On Solution Algorithms for Time-Dependent Quasi-Variational Inequalities with Gradient Constraints

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Outline

1. Motivation
2. The QVI
3. Reformulation as a minimization problem
4. Algorithms
5. Numerical Experiments

Motivation

- Granular cohesionless material that is poured onto a solid surface exhibits slopes that are not steeper than the angle of repose.
- materialspecific
- Example: Sand: 34°

[source: www.meyers-material.de]

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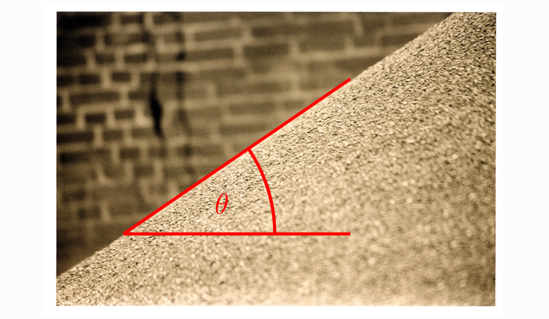
- Example: Sand: $34^\circ = \theta$



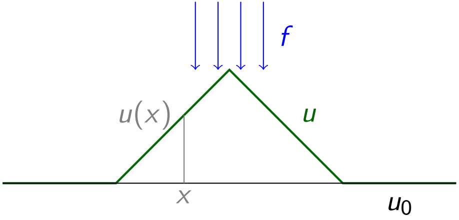
[source: www.ilonagorling.de]

Motivation

- underlaying solid surface can exhibit slopes that are steeper than θ

steep slopes [source: www.berlinonline.de]

Physical Modeling



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u_0 – initial surface
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Illustration in one dimension

Let $\Omega \subset \mathbb{R}^2$ and $T > 0$. For $t \in (0, T)$ and $x \in \Omega$.

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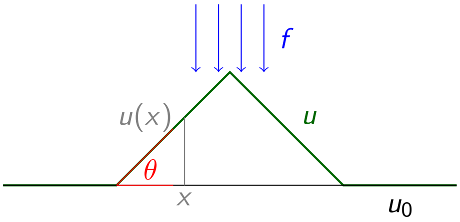

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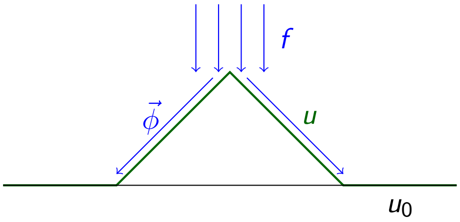

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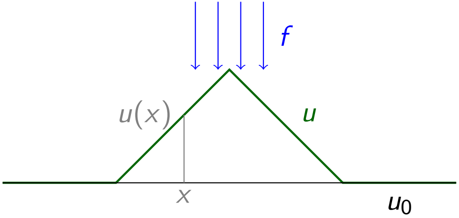

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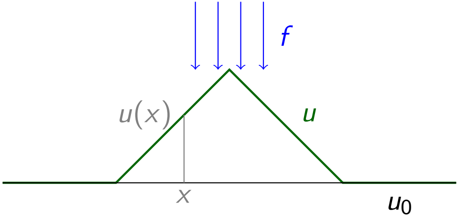

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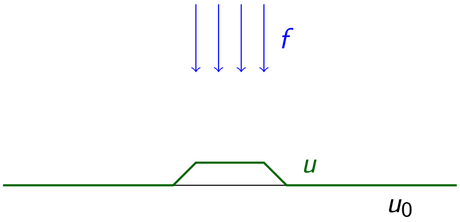

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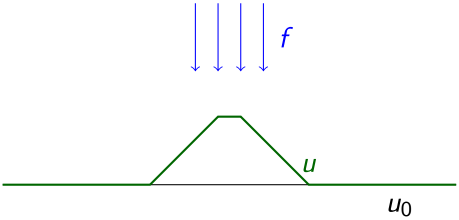

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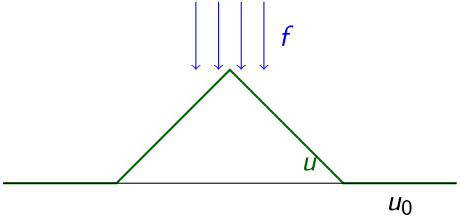

