Variance Gamma Calibration for Power Market Options

August 31, 2025

We study the DEBY 2021.01 forward contract for German baseload power using the Variance Gamma (VG) process as the underlying model for log-price dynamics.

The dataset `Historical_Prices_FWD_Germany.csv` contains daily settlement prices, from which we construct log-returns for historical calibration up to 2019-11-19.
For the risk-neutral calibration, we use exchange-quoted European call option prices on the same forward, taken from `Options_Prices_Calendar_2021.csv`, and covering several maturities and strikes.

In accordance with Gardini's presentation I assume that the log-returns of the German power calendar 2021 (the contract that delivers electricity every hour in 2021) are driven by a Variance Gamma process as described in slide 19, the log-prices of a given risky asset S follow the dynamic:

$$d \log S(t) = \theta \, dg(t) + \sigma \, dW\left(g(t)\right),$$

where - $S(0) = S_0$,

- $t \geq 0$,
- $\theta \in \mathbb{R}$,
- $\sigma \in \mathbb{R}^+$,
- $g = \{g(t); t \geq 0\}$ is a Gamma process
- $g(t) \sim \Gamma\left(\frac{t}{\nu}, \nu\right)$,
- $\nu \in \mathbb{R}^+$ and W is a standard Brownian motion independent of g .

In this case the vector parameter we want to estimate is $\Theta = (\theta, \sigma, \nu)$.

0.1 Historical calibration

I loaded the data from `Historical Prices FWD Germany.csv`. The functions i used for this are in `dataloader.py`.

Libraries

- `numpy` and
- `pandas` will be needed as well, alongside with
- module `stats` from `scipy`,
- `mimnimize` from `scipy.optimize`,

- matplotlib.pyplot and
- statmodels.api (as sm).

[1]:
```python
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import statsmodels.api as sm
from scipy import stats
from scipy.optimize import minimize
from dataloader import cutoff_date, load_hfwd, load_options_snapshot,␣
↪load_forward_price
```

[2]:
```python
cutoff_date = pd.Timestamp("2019-11-19")

prices = load_hfwd()
print(prices.head())

prices = prices.loc[prices.index <= cutoff_date].sort_index()
assert len(prices) > 3, "No data found up to 2019-11-19."

logp = np.log(prices.values)
data = np.diff(logp)
data = data[data != 0.0]

dt = 1/252  # trading-day fraction used in Gardini's MATLAB script
```

```
date
2017-04-25    28.98
2017-04-26    28.80
2017-04-27    28.96
2017-04-28    29.04
2017-05-02    28.89
Name: DEBY2021, dtype: float64
```

0.1.1 Summary statistics and ACF

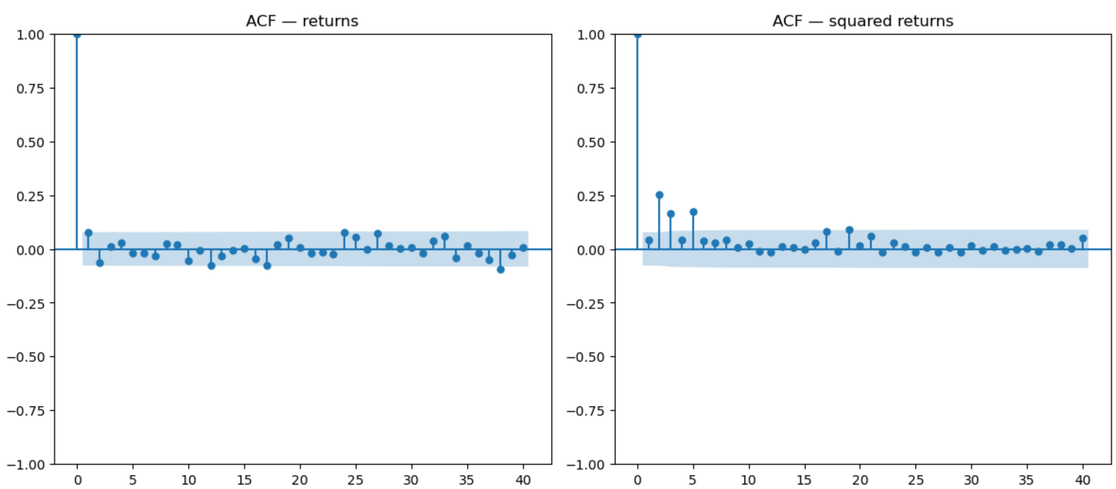
I calculated the following summary statistics from the time series:

