# STATS 701: THEORY OF RL

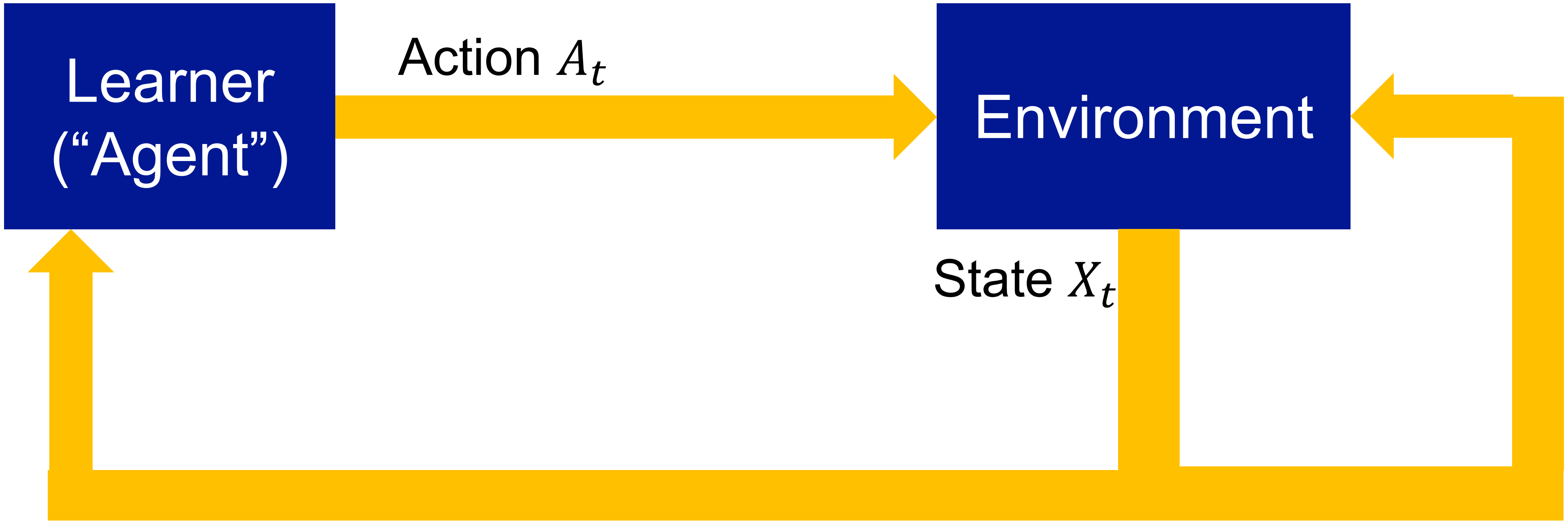
WINTER 2021

# ADVERSARIAL MDPS

Ambuj Tewari

slide credits: Gergely Neu @ Universitat Pompeu Fabra, Barcelona

# MARKOV DECISION PROCESSES



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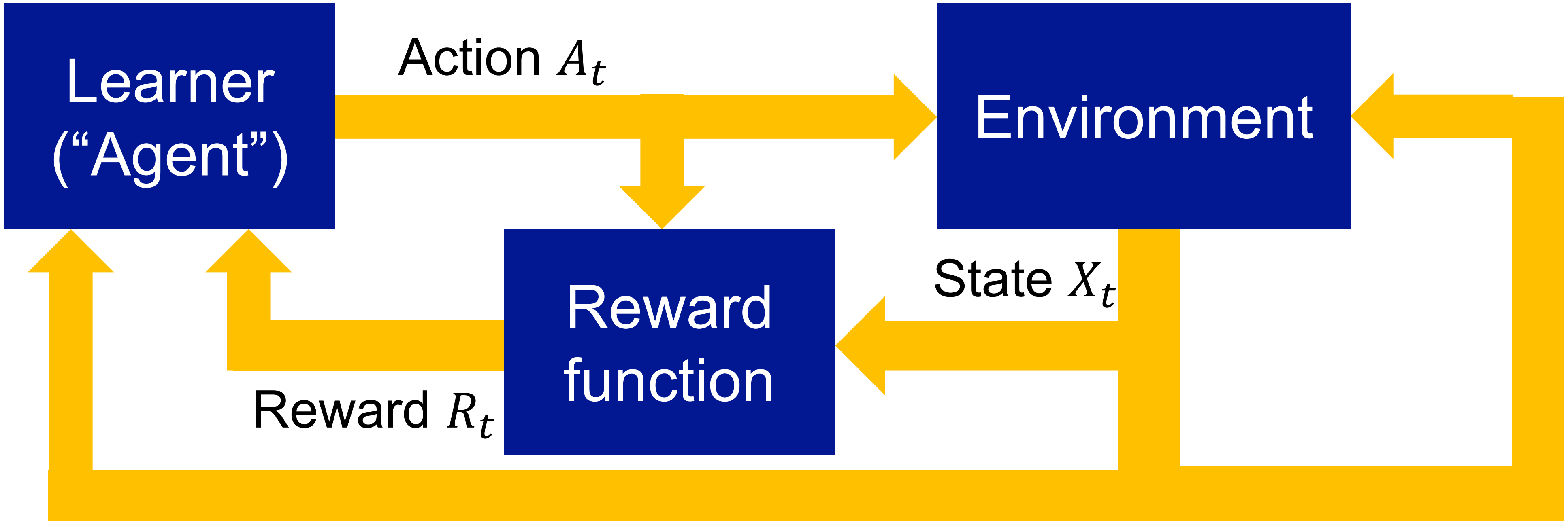

