STATS 701: THEORY OF RL

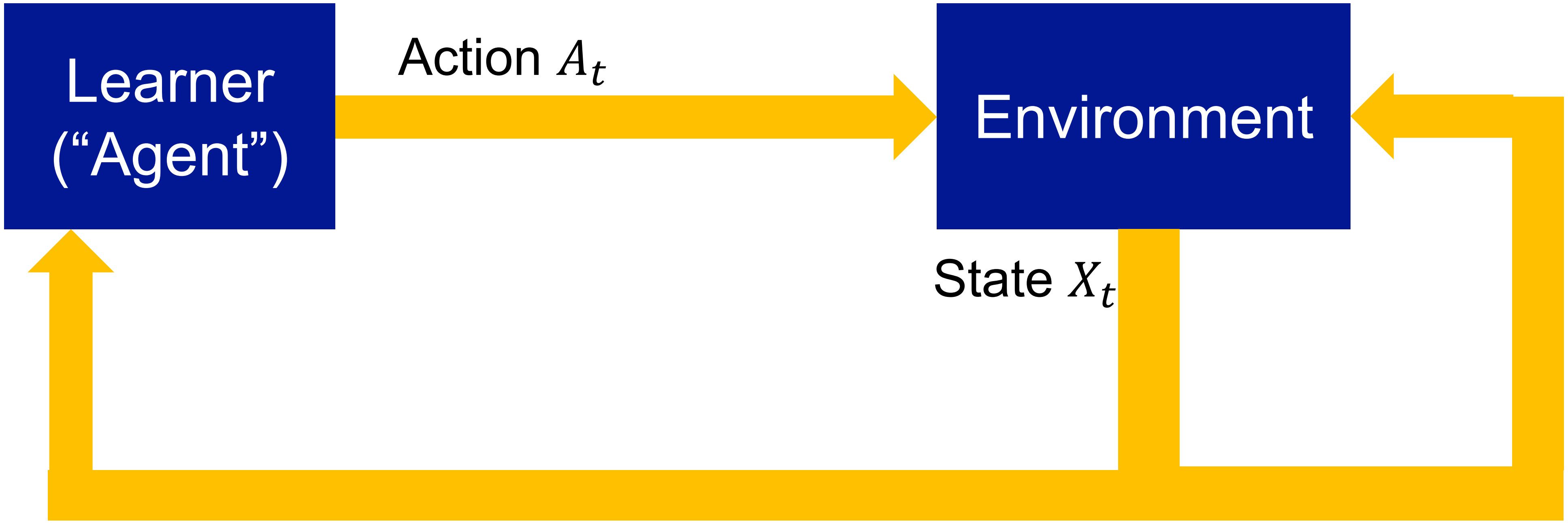
WINTER 2021

ADVERSARIAL MDPS

Ambuj Tewari

slide credits: Gergely Neu @ Universitat Pompeu Fabra, Barcelona

MARKOV DECISION PROCESSES



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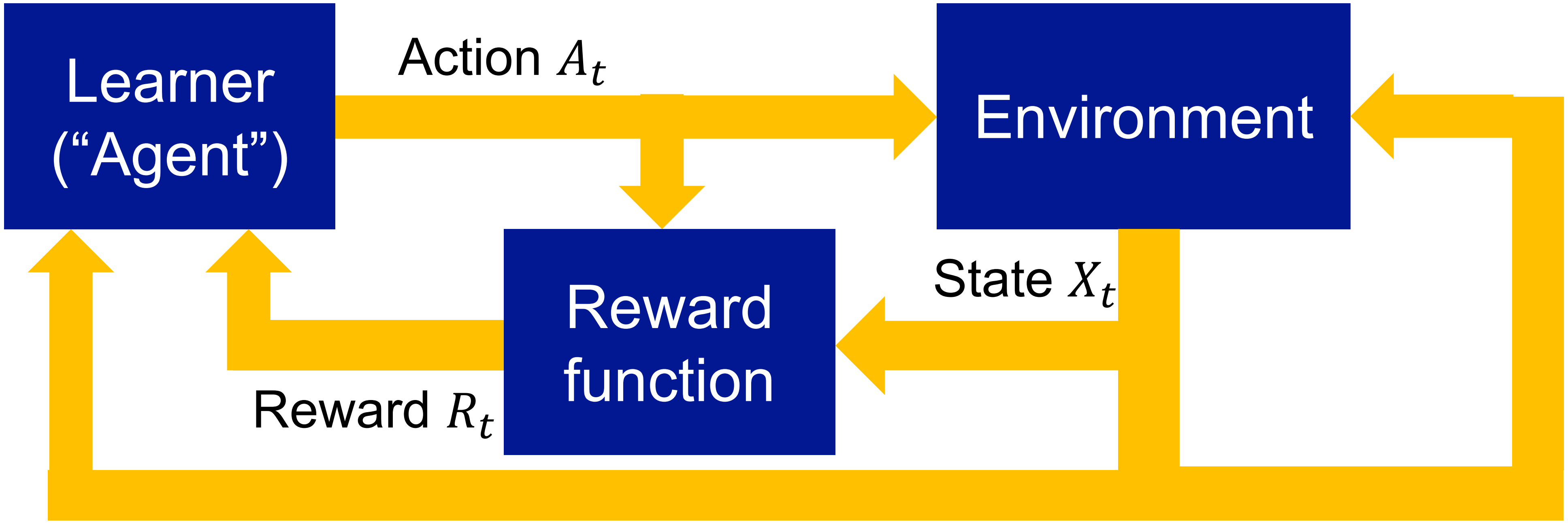