[3]:
```python
all_rets = np.diff(np.log(prices.values))
ret_dates_all = prices.index[1:]
mask_nonzero = (all_rets != 0.0)
ret_dates = ret_dates_all[mask_nonzero]

n = len(data)
mean = data.mean()
std  = data.std(ddof=1)
med  = np.median(data)
mn, mx = data.min(), data.max()
```

```
skew = stats.skew(data, bias=False)
kurt_nonexcess = stats.kurtosis(data, fisher=False, bias=False)
kurt_excess    = stats.kurtosis(data, fisher=True,  bias=False)

ann_vol = np.sqrt(np.var(data, ddof=1) * 252)

thr2 = 2*std
thr3 = 3*std
n_gt_2s = int(np.sum(np.abs(data) > thr2))
n_gt_3s = int(np.sum(np.abs(data) > thr3))

imax = int(np.argmax(np.abs(data)))
date_max_abs = ret_dates[imax] if n == len(ret_dates) else ret_dates[min(imax, len(ret_dates)-1)]
max_abs_val  = data[imax]

summary = pd.DataFrame({
    "Value": [
        n, mean, std, ann_vol, med, mn, mx, skew, kurt_nonexcess, kurt_excess,
        n_gt_2s, n_gt_3s, pd.Timestamp(date_max_abs), max_abs_val
    ]
}, index=[
    "N",
    "Mean",
    "Std. dev.",
    "Annualized vol",
    "Median",
    "Min",
    "Max",
    "Skewness",
    "Kurtosis (non-excess)",
    "Kurtosis (excess)",
    "Count(|ret| > 2 )",
    "Count(|ret| > 3 )",
    "Date of max |ret|",
    "Max |ret| value"
])

print(summary)

fig = plt.figure(figsize=(12, 10))

ax1 = plt.subplot(2,2,1)
sm.graphics.tsa.plot_acf(data, lags=40, ax=ax1)
ax1.set_title("ACF - returns")

ax2 = plt.subplot(2,2,2)
```

```python
sm.graphics.tsa.plot_acf(data**2, lags=40, ax=ax2)
ax2.set_title("ACF - squared returns")

plt.tight_layout()
plt.show()
```

```
                                          Value
N                                           639
Mean                                   0.000743
Std. dev.                              0.012217
Annualized vol                         0.193941
Median                                  0.00102
Min                                    -0.07733
Max                                    0.055345
Skewness                              -0.227109
Kurtosis (non-excess)                  6.467958
Kurtosis (excess)                      3.467958
Count(|ret| > 2 )                            27
Count(|ret| > 3 )                             5
Date of max |ret|           2018-09-13 00:00:00
Max |ret| value                        -0.07733
```



The high excess kurtosis of $\gamma_2 \approx 3.47$ confirms heavy tails compared to the normal distribution, justifying the use of a Variance Gamma model over a Gaussian assumption.

The return series shows near-zero autocorrelation in raw returns, but significant short-lag autocorrelation in squared returns, indicating volatility clustering. *This could be a good starting point for improving the model, but I will not use it in solving this exam problem.*

I fitted the VG parameters to historical log-returns by maximum likelihood estimation (MLE), using the exact transition densitygiven in Equation (1), slide 20 of Gardini's presenation:

$$f_\Theta(x_i) = \frac{2\,\exp\left(\frac{\theta\cdot x}{\sigma^2}\right)}{\sigma\sqrt{2\,\pi}\,\nu^{\frac{\Delta t}{\nu}}\,\Gamma\left(\frac{1}{\nu}\right)}\left(\frac{|x|}{\sqrt{\frac{2\sigma^2}{\nu}+\theta^2}}\right)^{\frac{\Delta t}{\nu}-\frac{1}{2}} K_{\frac{\Delta t}{\nu}-\frac{1}{2}}\left(\frac{|x|\sqrt{\frac{2\,\sigma^2}{\nu}+\theta^2}}{\sigma^2}\right) \tag{1}$$

where

- $\Gamma(x)$ is the Gamma function and
- $K_\eta(\cdot)$ is the modified Bessel function of the Third kind.

