PAC-Exploration in Reinforcement Learning with Function Approximation

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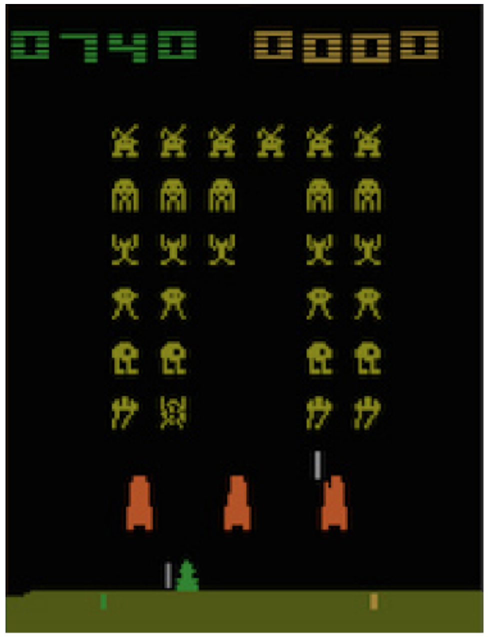
RL applications



Key aspects of RL

Bellman Equations

(Dynamic Programming)

Temporal credit assignment

Generalization

(Supervised Learning)

Function Approx.

Exploration

(Multi-Armed Bandit)

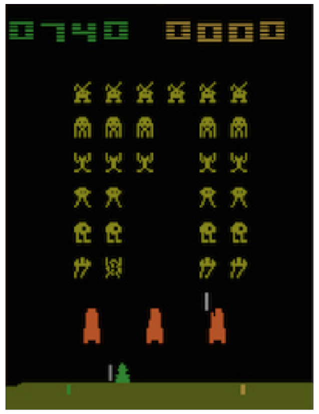
Optimism

Bellman Equations

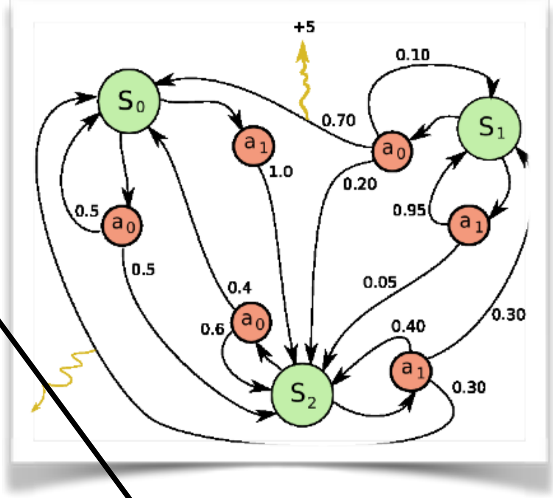
(Dynamic Programming)

Temporal credit assignment

(Approx. DP)



(PAC-MDP)



?

Generalization

(Supervised Learning)

Function Approx.

(Contextual Bandit)

Exploration

(Multi-Armed Bandit)

Optimism

Bellman Equations

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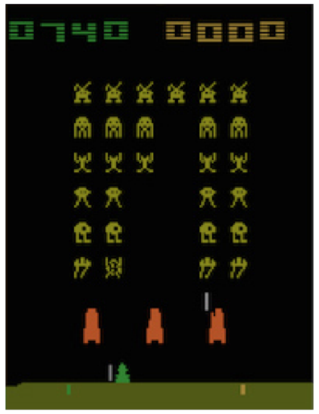
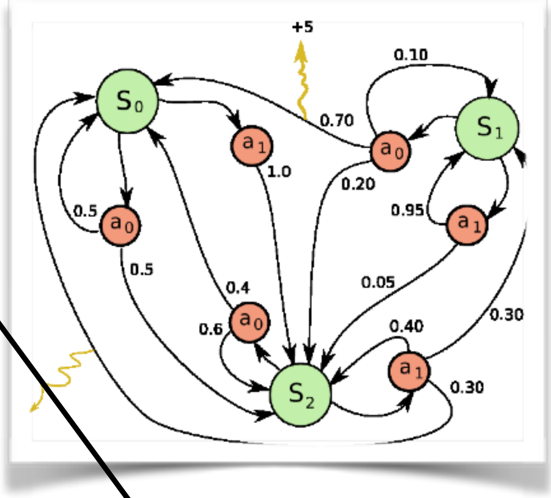


(PAC-MDP)



Contextual Decision Processes

Bellman Rank

Generalization

(Contextual Bandit)

Exploration

(Supervised Learning)

Function Approx.

(Multi-Armed Bandit)

Optimism

Contextual Decision Processes

Finite action space $\mathcal{A}$, context space $\mathcal{X}$, horizon H

For every episode (stochastic and stationary)

- x_1 is drawn, and the learner chooses a_1.
- r_2 , x_2 are drawn, and the learner chooses a_2.
- r_3 , x_3 are drawn, and the learner chooses a_3.
- …
- r_H is drawn, and episode ends. (Next episode starts)

Policy $\pi : \mathcal{X} \to \mathcal{A}$

Goal: maximize $V^\pi = \mathbb{E}\left[\sum_{h=1}^{H} r_h \,\middle|\, a_{1:H} \sim \pi\right]$

What are contexts?

- Similar to features (a design choice).
- The most detailed choice of context: full interaction history with the environment in this episode.
- Often can be simpler, e.g., when the problem is (short-order) Markov.
- For tabular MDPs: context = (state, time-step).

Policy vs Value function

- A policy $\pi : \mathcal{X} \to \mathcal{A}$ tells you what to do
- Good policy achieves high value

- A value function $f : \mathcal{X} \times \mathcal{A} \to \mathbb{R}$ tells you how much value you would get in the long term
  - induces a greedy policy $\pi_f = (x \mapsto \arg\max_{a \in \mathcal{A}} f(x, a))$
- Good value function…
  - induces a good policy
  - predicts its long-term value accurately

RL with value-function approximation

Given $\mathcal{F} \subset (\mathcal{X} \times \mathcal{A} \to \mathbb{R})$, learner identifies $f \in \mathcal{F}$

- $\log |\mathcal{F}|$ is small
- ∃ f that satisfies *Bellman Equations* (“valid”)

$$\forall f' \in \mathcal{F}, h \in [H]$$

$$\mathbb{E}_{\substack{a_{1:h-1} \sim \pi_{f'} \\ a_{h:h+1} \sim \pi_f}} [f(x_h, a_h) - r_h - f(x_{h+1}, a_{h+1})] = 0$$

the optimal value we aim at is $V^\star_{\mathcal{F}} = \sup_{\text{valid } f} V^{\pi_f}$

PAC Learning

- Ideally, we want to identify a near-optimal policy after acquiring

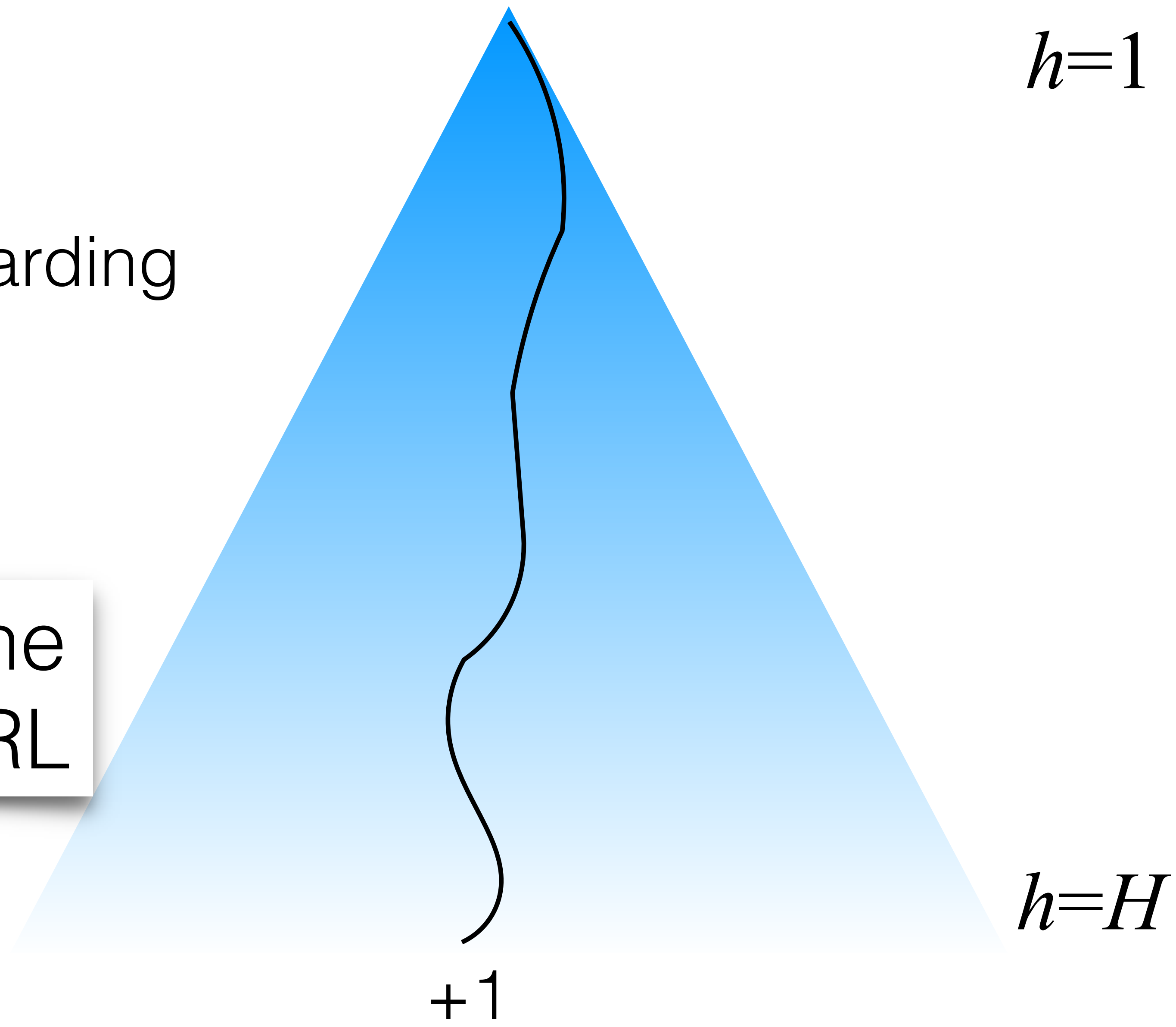
$$poly(|\mathcal{A}|, H, \log|\mathcal{F}|, 1/\delta, 1/\epsilon)$$