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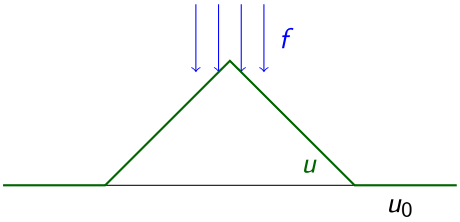

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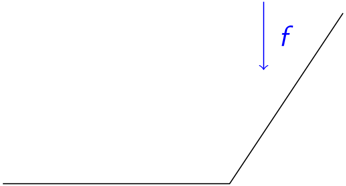

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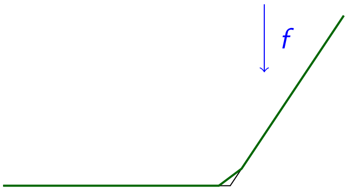

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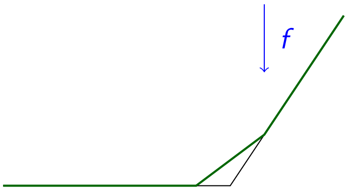

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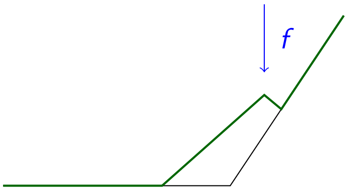

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QVI with Gradient Constraint

- Gradient constraint

$$M(u)(x) = \begin{cases} \alpha & u(x) > u_0(x) \\ \max(\alpha, |\nabla u_0(x)|) & u(x) = u_0(x). \end{cases}$$

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- Constraint set

$$K(u) = \left\{ v \in H_0^1(\Omega) \middle| |\nabla v| \leq M(u), \text{ a.e. in } \Omega \right\}$$

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From (2)-(4) a Quasi–variational inequality can be derived [Prigozhin 94]:

Problem 1 (QVI)

Find $u \in W(0, T)$ s.t. $u(x, 0) = u_0(x)$ and for a.e. $t \in (0, T)$, $u(t) \in K(u(t))$ and

$$(u_t(x,t) - f(x,t), v(x,t) - u(x,t))_{L^2(\Omega)} \geq 0, \quad \textit{for all } v \in K(u(t))$$

where $W(0, T) = \{y \mid y \in L^2(0, T; H_0^1(\Omega)) : y_t \in L^2(0, T; H^{-1}(\Omega))\}$

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- $M(u)$ constrains the gradient.

Towards Implementation

Semidiscretization by applying implicit Euler yields

Problem 2 (Stationary QVIs)

find $\{u_n\}_n = 0 \dots N$ *s.t. for* $n = 0, \dots, N-1$, u_{n+1} *solves*

$$\left(\frac{u - u_n}{\tau} - f_n, v - u\right)_{L^2(\Omega)} \geq 0, \quad \forall v \in K(u)$$

- Existence of solution in case of regularization of M [Rodrigues, Santos]

Equivalence for Variational Inequality

- In the case of Varitional Inequality for $u \in K$ one often has the equivalence between

$$(\nabla J(u), v - u) \geq 0, \quad \forall v \in K$$

  and

$$\min J(u) \text{ over } K$$

- This does generally not hold for QVIs.

Counterexample

The QVI is **not** equivalent to the minimization problem

$$\begin{aligned} &\min J(u) = \frac{1}{2\tau}\|u - u_n\|^2 - (f_n, u) \\ &\text{s.t. } u \in K(u) \end{aligned} \tag{1}$$

- Counterexample exists based on following proposition

Proposition 1

Let u^* be a solution of the QVI. Further suppose that

1. $\{x \in \Omega : u^* > u_n(x)\} \neq \emptyset$ *and*
2. $\{x \in \Omega : u^* > u_n(x)\} \cap \{x \in \Omega : f(x) > 0\} = \emptyset.$

Then u^* does not solve (1).