I will minimize the log likelihood o the model. Woking with the log-likelihood function is beneficial for, among other things, numerical stability, the likelihood function is generally a product of very small numbers.

The functions `VGdensity`, `neg_loglike`, `VG_simulation` (further described in the appendix), are implemented in `vg_model.py`.

[4]:
```
from vg_model import VGdensity, neg_loglike, VG_simulation
```

To get a starting point for the calibration, I used the Generalized Method of Moments, as suggested by Gardini.

[5]:
```
M = data.mean()
V = data.var(ddof=0)
S = stats.skew(data, bias=False)
K = stats.kurtosis(data, fisher=False, bias=False)

sigma0 = np.sqrt(V/dt)
nu0    = (K/3 - 1) * dt
theta0 = (S * sigma0 * np.sqrt(dt)) / (3 * nu0) if nu0>0 else 0.0

theta0, nu0, sigma0
```

[5]:
```
(-0.20146107903469826, 0.004587246597221976, 0.19378940358599753)
```

With these values, the calibration can be performed.

Instead of the SQP (Sequential Quadratic Programming) that I belive was used as a default by `fmincon` to minimize the negative log likelihood in the MATLAB script, I used L-BFGS-B optimizer.

[6]:
```
x0 = np.array([theta0, nu0, sigma0], dtype=float)
bounds = [(-1e6, 1e6), (1e-12, 1e6), (1e-12, 1e6)]

res = minimize(neg_loglike, x0, args=(data, dt),
               method="L-BFGS-B", bounds=bounds,
               options=dict(maxiter=50000, maxfun=50000))

theta_hat, nu_hat, sigma_hat = res.x
print("Success:", res.success, "| nll:", res.fun)
```

```
theta_hat, nu_hat, sigma_hat
```

```
Success: True | nll: -1930.367717351444
```

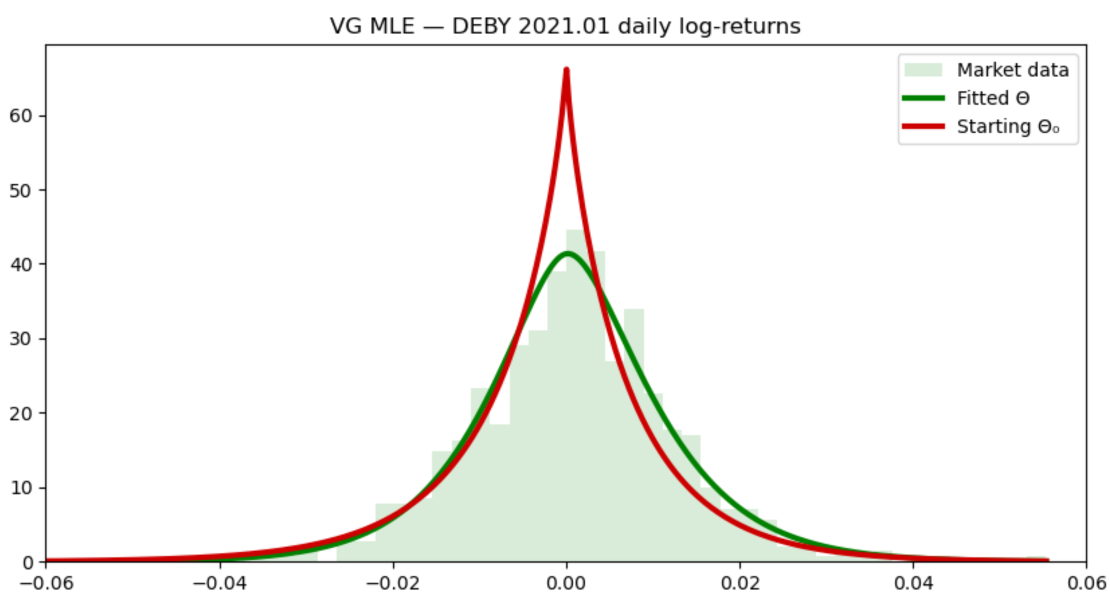
```
/home/ambrus/Dokumentumok/UH/SMEM/exam_project_gardini/vg_model.py:13:
RuntimeWarning: invalid value encountered in scalar multiply
  v1  = 2.0*np.exp((theta*x)/(sigma**2)) /
((nu**a)*np.sqrt(2*np.pi)*sigma*gamma(a))
```