episodes of data. (note: no $|\mathcal{X}|$)

- But, there is a lower bound exponential in $H\ldots$

Deterministic, complete tree

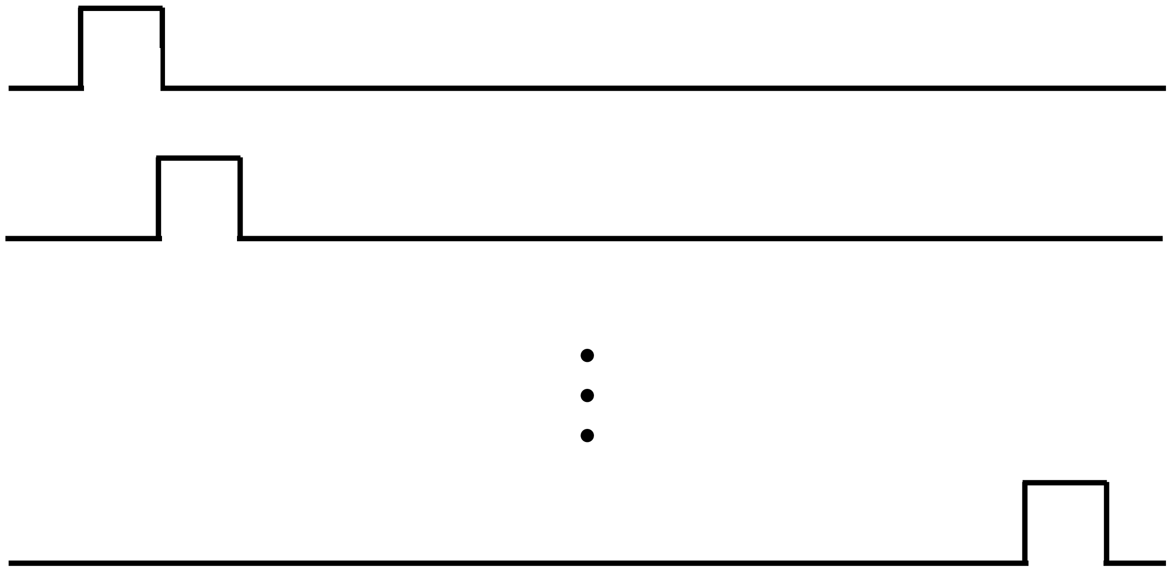
One of the $|\mathcal{A}|^H$ leaves is rewarding



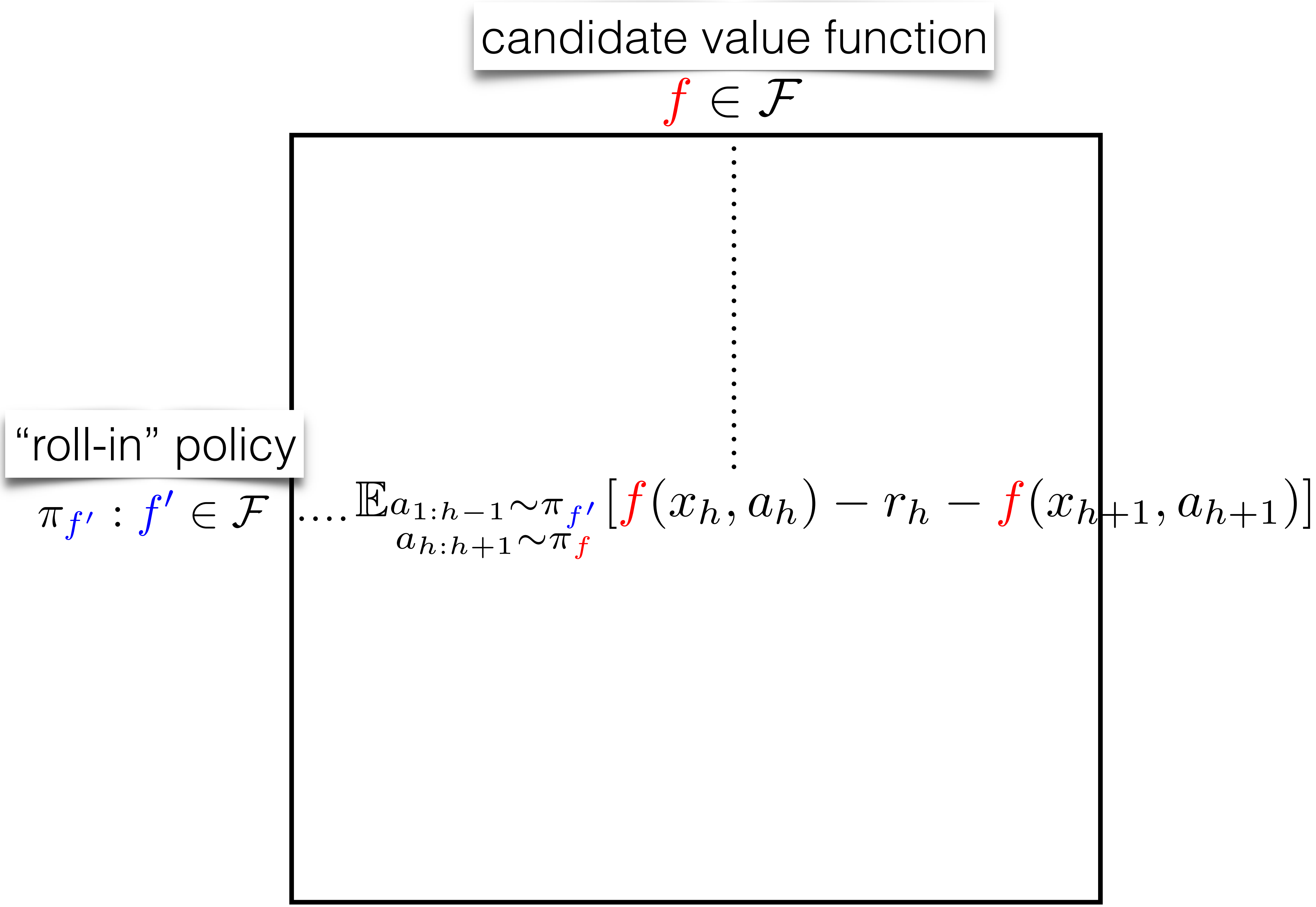
need a new measure for the difficulty of exploration in RL

Size of function space

$\log|\mathcal{F}| = H \log|\mathcal{A}|$



Bellman Rank = rank of Bellman error matrix (largest over all level h)