Towards Implementation

Let u^* be a solution to the QVI

- Then

$$u^* = \operatorname{argmin} J(u), \qquad \text{where } J(u) = \frac{1}{2\tau}\|u - u_n\|^2 - (f_n, u),$$
$$\text{s.t. } u \in K(u^*)$$

- this motivates fixed point iteration [Hintermüller, Rautenberg]

$$u^{k+1} = \operatorname*{argmin}_{u^{k+1} \in K(u^k)} J(u)$$

- series of convex minimization problems

Algorithm 1: Variable Splitting

- Idea: Variable Splitting [Hintermüller, Rasch]

$$\min J(u) = \frac{1}{2\tau}\|u - u_n\|^2 - (f_n, u)$$
$$\text{s.t. } u \in H_0^1(\Omega),\ \boldsymbol{q} = \nabla u,\ \boldsymbol{q} \in \tilde{K}(u^k)$$

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- Moreau–Yosida regularization yields unconstraint problem

$$\min J_\gamma(u, q) = \frac{1}{2\tau}\|u - u_n\|^2_{L^2(\Omega)} - (f_n, u) - \frac{\gamma}{2}\|(\nabla u - q)^+\|^2_{L^2(\Omega)}$$

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- Solving via Alternating Minimization Scheme

$$\boldsymbol{q}^{k+1}_{\gamma\, j+1} := \operatorname*{argmin}_{\boldsymbol{q} \in \tilde{K}(u^k)} J_\gamma(u^{k+1}_{\gamma\, j}, \boldsymbol{q}) \quad \text{(Explicit Projection)}$$
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- Linear convergence with mesh dependent linear factor ≈ 1

Algorithm 2: Semismooth Newton

- Moreau–Yosida regularization of $I_{K(u^k)}$ yields unconstrained minimization problem

$$\min J_\gamma(u) = \frac{1}{2\tau}\|u - u_n\| - (f_n, u) + \frac{\gamma}{2}\|(|\nabla u| - M(u^k))^+\|^2_{L^2(\Omega)}$$

- Ensure regularity of the differential operator by introducing $\frac{\varepsilon}{2}\|\nabla u\|_{L^2(\Omega)}$

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Proposition 2

For $\gamma \to \infty$ and $\varepsilon \to 0$, the sequence of the unique minimizers of $J_{\gamma,\varepsilon}(u)$ converges weakly to the unique minimizer of $J(u)$.

Algorithm 2: Semismooth Newton

For the minimization problem min $J_{\gamma\varepsilon}(u)$, the necessary and sufficient first–order condition for optimality of a point u for this problem is [Hintermüller, Rasch]

$$0 = F_{\gamma,\varepsilon}(u) := u - \varepsilon\Delta u - g - \gamma\, div\,(\mathbf{a}(u)\nabla u) \tag{2}$$

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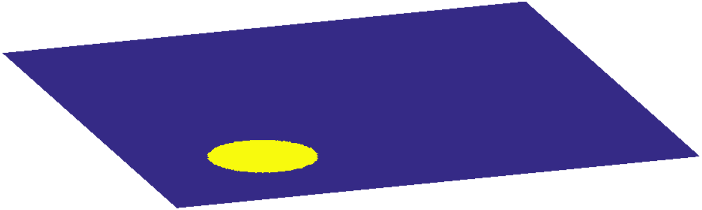
- local superlinear convergent

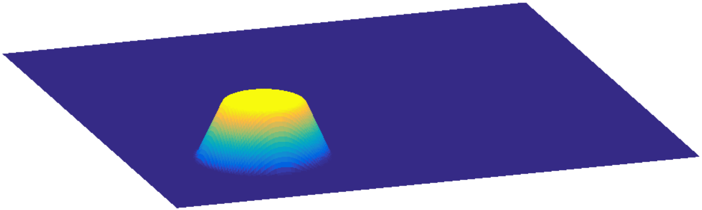
Evolving Cone (Algorithm 1)



$n = 0$

$\alpha = 1$, 66049 nodes, 200 time steps; took 227 min on a Core2Duo–processor

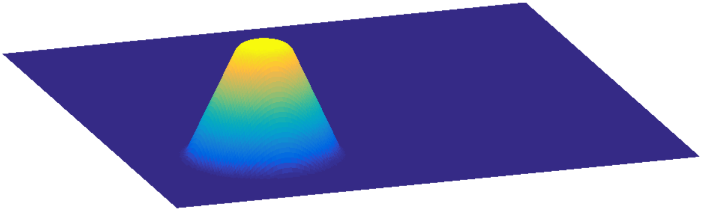
Evolving Cone (Algorithm 1)