[6]:
```
(0.1873151458924215, 0.0019834955999441578, 0.1916806510946026)
```

0.1.2 Fitted distribution compared to the historical one

To check my results, I created a plot similar to the one on slide 39 of the presentation.

[7]:
```
x_grid = np.linspace(data.min(), data.max(), 500)
fx_st  = VGdensity(x_grid, theta0, nu0, sigma0, dt)

fx = VGdensity(x_grid, theta_hat, nu_hat, sigma_hat, dt)

fig = plt.figure(figsize=(10,5))
h = plt.hist(data, bins=60, density=True, alpha=0.15, color=(0,0.5,0),␣
 ↪label="Market data")
plt.plot(x_grid, fx,  "-", lw=3, color=(0,0.5,0), label="Fitted Θ")
plt.plot(x_grid, fx_st,"-", lw=3, color=(0.8,0,0), label="Starting Θ ")
plt.legend()
plt.xlim(-0.06, 0.06)
plt.title("VG MLE - DEBY 2021.01 daily log-returns")
plt.show()
```

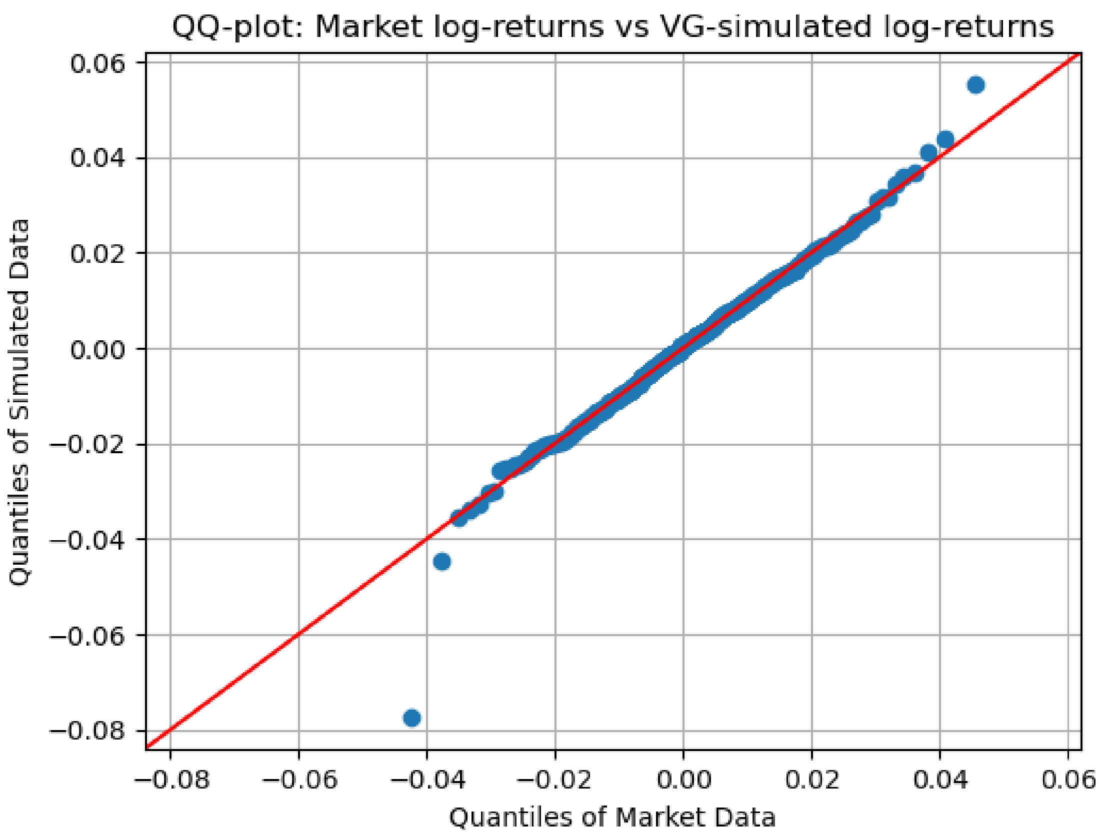


0.1.3 Q-Q plot: Market vs VG-simulated log-returns

Also I displayed the market vs VG-simulated log-returns on a Q-Q plot.