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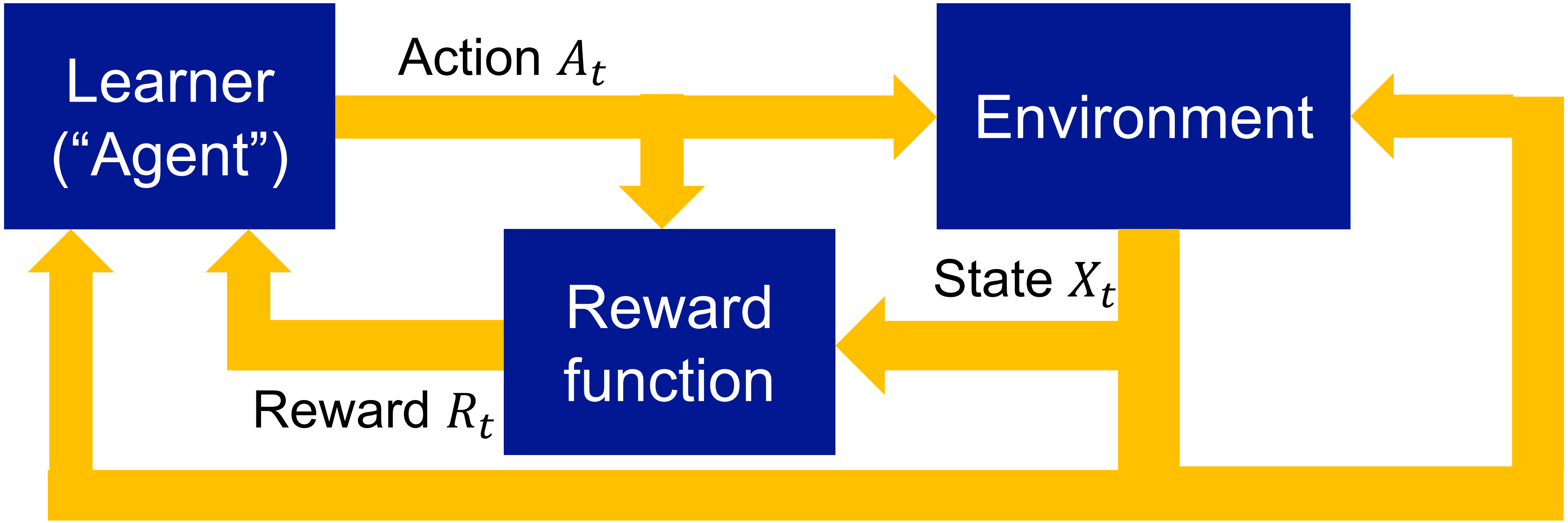
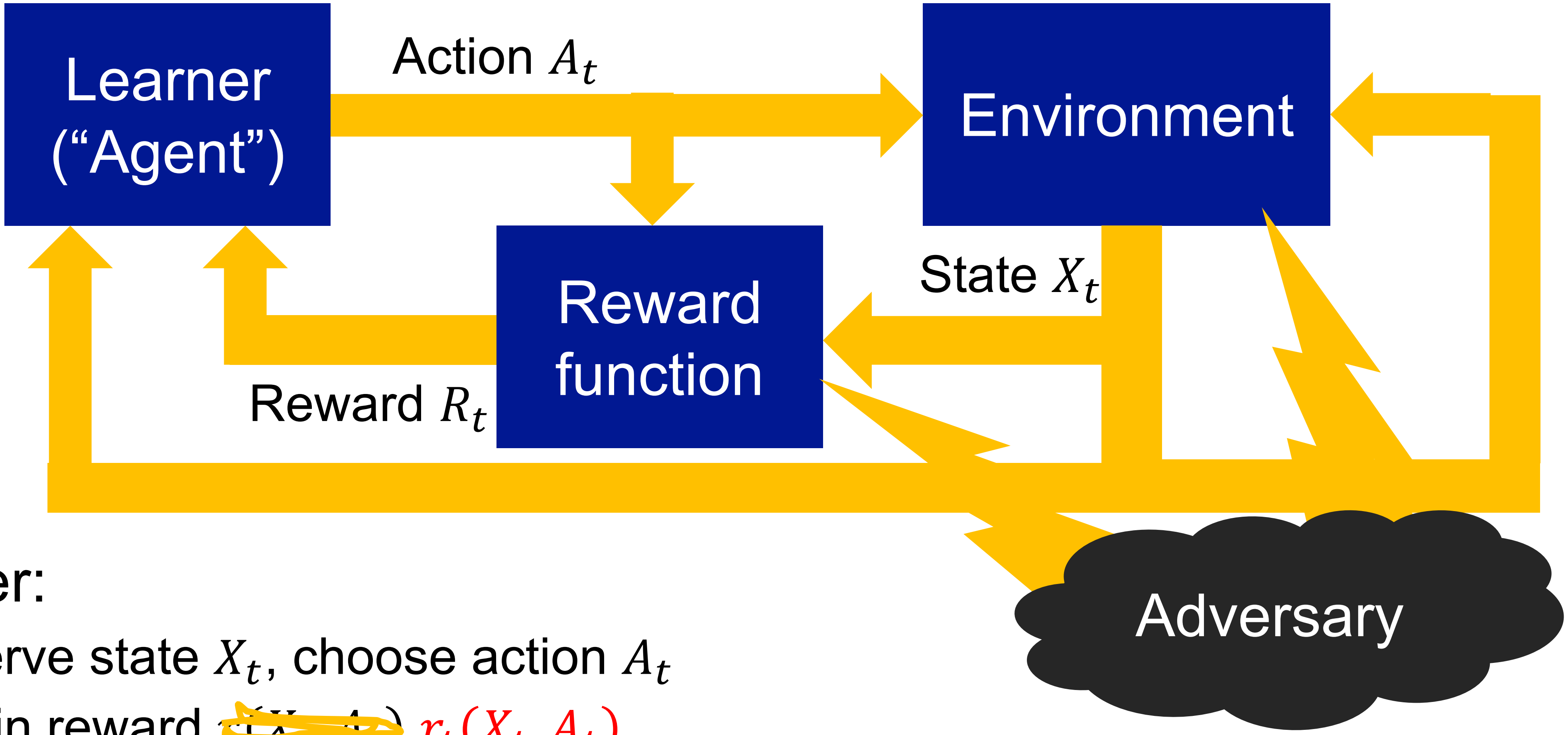



- Learner:
  - Observe state $X_t$, choose action $A_t$
  - Obtain reward $r(X_t, A_t)$
- Environment: Draw next state $X_{t+1} \sim P(\cdot\,|X_t, A_t)$

# ADVERSARIAL
# MARKOV DECISION PROCESSES

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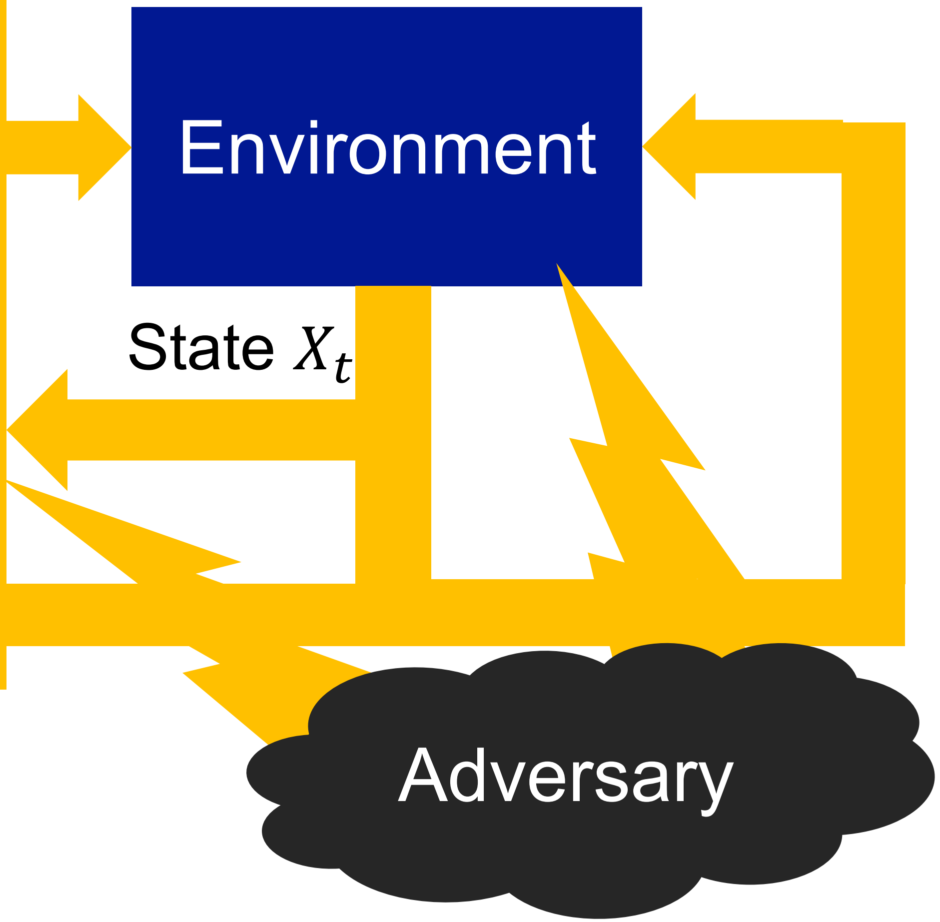
# ADVERSARIAL
# MARKOV DECISION PROCESSES

**This lecture:**
what is achievable when an external adversary is allowed to change the reward function and the transition function over time?

