Implementation report on the paper '*Kinematic Analysis of the 6-R Manipulator of General Geometry*' by Raghavan and Roth

Ambuj Shahi

1 Introduction

This report is on the paper titled 'Kinematic Analysis of the 6-R Manipulator of General Geometry' by Raghavan and Roth [1]. The aforementioned paper presents a solution to the *inverse kinematics problem* for the 6-R serial manipulator of general geometry. Such a manipulator has 6 revolute (R) joints arranged in series. The link lengths, twist angles and offsets, may be arbitrary real numbers. The inverse kinematics problem for six-link manipulators is a central problem in automatic robot control. Given the pose of the end effector (the position and orientation), the problem is to compute the joint parameters for that pose. The complexity of this problem is a function of the geometry of the manipulator. While the solution can be expressed in closed-form for a variety of special cases, such as when three consecutive axes intersect at a common point, no such formulation was known for the general case at the time when this paper was written. In this report, a detailed description on the solution methodology adopted by Raghavan and Roth [1] has been presented. When stated in mathematical terms, this problem reduces to the solution of a principal system of multivariate polynomials (i.e. n polynomials in n unknowns). The solution procedure presented here is based on *dialytic elimination*. This procedure exploits the structure of the ideal of the closure equations of the 6-R manipulator and results in a compact elimination algorithm. The result is a 16^{th} degree polynomial in the tangent of the half-angle of one of the joint variables.

2 Inverse Kinematics

In this report, the standard Denavit-Hartenberg formalism [2] has been used to model the 6-R manipulator. Each link is represented by the line along its joint axis and the common normal to the next joint axis. The links of the 6-R manipulator are numbered from 1 to 7. The base link is 1, and the outermost link or hand is 7. A coordinate system is attached to each link for describing the relative arrangements among the various links. The coordinate system attached to the i^{th} link is numbered i. More details of the model are given in [3, 4]. The 4 x 4 transformation matrix relating $i+1^{th}$ coordinate system to i^{th} coordinate system [3] is:

$$\boldsymbol{A}_i = \begin{pmatrix} c_i & -s_i\lambda_i & s_i\mu_i & a_i c_i \\ s_i & c_i\lambda_i & -c_i\mu_i & a_i s_i \\ 0 & \mu_i & \lambda_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

where, $s_i = \sin\theta_i$, $c_i = \cos\theta_i$, $\mu_i = \sin\alpha_i$, $\lambda_i = \cos\alpha_i$,

a_i is the length of link $i+1$,

α_i is the twist angle between the axes of joints i and $i+1$,

d_i is the offset distance at joint i,

θ_i is the joint rotation angle at joint i.

For a given robot with revolute joints we are given the a_i's, d_i's, μ_i's and λ_i's. For the inverse kinematics problem we are also given the pose of the end-effector, attached to link 7. This pose is described with respect to the base link or link 1. Here the pose is represented as:

$$\boldsymbol{A}_{hand} = \begin{pmatrix} l_x & m_x & n_x & \rho_x \\ l_y & m_y & n_y & \rho_y \\ l_z & m_z & n_z & \rho_z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The problem of inverse kinematics corresponds to computing the joint angles, θ_1, θ_2, θ_3, θ_4, θ_5 and θ_6 such that:

$$\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3\boldsymbol{A}_4\boldsymbol{A}_5\boldsymbol{A}_6 = \boldsymbol{A}_{hand} \tag{2}$$

3 Raghavan and Roth's solution

In this section, the approach followed by Raghavan and Roth [1] has been briefly described. They reduce the multivariate system to a degree 16 polynomial in $\tan\left(\frac{\theta_3}{2}\right)$, such that the joint angle θ_3 can be computed from its roots. The other joint angles are computed from substitution and solving for some intermediate equations.