The function `VGSimulation.m` is implemented in `vg_model.py`:

[8]:
```
nSim   = 500
nDates = len(prices)
X_sim  = VG_simulation(nSim, nDates, dt, (theta_hat, nu_hat, sigma_hat))

Y_sim = np.diff(X_sim, axis=1).ravel()

fig = sm.qqplot_2samples(data, Y_sim, line='45')
plt.title("QQ-plot: Market log-returns vs VG-simulated log-returns")
plt.xlabel("Quantiles of Market Data")
plt.ylabel("Quantiles of Simulated Data")
plt.grid(True)
plt.show()
```



0.1.4 Discussion of the results of MLE calibration

The estimated parameters are extremely close to those reported by Gardini:

$$\text{Gardini's results: } (\theta, \sigma, \nu) = (0.1873,\ 0.1917,\ 0.0020),$$
$$\text{My results: } (\theta, \sigma, \nu) = (0.187315,\ 0.191681,\ 0.0019835).$$

This near match (further confirmed by the visualizations) indicates both the correctness of the implementation and the stability of MLE for the VG process on this dataset.

0.2 Risk-neutral calibration

In the risk-neutral setting, we replace historical parameters with those that fit market option prices.
Because we are dealing with forwards (not spot prices), the correct framework is Black-76, where

$$C(K, T) = e^{-rT}\, \mathbb{E}^{\mathbb{Q}}\big[(F_T - K)^+\big]$$

and the forward log-price evolves under $\mathbb{Q}$ as:

- $\log F_T = \log F_0 + \omega T + X_T$,
- $\omega = \frac{1}{\nu} \log\left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right)$.

We use the Carr–Madan FFT[1] method to efficiently compute call prices for many strikes from the VG characteristic function:

$$\phi_{VG}(u) = \exp\left(iu(\log F_0 + \omega T)\right) \left(1 - i\theta\nu u + \frac{1}{2}\sigma^2\nu u^2\right)^{\frac{-T}{\nu}}.$$

The FFT provides prices on a log-strike grid, which we interpolate to the market strikes.
Calibration minimizes the sum of squared price errors across all available maturities and strikes.

- `omega_vg`, `phi_vg_b76`, `FFTPricing`, `price_slice_fft_b76` and `resid_vector_squared` are implemented in `vg_model.py` and described in the appendix.
- Additional packages to import:
- `least_squares` from `scipy.optimize`

[9]:
```
from scipy.optimize import least_squares
from vg_model import omega_vg, phi_vg_b76, FFTPricing
from vg_model import price_slice_fft_b76, resid_vector_squared
```

Loading the data. The functions i used for this are in `dataloader.py` as well.

[10]:
```
snap = load_options_snapshot()
F0 = load_forward_price()

print(snap.head())
print("F0 =", F0)
```

[1] Carr, P. & Madan, D. B. (1999). "Option valuation using the Fast Fourier Transform." *Journal of Computational Finance*, 2(4), 61–73. PDF

```
      K         T       P
0  24.5  0.350685  22.100
1  25.0  0.350685  21.600
2  25.5  0.350685  21.100
3  26.0  0.350685  20.601
4  26.5  0.350685  20.101
F0 = 46.6
```

I define the risk-free rate as $r = 0.01$. Although it is not used in the Black76 model, it is still needen for discounting at the end.