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# PERFORMANCE MEASURE: REGRET

## Regret

$$\mathfrak{Reg}_T(\pi) = \sum_{t=1}^{T} \mathbb{E}\Big[r_t\big(X_t^*, \pi(X_t^*)\big) - r_t(X_t, A_t)\Big],$$

where $X_1^*, X_2^*, \ldots$ is the sequence of states that would have been generated by running comparator policy $\pi$ through the dynamics $P_1, P_2, \ldots$

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Goal: sublinear regret

$$\lim_{T\to\infty} \max_{\pi} \frac{\mathfrak{Reg}_T(\pi)}{T} = 0$$

# OUTLINE

- Hardness results
  - Non-oblivious adversaries
  - Arbitrarily changing dynamics
- Arbitrarily changing reward functions
  - Some common ideas
  - Two algorithm families

# SOME HARDNESS RESULTS

# NON-OBLIVIOUS ADVERSARIES

Non-oblivious adversary:
can take history $\mathcal{H}_t = X_t, A_{t-1}, X_{t-1}, A_{t-2}, \ldots$
into account when selecting $r_t$ and $P_t$

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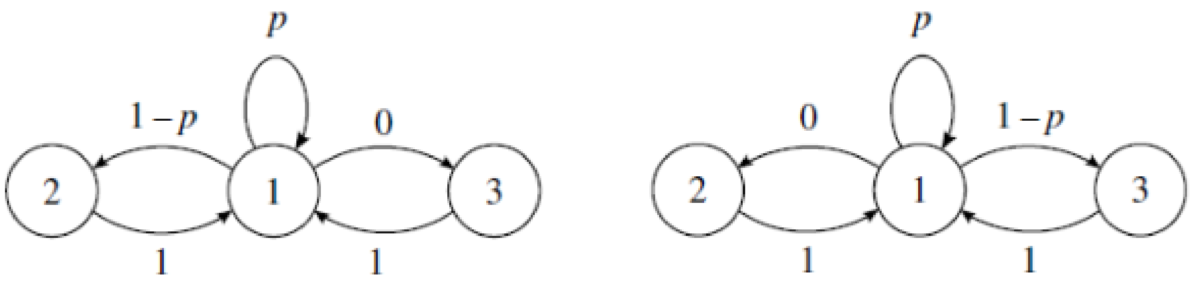
Theorem
(Yu, Mannor and Shimkin, 2009)
No algorithm can guarantee sublinear regret against a non-oblivious adversary

# PROOF

Simple counterexample by Yu, Mannor and Shimkin (2009):




(a) Transition model if the agent chooses to go left.

(b) Transition model if the agent chooses to go right.

# PROOF

Simple counterexample by Yu, Mannor and Shimkin (2009):

- Reward is function of state
- $r_t(\text{default}) = 0$
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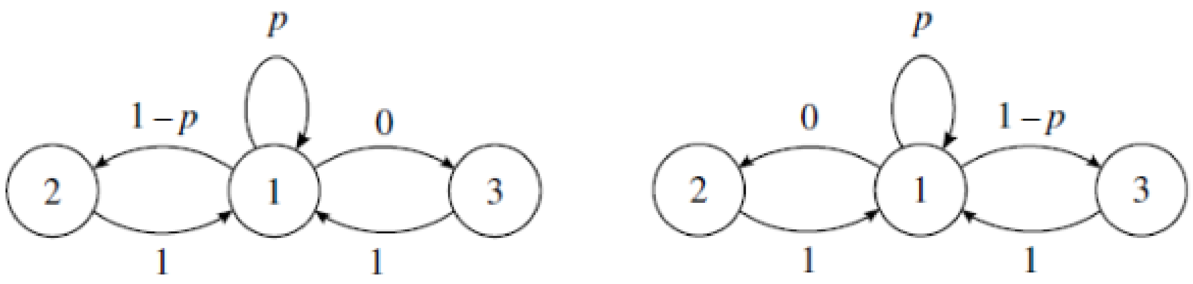


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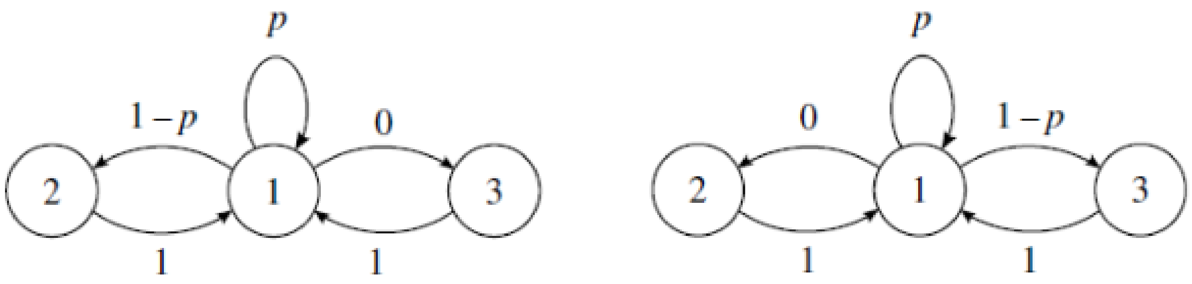


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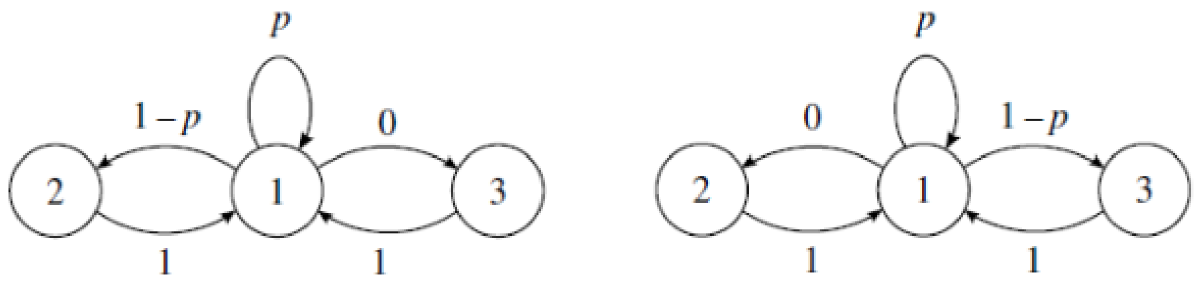