candidate value function

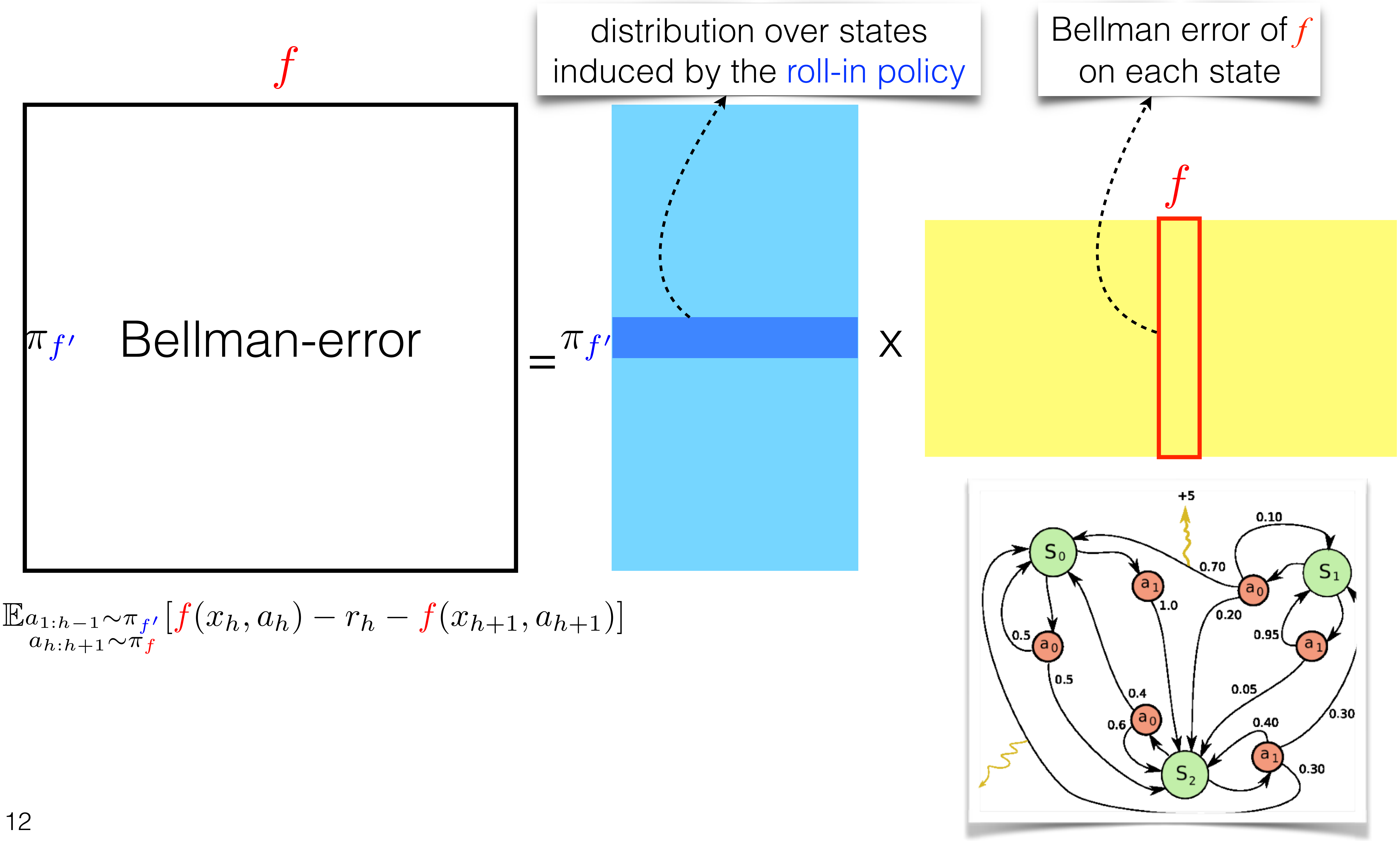
$f \in \mathcal{F}$

"roll-in" policy

$\pi_{f'} : f' \in \mathcal{F}$

$$\ldots\mathbb{E}_{\substack{a_{1:h-1}\sim\pi_{f'}\\ a_{h:h+1}\sim\pi_{f}}}[f(x_h, a_h) - r_h - f(x_{h+1}, a_{h+1})]$$

Tabular MDP has low Bellman Rank

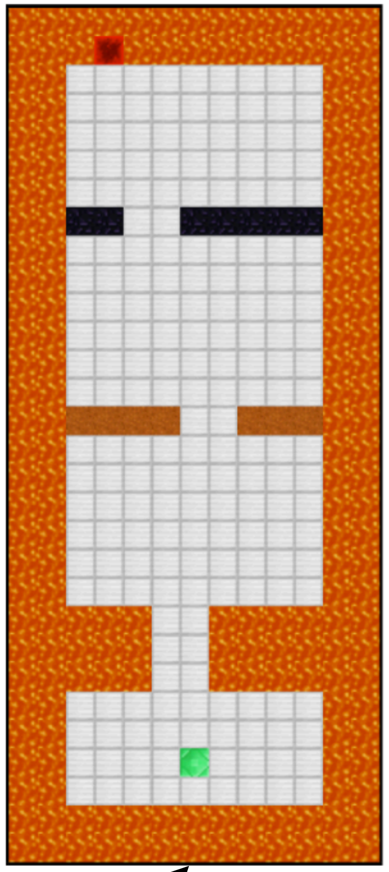


distribution over states induced by the roll-in policy

Bellman error of f on each state

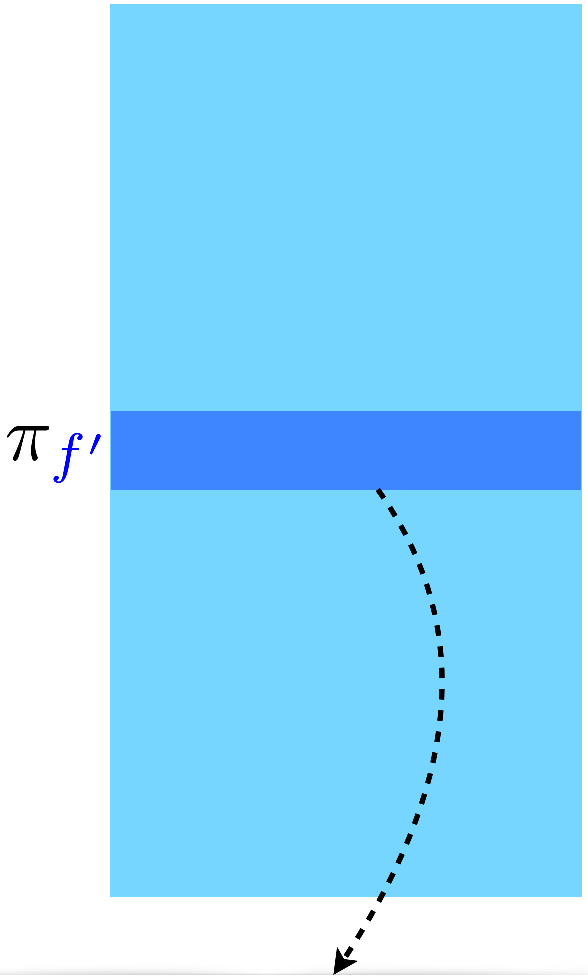
$$\mathbb{E}_{\substack{a_{1:h-1}\sim\pi_{f'} \\ a_{h:h+1}\sim\pi_{f}}}\left[f(x_h, a_h) - r_h - f(x_{h+1}, a_{h+1})\right]$$

“Visual grid-world” has low Bellman Rank



small hidden state space

rich & complex observations (and near-Markovian)



distribution over hidden-states induced by the roll-in policy

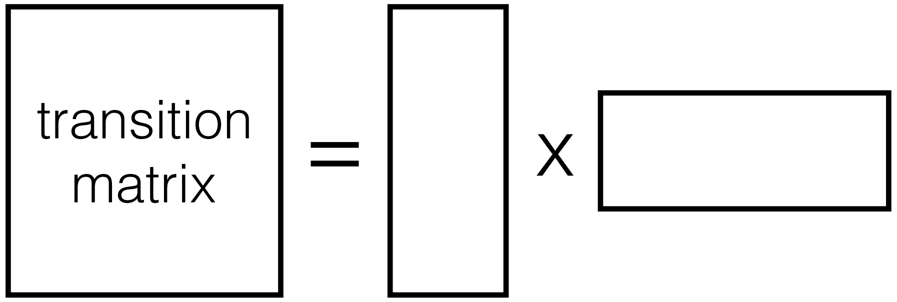
Figure from [Johnson et al. IJCAI’16]

RL settings that yield low Bellman Rank (M)

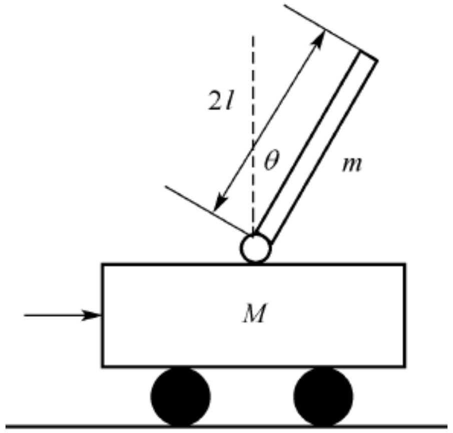
rich-obs POMDPs w/
reactive value function:
M = #hidden states

$$P_{\mathcal{T}|h}$$

PSRs w/ similar set-up:
M = poly(linear dim.)



large MDPs w/
low-rank dynamics:
M = rank of transition matrix



LQR control*:
M = poly(#state variables)