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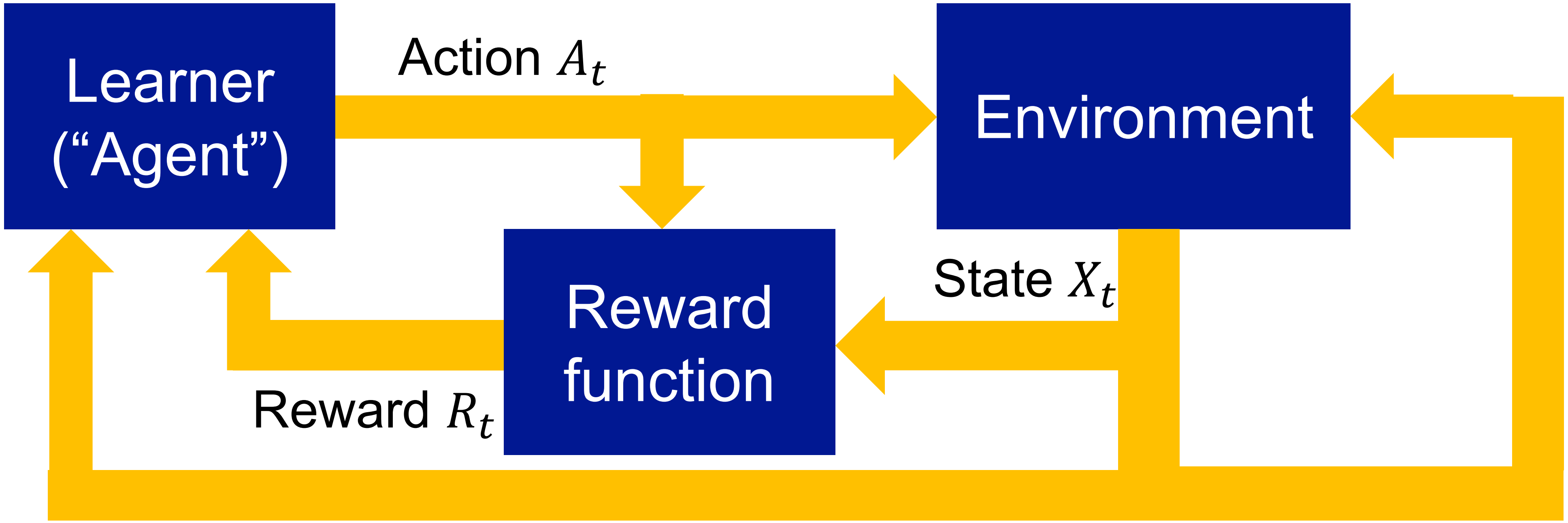
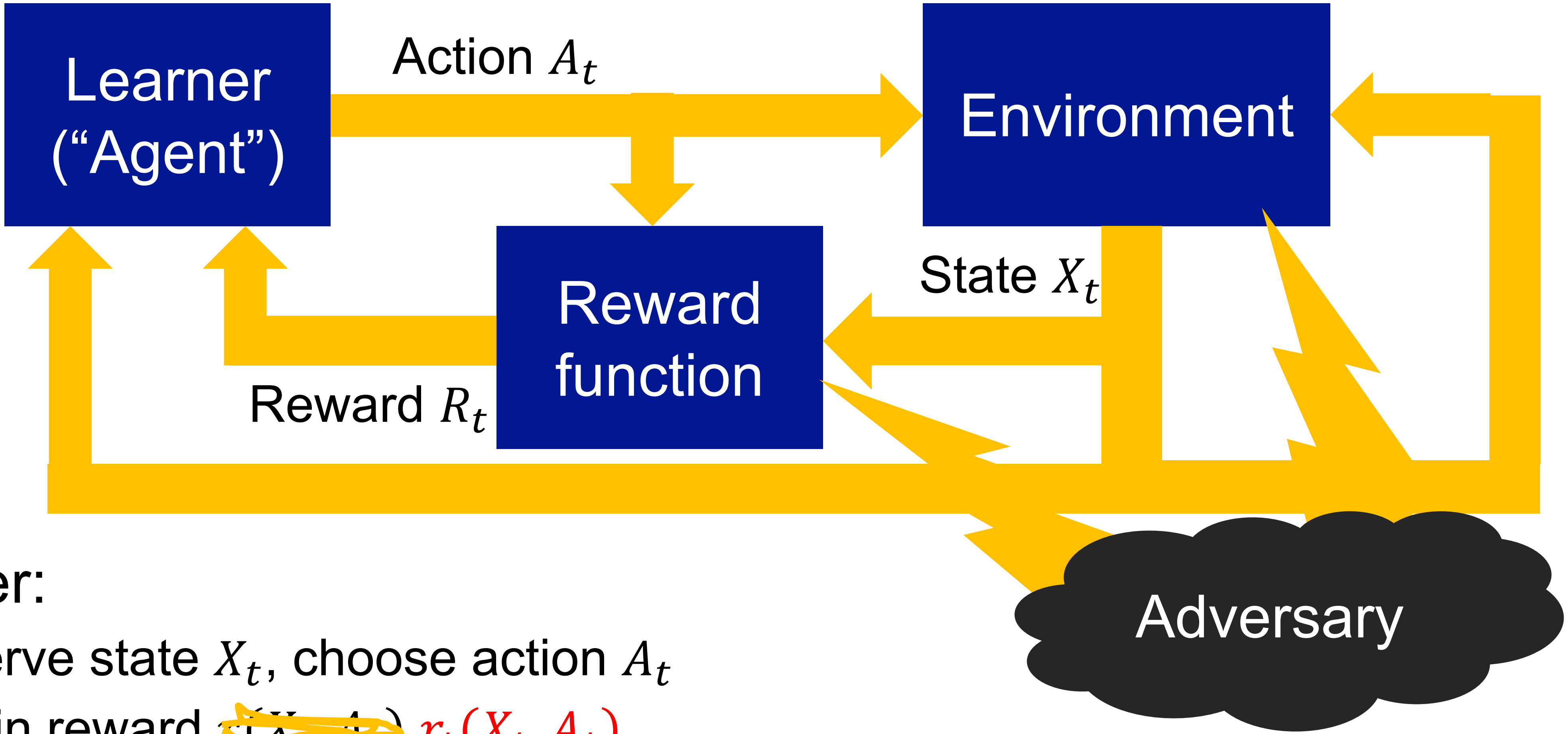


- Learner:
  - Observe state X_t, choose action A_t
  - Obtain reward $r(X_t, A_t)$
- Environment: Draw next state $X_{t+1} \sim P(\cdot\,|X_t, A_t)$

ADVERSARIAL MARKOV DECISION PROCESSES



- Learner:
  - Observe state X_t, choose action A_t
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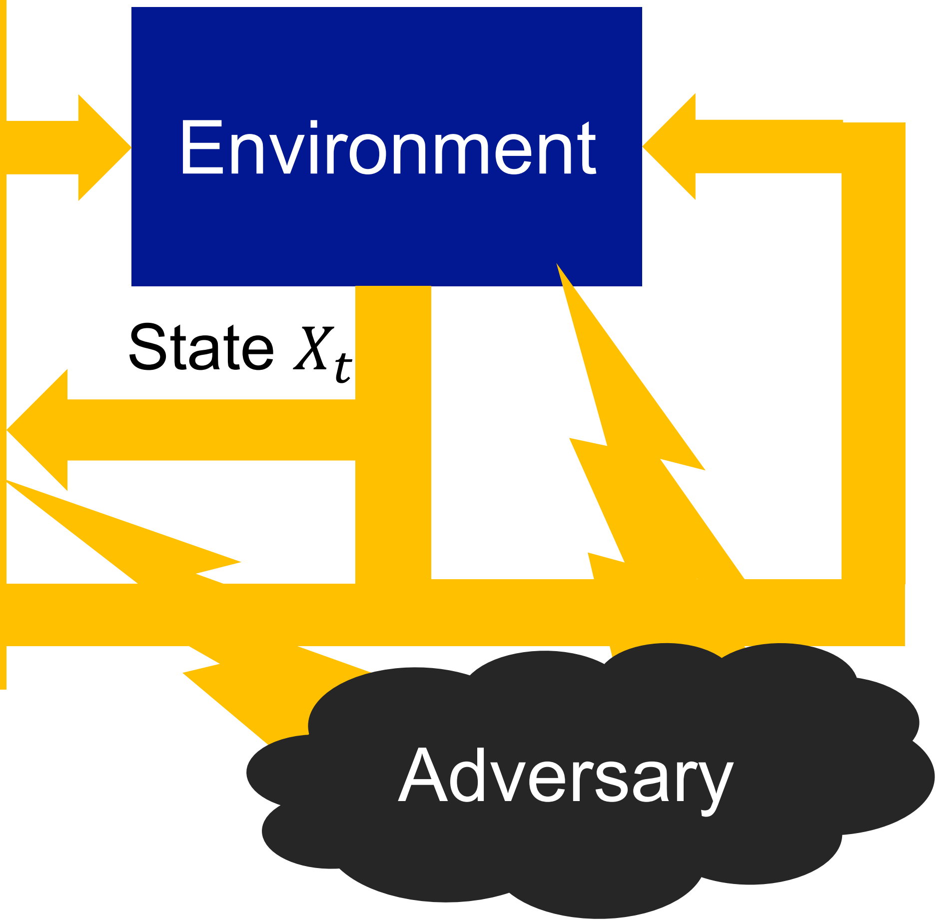
ADVERSARIAL
MARKOV DECISION PROCESSES

This lecture:
what is achievable when an external adversary is allowed to change the reward function and the transition function over time?