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But there is a policy $\pi$ with $\mathbb{E}\left[\sum_t r_t\left(X_t^*, \pi(X_t^*)\right)\right] \geq \left(\frac{1}{2} - p\right) T$

Either $\pi(1) = \text{left}$ or $\pi(1) = \text{right}$

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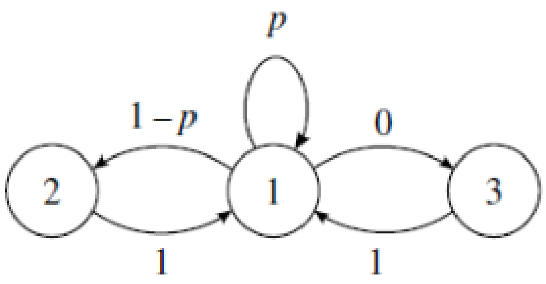


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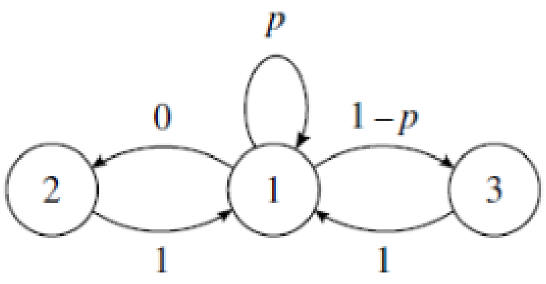


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$r_t(X_t) = 0$ for all $t$!

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Either $\pi(1) = \text{left}$ or $\pi(1) = \text{right}$

$$\mathfrak{Reg}_T(\pi) \geq \left(\frac{1}{2} - p\right) T$$

# OBLIVIOUS ADVERSARIES

Non-oblivious adversary:
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# OBLIVIOUS ADVERSARIES

**Oblivious adversary:**
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"Adversary $\approx$ nature":
it can (mis)behave arbitrarily, but doesn't care about what you do

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"Adversary $\approx$ nature":
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**Can we guarantee sublinear regret now?**

# LEARNING WITH CHANGING TRANSITIONS IS HARD

Learning against an oblivious adversary can still be computationally hard when the transition function is allowed to change!

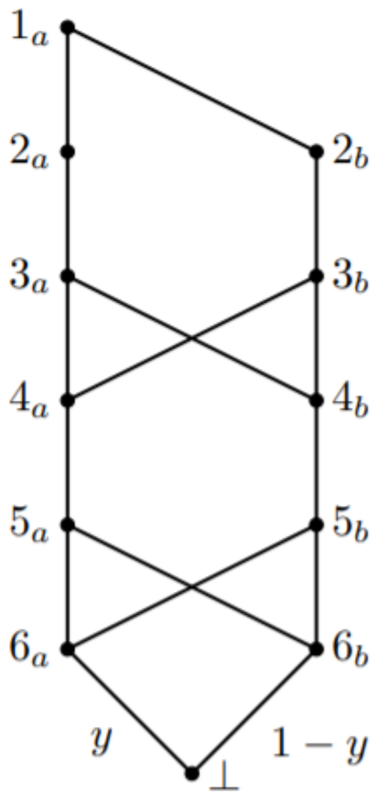
**Theorem**

(Abbasi-Yadkori et al., 2013)

There is an adversarial MDP where achieving sublinear regret is computationally hard.

# PROOF CONSTRUCTION

- **Idea:** learning of noisy parities can be formulated as an MDP with changing transition functions & rewards!
- $O(\text{poly}(n)T^{1-\alpha})$ regret $\Rightarrow O\left(\frac{poly(n)}{\varepsilon^{1/\alpha}}\right)$ excess risk, conjectured to be computationally hard to achieve
- Construction: an instance $x \in \{0,1\}^n$ corresponds to a deterministic transition graph with rewards determined by the label $y$



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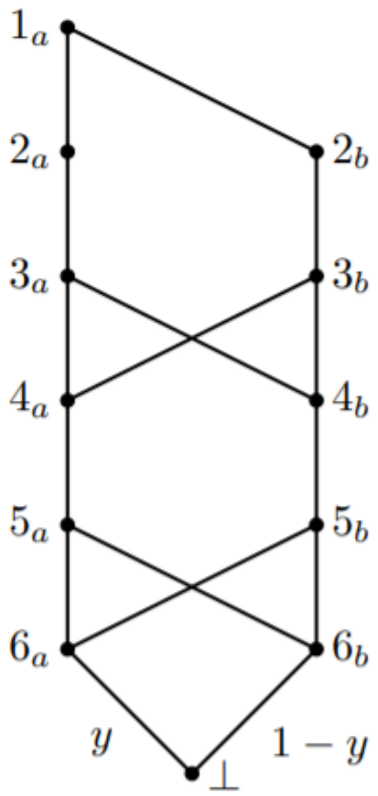


Corresponds to an oblivious adversary that picks $(P_t, r_t)$ jointly!

# SLOWLY CHANGING MDPS

Very recent work by Gajane et al. (2019), Cheung et al. (2020):

- define reward and transition variation as

$$V_T^r = \sum_{t=1}^{T} \max_{x,a} |r_t(x,a) - r_{t+1}(x,a)|$$

$$V_T^P = \sum_{t=1}^{T} \max_{x,a} \|P_t(\cdot\,|x,a) - P_{t+1}(\cdot\,|x,a)\|_1$$

- regret bounds of $O\left((V_T^P + V_T^r)^{1/3} T^{2/3}\right)$ are possible
- algorithm: UCRL + forgetting old data

# ALGORITHMS FOR MDPS WITH ADVERSARIAL REWARDS

# WHERE IT ALL STARTED…

## Experts in a Markov Decision Process

**Eyal Even-Dar**
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NeurIPS 2005

**MATHEMATICS OF OPERATIONS RESEARCH**
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ISSN 0364-765X | EISSN 1526-5471 | 09 | 3403 | 0726

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## Online Markov Decision Processes

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Math of OR 2009

# FORMAL PROTOCOL

**Online learning in a fixed MDP**

For each round $t = 1,2, \dots, T$

- Learner observes state $X_t \in \mathcal{X}$

- Learner takes action $A_t \in \mathcal{A}$
- Adversary selects reward function $r_t: \mathcal{X} \times \mathcal{A} \to [0,1]$
- Learner earns reward $R_t = r_t(X_t, A_t)$
- Learner observes feedback
  - Full information: $r_t$
  - Bandit feedback: $R_t$
- Environment produces new state $X_{t+1} \sim P(\cdot \mid X_t, A_t)$