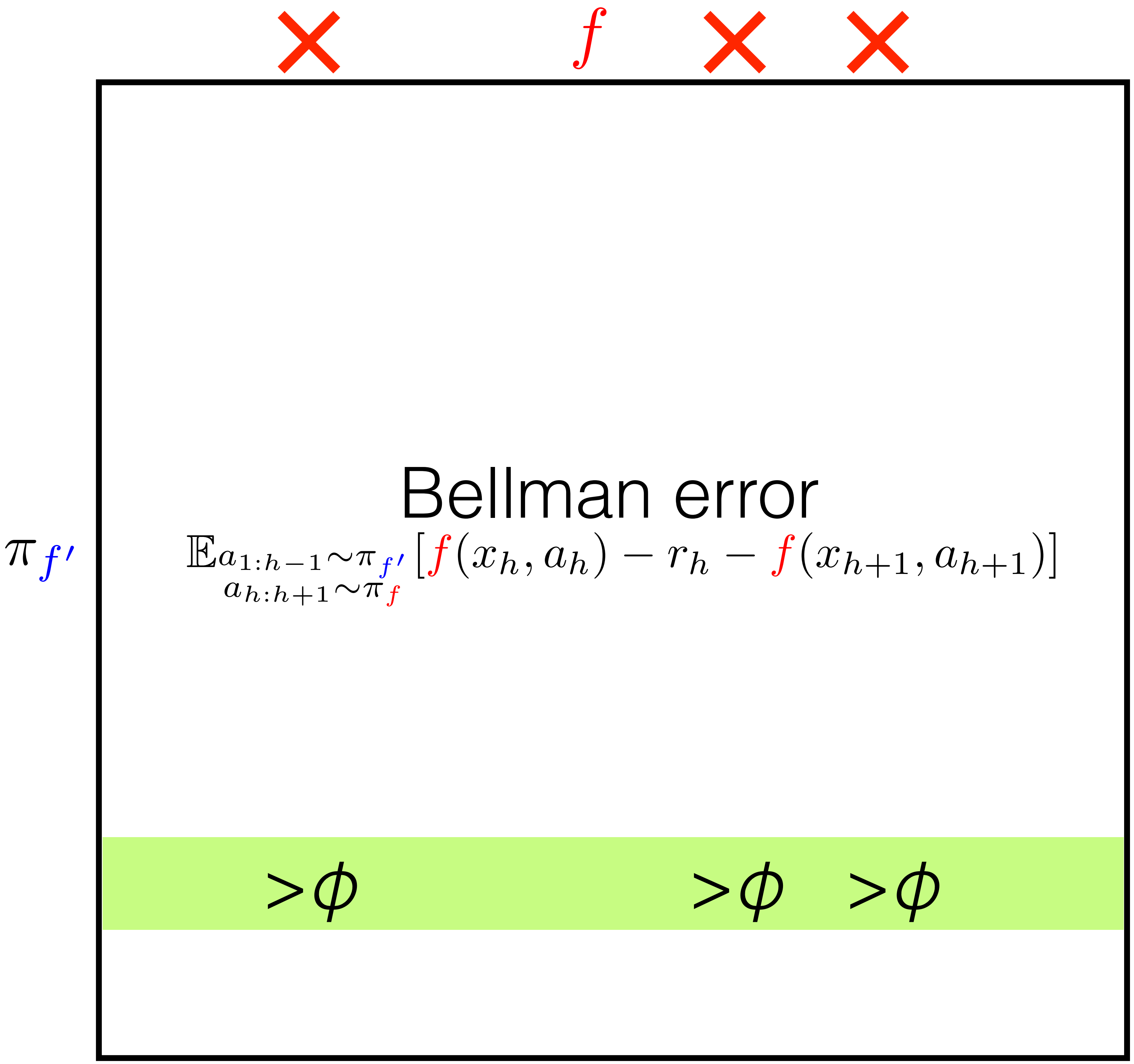
$$Q^\star$$

state abstraction
that preserves Q*:
M = poly(#abstract states)

.. can measure any
(process, function space)

*our algorithm does not directly
apply to continuous actions

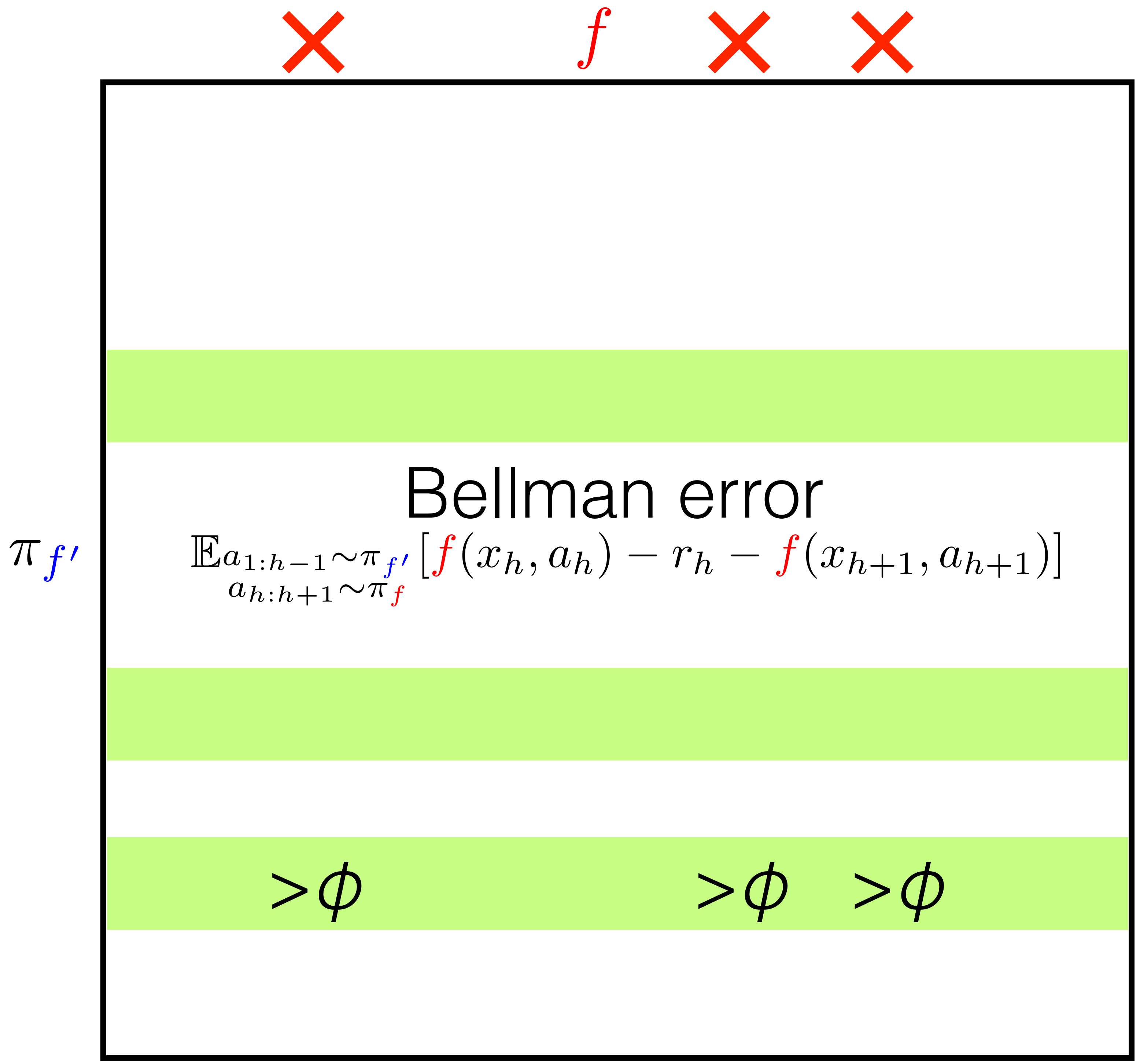
Overview of the algorithm



$$\mathbb{E}_{\substack{a_{1:h-1}\sim\pi_{f'}\\ a_{h:h+1}\sim\pi_{f}}}[f(x_h,a_h) - r_h - f(x_{h+1},a_{h+1})]$$

