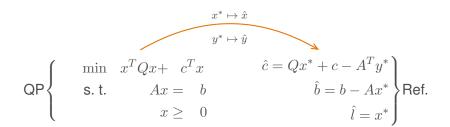
- $\hat{x}$  ( $\hat{y}$ ) is primal (dual) feasible for (2) within an absolute tolerance  $\varepsilon_P > 0$  ( $\varepsilon_D > 0$ ) iff  $x^* + \frac{\hat{x}}{\Delta_D} (y^* + \frac{\hat{y}}{\Delta_D})$  is primal (dual) feasible for (1) within  $\varepsilon_P/\Delta_P$  ( $\varepsilon_D/\Delta_D$ ).
- $\hat{x}$ ,  $\hat{y}$  satisfy complementary slackness for (2) within an absolute tolerance  $\varepsilon_S > 0$  iff  $x^* + \frac{\hat{x}}{\Delta_D}$ ,  $y^* + \frac{\hat{y}}{\Delta_D}$  satisfy complementary slackness for (1) within  $\varepsilon_S/(\Delta_P\Delta_D)$ .

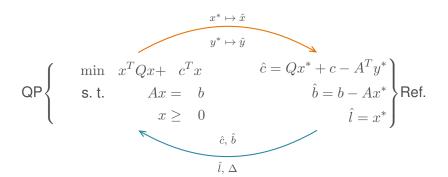


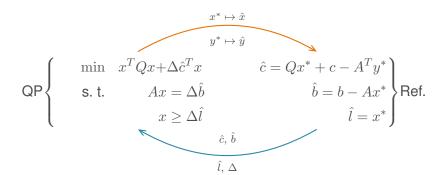
$$\mathsf{QP} \left\{ \begin{array}{rll} & \min & x^TQx + & c^Tx \\ & \mathsf{s.\,t.} & Ax = & b \\ & & x \geq & 0 \end{array} \right.$$

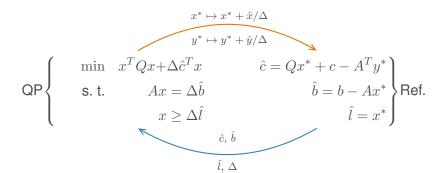












# The Implementation

- Extension of SoPlex LP refinment framework<sup>3</sup>
- Use of qpOASES to solve QPs<sup>4</sup>
- GMP for rational calculations (operator overloading)
- Templates for different precision oparations

<sup>&</sup>lt;sup>4</sup> Ferreau, Hans Joachim, et al. qpOASES: A parametric active-set algorithm for quadratic programming. Mathematical Programming Computation 6.4 (2014): 327-363.



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- Use of qpOASES to solve QPs<sup>4</sup>
- GMP for rational calculations (operator overloading)
- Templates for different precision oparations
- Rational (and floating point) reading and writing of QPS
- Adaptive scaling and solving
- Sparse solving with qpOASES

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Solver	Time	Iter	Ref	Fails	Tol. reached	Incons.
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Results for Maros Mészáros subset: Problems with less than 1000 variables and constraints (73). \*Tolerance  $\approx 10^{-9}$ , \*\*Tolerance  $10^{-100}$ 





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QPIR RELX**	0.14	26	8	3	67	3

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Relaxed settings: MPC + NZCTests + DriftCorrection + Ramping; Reliable + numRefinmentSteps 10





# **Future Work**

- · Solve basic system in rational precision
- · Handle infeasible and unbounded QPs
- Compare with existing solver for geometric optimization<sup>5</sup>

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Solve  $\min\{\frac{1}{2}x^TQx + c^Tx \| Ax = b, x \geq 0\}$  approximately and get basis  $\mathcal{B}$  and  $x^1$ ,  $y^1$  returned as optimal. Define maximal scaling per iteration  $\alpha$  and tolerances  $\varepsilon_P, \varepsilon_D$ . For  $k = 1, 2, ... k_{max}$  do:

- $\hat{b} = b Ax^k$
- $\delta_P = \max\{\|\hat{b}\|_{\infty}, \max_i -x_i^k\}$
- $\hat{c} = Qx^k + c A^Ty^k$
- $\delta_D = \max\{\|\hat{c}_{\mathcal{B}}\|_{\infty}, \max_{i \notin \mathcal{B}} -\hat{c}_i\}$
- $\Delta = \min\{1/\delta_P, 1/\delta_D, \alpha\Delta\}$
- If  $\delta_P \leq \varepsilon_P$  and  $\delta_D \leq \varepsilon_D$ : break
- Else:
  - Solve  $\min\{\frac{1}{2}x^TQx + \Delta\hat{c}^Tx || Ax = \Delta\hat{b}, x \ge -\Delta x^k\}$  (Hotstart!)
  - Get  $\mathcal{B}$ ,  $\tilde{x}^*$ , and  $\tilde{y}^*$
  - Update  $x^{k+1} = x^k + \frac{\tilde{x}^*}{\Delta}$  and  $y^{k+1} = y^k + \frac{\tilde{y}^*}{\Delta}$
  - $x_i^{k+1} = 0 \ \forall \ i \notin \mathcal{B}$