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PERFORMANCE MEASURE: REGRET

Regret

$$\mathfrak{Reg}_T(\pi) = \sum_{t=1}^{T} \mathbb{E}\left[r_t\left(X_t^*, \pi(X_t^*)\right) - r_t(X_t, A_t)\right],$$

where $X_1^*, X_2^*, \ldots$ is the sequence of states that would have been generated by running comparator policy π through the dynamics $P_1, P_2, \ldots$

PERFORMANCE MEASURE: REGRET

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Goal: sublinear regret

$$\lim_{T\to\infty} \max_{\pi} \frac{\mathfrak{Reg}_T(\pi)}{T} = 0$$

OUTLINE

- Hardness results
  - Non-oblivious adversaries
  - Arbitrarily changing dynamics
- Arbitrarily changing reward functions
  - Some common ideas
  - Two algorithm families

SOME HARDNESS RESULTS

NON-OBLIVIOUS ADVERSARIES

Non-oblivious adversary:
can take history $\mathcal{H}_t = X_t, A_{t-1}, X_{t-1}, A_{t-2}, \ldots$
into account when selecting r_t and P_t

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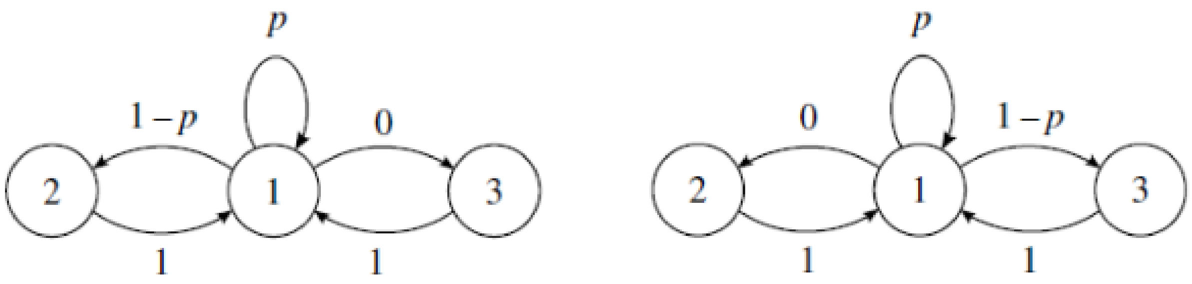
Theorem
(Yu, Mannor and Shimkin, 2009)
No algorithm can guarantee sublinear regret against a non-oblivious adversary

PROOF

Simple counterexample by Yu, Mannor and Shimkin (2009):



(a) Transition model if the agent chooses to go left.

(b) Transition model if the agent chooses to go right.

PROOF

Simple counterexample by Yu, Mannor and Shimkin (2009):

- Reward is function of state
- $r_t(\text{default}) = 0$
- $r_t(\text{left}) = 1$ if $A_{t-1} = \text{right}$
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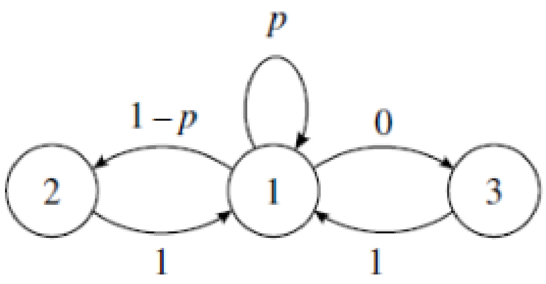

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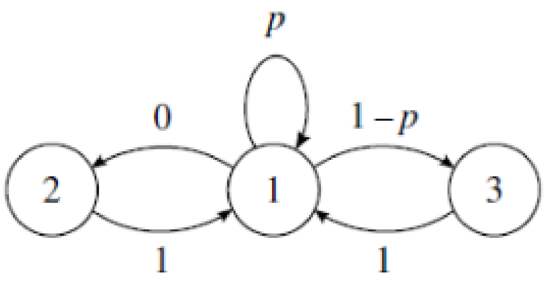

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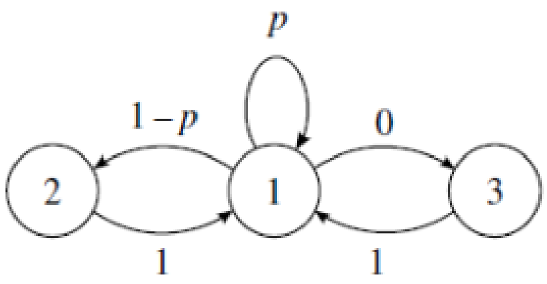

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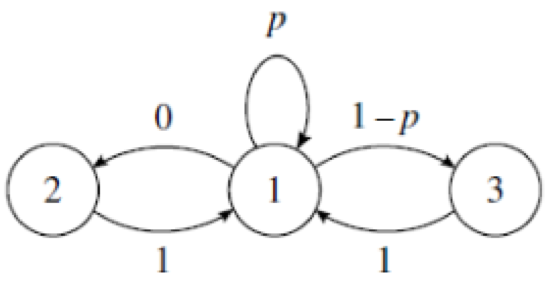

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$r_t(X_t) = 0$ for all t!