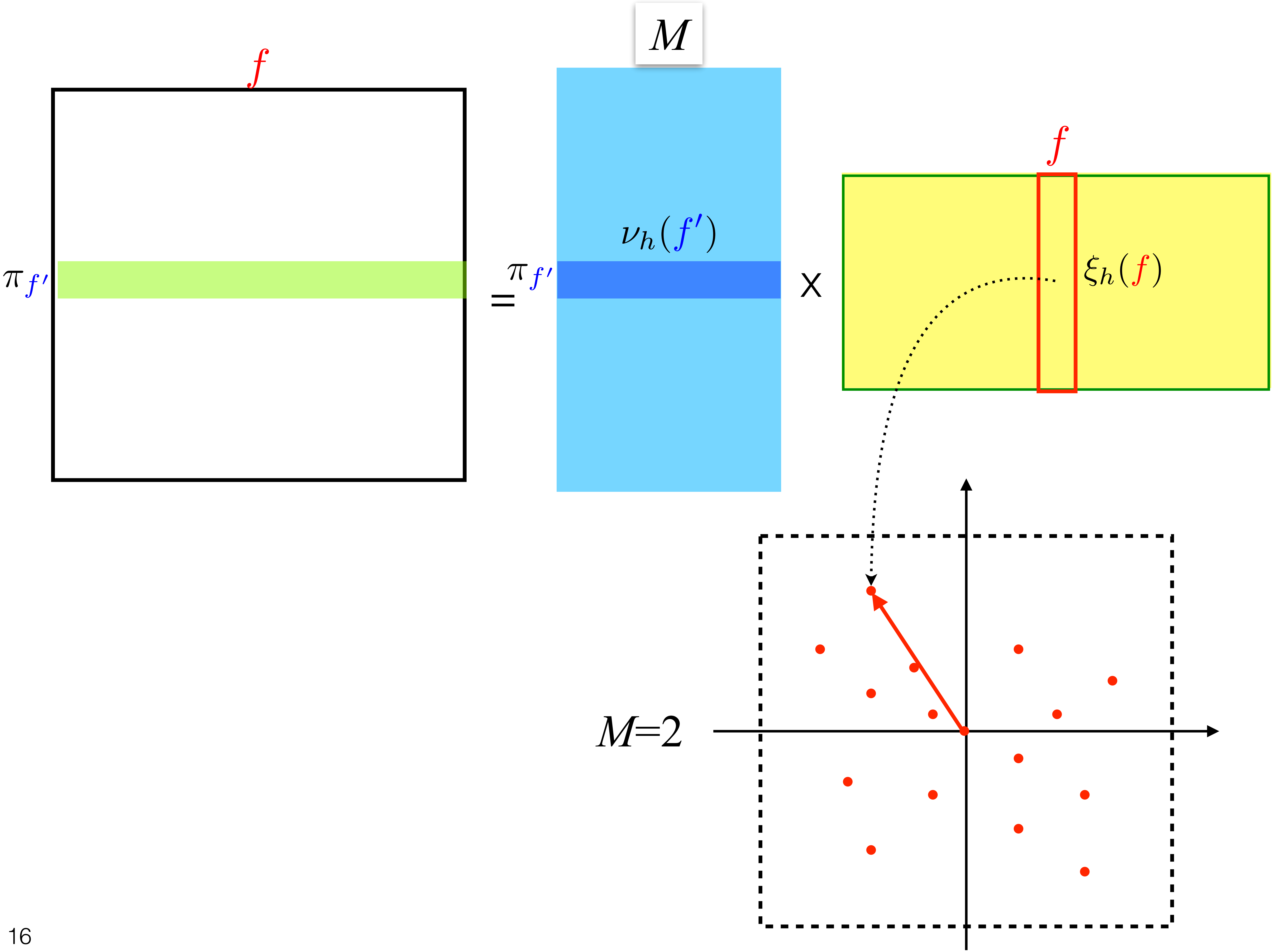
1. Pick an exploration policy
2. Estimate all entries in the corresponding row
3. Eliminate columns whose error > 0 (w/ statistical significance)

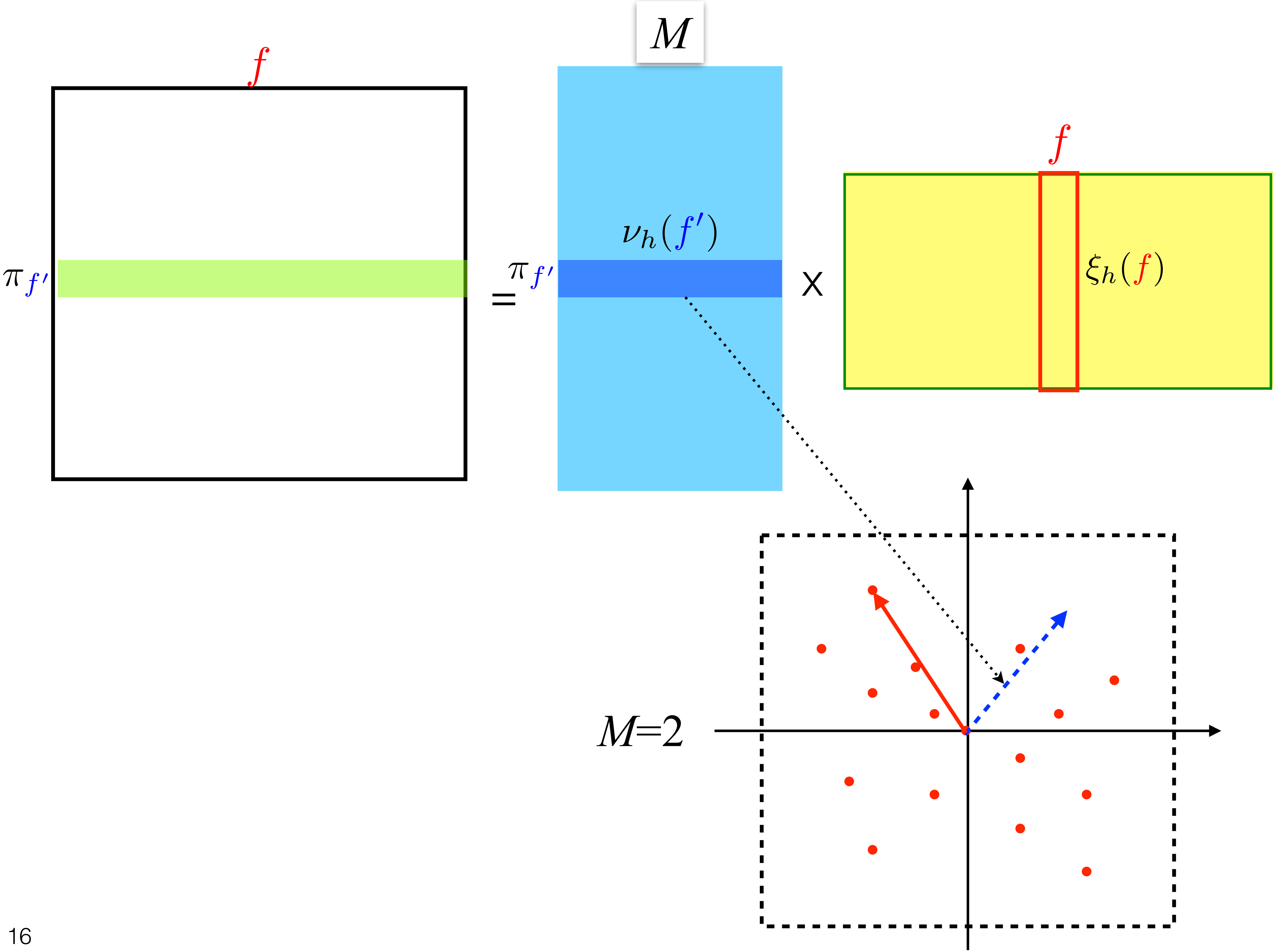
Overview of the algorithm


✕
f
✕
✕
$\pi_{f'}$
Bellman error
$\mathbb{E}_{\substack{a_{1:h-1}\sim\pi_{f'} \\ a_{h:h+1}\sim\pi_{f}}}[f(x_h, a_h) - r_h - f(x_{h+1}, a_{h+1})]$
$>\phi$
$>\phi$
$>\phi$

1. Pick an exploration policy
2. Estimate all entries in the corresponding row
3. Eliminate columns whose error > 0 (w/ statistical significance)
4. Repeat