$n = 20$

$\alpha = 1$, 66049 nodes, 200 time steps; took 227 min on a Core2Duo–processor

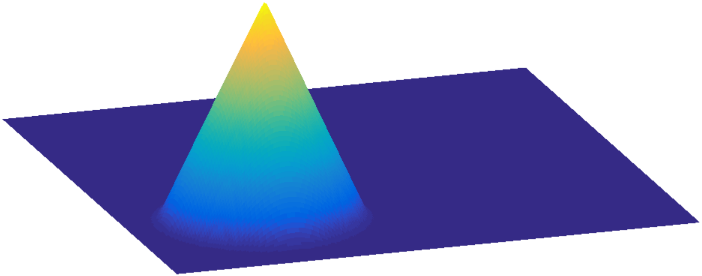
Evolving Cone (Algorithm 1)



$n = 40$

$\alpha = 1$, 66049 nodes, 200 time steps; took 227 min on a Core2Duo–processor

Evolving Cone (Algorithm 1)



$n = 100$

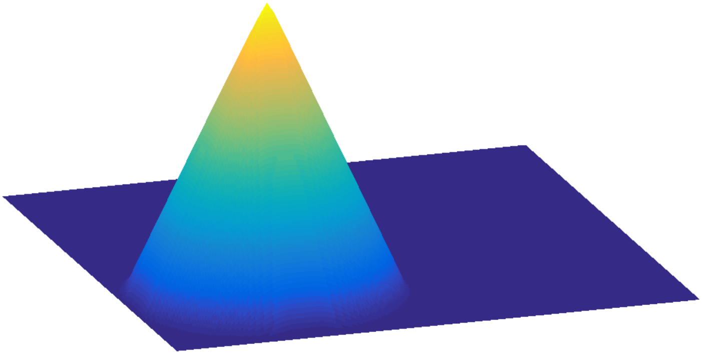
$\alpha = 1$, 66049 nodes, 200 time steps; took 227 min on a Core2Duo–processor

Evolving Cone (Algorithm 1)

$n = 200$

$\alpha = 1$, 66049 nodes, 200 time steps; took 227 min on a Core2Duo–processor

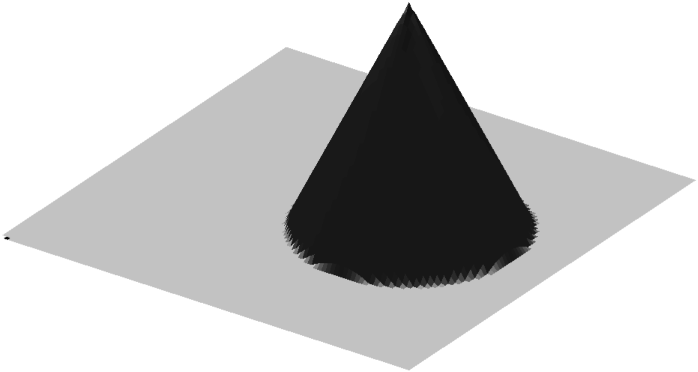
Evolving Cone (Algorithm 1)



$n = 300$

$\alpha = 1$, 66049 nodes, 200 time steps; took 227 min on a Core2Duo–processor

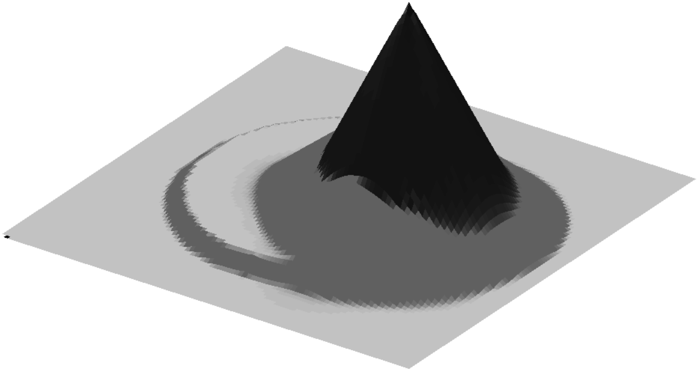
Example with Steep Slopes (Algorithm 1)