3.1 Derivation of $p_1, p_2, p_3, l_1, l_2, l_3$

Raghavan and Roth rearrange the matrix Eq. (2), as:

$$\boldsymbol{A}_3\boldsymbol{A}_4\boldsymbol{A}_5 = \boldsymbol{A}_2{}^{-1}\boldsymbol{A}_1{}^{-1}\boldsymbol{A}_{hand}\boldsymbol{A}_6{}^{-1} \tag{3}$$

As a result the entries of the left hand side matrix are functions of θ_3, θ_4 and θ_5 and the entries of the right hand side matrlx are functions of θ_1, θ_2 and θ_6. This lowers their degrees and reduces the symbolic complexity of the resulting expressions. When the matrix multiplications are carried out, Eq. (3) takes the form:

$$
\begin{pmatrix}
f_{11}(\theta_3,\theta_4,\theta_5) & f_{12}(\theta_3,\theta_4,\theta_5) & f_{13}(\theta_3,\theta_4,\theta_5) & f_{14}(\theta_3,\theta_4,\theta_5) \\
f_{21}(\theta_3,\theta_4,\theta_5) & f_{22}(\theta_3,\theta_4,\theta_5) & f_{23}(\theta_3,\theta_4,\theta_5) & f_{24}(\theta_3,\theta_4,\theta_5) \\
f_{31}(\theta_3,\theta_4,\theta_5) & f_{32}(\theta_3,\theta_4,\theta_5) & f_{33}(\theta_3,\theta_4,\theta_5) & f_{34}(\theta_3,\theta_4,\theta_5) \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
f'_{11}(\theta_1,\theta_2,\theta_6) & f'_{12}(\theta_1,\theta_2,\theta_6) & f'_{13}(\theta_1,\theta_2,\theta_6) & f'_{14}(\theta_1,\theta_2,\theta_6) \\
f'_{21}(\theta_1,\theta_2,\theta_6) & f'_{22}(\theta_1,\theta_2,\theta_6) & f'_{23}(\theta_1,\theta_2,\theta_6) & f'_{24}(\theta_1,\theta_2,\theta_6) \\
f'_{31}(\theta_1,\theta_2,\theta_6) & f'_{32}(\theta_1,\theta_2,\theta_6) & f'_{33}(\theta_1,\theta_2,\theta_6) & f'_{34}(\theta_1,\theta_2,\theta_6) \\
0 & 0 & 0 & 1
\end{pmatrix} \tag{4}
$$

Equation (4) shows the variables appearing in the individual entries in Eq. (3). The 6 scalar equations obtained from columns 3 and 4 of Eq. (3) are devoid of θ_6. These equations though linearly independent are governed by the constraint that the magnitude of the column 3 vector is unity. Starting from these six equations a univariate polynomial is obtained which will vanish at their common zeros, by eliminating 4 of the 5 variables. The 6 equations are:

$$c_3 f_1 + s_3 f_2 = c_2 h_1 + s_2 h_2 - a_2, \tag{5}$$

$$s_3 f_1 - c_3 f_2 = -\lambda_2(s_2 h_1 - c_2 h_2) + \mu_2(h_3 - d_2), \tag{6}$$

$$f_3 = \mu_2(s_2 h_1 - c_2 h_2) + \lambda_2(h_3 - d_2), \tag{7}$$

$$c_3 r_1 + s_3 r_2 = c_2 n_1 + s_2 n_2, \tag{8}$$

$$s_3 r_1 - c_3 r_2 = -\lambda_2(s_2 n_1 - c_2 n_2) + \mu_2 n_3, \tag{9}$$

$$r_3 = \mu_2(s_2 n_1 - c_2 n_2) + \lambda_2 n_3, \tag{10}$$

where, $f_i's, h_i's, r_i's$ and $n_i's, i = 1, 2$ and 3
are functions of the input variables. The Eqs. (5-10), are rearranged, in terms of variables, to obtain 6 new equations, $p_1, p_2, p_3, l_1, l_2, l_3$ as functions of $f_i's, h_i's, r_i's$ and $n_i's$, as shown in [1]. After the arrangement the left hand side of p_i and l_i is a linear combination of $1, c_5, s_5, c_4, s_4, c_4c_5, c_4s_5, s_4c_5, s_4s_5$ and the right hand side is a linear combination of $c_2, s_2, c_1, s_1, c_1c_2, c_1s_2, s_1c_2, s_1s_2$. However, the coefficients used to express the left hand side as a linear combination are functions of s_3 and c_3.