*There are H such matrices


f
$\pi_{f'}$
=
$\pi_{f'}$
M
$\nu_h(f')$
X
f
$\xi_h(f)$
M=2


f
$\pi_{f'}$
$=$
$\pi_{f'}$
M
$\nu_h(f')$
X
f
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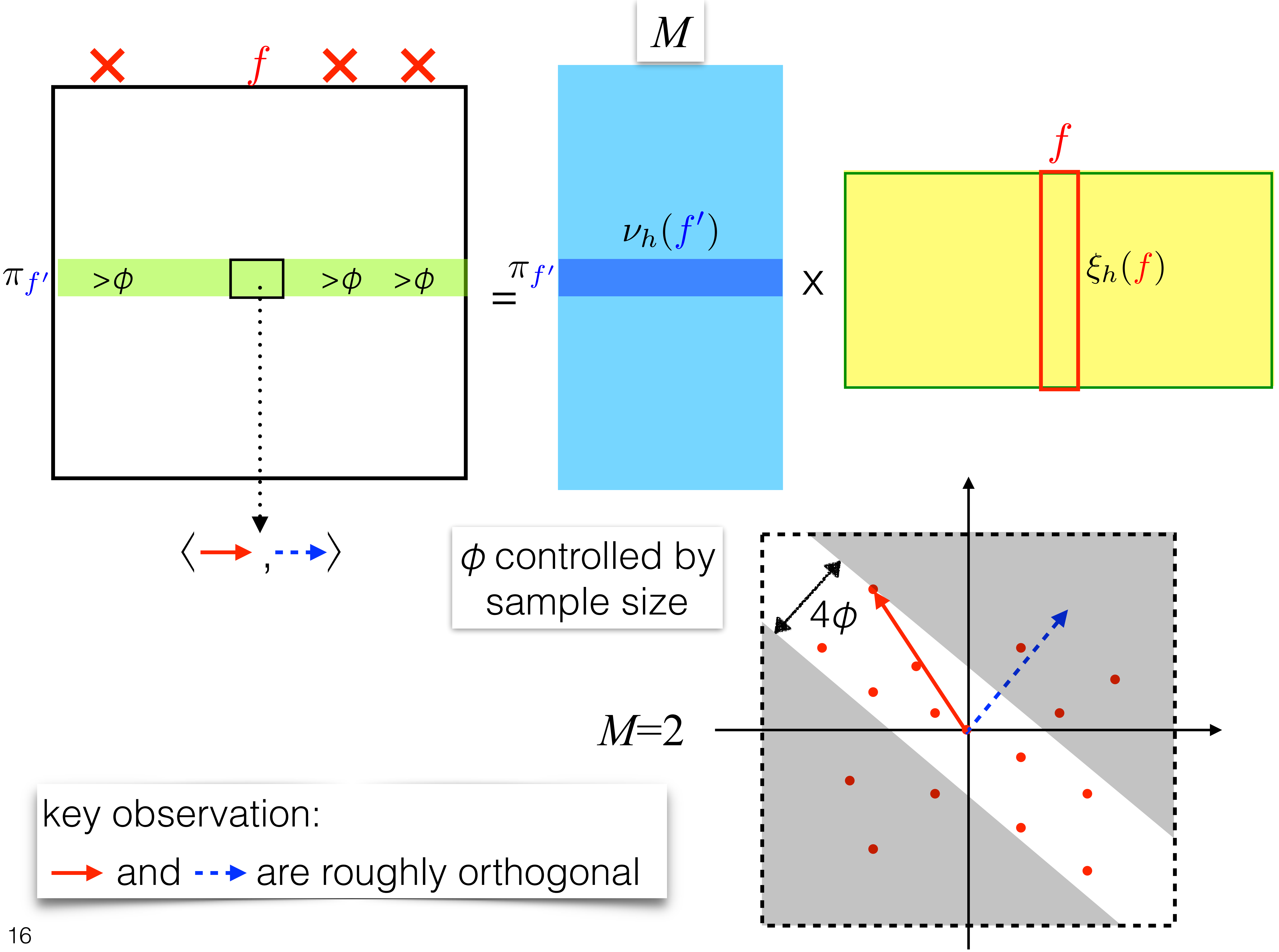
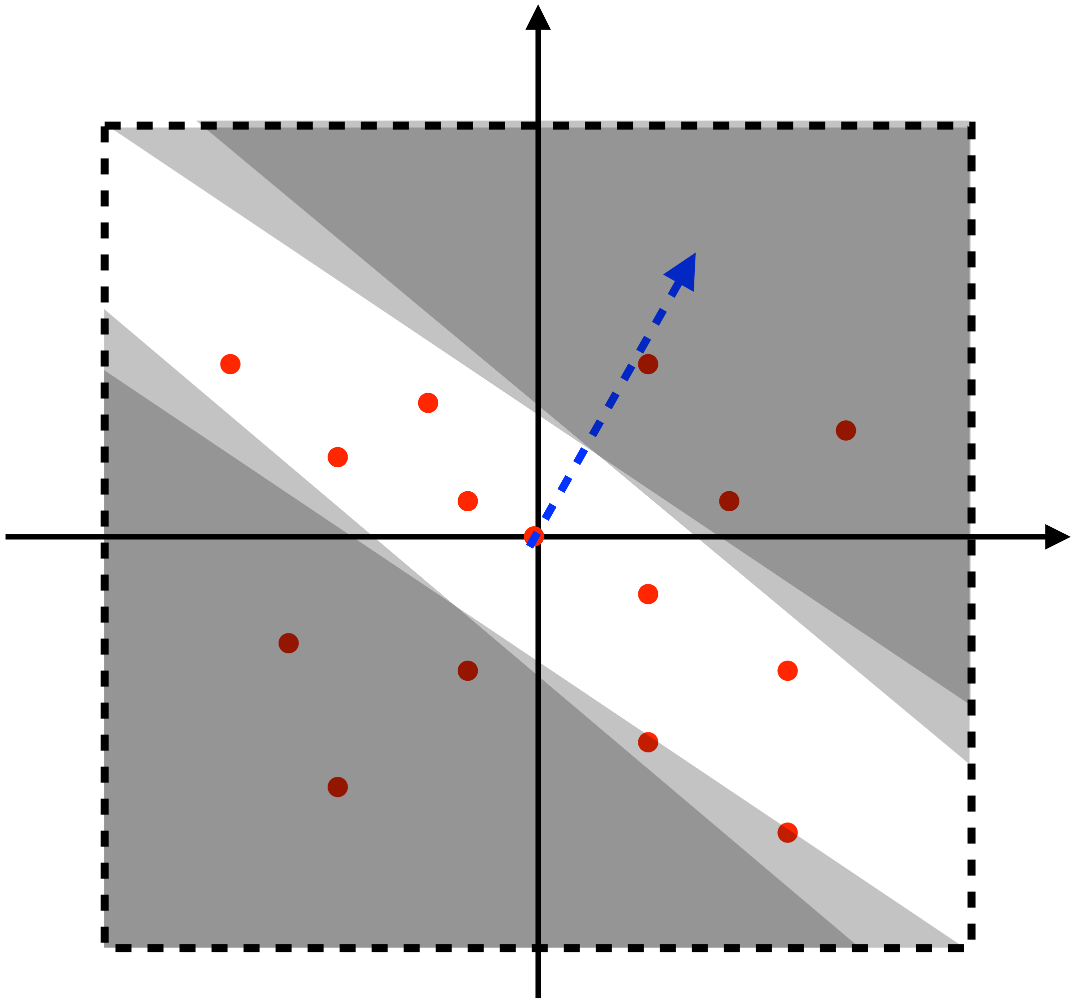


ϕ controlled by sample size

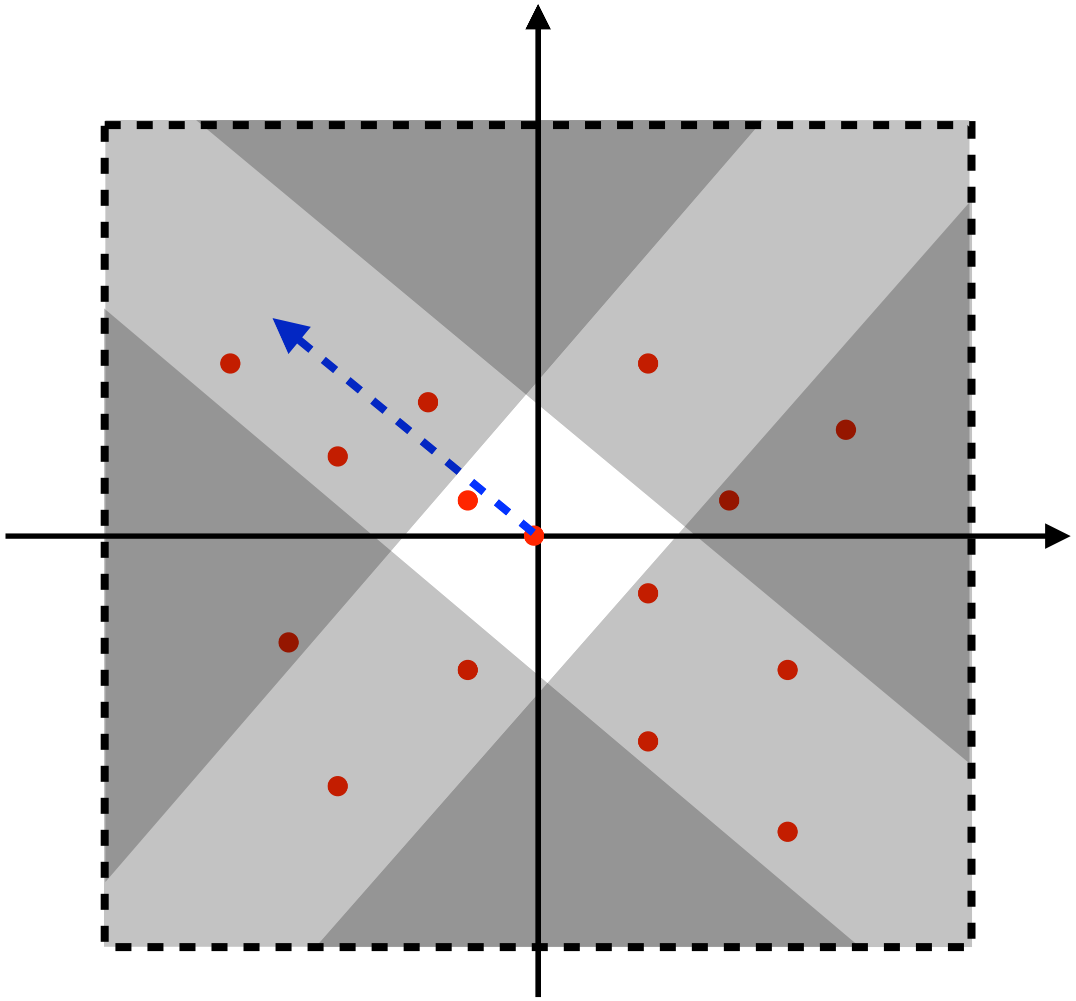
key observation:

→ and ⇢ are roughly orthogonal



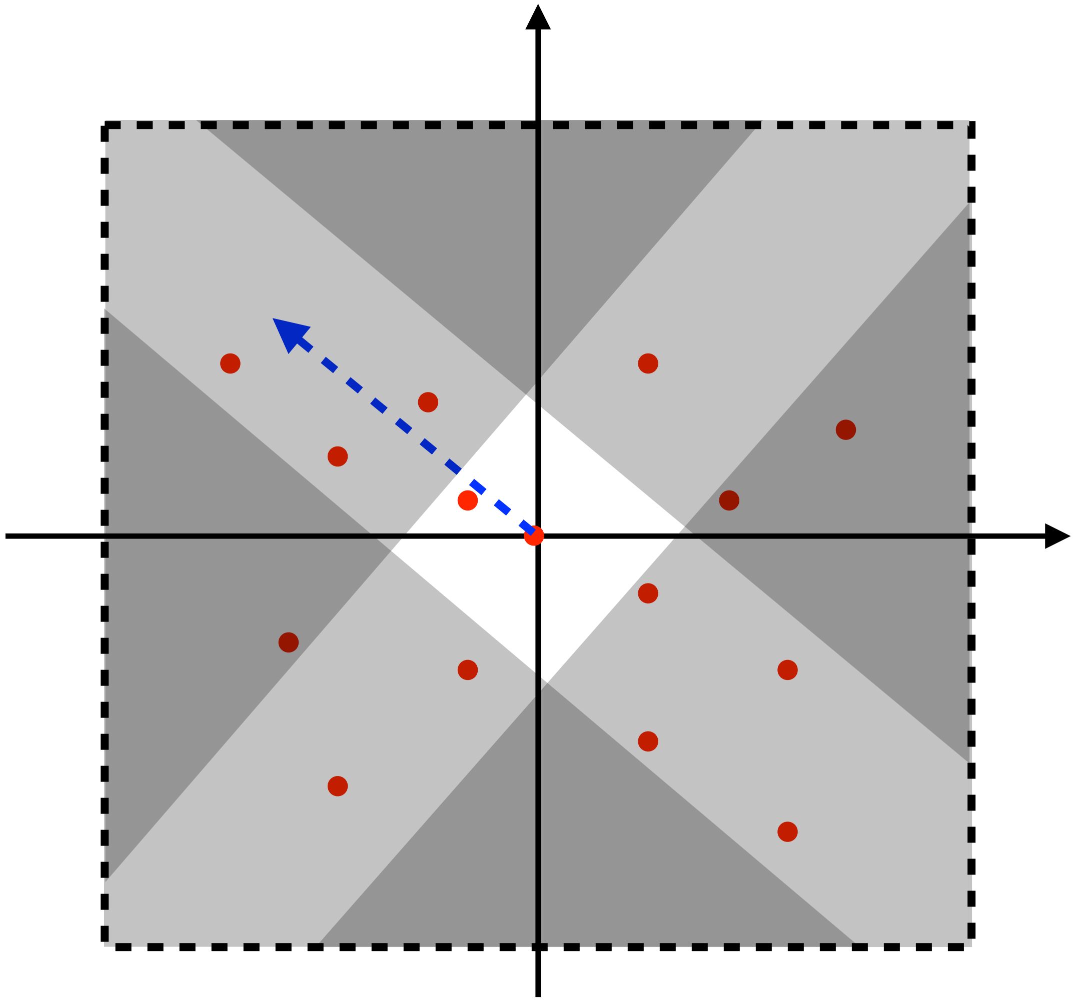
inefficient exploration

- new distribution is similar to previous ones
- area of while space shrinks slowly



efficient exploration

- new distribution is different from previous ones
- area of while space shrinks quickly



efficient exploration

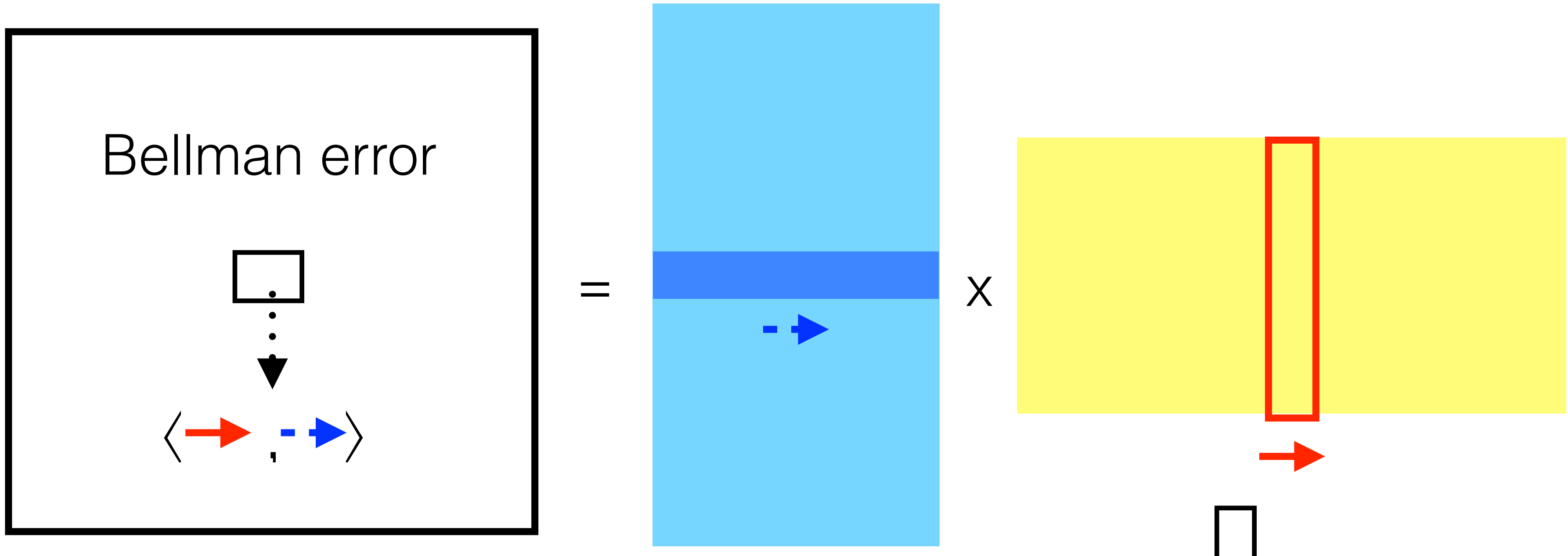
algorithm

- new distribution is different from previous ones

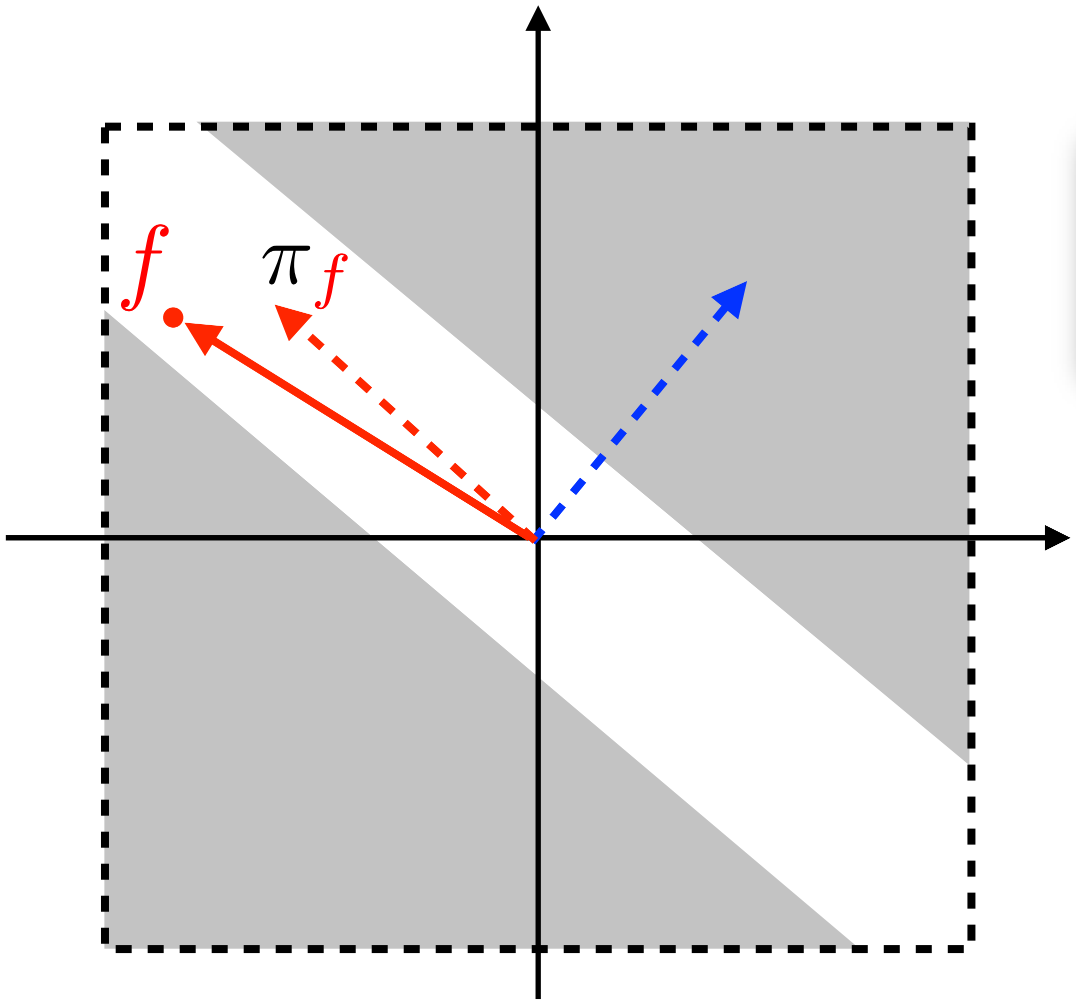
analysis

- area of while space shrinks quickly

Exploration strategy: pick f optimistically and explore with π_f



⇩



key observation:

→ and ⇢ are roughly orthogonal

⟨→ , ⇢⟩ is large (parallel)

⇩

⇢ and ⇢ are orthogonal

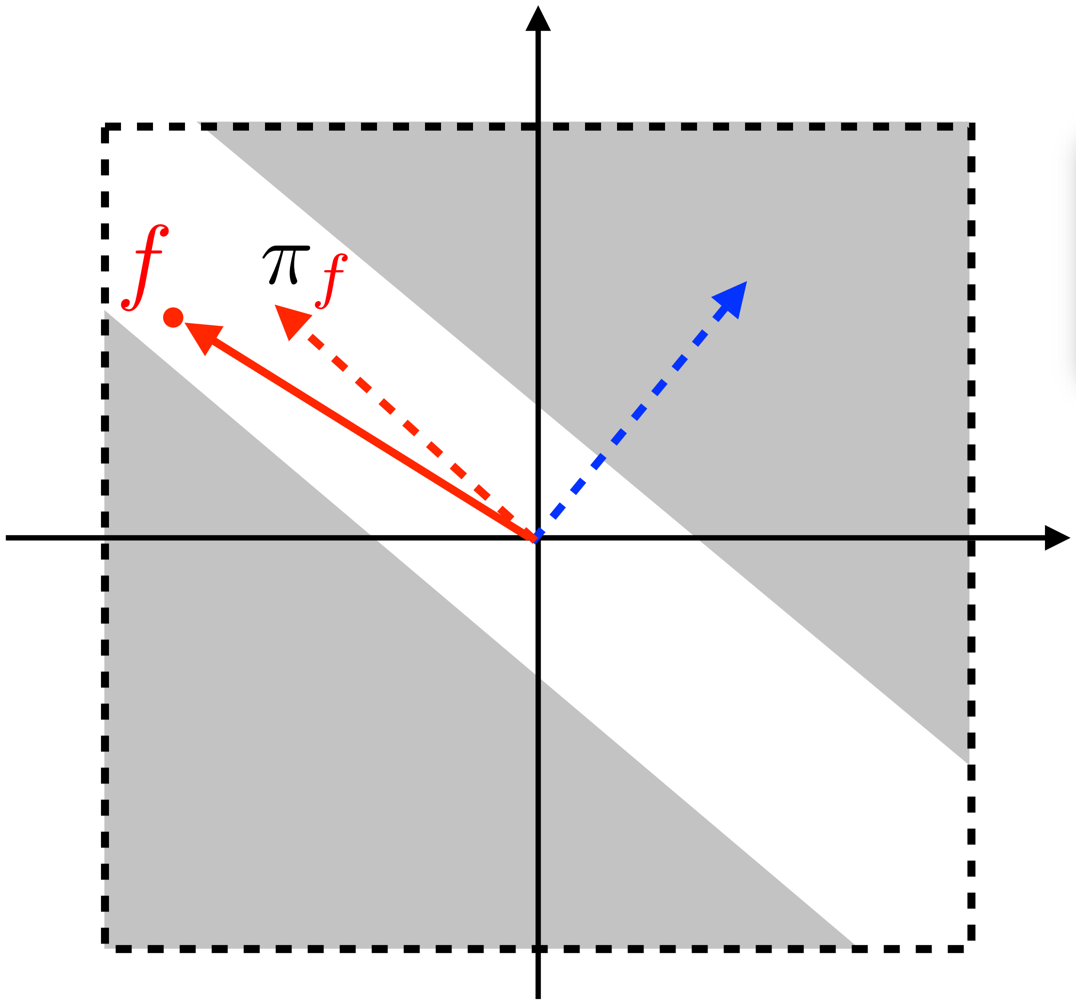
Exploration strategy: pick f optimistically and explore with π_f

Exploration Lemma: for any f,

⟨→ , ⇢⟩

$$\mathbb{E}[\max_{a\in\mathcal{A}} f(x_1,a)] - V^{\pi_f} = \sum_{h=1}^{H} \mathbb{E}_{\substack{a_{1:h-1}\sim\pi_f \\ a_{h:h+1}\sim\pi_f}} \left[f(x_h,a_h) - r_h - f(x_{h+1},a_{h+1}) \right]$$

⇩



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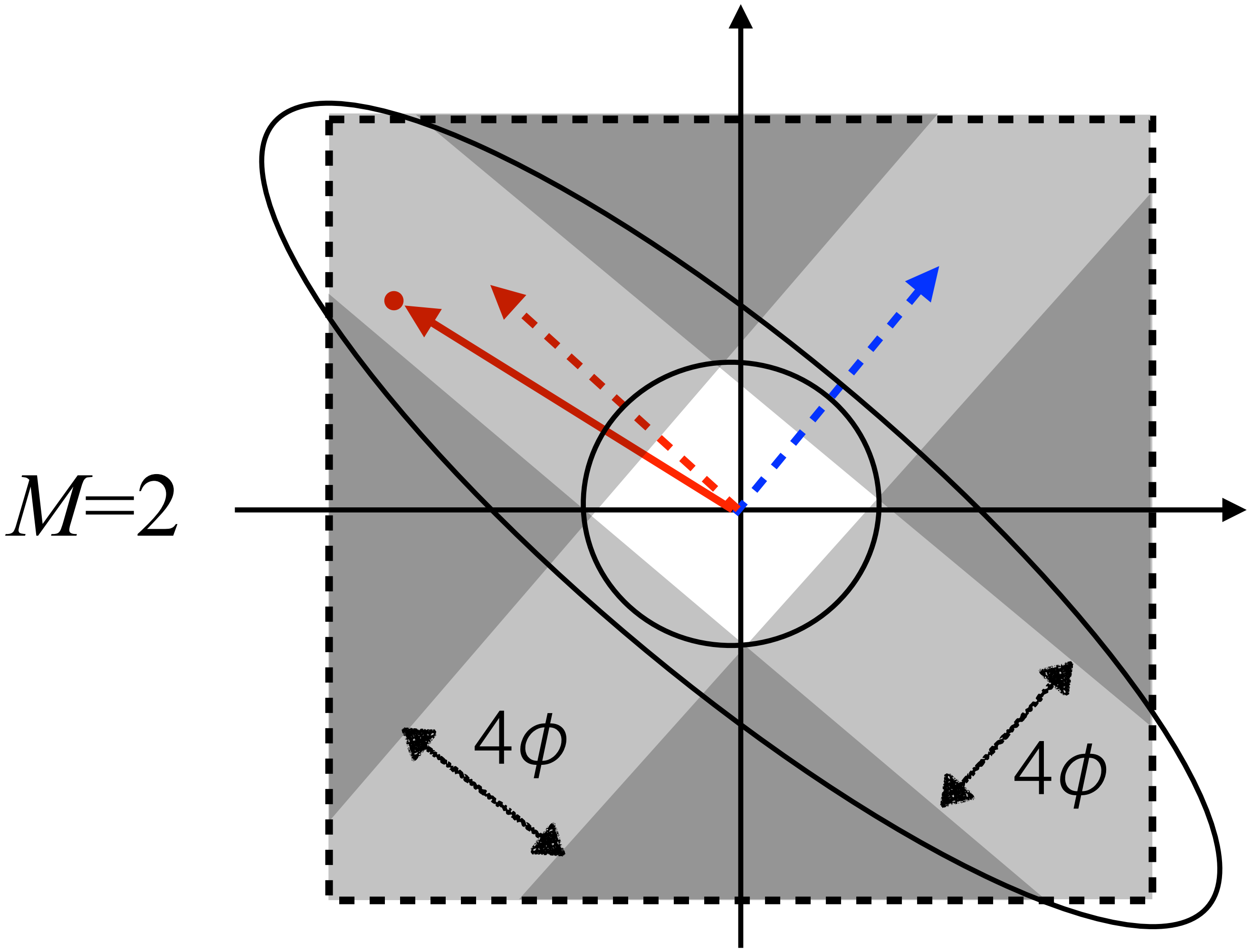
⇢ and ⇢ are orthogonal



Adaptation of [Todd,1982]:
Ellipsoid volume shrinks exponentially if

$$|\langle \longrightarrow , \dashrightarrow \rangle| \geq 3\sqrt{M} \times 2\phi$$

$\Uparrow$ controlled by sub-optimality

$\Uparrow$ controlled by sample size

Sample complexity

We can identify a policy ε-suboptimal compared to $V^{\star}_{\mathcal{F}}$ with probability at least 1-δ, after acquiring this many episodes of data:

$$\tilde{O}\left(\frac{M^2 H^3 |\mathcal{A}|}{\epsilon^2} \log(|\mathcal{F}|/\delta)\right)$$