$n = 0$

$\alpha = 0.4$, 1021 nodes, 200 time steps; took 53 min

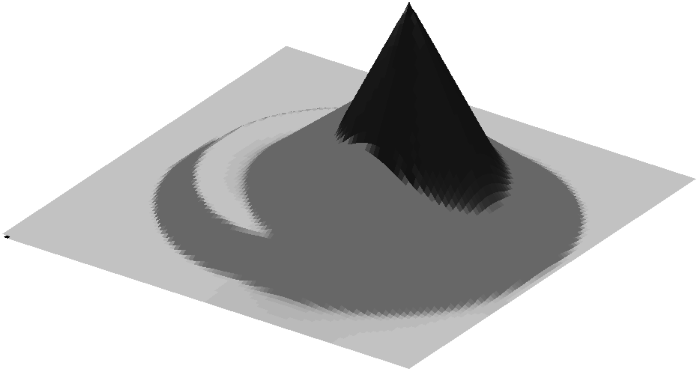
Example with Steep Slopes (Algorithm 1)



$n = 50$

$\alpha = 0.4$, 1021 nodes, 200 time steps; took 53 min

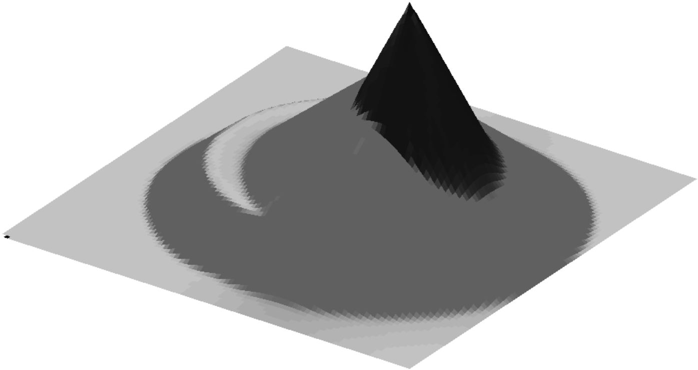
Example with Steep Slopes (Algorithm 1)



$n = 100$

$\alpha = 0.4$, 1021 nodes, 200 time steps; took 53 min

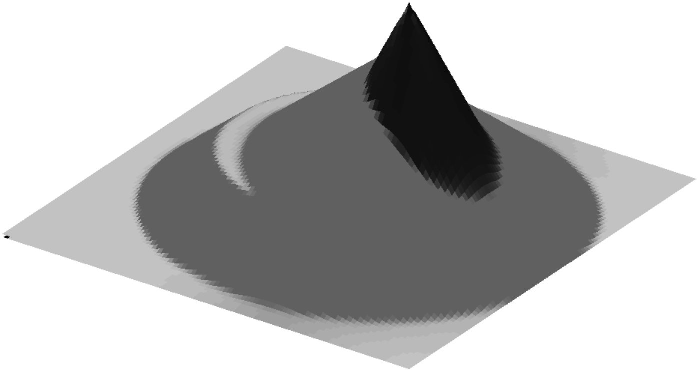
Example with Steep Slopes (Algorithm 1)



$n = 150$

$\alpha = 0.4$, 1021 nodes, 200 time steps; took 53 min

Example with Steep Slopes (Algorithm 1)