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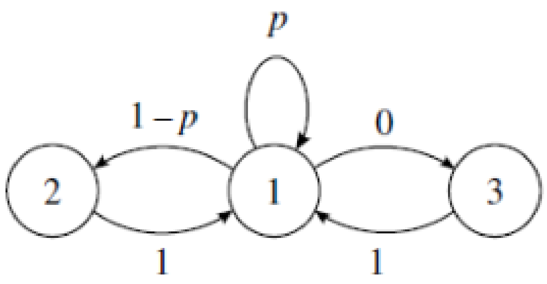

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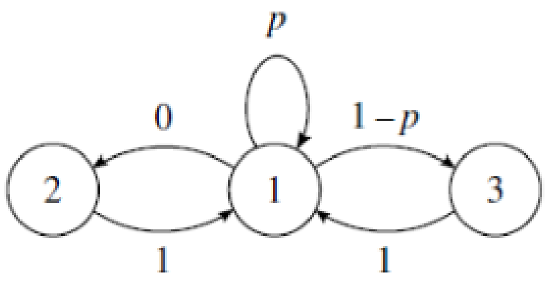

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But there is a policy π with $\mathbb{E}\left[\sum_t r_t\left(X_t^*, \pi(X_t^*)\right)\right] \geq \left(\frac{1}{2} - p\right) T$

Either $\pi(1) = \text{left}$ or $\pi(1) = \text{right}$

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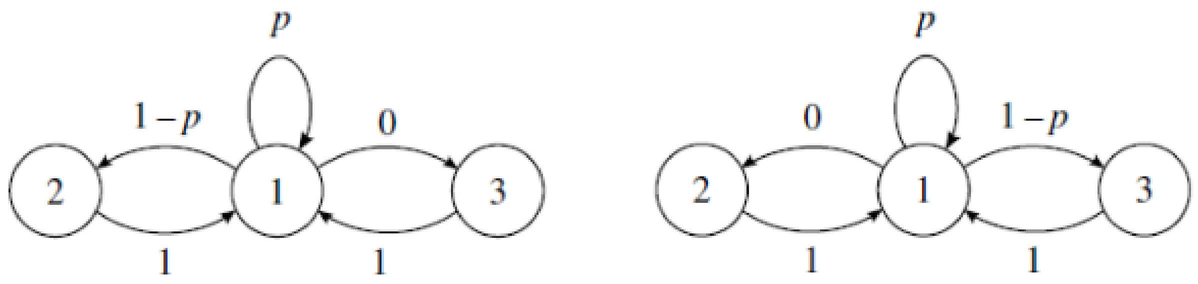

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$$\mathfrak{Reg}_T(\pi) \geq \left(\frac{1}{2} - p\right) T$$

OBLIVIOUS ADVERSARIES

Non-oblivious adversary:
can take history $\mathcal{H}_t = X_t, A_{t-1}, X_{t-1}, A_{t-2}, \ldots$
into account when selecting r_t and P_t

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Oblivious adversary:
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"Adversary $\approx$ nature":
it can (mis)behave arbitrarily, but doesn't care about what you do

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Can we guarantee sublinear regret now?

LEARNING WITH CHANGING TRANSITIONS IS HARD

Learning against an oblivious adversary can still be computationally hard when the transition function is allowed to change!

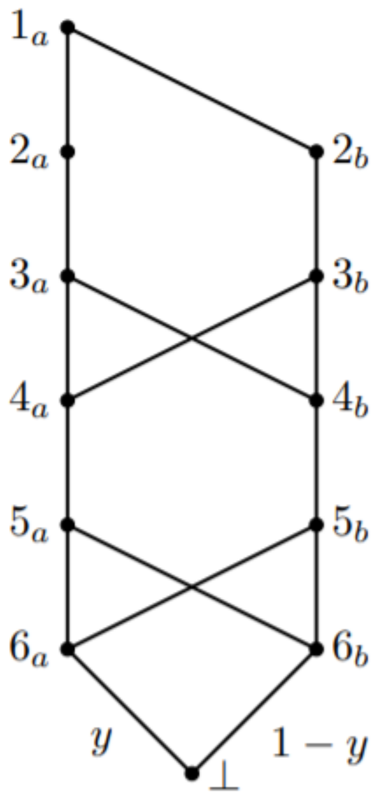
Theorem

(Abbasi-Yadkori et al., 2013)

There is an adversarial MDP where achieving sublinear regret is computationally hard.

PROOF CONSTRUCTION

- **Idea:** learning of noisy parities can be formulated as an MDP with changing transition functions & rewards!
- $O(\text{poly}(n)T^{1-\alpha})$ regret $\Rightarrow O\left(\frac{poly(n)}{\varepsilon^{1/\alpha}}\right)$ excess risk, conjectured to be computationally hard to achieve
- Construction: an instance $x \in \{0,1\}^n$ corresponds to a deterministic transition graph with rewards determined by the label y



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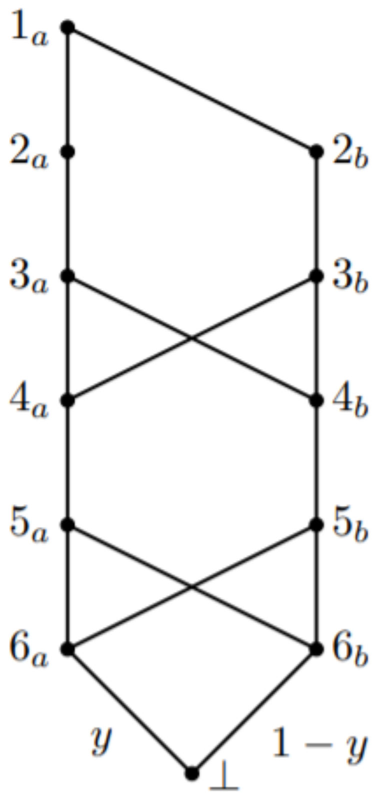
Corresponds to an oblivious adversary that picks (P_t, r_t) jointly!



SLOWLY CHANGING MDPS

Very recent work by Gajane et al. (2019), Cheung et al. (2020):

- define reward and transition variation as

$$V_T^r = \sum_{t=1}^{T} \max_{x,a} |r_t(x,a) - r_{t+1}(x,a)|$$

$$V_T^P = \sum_{t=1}^{T} \max_{x,a} \|P_t(\cdot\,|x,a) - P_{t+1}(\cdot\,|x,a)\|_1$$

- regret bounds of $O\left((V_T^P + V_T^r)^{1/3} T^{2/3}\right)$ are possible
- algorithm: UCRL + forgetting old data

ALGORITHMS FOR MDPS WITH ADVERSARIAL REWARDS

WHERE IT ALL STARTED…

Experts in a Markov Decision Process

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Online Markov Decision Processes

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Math of OR 2009

FORMAL PROTOCOL

Online learning in a fixed MDP

For each round $t = 1,2, \ldots, T$

- Learner observes state $X_t \in \mathcal{X}$
- Learner takes action $A_t \in \mathcal{A}$
- Adversary selects reward function $r_t : \mathcal{X} \times \mathcal{A} \to [0,1]$
- Learner earns reward $R_t = r_t(X_t, A_t)$
- Learner observes feedback
  - Full information: r_t
  - Bandit feedback: R_t
- Environment produces new state $X_{t+1} \sim P(\cdot \mid X_t, A_t)$