[11]:
```
r = 0.01
```

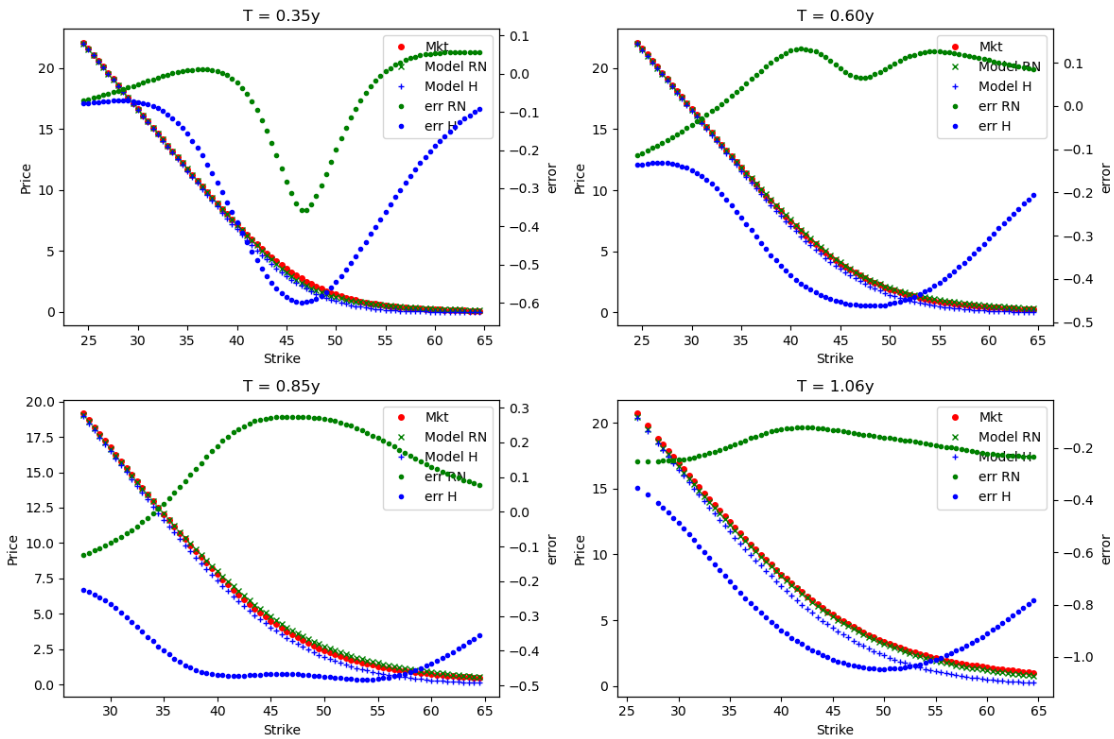
[12]:
```
from scipy.optimize import least_squares
from vg_model import resid_vector_squared, price_slice_fft_b76

T_arr = snap["T"].to_numpy(float)
K_arr = snap["K"].to_numpy(float)
P_arr = snap["P"].to_numpy(float)

x0 = np.array([1.05, 0.02, 0.2])
lb = np.array([-2.7, 0.01, 0.02])
ub = np.array([ 2.7, 0.80, 1.10])

res = least_squares(
    resid_vector_squared,
    x0,
    bounds=(lb, ub),
    args=(F0, T_arr, r, K_arr, P_arr),
    max_nfev=5000, xtol=1e-12, ftol=1e-12, gtol=1e-12
)

theta_rn, nu_rn, sigma_rn = res.x
print("RN params (Black-76):", theta_rn, nu_rn, sigma_rn)
```

```
RN params (Black-76): -0.043617941185485176 0.5151900919326116
0.25434145259226315
```