# FORMAL PROTOCOL

**Online learning in a fixed MDP**

For each round $t = 1,2,\dots,T$

- Learner observes state $X_t \in \mathcal{X}$
- Learner selects stochastic policy $\pi_t$
- Learner takes action $A_t \sim \pi_t(\cdot|X_t)$
- Adversary selects reward function $r_t : \mathcal{X} \times \mathcal{A} \to [0,1]$
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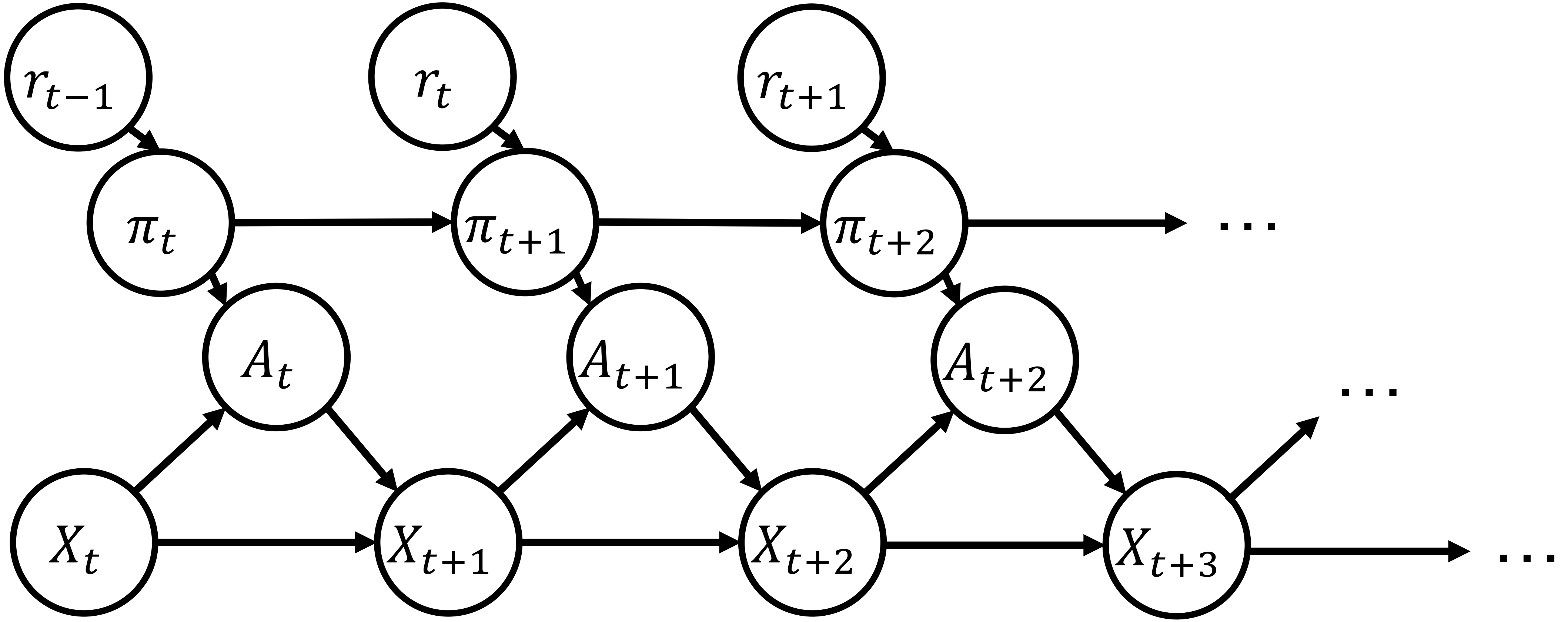
Stochastic policy: $\pi(a|x) = \mathbb{P}[A_t = a | X_t = x]$

# TEMPORAL DEPENDENCES

**Main challenge:** dependence between consecutive time steps




NB this graph is accurate for full information feedback; bandit is a bit more complicated

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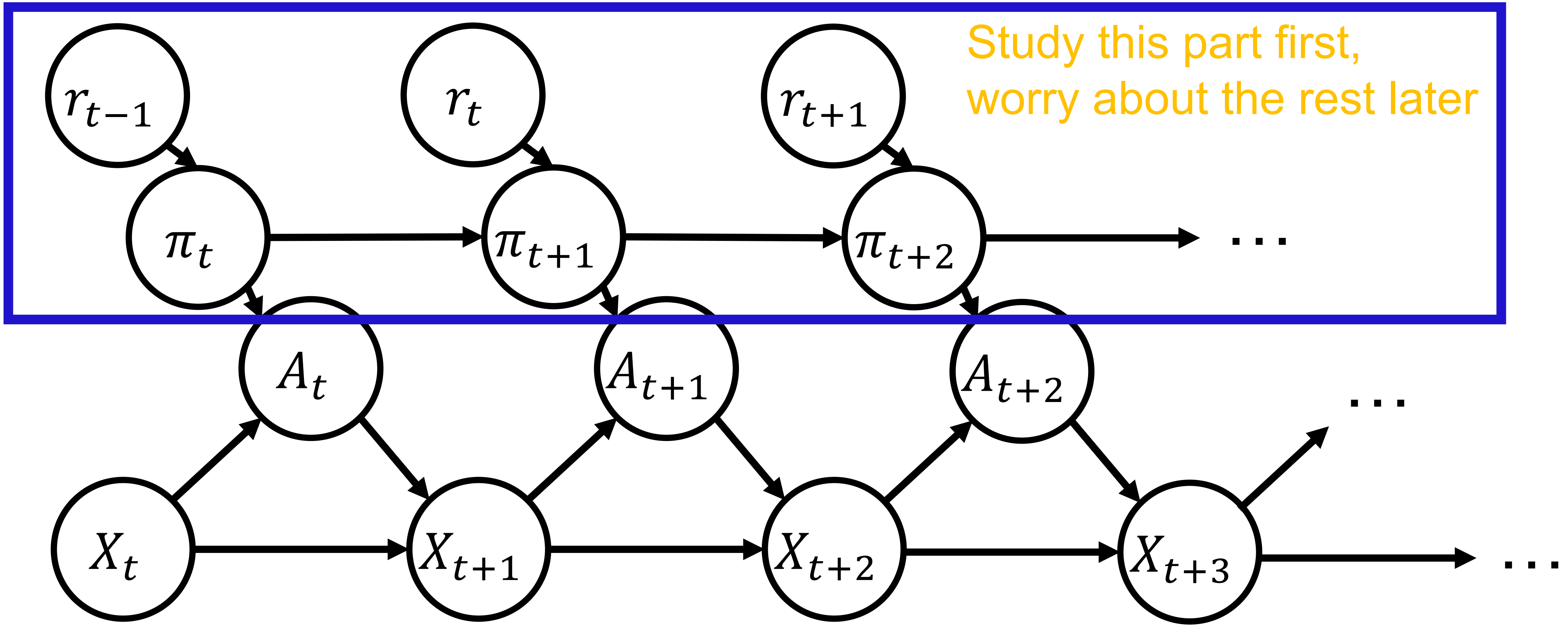