FORMAL PROTOCOL

Online learning in a fixed MDP

For each round $t = 1,2,\dots,T$

- Learner observes state $X_t \in \mathcal{X}$
- Learner selects stochastic policy π_t
- Learner takes action $A_t \sim \pi_t(\cdot|X_t)$
- Adversary selects reward function $r_t: \mathcal{X} \times \mathcal{A} \to [0,1]$
- Learner earns reward $R_t = r_t(X_t, A_t)$
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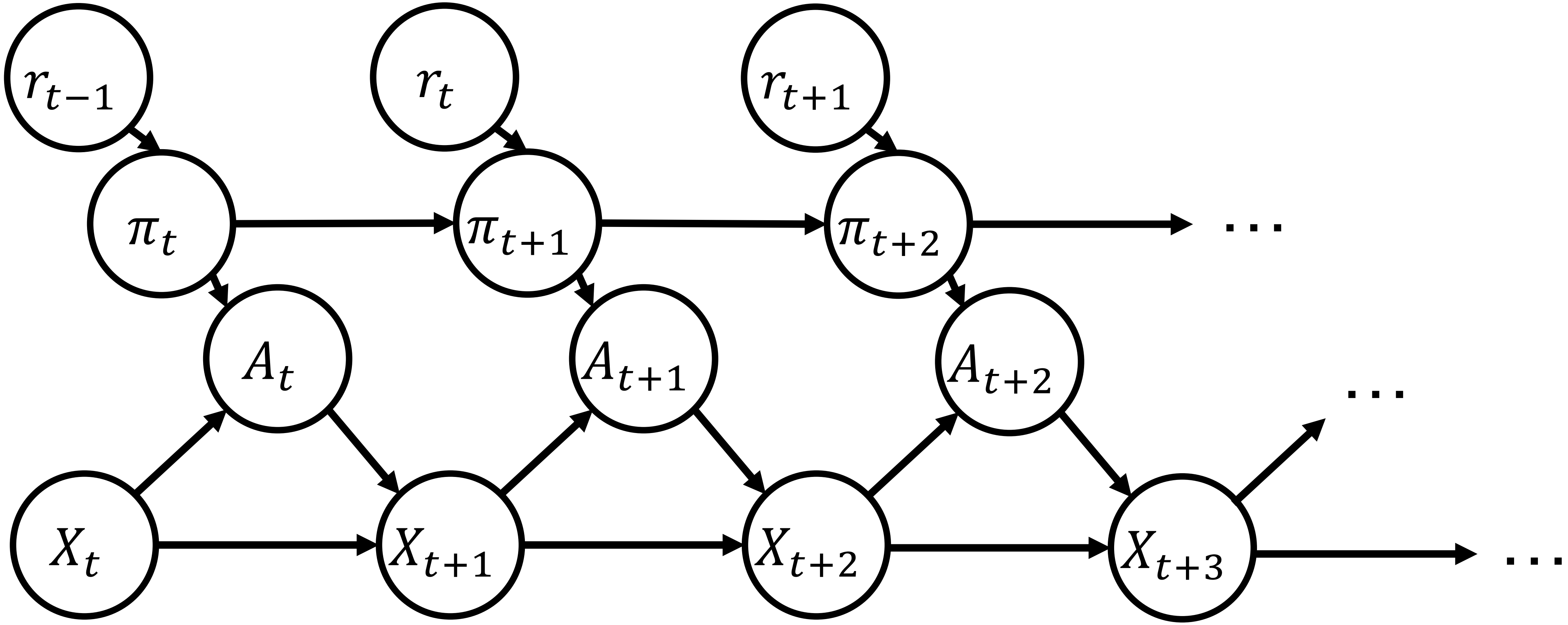
Stochastic policy: $\pi(a|x) = \mathbb{P}[A_t = a | X_t = x]$

TEMPORAL DEPENDENCES

Main challenge: dependence between consecutive time steps



NB this graph is accurate for full information feedback; bandit is a bit more complicated

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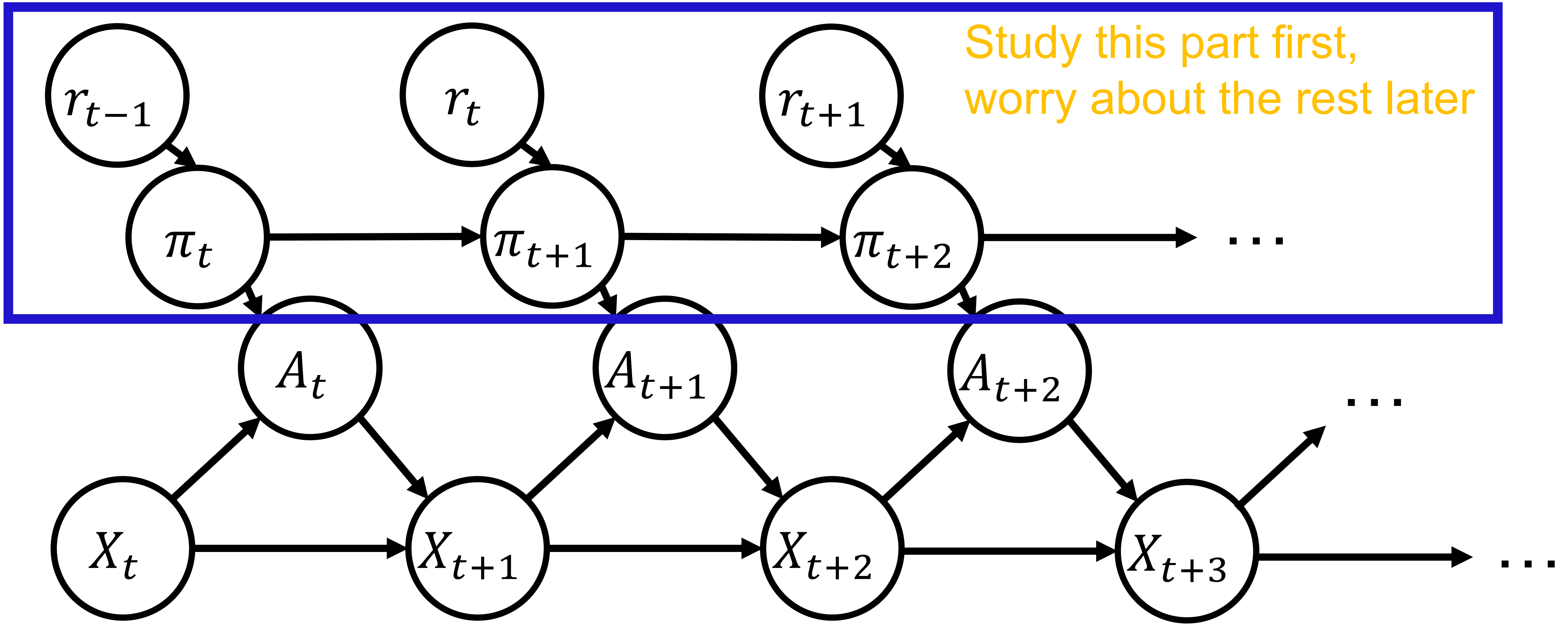