3.2 The Ideal of $p_1, p_2, p_3, l_1, l_2, l_3$

The *Ideal* generated by a set of polynomials $q_1, q_2, ..., q_r$ in the variables $x_1, x_2, ..., x_r$ is the set of all elements of the form $q_1\beta_1, q_2\beta_2, ..., q_r\beta_r$, where $\beta_1, \beta_2, ..., \beta_r$ are arbitrary elements of the set of all polynomials in $x_1, x_2, ..., x_r$. Consider $\boldsymbol{p} = (p_1, p_2, p_3)^T$ and $\boldsymbol{l} = (l_1, l_2, l_3)^T$ as 3×1 vectors. According to [1], the left and right hand sides of the following equations have the same power products as the left and right hand sides of p_i and l_i :

$$\boldsymbol{p} \cdot \boldsymbol{p}, \boldsymbol{p} \cdot \boldsymbol{l}, \boldsymbol{p} \times \boldsymbol{l}, (\boldsymbol{p} \cdot \boldsymbol{p})\boldsymbol{l} - 2(\boldsymbol{p} \cdot \boldsymbol{l})\boldsymbol{p} \tag{11}$$

By power products, the monomials of a polynomial is meant (e.g., the power products of the polynomial $5x^2y + 3xz + 9y + 4z = 0$ are x^2y, xz, y and z.).

The ideal generated by the component equations of $\boldsymbol{p}$ and $\boldsymbol{l}$ has the following interesting properties:

Property 1: $\boldsymbol{p} \cdot \boldsymbol{p}$ has the same power products as $\boldsymbol{p}$ and $\boldsymbol{l}$. In this report, we have used a computer algebra package called Wolfram $Mathematica^{®}$ to prove this property through symbolic computations.

Property 2: $\boldsymbol{p} \cdot \boldsymbol{l}$ has the same power products as $\boldsymbol{p}$ and $\boldsymbol{l}$. This property has also been proven using Wolfram $Mathematica^{®}$.

Property 3: $\boldsymbol{p} \times \boldsymbol{l}$ has the same power products as $\boldsymbol{p}$ and $\boldsymbol{l}$. This property has also been proven using Wolfram $Mathematica^{®}$.

Property 4: $(\boldsymbol{p} \cdot \boldsymbol{p})\boldsymbol{l} - 2(\boldsymbol{p} \cdot \boldsymbol{l})\boldsymbol{p}$ has the same power products as $\boldsymbol{p}$ and $\boldsymbol{l}$. The proof given in [1] for the 1^{st} two components of $(\boldsymbol{p} \cdot \boldsymbol{p})\boldsymbol{l} - 2(\boldsymbol{p} \cdot \boldsymbol{l})\boldsymbol{p}$ is not that straight forward and the full steps to prove these properties has been outlined below, to get the two equations (23) and (24) of [1].

- *Property 4a*: It is required to prove that $(p_1^2 + p_2^2 + p_3^2)l_1 - 2(p_1l_1 + p_2l_2 + p_3l_3)p_1$ has the same power products as $\boldsymbol{p}$ and $\boldsymbol{l}$. The simple LHS and RHS expansions of $(p_1^2 + p_2^2 + p_3^2)l_1 - 2(p_1l_1 + p_2l_2 + p_3l_3)p_1$ using Eqs. (10-15), of [1] does not lead directly to Eq. (23) of [1]. In this equation, the RHS has been simplified (some of its terms has been transferred to its LHS) so as to get the RHS, which is a function of θ_1 and θ_2, to zero, further down the steps in Eq. (28) of [1], thus eliminating these two joint-variables. Here δ_1 and δ_2 have been used as substitutions for $p^2+q^2+(r-d_1)^2-a_1^2$ and $pu+qv+(r-d_1)w$ respectively. Through some algebraic manipulations, it can be shown that $\delta_1 = h_1^2+h_2^2+h_3^2+2a_1h_1$ and $\delta_2 = (h_1+a_1)n_1+h_2n_2+h_3n_3$. Through an inspection of the LHS of Eq. (23) of [1] it can be seen that there are a few terms involving δ_1 and δ_2 in that side of the equation which could have come only from the RHS of that equation as only the RHS of $\boldsymbol{p}$ and $\boldsymbol{l}$ are functions of $h_i's$ and $n_i's$ and as shown earlier δ_1 and δ_2 are also functions of $h_i's$ and $n_i's$ only. The collection of all the terms involving δ_1 and δ_2 from the LHS results in the expression:

$$2\delta_2(c_3f_1+s_3f_2+a_2)-\delta_1(c_3r_1+s_3r_2), \tag{12}$$

which, using Eqs. (10) and (13) of [1], results into:

$$2\delta_2(c_2h_1+s_2h_2)-\delta_1(c_2n_1+s_2n_2) \tag{13}$$

Taking Eq. (12) to the RHS of Eq. (23) of [1] and noting that the last two terms on both the sides of that equation cancel out (using Eqs. (10) and (13) of [1]), one finally gets the expression which is obtained by simply expanding the RHS of $(p_1^2+p_2^2+p_3^2)l_1-2(p_1l_1+p_2l_2+p_3l_3)p_1$ using Eqs. (10-15), of [1] and the left over terms in the LHS of that equation now represents the expression which is obtained by simply expanding the LHS of $(p_1^2+p_2^2+p_3^2)l_1-2(p_1l_1+p_2l_2+p_3l_3)p_1$ using Eqs. (10-15), of [1]. Once the expressions on both sides of the Eq. (23) of [1] has been fully obtained, the property about the power product of this equation can now be proven through symbolic computations in any computer algebra package.

- *Property 4b*: For this property, it is required to prove that $(p_1^2+p_2^2+p_3^2)l_2-2(p_1l_1+p_2l_2+p_3l_3)p_2$ has the same power products as $\boldsymbol{p}$ and $\boldsymbol{l}$. Following a similar procedure as adopted earlier in proving *Property 4a*, through an inspection of the LHS of Eq. (24) of [1], it can be seen that there are a few terms involving δ_1 and δ_2. The collection of all the terms involving δ_1

and δ_2 from the LHS results in the expression:

$$2\delta_2(\mu_2 f_3 - \lambda_2(s_3 f_1 - c_3 f_2)) - \delta_1(\mu_2 r_3 - \lambda_2(s_3 r_1 - c_3 r_2)), \qquad (14)$$

which, using Eqs. (11) and (14) of [1], results into:

$$2\delta_2(s_2 h_1 - c_2 h_2) - \delta_1(s_2 n_1 - c_2 n_2) \qquad (15)$$

Taking Eq. (15) to the RHS of Eq. (24) of [1] and noting that the last two terms on both the sides of that equation cancel out (using Eqs. (10) and (13) of [1]), one finally gets the expression which is obtained by simply expanding the RHS of $(p_1^2 + p_2^2 + p_3^2)l_2 - 2(p_1 l_1 + p_2 l_2 + p_3 l_3)p_2$ using Eqs. (10) - (15), of [1] and the left over terms in the LHS of that equation now represents the expression which is obtained by simply expanding the LHS of $(p_1^2 + p_2^2 + p_3^2)l_2 - 2(p_1 l_1 + p_2 l_2 + p_3 l_3)p_2$ using Eqns. (10-15), of [1]. Once the expressions on both sides of the Eq. (24) of [1] has been fully obtained, the property about the power product of this equation can now be proven through symbolic computations in any computer algebra package.

- *Property 4c*: For this property, it is required to prove that $(p_1^2 + p_2^2 + p_3^2)l_3 - 2(p_1 l_1 + p_2 l_2 + p_3 l_3)p_3$ has the same power products as $\boldsymbol{p}$ and $\boldsymbol{l}$. The simple LHS and RHS expansions of $(p_1^2 + p_2^2 + p_3^2)l_3 - 2(p_1 l_1 + p_2 l_2 + p_3 l_3)p_3$ using Eqs. (10-15), of [1] leads directly to Eq. (25) of [1]. The property about the power product of this equation can now be proven through symbolic computations in any computer algebra package.

The following set of linearly independent equations have now been obtained, all of which have the same power products:

Vector	No. of Scalar Equations
$\boldsymbol{p}$	3
$\boldsymbol{l}$	3
$\boldsymbol{p} \cdot \boldsymbol{p}$	1
$\boldsymbol{p} \cdot \boldsymbol{l}$	1
$\boldsymbol{p} \times \boldsymbol{l}$	3
$(\boldsymbol{p} \cdot \boldsymbol{p})\boldsymbol{l} - 2(\boldsymbol{p} \cdot \boldsymbol{l})\boldsymbol{p}$	3
	(Total) 14

These 14 equations may be written in the matrix form as:

$$\boldsymbol{P} \begin{pmatrix} s_4 s_5 \\ s_4 c_5 \\ c_4 s_5 \\ c_4 c_5 \\ s_4 \\ c_4 \\ s_5 \\ c_5 \\ 1 \end{pmatrix} = \boldsymbol{Q} \begin{pmatrix} s_1 s_2 \\ s_1 c_2 \\ c_1 s_2 \\ c_1 c_2 \\ s_1 \\ c_1 \\ s_2 \\ c_2 \end{pmatrix}, \tag{16}$$

where $\boldsymbol{P}$ is a 14×9 matrix whose entries are linear combinations of $s_3, c_3, 1$ and $\boldsymbol{Q}$ is a 14×8 matrix whose entries are all constants. One can now proceed to eliminate variables sequentially from the above equations.