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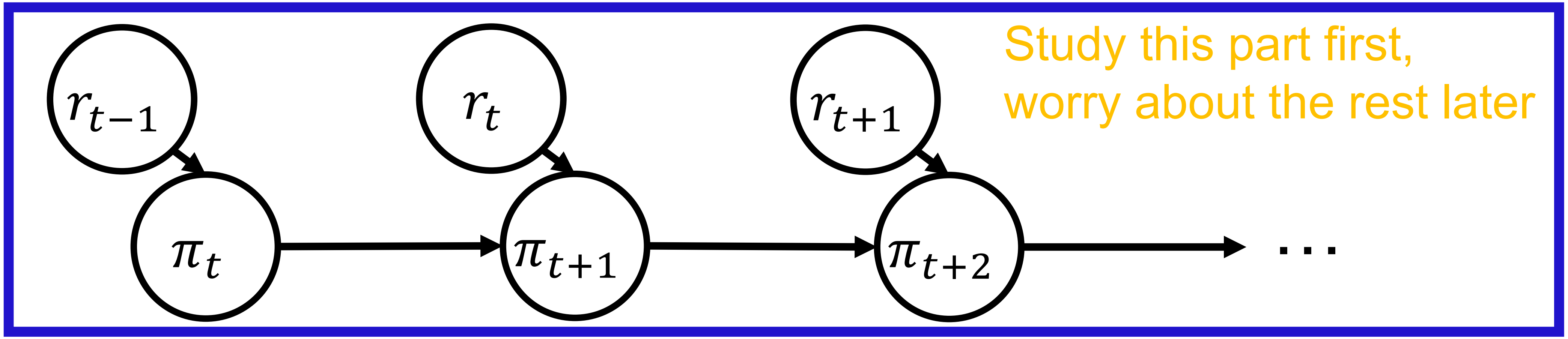


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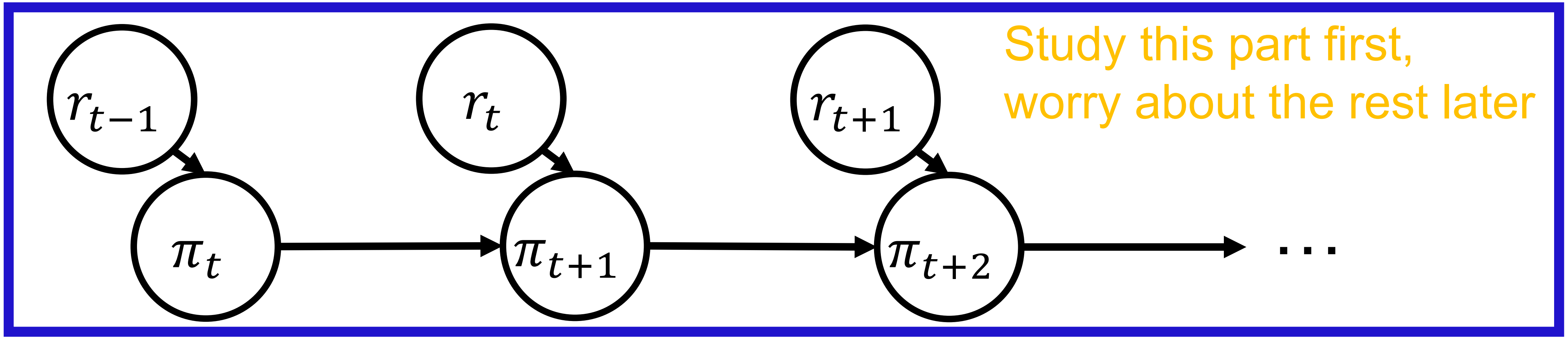


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**Def:** stationary distribution of policy $\pi$:

$$\mu_\pi(x,a) = \lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} \mathbb{I}_{\{X_k=x, A_k=a\}}$$

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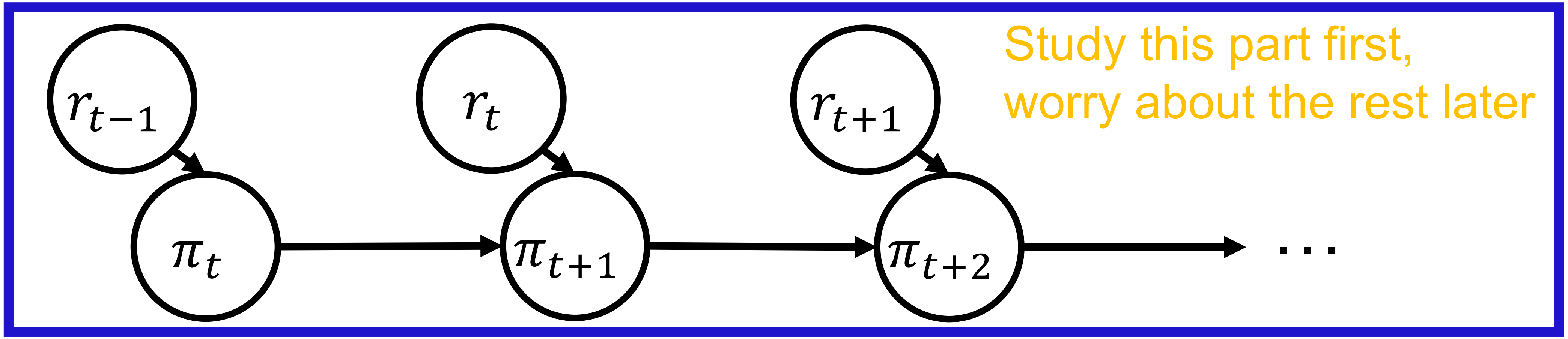


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**Assumption:** 1-step mixing $\forall \pi$

$$\|(\nu - \nu')P_\pi\|_1 \leq e^{1/\tau} \|\nu - \nu'\|_1$$

# REGRET DECOMPOSITION

- Define
  $\nu_t(x,a) = \mathbb{P}[X_t = x, A_t = a]$ and $\nu_t^*(x,a) = \mathbb{P}[X_t^* = x, A_t^* = a]$
  $\mu_t = \mu_{\pi_t}$, stationary distribution induced by policy $\pi_t$
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$$\mathfrak{Reg}_T(\pi^*) = \sum_{t=1}^{T} \mathbb{E}\left[r_t\left(X_t^*, \pi^*(X_t^*)\right) - r_t(X_t, A_t)\right] = \sum_{t=1}^{T} \langle \nu_t^* - \nu_t, r_t \rangle$$

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$$= \sum_{t=1}^{T} \langle \nu_t^* - \mu^*, r_t \rangle + \sum_{t=1}^{T} \langle \mu^* - \mu_t, r_t \rangle + \sum_{t=1}^{T} \langle \mu_t - \nu_t, r_t \rangle$$

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$$= \sum_{t=1}^{T} \langle \nu_t^* - \mu^*, r_t \rangle + \underbrace{\sum_{t=1}^{T} \langle \mu^* - \mu_t, r_t \rangle}_{\text{“stationarized regret”}} + \sum_{t=1}^{T} \langle \mu_t - \nu_t, r_t \rangle$$

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$$= \underbrace{\sum_{t=1}^{T} \langle \nu_t^* - \mu^*, r_t \rangle}_{\text{“comparator drift”}} + \underbrace{\sum_{t=1}^{T} \langle \mu^* - \mu_t, r_t \rangle}_{\text{“stationarized regret”}} + \underbrace{\sum_{t=1}^{T} \langle \mu_t - \nu_t, r_t \rangle}_{\text{“learner drift”}}$$