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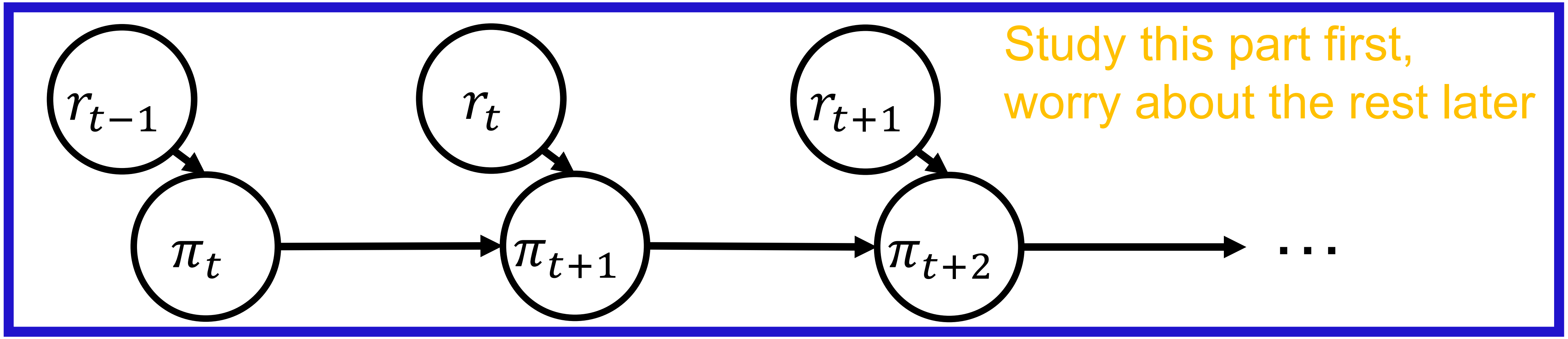

“Pretend that every policy reaches its stationary distribution immediately!”

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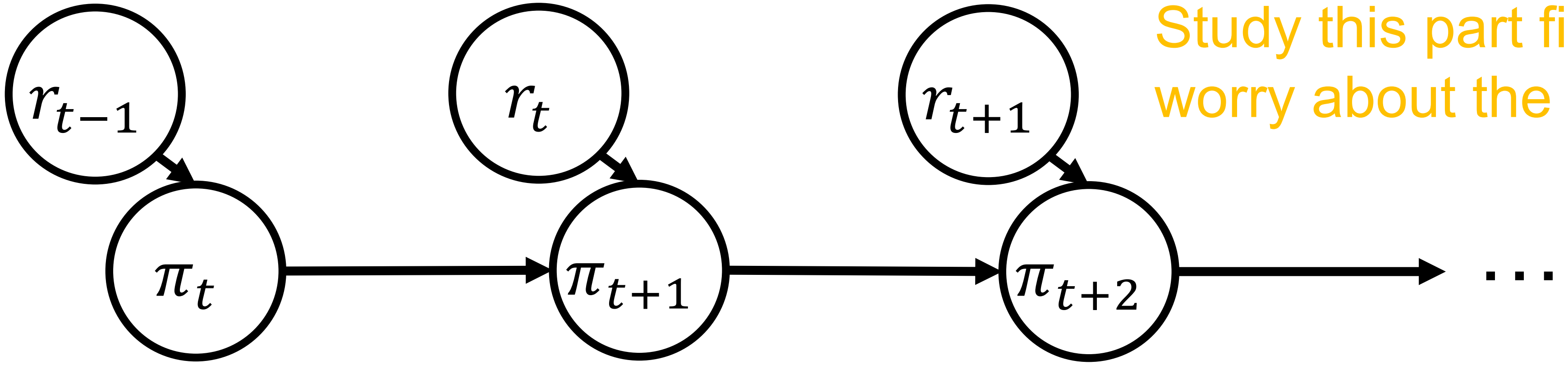

Study this part first,
worry about the rest later

"Pretend that every policy reaches its stationary distribution immediately!"

Def: stationary distribution of policy π:

$$\mu_\pi(x,a) = \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \mathbb{I}_{\{X_k = x, A_k = a\}}$$

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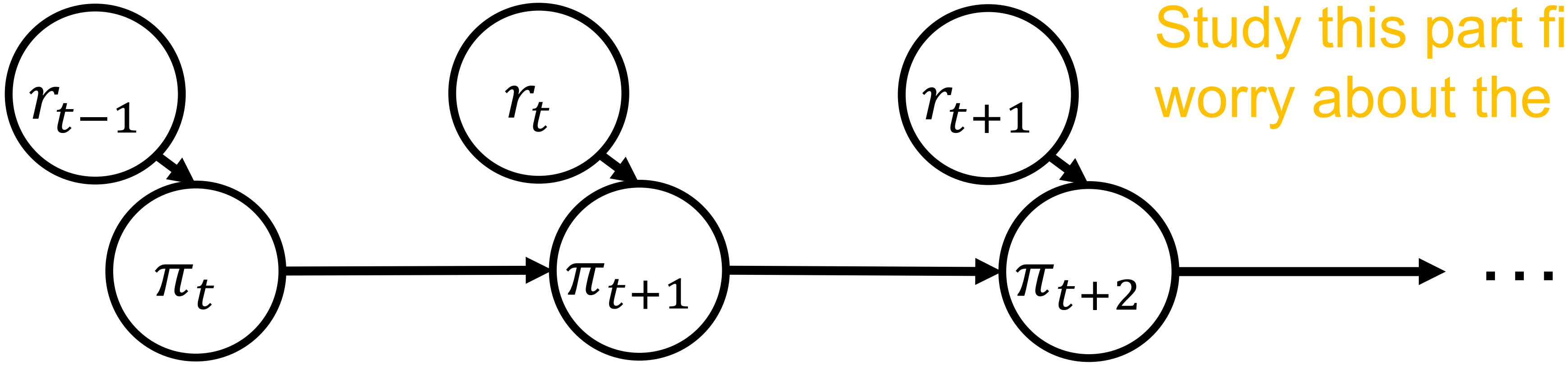

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Assumption: 1-step mixing $\forall\pi$

$$\|(\nu-\nu')P_\pi\|_1 \le e^{1/\tau}\|\nu-\nu'\|_1$$

REGRET DECOMPOSITION

- Define
  $\nu_t(x,a) = \mathbb{P}[X_t = x, A_t = a]$ and $\nu_t^*(x,a) = \mathbb{P}[X_t^* = x, A_t^* = a]$
  $\mu_t = \mu_{\pi_t}$, stationary distribution induced by policy π_t
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$$\mathfrak{Reg}_T(\pi^*) = \sum_{t=1}^{T} \mathbb{E}\left[r_t\left(X_t^*, \pi^*(X_t^*)\right) - r_t(X_t, A_t)\right] = \sum_{t=1}^{T} \langle \nu_t^* - \nu_t, r_t \rangle$$

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$$= \sum_{t=1}^{T} \langle \nu_t^* - \mu^*, r_t \rangle + \sum_{t=1}^{T} \langle \mu^* - \mu_t, r_t \rangle + \sum_{t=1}^{T} \langle \mu_t - \nu_t, r_t \rangle$$

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$$= \sum_{t=1}^{T} \langle \nu_t^* - \mu^*, r_t \rangle + \underbrace{\sum_{t=1}^{T} \langle \mu^* - \mu_t, r_t \rangle}_{\text{“stationarized regret”}} + \sum_{t=1}^{T} \langle \mu_t - \nu_t, r_t \rangle$$

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$$= \underbrace{\sum_{t=1}^{T} \langle \nu_t^* - \mu^*, r_t \rangle}_{\text{“comparator drift”}} + \underbrace{\sum_{t=1}^{T} \langle \mu^* - \mu_t, r_t \rangle}_{\text{“stationarized regret”}} + \underbrace{\sum_{t=1}^{T} \langle \mu_t - \nu_t, r_t \rangle}_{\text{“learner drift”}}$$