3.3 Elimination of θ_1 and θ_2

Raghavan and Roth use 8 of the 14 equations (Eqs. (10), (11), (13), (14), (21), (22), (23) and (24) in [1]) in Eq. (16) to eliminate the right hand side terms, expressed as functions of θ_1 and θ_2, in terms of the left hand side, expressed as functions of θ_3, θ_4 and θ_5. As a result, they obtain the relation:

$$
\boldsymbol{\Sigma}\begin{pmatrix} s_4 s_5 \\ s_4 c_5 \\ c_4 s_5 \\ c_4 c_5 \\ s_4 \\ c_4 \\ s_5 \\ c_5 \\ 1 \end{pmatrix} = \mathbf{0}, \tag{17}
$$

where $\boldsymbol{\Sigma}$ is a 6×9 matrix, whose entries are linear combinations of s_3, c_3 and 1.

3.4 Elimination of θ_4 and θ_5

Making the following substitutions in Eq. (17):

$$
s_4 \to \frac{2x_4}{1+x_4^2},\ c_4 \to \frac{1-x_4^2}{1+x_4^2},\ s_5 \to \frac{2x_5}{1+x_5^2},\ c_5 \to \frac{1-x_5^2}{1+x_5^2}
$$

where, $x_4 = \tan\left(\frac{\theta_4}{2}\right), x_5 = \tan\left(\frac{\theta_5}{2}\right)$

Each equation is now multiplied by $(1+x_4^2)$ and $(1+x_5^2)$ to clear denominators.

Equation (17) then takes the form:

$$
\boldsymbol{\Sigma}'\begin{pmatrix} x_4^2 x_5^2 \\ x_4^2 x_5 \\ x_4^2 \\ x_4 x_5^2 \\ x_4 x_5 \\ x_4 \\ x_5^2 \\ x_5 \\ 1 \end{pmatrix} = \mathbf{0}, \tag{18}
$$

where $\boldsymbol{\Sigma}'$ is a 6×9 matrix whose entries are linear combinations of s_3, c_3 and 1.

Making the following substitutions in Eq. (18):

$$s_3 \rightarrow \frac{2x_3}{1+x_3^2}, c_3 \rightarrow \frac{1-x_3^2}{1+x_3^2}$$

The first 4 scalar equations in Eq. (18) is now multiplied by $(1+x_3^2)$ to clear denominators. The resulting equation is of the form:

$$\mathbf{\Sigma}'' \begin{pmatrix} x_4^2 x_5^2 \\ x_4^2 x_5 \\ x_4^2 \\ x_4 x_5^2 \\ x_4 x_5 \\ x_4 \\ x_5^2 \\ x_5 \\ 1 \end{pmatrix} = \mathbf{0}, \tag{19}$$

where $\mathbf{\Sigma}''$ is a 6×9 matrix. The entries in the first 4 rows of $\mathbf{\Sigma}''$ are quadratic polynomials in x_3. The entries in the last two rows are rational functions of x_3, the numerators being quadratic polynomials in x_3, the denominators being $(1+x_3^2)$. It is noteworthy that the determinant of the 6×6 array comprised of any set of six columns of $\mathbf{\Sigma}''$ is always an 8^{th} degree polynomial *and not a rational function.*

The system given above is not a square system and it is converted into a square system using dialytic elimination [5]. Multiplying Eq. (19) by x_4, one gets the following equation:

$$\mathbf{\Sigma}'' \begin{pmatrix} x_4^3 x_5^2 \\ x_4^3 x_5 \\ x_4^3 \\ x_4^2 x_5^2 \\ x_4^2 x_5 \\ x_4^2 \\ x_4 x_5^2 \\ x_4 x_5 \\ x_4 \end{pmatrix} = \mathbf{0}. \tag{20}$$