# THE DRIFT TERMS

- For the comparator, fast mixing is guaranteed by assumption:

$$\sum_{t=1}^{T} \langle \nu_t^* - \mu^*, r_t \rangle \leq \sum_{t=1}^{T} \|\nu_t^* - \mu^*\|_1 \leq \sum_{t=1}^{T} e^{-t/\tau} \|\nu_1^* - \mu^*\|_1 \leq 2\tau + 2$$

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- The other term is small if the policies change slowly:

**Lemma**

If $\max_x \|\pi_t(\cdot\,|x) - \pi_{t-1}(\cdot\,|x)\|_1 \leq \varepsilon$ for all $t$, then

$$\sum_{t=1}^{T} \|\mu_t - \nu_t\|_1 \leq (\tau+1)^2 \varepsilon T + 2e^{-T/\tau}$$

"$\nu_t$ tracks $\mu_t$ if policies change slowly"

# ALGORITHMS FOR MDPS WITH ADVERSARIAL REWARDS

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| Local-to-global regret decomposition | Reduction to online linear optimization |

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# LOCAL-TO-GLOBAL REGRET DECOMPOSITION

- Idea by Even-Dar, Kakade and Mansour (2005,2009) based on the performance difference lemma:

**Lemma**

Let $\pi, \pi'$ be two arbitrary policies, $r$ a reward function and $Q^\pi$ and $V^\pi$ be the value functions corresponding to $r$ and $\pi$. Then,

$$\langle \mu_{\pi'} - \mu_\pi, r \rangle = \langle \mu_{\pi'}, Q^\pi - V^\pi \rangle$$

# LOCAL-TO-GLOBAL REGRET DECOMPOSITION

Apply with $r = r_t$, $\pi = \pi_t$ and $\pi' = \pi^*$:

$$\langle \mu^* - \mu_t, r \rangle = \sum_x \mu^*(x) \sum_a \big(\pi^*(a|x) - \pi_t(a|x)\big) Q_t(x, a)$$

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Q-function of $\pi_t$ with reward function $r_t$

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Stationarized regret can be written as:

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# LOCAL-TO-GLOBAL REGRET DECOMPOSITION

Q-function of $\pi_t$ with reward function $r_t$

Apply with $r = r_t$, $\pi = \pi_t$ and $\pi' = \pi^*$:

$$\langle \mu^* - \mu_t, r \rangle = \sum_x \mu^*(x) \sum_a \big(\pi^*(a|x) - \pi_t(a|x)\big) Q_t(x,a)$$

Stationarized regret can be written as:

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Local regret in state $x$ with reward function $Q_t(x,\cdot)$

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**Algorithm idea:**
run a local regret-minimization algorithm in each state $x$ with reward function $Q_t(x,\cdot)$!

Local regret in state $x$ with reward function $Q_t(x,\cdot)$

# THE MDP-EXPERT ALGORITHM

MDP-E

For each round $t = 1,2, \dots, T$

- Observe state $X_t$
- Take action $A_t \sim \pi_t(\cdot\,|X_t)$
- Observe reward function $r_t$
- Calculate value functions as solution to $Q_t(x,a) = r_t - \langle \mu_t, r_t \rangle + \sum_{x'} P(x'|x,a) V_t(x')$
- For all $x$, feed $Q_t(x,\cdot)$ to expert algorithm $\mathfrak{Alg}(x)$

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- For all $x$, feed $Q_t(x,\cdot)$ to expert algorithm $\mathfrak{Alg}(x)$
- **Example:** $\mathfrak{Alg}$ = Exponential weights

$$\pi_{t+1}(a|x) \propto \pi_t(a|x) \cdot e^{\eta Q_t(x,a)}$$

# GUARANTEES FOR MDP-E

**Theorem**

(Even-Dar et al., 2009, Neu et al., 2014)

If $\mathfrak{Alg}(x)$ guarantees a regret bound of $B_T$ for rewards bounded in $[0,1]$, the stationarized regret of MDP-E satisfies

$$\sum_{t=1}^{T} \langle \mu^* - \mu_t, r \rangle \leq \tau B_T$$

**Proof** is obvious given the regret decomposition.

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**Theorem**

If $\mathfrak{Alg}(x)$=EWA, the regret of MDP-E satisfies

$$\mathfrak{Reg}_T = O\left(\sqrt{\tau^3 T \log|\mathcal{A}|}\right)$$

**Proof** is obvious given the regret decomposition.

# BANDIT FEEDBACK

Addressed in Neu, György, Szepesvári and Antos (2010,2014): replace $r_t$ by an unbiased estimator

$$\hat{r}_t(x,a) = \frac{r_t(x,a)}{\mu_t^N(x,a)} \mathbb{I}\{(X_t, A_t) = (x,a)\},$$

with $\mu_t^N(x,a) = \mathbb{P}[(X_t, A_t) = (x,a) | \mathcal{H}_{t-N}]$

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**Theorem**

If $\mathfrak{Alg}(x)$=EWA, the regret of MDP-Exp3 satisfies

$$\mathfrak{Reg}_T = O\left(\sqrt{\tau^3 T |\mathcal{A}| \log|\mathcal{A}| / \beta}\right)$$

**Assumption:** $\mu_\pi(x) \geq \beta$ for all $\pi, x$

# ALGORITHMS FOR MDPS WITH ADVERSARIAL REWARDS

| | |
|---|---|
| Local-to-global regret decomposition | Reduction to online linear optimization |

# ONLINE LINEAR OPTIMIZATION

**Notice:** stationarized regret = regret in an OLO problem!

$$\sum_{t=1}^{T} \langle \mu^* - \mu_t, r_t \rangle$$

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**Algorithm idea:**

run an OLO algorithm with the set of all stationary distributions as decision set!

$$\mathcal{U} = \left\{ \mu \in \Delta_{\mathcal{X}\times\mathcal{A}} : \sum_{a} \mu(x,a) = \sum_{x',a'} P(x|x',a')\mu(x',a') \right\}$$

# ONLINE MIRROR DESCENT

- In each round, update stationary distribution

$$\mu_{t+1} = \arg\max_{\mu \in \mathcal{U}} \left\{ \langle \mu, r_t \rangle - \frac{1}{\eta} D(\mu | \mu_t) \right\}$$

and extract policy $\pi_{t+1}(a|x) \propto \mu_{t+1}(x, a)$

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and extract policy $\pi_{t+1}(a|x) \propto \mu_{t+1}(x, a)$
- Choosing the regularizer:
  - Relative entropy: $D(\mu|\nu) = \sum_{x,a} \mu(x, a) \log \frac{\mu(x,a)}{\nu(x,a)}$
  
    ⇒ "Online Relative Entropy Policy Search" (Zimin and Neu, 2013, Dick, György and Szepesvári, 2014)
  - Conditional relative entropy: $D(\mu|\nu) = \sum_{x,a} \mu(x, a) \log \frac{\pi_\mu(a|x)}{\pi_\nu(a|x)}$
  