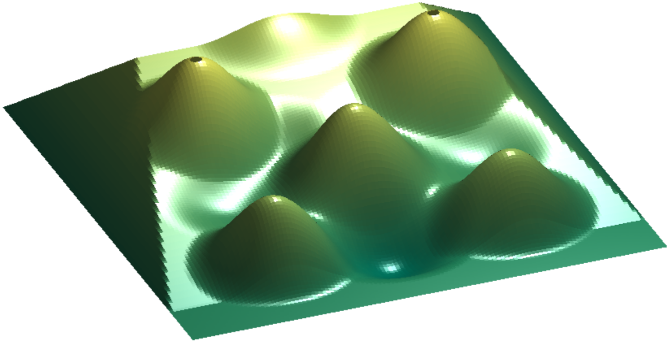


$n = 200$

$\alpha = 0.4$, 1021 nodes, 200 time steps; took 53 min

Further application: Water drainage

Application: Modelling the flow of water by using small $\alpha > 0$

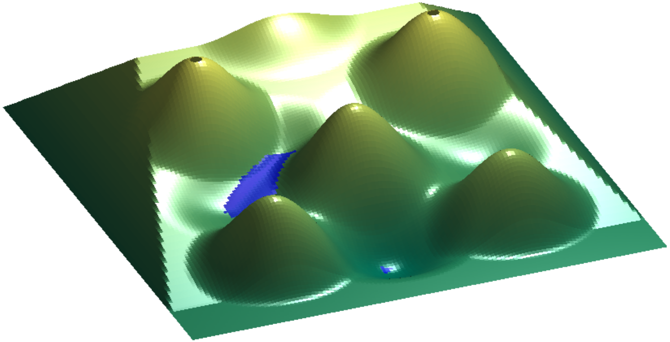


$n = 0$

$\alpha = 0.125$, 20769 nodes, 30 time steps; took 162 min

Further application: Water drainage

Application: Modelling the flow of water by using small $\alpha > 0$

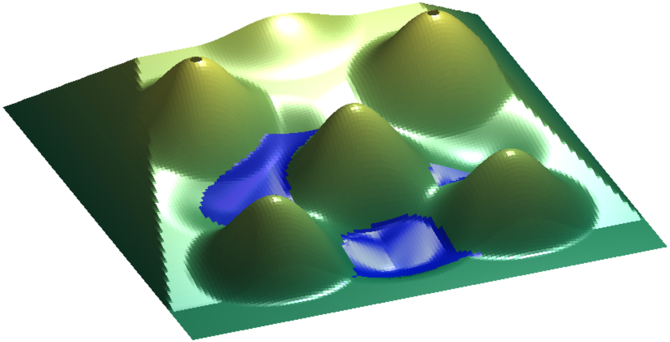


$n = 2$

$\alpha = 0.125$, 20769 nodes, 30 time steps; took 162 min

Further application: Water drainage

Application: Modelling the flow of water by using small $\alpha > 0$

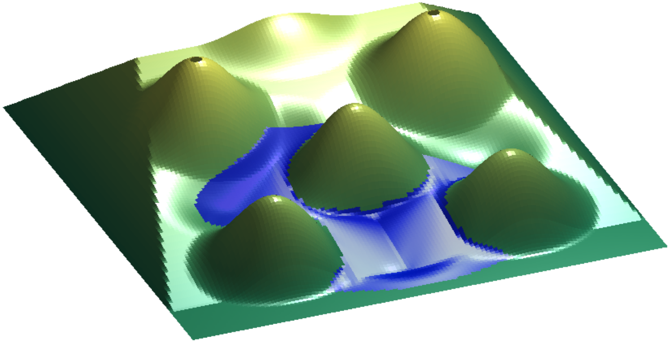


$n = 10$

$\alpha = 0.125$, 20769 nodes, 30 time steps; took 162 min

Further application: Water drainage

Application: Modelling the flow of water by using small $\alpha > 0$

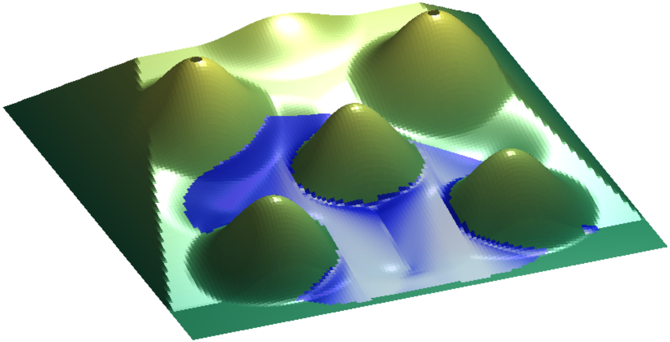


$n = 18$

$\alpha = 0.125$, 20769 nodes, 30 time steps; took 162 min

Further application: Water drainage

Application: Modelling the flow of water by using small $\alpha > 0$



$n = 30$

$\alpha = 0.125$, 20769 nodes, 30 time steps; took 162 min

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