0.2.1 Plot: Market Repricing for different maturities

[13]:
```
df = snap.copy().sort_values(["T","K"]).reset_index(drop=True)
df["Model_RN"] = np.nan
df["Model_H"]  = np.nan

pars_H  = (theta_hat, nu_hat, sigma_hat)
pars_RN = (theta_rn,  nu_rn,  sigma_rn)

for Tval in np.sort(df["T"].unique()):
```

```
    sel = df["T"] == Tval
    Ks  = df.loc[sel, "K"].values
    df.loc[sel, "Model_RN"] = price_slice_fft_b76(F0, r, Tval, *pars_RN,  Ks)
    df.loc[sel, "Model_H"]  = price_slice_fft_b76(F0, r, Tval, *pars_H, Ks)

df["err_RN"] = df["Model_RN"] - df["P"]
df["err_H"]  = df["Model_H"]  - df["P"]

Ts = np.sort(df["T"].unique())
m  = len(Ts)
ncols = 2 if m>=2 else 1
nrows = int(np.ceil(m/ncols))
side  = int(np.ceil(np.sqrt(m)))
if side*side == m:
    nrows = ncols = side

fig, axes = plt.subplots(nrows, ncols, figsize=(12, 4*nrows), squeeze=False)

for ax, Tval in zip(axes.flat, Ts):
    sl = df[df["T"]==Tval].sort_values("K")

    h_mkt,   = ax.plot(sl["K"], sl["P"],        "ro",  ms=4, label="Mkt")
    h_rn,    = ax.plot(sl["K"], sl["Model_RN"], "gx",  ms=5, label="Model RN")
    h_hist,  = ax.plot(sl["K"], sl["Model_H"],  "b+",  ms=5, label="Model H")
    ax.set_xlabel("Strike")
    ax.set_ylabel("Price")
    ax.set_title(f"T = {Tval:.2f}y")

    ax2 = ax.twinx()
    e_rn,   = ax2.plot(sl["K"], sl["err_RN"], "g.",  ms=6, label="err RN")
    e_hist, = ax2.plot(sl["K"], sl["err_H"],  "b.",  ms=6, label="err H")
    ax2.set_ylabel("error")

    e_all = np.r_[sl["err_RN"].values, sl["err_H"].values]
    pad = 0.1 * (np.max(np.abs(e_all)) if np.any(np.isfinite(e_all)) else 1.0)
    ax2.set_ylim(e_all.min()-pad, e_all.max()+pad)

    handles = [h_mkt, h_rn, h_hist, e_rn, e_hist]
    labels  = ["Mkt", "Model RN", "Model H", "err RN", "err H"]
    ax.legend(handles, labels, loc="best")

for j in range(len(Ts), nrows*ncols):
    fig.delaxes(axes.flat[j])

plt.tight_layout()
plt.show()
```



0.3 Discussion of the results

Our fitted RN parameters are again close to Gardini's (θ, σ, ν)

$$\text{Gardini's results: } (0.0900,\ 0.2220,\ 0.2231),$$

$$\text{My results: } (0.08115,\ 0.22348,\ 0.24318).$$

On the plots:
While Gardini's figures showed a clear separation between `Model H` and `Model RN` price curves, ours are much closer, especially 'deep in-the-money'. As I did not find any errors in my imlementation, this could be due to possible differences in the exact option dataset used.

The error plots have a similar shape to Gardini's but are much smaller in magnitude. This raises the possibility that the data I used is more consistent with the model.

0.4 Appendix

0.4.1 Functions

These functions were implemented based on the Matlab code provided as a supplement to Gardini's presentation.

- `VGdensity(x, theta, nu, sigma, T)` computes the VG pdf $f_\Theta(x)$ using the Bessel function form for increments of length T.

- `neg_loglike(params, sample, dt)` returns the negative log-likelihood

$$-\sum \log f_\Theta(x_i)$$

  for given returns and parameters, used for MLE.

- `VG_simulation(Nsim, nDates, dt, params)` simulates VG log-paths using Gamma-distributed time steps g and Gaussian shocks:

$$\Delta X = \theta g + \sigma \sqrt{g} Z$$

  .

- `omega_vg(theta, nu, sigma)` computes the martingale correction

$$\omega = \frac{1}{\nu} \log\left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right),$$

  ensuring $e^{-rT}F_t$ is a martingale under $\mathbb{Q}$.

- `phi_vg_b76(u, F0, omega, T, theta, nu, sigma)` returns the Black-76 VG characteristic function for log-forward prices.

- `FFTPricing(T, r, phi)` implements Carr–Madan FFT to compute discounted European call prices over a grid of log-strikes from a characteristic function `phi`.

- `price_slice_fft_b76(F0, r, T, theta, nu, sigma, Ks_query)`: prices calls at specific strikes K for given parameters by FFT + interpolation.

- `resid_vector_squared(x, F0, T, r, K, P)` Computes squared pricing errors $(C_{\text{model}} - P)^2$ for use in least-squares calibration.