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# THE ONLINE REPS ALGORITHM

**O-REPS**

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$$\min_V \log \sum_{x,a} \mu_t(x,a) e^{\eta\left(r_t(x,a)+\sum_{x'} P(x'|x,a)V(x')-V(x)\right)}$$

- Update stationary distribution as

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Algorithm inspired by Peters, Mülling and Altün (2010)

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Unconstrained convex minimization

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Algorithm inspired by Peters, Mülling and Altün (2010)

# GUARANTEES FOR O-REPS

## Theorem

(Zimin and Neu, 2013, Dick et al. 2014)

The stationarized regret of O-REPS satisfies

$$\sum_{t=1}^{T} \langle \mu^* - \mu_t, r \rangle \leq \sqrt{T \log|\mathcal{X}||\mathcal{A}|}$$

## Theorem

The regret of O-REPS satisfies

$$\mathfrak{Reg}_T = O\left(\sqrt{\tau T \log|\mathcal{X}||\mathcal{A}|}\right)$$

**Proof** is based on standard OLO analysis.

# BANDIT FEEDBACK

Addressed in Zimin and Neu (2013) in **episodic MDPs**:
replace $r_t$ by an unbiased estimator

$$\hat{r}_t(x,a) = \frac{r_t(x,a)}{q_t(x,a)} \mathbb{I}\{(x,a) \text{ visited in episode } t\},$$

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**Theorem**

If $\mathfrak{Alg}(x)$=EWA, the regret of MDP-Exp3 satisfies

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| Local-to-global regret decomposition | Reduction to online linear optimization |
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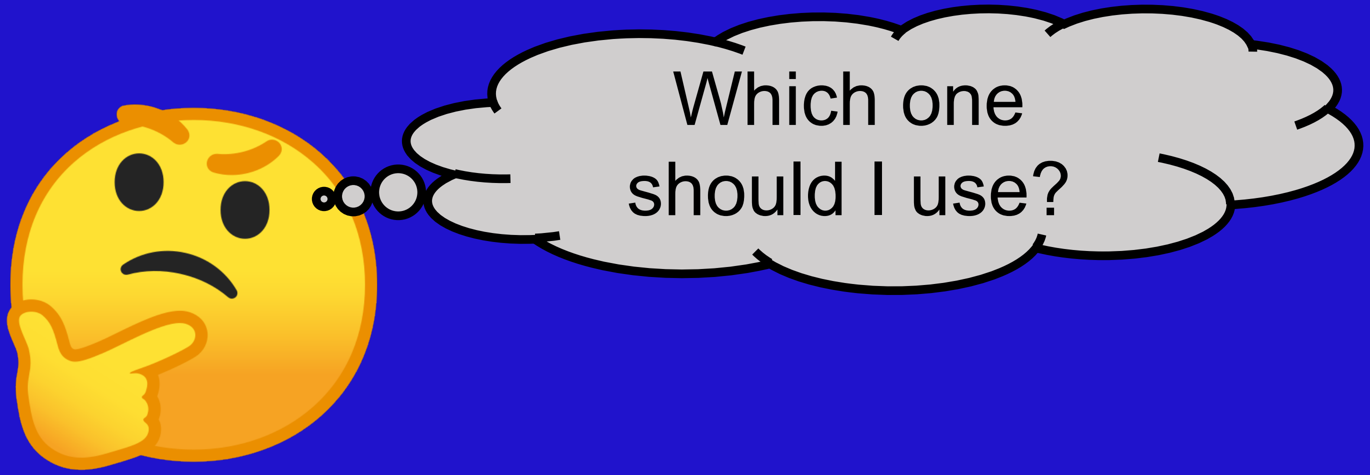


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# COMPARISON OF GUARANTEES

| | MDP-E | O-REPS |
|---|---|---|
| Full information | $\sqrt{\tau^3 T \log\lvert\mathcal{A}\rvert}$ | $\sqrt{\tau T \log\lvert\mathcal{X}\rvert\lvert\mathcal{A}\rvert}$ |
| Bandit feedback | $\sqrt{\tau^3 \lvert\mathcal{A}\rvert T \log\lvert\mathcal{A}\rvert / \beta}$ | ??? |
| Full information (episodic case) | $H^2\sqrt{T \log\lvert\mathcal{A}\rvert}$ | $H\sqrt{T \log\lvert\mathcal{X}\rvert\lvert\mathcal{A}\rvert}$ |
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+ MDP-E works well with function approximation for Q-function

+ O-REPS can easily handle model constraints and extensions

# MDP-E WITH FUNCTION APPROXIMATION

MDP-E only needs a good approximation of the action-value function $\hat{Q}_t \approx Q^{\pi_t}$ to define its policy

$$\pi_{t+1}(a|x) \propto \exp\left(\eta \sum_{k=1}^{t} \hat{Q}_k(x,a)\right)$$

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- POLITEX (Abbasi-Yadkori et al., 2019):
  use LSPE to estimate $Q^{\pi_t}$ with linear FA
  regret = $O(T^{3/4} + \varepsilon_0 T)$
- OPPO (Cai et al., 2019)
  use LSPE to estimate $Q^{\pi_t}$ with **realizable** linear FA
  regret = $O(\sqrt{T})$

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MDP-E only needs a good approximation of the action-value function $\hat{Q}_t \approx Q^{\pi_t}$ to define its policy

$$\pi_{t+1}(a|x) \propto \exp\left(\eta \sum_{k=1}^{t} \hat{Q}_k(x,a)\right)$$

+ MDP-E is essentially identical to the "Trust-Region Policy Optimization" (TRPO) algorithm of Schulman et al. (2015), as shown by Neu, Jonsson and Gómez (2017)!!!

# O-REPS WITH UNCERTAIN MODELS

O-REPS can easily accommodate uncertainties in the transition model by extending the decision set:

$$\mathcal{U} = \left\{ \mu \in \Delta_{\mathcal{X}\times\mathcal{A}} : \sum_{a} \mu(x,a) = \sum_{x',a'} P(x|x',a')\mu(x',a') \right\}$$

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Confidence set of transition models

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Confidence set of transition models

UC-O-REPS by Rosenberg and Mansour (2019)

Extended to bandit feedback by Jin et al. (2020):

$$\hat{r}_t(x,a) = \frac{r_t(x,a)}{u_t(x,a)} \mathbb{I}\{(x,a) \text{ visited in episode } t\},$$

with $u_t(x,a) > q_t(x,a) = \mathbb{P}[(x,a) \text{ visited in episode } t | \mathcal{H}_{t-1}]$ w.h.p.

# OUTLOOK

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- Open problems:
  - Lower bounds? Right scaling with $\tau$? Is uniform mixing necessary?
  - Large state spaces and function approximation?
  - Practical algorithms?

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Relevance to practice of RL?

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- Open problems:
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  - Practical algorithms?

## Relevance to practice of RL?

- Online learning algorithms are robust! Main tool: regularization
- Better understanding of regularization tools $\Rightarrow$ better algorithms!
- Remember: TRPO = MDP-E!

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