Equations (19) and (20), taken together may be written in matrix form as:

$$\begin{pmatrix} \mathbf{\Sigma}'' & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}'' \end{pmatrix} \begin{pmatrix} x_4^3x_5^2 \\ x_4^3x_5 \\ x_4^3 \\ x_4^2x_5^2 \\ x_4^2x_5 \\ x_4^2 \\ x_4x_5^2 \\ x_4x_5 \\ x_4 \\ x_5^2 \\ x_5 \\ 1 \end{pmatrix} = \mathbf{0}. \tag{21}$$

Equation (21) constitutes a set of 12 linearly independent equations in the 12 terms $x_4^3x_5^2, x_4^3x_5, x_4^3, x_4^2x_5^2, x_4^2x_5, x_4^2, x_4x_5^2, x_4x_5, x_4, x_5^2, x_5, 1$. Let, the 12×12 coefficient matrix in Eq. (21) be named as Eq. (22).

$$\begin{pmatrix} \mathbf{\Sigma}'' & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}'' \end{pmatrix} \tag{22}$$

Equation (21) is clearly an over-constrained linear system. In order for this system to have a non-trivial solution, the coefficient matrix must be singular. The determinant of the coefficient matrix is a 16^{th} degree polynomial in x_3. The roots of this polynomial give the values of x_3 corresponding to the 16 solutions of the inverse kinematics problem. For each value of x_3 thus obtained, the corresponding value of θ_3 may be computed using the formula $\theta_3 = 2\arctan(x_3)$, where arctan is the single argument inverse tangent function.

3.5 The Remaining Joint Variables

For each value of θ_3, the remaining joint variables may be computed in the following manner: At first, the numerical value of θ_3 is substituted in the coefficient matrix of Eq. (21). Then 11 independent members of Eq. (21) is used to solve for the 11 terms $x_4^3x_5^2, x_4^3x_5, x_4^3, x_4^2x_5^2, x_4^2x_5, x_4^2, x_4x_5^2, x_4x_5, x_4, x_5^2, x_5$. The numerical values of x_4 and x_5 may be used to compute θ_4 and θ_5. The numerical values for θ_3, θ_4, and θ_5 is then substituted in Eq. (16). Then 8 linearly independent members of the resulting equation is used to solve for $s_1s_2, s_1c_2, c_1s_2, c_1c_2, s_1, c_1, s_2, c_2$. Then the numerical values of s_1 and c_1 is used to obtain a unique value for θ_1. Similarly, θ_2 may be computed using the numerical values of s_2 and c_2. Finally, substituting values for θ_1, θ_2, θ_3, θ_4, and θ_5 in the $(1,1)$ and $(2,1)$ elements of the following equation:

$$\boldsymbol{A}_6 = \boldsymbol{A}_5{}^{-1}\boldsymbol{A}_4{}^{-1}\boldsymbol{A}_3{}^{-1}\boldsymbol{A}_2{}^{-1}\boldsymbol{A}_1{}^{-1}\boldsymbol{A}_{hand}, \tag{23}$$

yields two linear equations in s_6 and c_6. After solving for s_6 and c_6 one may use their values to determine a unique corresponding value for θ_6.

3.6 Numerical Example

The numerical example that has been implemented in this report is the same as that in the paper by Raghavan and Roth [1]. In that paper, the link parameters for the 6-R manipulator was taken as:

a_i	d_i	α_i(deg)
0.8	0.9	20.0
1.2	3.7	31.0
0.33	1.0	45.0
1.8	0.5	81.0
0.6	2.1	12.0
2.2	0.63	100.0

The end-effector location at the goal position was taken as:

$$\boldsymbol{A}_{hand} = \begin{pmatrix} 0.354937475307970 & 0.461639573991743 & -0.812962663562556 & 6.82151837150213 \\ 0.876709605247149 & 0.137616185817977 & 0.460914366741046 & 1.46146704002829 \\ 0.324653132880913 & -0.876327957516839 & -0.355878707125018 & 5.36950521368663 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The numerical value of the matrices ζ and σ in Eq. (26) of the paper by Raghavan and Roth [1], as obtained by us fully matches with the value given in the paper. L.H.S. of Eqs. (28, 29) of the paper [1] also matches with the expressions obtained by us. The aforementioned implementations has been done using Wolfram *Mathematica*®.