THE DRIFT TERMS

- For the comparator, fast mixing is guaranteed by assumption:

$$\sum_{t=1}^{T} \langle \nu_t^* - \mu^*, r_t \rangle \leq \sum_{t=1}^{T} \|\nu_t^* - \mu^*\|_1 \leq \sum_{t=1}^{T} e^{-t/\tau} \|\nu_1^* - \mu^*\|_1 \leq 2\tau + 2$$

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- The other term is small if the policies change slowly:

Lemma

If $\max_x \|\pi_t(\cdot\,|x) - \pi_{t-1}(\cdot\,|x)\|_1 \leq \varepsilon$ for all t, then

$$\sum_{t=1}^{T} \|\mu_t - \nu_t\|_1 \leq (\tau + 1)^2 \varepsilon T + 2e^{-T/\tau}$$

"ν_t tracks μ_t if policies change slowly"

ALGORITHMS FOR MDPS WITH ADVERSARIAL REWARDS

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Local-to-global regret decomposition	Reduction to online linear optimization

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Local-to-global regret decomposition

Reduction to online linear optimization

LOCAL-TO-GLOBAL REGRET DECOMPOSITION

- Idea by Even-Dar, Kakade and Mansour (2005,2009) based on the performance difference lemma:

Lemma

Let π, π' be two arbitrary policies, r a reward function and Q^π be the (differential) value functions corresponding to π. Then,

$$\langle \mu_{\pi'} - \mu_\pi, r \rangle = \sum_x \mu_{\pi'}(x) \sum_a \big(\pi'(a|x) - \pi(a|x)\big) Q_\pi(x, a)$$

LOCAL-TO-GLOBAL REGRET DECOMPOSITION

Apply with $r = r_t$, $\pi = \pi_t$ and $\pi' = \pi^*$:

$$\langle \mu^* - \mu_t, r \rangle = \sum_x \mu^*(x) \sum_a \big(\pi^*(a|x) - \pi_t(a|x)\big) Q_t(x, a)$$

LOCAL-TO-GLOBAL REGRET DECOMPOSITION

Apply with $r = r_t$, $\pi = \pi_t$ and $\pi' = \pi^*$:

Q-function of π_t with reward function r_t

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Stationarized regret can be written as:

$$\sum_{t=1}^{T} \langle \mu^* - \mu_t, r \rangle = \sum_{t=1}^{T} \sum_x \mu^*(x) \sum_a \big(\pi^*(a|x) - \pi_t(a|x)\big) Q_t(x,a)$$

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LOCAL-TO-GLOBAL REGRET DECOMPOSITION

Apply with $r = r_t$, $\pi = \pi_t$ and $\pi' = \pi^*$:

Q-function of π_t with reward function r_t

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Algorithm idea:
run a local regret-minimization algorithm in each state x with reward function $Q_t(x,\cdot)$!

THE MDP-EXPERT ALGORITHM

MDP-E

For each round $t = 1,2,\ldots,T$

- Observe state X_t
- Take action $A_t \sim \pi_t(\cdot\,|X_t)$
- Observe reward function r_t
- Calculate value functions as solution to
  $Q_t(x,a) = r_t - \langle \mu_t, r_t \rangle + \sum_{x'} P(x'|x,a) V_t(x')$
- For all x, feed $Q_t(x,\cdot)$ to expert algorithm $\mathfrak{Alg}(x)$

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- For all x, feed $Q_t(x,\cdot)$ to expert algorithm $\mathfrak{Alg}(x)$
- **Example:** $\mathfrak{Alg}$ = Exponential weights

$$\pi_{t+1}(a|x) \propto \pi_t(a|x) \cdot e^{\eta Q_t(x,a)}$$

GUARANTEES FOR MDP-E

Theorem

(Even-Dar et al., 2009, Neu et al., 2014)

If $\mathfrak{Alg}(x)$ guarantees a regret bound of B_T for rewards bounded in $[0,1]$, the stationarized regret of MDP-E satisfies

$$\sum_{t=1}^{T} \langle \mu^* - \mu_t, r \rangle \leq \tau B_T$$

Proof is obvious given the regret decomposition.

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Theorem

If $\mathfrak{Alg}(x)$=EWA, the regret of MDP-E satisfies

$$\mathfrak{Reg}_T = O\left(\sqrt{\tau^3 T \log|\mathcal{A}|}\right)$$

Proof is obvious given the regret decomposition.

BANDIT FEEDBACK

Addressed in Neu, György, Szepesvári and Antos (2010,2014): replace r_t by an unbiased estimator

$$\hat{r}_t(x,a) = \frac{r_t(x,a)}{\mu_t^N(x,a)} \mathbb{I}\{(X_t, A_t) = (x,a)\},$$

with $\mu_t^N(x,a) = \mathbb{P}[(X_t, A_t) = (x,a) | \mathcal{H}_{t-N}]$

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Theorem

If $\mathfrak{Alg}(x)$=EWA, the regret of MDP-Exp3 satisfies

$$\mathfrak{Reg}_T = O\left(\sqrt{\tau^3 T |\mathcal{A}| \log|\mathcal{A}| / \beta}\right)$$

Assumption: $\mu_\pi(x) \geq \beta$ for all π, x

ALGORITHMS FOR MDPS WITH ADVERSARIAL REWARDS

Local-to-global regret decomposition	Reduction to online linear optimization

ONLINE LINEAR OPTIMIZATION

Notice: stationarized regret = regret in an OLO problem!

$$\sum_{t=1}^{T} \langle \mu^* - \mu_t, r_t \rangle$$

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Algorithm idea:

run an OLO algorithm with the set of all stationary distributions as decision set!

$$\mathcal{U} = \left\{ \mu \in \Delta_{\mathcal{X}\times\mathcal{A}} : \sum_{a} \mu(x,a) = \sum_{x',a'} P(x|x',a')\mu(x',a') \right\}$$

ONLINE MIRROR DESCENT

- In each round, update stationary distribution

$$\mu_{t+1} = \arg\max_{\mu \in \mathcal{U}} \left\{ \langle \mu, r_t \rangle - \frac{1}{\eta} D(\mu | \mu_t) \right\}$$

and extract policy $\pi_{t+1}(a|x) \propto \mu_{t+1}(x, a)$

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- Choosing the regularizer:
  - Relative entropy: $D(\mu|\nu) = \sum_{x,a} \mu(x,a) \log \frac{\mu(x,a)}{\nu(x,a)}$

    ⇒ "Online Relative Entropy Policy Search" (Zimin and Neu, 2013, Dick, György and Szepesvári, 2014)
  - Conditional relative entropy: $D(\mu|\nu) = \sum_{x,a} \mu(x,a) \log \frac{\pi_\mu(a|x)}{\pi_\nu(a|x)}$

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Algorithm inspired by Peters, Mülling and Altün (2010)

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Unconstrained convex minimization

- Update stationary distribution as

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Algorithm inspired by Peters, Mülling and Altün (2010)

GUARANTEES FOR O-REPS

Theorem

(Zimin and Neu, 2013, Dick et al. 2014)

The stationarized regret of O-REPS satisfies

$$\sum_{t=1}^{T} \langle \mu^* - \mu_t, r \rangle \leq \sqrt{T \log |\mathcal{X}| |\mathcal{A}|}$$

Theorem

The regret of O-REPS satisfies

$$\mathfrak{Reg}_T = O\left(\sqrt{\tau T \log |\mathcal{X}| |\mathcal{A}|}\right)$$

Proof is based on standard OLO analysis.