The 16^{th} degree polynomial in x_3 obtained from the determinant of coefficient matrix (22) is:

$$x_3^{16} + 0.510362x_3^{15} - 0.214998x_3^{14} + 4.75325x_3^{13} + 1.77453x_3^{12} + 9.80134x_3^{11} + 14.1216x_3^{10} - 2.58323x_3^{9} + 15.8339x_3^{8} - x_3^{7} + x_3^{6} + 2.6547x_3^{5} - 1.24203x_3^{4} + 7.11005x_3^{3} + 1.5436x_3^{2} + 2.00539x_3 + 1.08705 = 0$$

This polynomial has only 2 real roots, which are $x_3 = -0.4142$ and $x_3 = -0.7271$. This implies that for this particular example, there are at most 2 configurations in which the manipulator can reach the goal position. The values of θ_3 corresponding to $x_3 = -0.4142$ and $x_3 = -0.7271$ are respectively -45.0009 deg and -72.0439 deg.

For θ_3 = -45.0009 deg and -72.0439 deg, one can now follow the rest of the procedure in [1] for computing the other joint variables, which have been obtained to be: for θ_3 = -45.0009 deg, (θ_1, θ_2, θ_4, θ_5, θ_6) = (14.0016, 29.6986, 71.0009, -62.9967, 9.99674), all in degrees. Thus one solution of the inverse kinematics problem is the tuple (14.0016, 29.6986, -45.0009, 71.0009, -62.9967, 9.99674), all in degrees.

The solution tuple corresponding to the other real root ($x_3 = -0.7271$) of the 16^{th} degree polynomial in x_3, is (13.1334, 50.9693, -72.0439, 72.0652, -7.16093, -37.8847), all in degrees.

3.7 Numerical Problems in Raghavan and Roth's Solution

Although Raghavan and Roth present a constructive solution to the inverse kinematics problem, their algorithm suffers from numerical accuracy problems. For example, the algebraic manipulations in generating $p_1, p_2, p_3, l_1, l_2, l_3$ can introduce errors due to floating point arithmetic. Furthermore, the determinant expansion of the coefficient matrix (22) can introduce significant numerical errors such that $(1+x_3^2)^4$ may not exactly divide the determinant. The symbolic expansion of the determinant is relatively expensive for real time performance. Finally, the computation of real roots of polynomials of degree 16 can be ill-conditioned [6]. The floating point errors accumulated in the intermediate steps of the computation, and therefore in the coefficients of the degree 16 polynomial, can have a significant impact on the accuracy of the roots of the resulting polynomial.

In [7], Manocha and Canny present an algorithm and implementation for real time inverse kinematics for a general 6-R manipulator. They make use of the results presented in [1]. However, they perform matrix operations and reduce the problem to computing eigenvalues and eigenvectors of a matrix as opposed to computing a univariate polynomial in the tangent of the half-angle of a joint variable. The main advantage of this technique lies in its efficiency and numerical stability. The algorithms for computing eigenvalues and eigenvectors of a matrix are backward stable and fast implementations are available [8]. This is in contrast with expanding a symbolic determinant to compute a degree 16 polynomial and then computing its roots. The latter method is relatively slower and the problem of computing roots of such polynomials can be ill-conditioned [6]. The numerical stability of the operations used in their algorithm is well understood and they were able to come up with tight bounds on the accuracy of the solution. Furthermore, for almost all instances of the problem they were able to compute accurate solutions using 64 bit IEEE floating point arithmetic. Moreover, the average running time of the algorithm, as reported by them, was 11 milliseconds on an IBM RS/6000. In a few cases they were required to use sophisticated techniques like solving generalized eigenvalue system and the resulting algorithm took up to 25 milliseconds on the IBM RS/6000.

4 Conclusion

In the paper [1], a solution procedure for the inverse kinematics problem of the 6-R manipulator of general geometry was presented, which has been implemented here in this report. This procedure involved a sequential elimination of variables, resulting in a univariate polynomial of degree 16 in the tangent of the half-angle of one of the joint variables. Some new and interesting properties of the ideal of the closure equations of the 6-R manipulator were presented. The appearance of extraneous roots, which is common in sequential elimination methods was avoided by exploiting these properties during the elimination procedure.

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