BANDIT FEEDBACK

Addressed in Zimin and Neu (2013) in **episodic MDPs**:
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$$\hat{r}_t(x,a) = \frac{r_t(x,a)}{q_t(x,a)} \mathbb{I}\{(x,a) \text{ visited in episode } t\},$$

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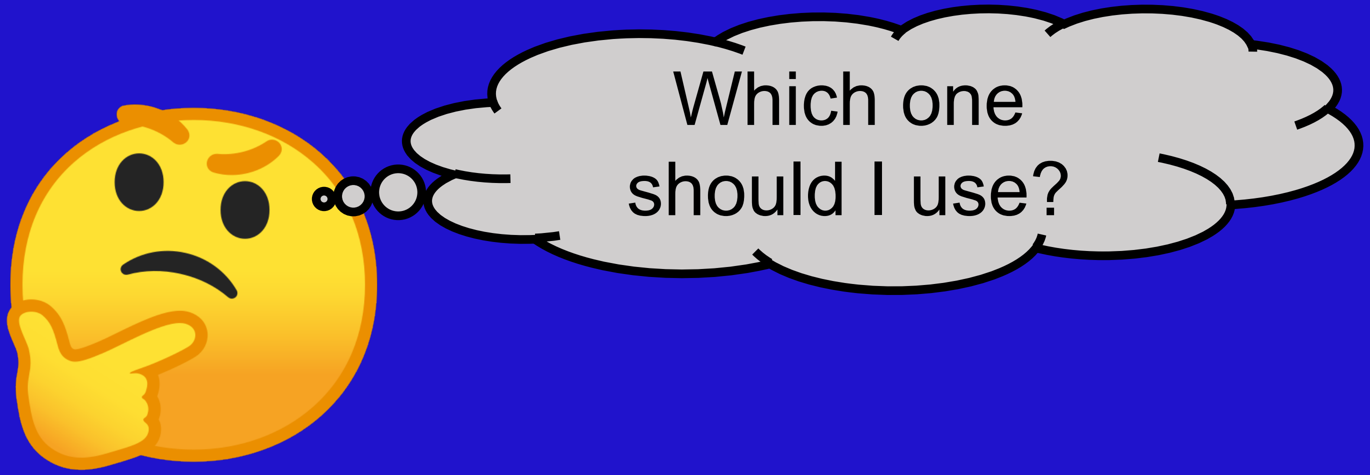
Local-to-global regret decomposition	Reduction to online linear optimization



ALGORITHMS FOR MDPS WITH ADVERSARIAL REWARDS

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COMPARISON OF GUARANTEES

	MDP-E	O-REPS
Full information	$\sqrt{\tau^3 T \log\lvert\mathcal{A}\rvert}$	$\sqrt{\tau T \log\lvert\mathcal{X}\rvert\lvert\mathcal{A}\rvert}$
Bandit feedback	$\sqrt{\tau^3 \lvert\mathcal{A}\rvert T \log\lvert\mathcal{A}\rvert / \beta}$	???
Full information (episodic case)	$H^2\sqrt{T \log\lvert\mathcal{A}\rvert}$	$H\sqrt{T \log\lvert\mathcal{X}\rvert\lvert\mathcal{A}\rvert}$
Bandit feedback (episodic case)	$H^2\sqrt{\lvert\mathcal{A}\rvert T \log\lvert\mathcal{A}\rvert / \beta}$	$\sqrt{H\lvert\mathcal{X}\rvert\lvert\mathcal{A}\rvert T \log\lvert\mathcal{X}\rvert\lvert\mathcal{A}\rvert}$

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Bandit feedback (episodic case)	$H^2\sqrt{\lvert\mathcal{A}\rvert T \log\lvert\mathcal{A}\rvert / \beta}$	$\sqrt{H\lvert\mathcal{X}\rvert\lvert\mathcal{A}\rvert T \log\lvert\mathcal{X}\rvert\lvert\mathcal{A}\rvert}$

+ MDP-E works well with function approximation for Q-function

+ O-REPS can easily handle model constraints and extensions

MDP-E WITH FUNCTION APPROXIMATION

MDP-E only needs a good approximation of the action-value function $\hat{Q}_t \approx Q^{\pi_t}$ to define its policy

$$\pi_{t+1}(a|x) \propto \exp\left(\eta \sum_{k=1}^{t} \hat{Q}_k(x,a)\right)$$

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- POLITEX (Abbasi-Yadkori et al., 2019):
  use LSPE to estimate Q^{π_t} with linear FA
  regret = $O(T^{3/4} + \varepsilon_0 T)$
- OPPO (Cai et al., 2019)
  use LSPE to estimate Q^{π_t} with **realizable** linear FA
  regret = $O\left(\sqrt{T}\right)$

MDP-E WITH FUNCTION APPROXIMATION

MDP-E only needs a good approximation of the action-value function $\hat{Q}_t \approx Q^{\pi_t}$ to define its policy

$$\pi_{t+1}(a|x) \propto \exp\left(\eta \sum_{k=1}^{t} \hat{Q}_k(x,a)\right)$$

+ MDP-E is essentially identical to the "Trust-Region Policy Optimization" (TRPO) algorithm of Schulman et al. (2015), as shown by Neu, Jonsson and Gómez (2017)!!!

O-REPS WITH UNCERTAIN MODELS

O-REPS can easily accommodate uncertainties in the transition model by extending the decision set:

$$\mathcal{U} = \left\{ \mu \in \Delta_{\mathcal{X}\times\mathcal{A}} : \sum_{a} \mu(x,a) = \sum_{x',a'} P(x|x',a')\mu(x',a') \right\}$$

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Confidence set of transition models

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Confidence set of transition models

UC-O-REPS by Rosenberg and Mansour (2019)

Extended to bandit feedback by Jin et al. (2020):

$$\hat{r}_t(x,a) = \frac{r_t(x,a)}{u_t(x,a)} \mathbb{I}\{(x,a) \text{ visited in episode } t\},$$

with $u_t(x,a) > q_t(x,a) = \mathbb{P}[(x,a) \text{ visited in episode } t | \mathcal{H}_{t-1}]$ w.h.p.

OUTLOOK

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- Open problems:
  - Lower bounds? Right scaling with τ? Is uniform mixing necessary?
  - Large state spaces and function approximation?
  - Practical algorithms?

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Relevance to practice of RL?

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  - Lower bounds? Right scaling with τ? Is uniform mixing necessary?
  - Large state spaces and function approximation?
  - Practical algorithms?

Relevance to practice of RL?

- Online learning algorithms are robust! Main tool: regularization
- Better understanding of regularization tools ⇒ better algorithms!
- Remember: TRPO = MDP-E!

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